Einstein metrics on compact simple Lie groups

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based on joint works with
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- Known results on Non-naturally reductive Einstein metrics on compact Lie groups
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- A discussion on Compact Lie groups $\mathrm{SO}(n)$ $(n \geq 7)$, $\mathrm{Sp}(n)$ $(n \geq 3)$
- By using generalized flag manifolds, compact Lie groups $\mathrm{SO}(n)$, $\mathrm{Sp}(n)$, G_2, F_4, E_6, E_7, E_8

Introduction

(M, g): Riemannian manifold

- (M, g) is called Einstein if the Ricci tensor $\mathrm{r}(g)$ of the metirc g satisfies $\underline{\mathrm{r}(g) = c\, g}$ for some constant c.

  We consider G-invariant Einstein metrics on a <u>homogeneous space G/H</u>.

- <u>General Problem:</u> Find G-invariant Einstein metrics on a homogeneous space G/H and classify them if it is not unique.

- Einstein homogeneous spaces can be diveded into three cases depending on Einstein constant c.
  Here we consider the case $c > 0$.

Examples of homogeneous Einstein manifolds
(we see that G/H is compact and $\pi_1(G/H)$ is finite)

- Sphere $(S^n = \mathrm{SO}(n+1)/\,\mathrm{SO}(n), g_0)$,
- Complex Projective space
  $(\mathbb{C}P^n = \mathrm{SU}(n+1)/(\mathrm{S}(\mathrm{U}(1) \times \mathrm{U(n)}), g_0)$,
- Symmetric spaces of compact type,
  isotropy irreducible spaces (in these cases G-invariant Einstein metrics is unique up to a constant multiple)
- In particular, compact semi-simple Lie group with a bi-invariant metric
- Generalized flag manifolds (Kähler C-spaces) (which admit Kähler Einstein metrics)
  (G-invariant Einstein metrics may not be unique as real manifold.)

Introduction (non-existence and existence)

- (Wang-Ziller 1986) There exist compact homogeneous space G/H with no G-invariant Einstein metrics.
- Let $G = \mathrm{SU}(4)$, $K = \mathrm{Sp}(2)$, $H = \mathrm{SU}(2)$ ($\mathrm{SU}(2)$ is a maxmal subgroup of $\mathrm{Sp}(2)$).
  Then G/H has no (G-)invariant Einstein metrics. Note that $\dim G/H = 12$.
- How about the case that $\dim G/H < 12$?
- (Böhm-Kerr (2006)) For a simply connected compact homogeneous space G/H of $\dim G/H \leq 11$, there exists always a G-invariant Einstein metric on G/H.

- Known results on small dimensions

  (Nikonorov, Rodionov (2003)) For a simply connected compact homogeneous space G/H of $\dim G/H \leq 7$, all G-invariant Einstein metrics has been determined on G/H, except $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

- (Wang-Ziller (1990))
  Homogeneous space $(\mathrm{SU}(2) \times \mathrm{SU}(2))/S^1$ has an Einstein metric. These spaces are the principal S^1-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$ which are all diffeomorphic to $S^2 \times S^3$, but as homogeneous spaces $(\mathrm{SU}(2) \times \mathrm{SU}(2))/S^1$, they are quite different. In fact, Wang and Ziller have shown that the moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many components by using these Einstein metrics.

- A compact simple Lie group with a bi-invariant metric (for example given by negative of Killing form) is Einstein.

- In 1979, D'Atri and Ziller obtained a great amount of Einstein metrics on a compact Lie group G, which are naturally reductive.

- Open problem: Find all left-invariant Einstein metrics on a compact simple Lie group G.
  How many are there? (finite or infinite)

- Even for $G = \mathrm{SU}(3)$, or $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, we do not know all left-invariant Einstein metrics on G. (finite or infinite)

- It is known that there exist at least two left invariant Einstein metrics on $\mathrm{SU}(2) \times \mathrm{SU}(2)$. One of these metrics is standard, the second metric ρ_J was found by G. Jensen.

- Nikonorov and Rodionov (2003) has computed the scalar curvature of left invariant metrics on $\mathrm{SU}(2) \times \mathrm{SU}(2)$. There is 15-parameters for the metrics and it seems to be difficult to obtain other critical points (Einstein metrics).

- Theorem (Nikonorov and Rodionov (2003)).
  Let g be a left-invariant Einstein metric on the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ which is Ad(S^1)-invariant with respect to a certain embedding $S^1 \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$. Then the metric g is isometric (up to a homothety) to one of the metrics above.

- (M,g) : a compact Riemannian manifold
  $I(M,g)$: the Lie group of all isometries of M (compact)

  A Riemannian manifold (M,g) is K-homogeneous if a closed subgroup K of $I(M,g)$ acts transitively on M.

  For a K-homogeneous Riemannian manifold (M,g), we write $M = K/L$, where L is the isotropy subgroup of K at a point o.

- $\mathfrak{k}$: the Lie algebra of K
  $\mathfrak{l}$: the subalgebra corresponding to L
  $\mathfrak{p}$: a complement subspace of $\mathfrak{k}$ to $\mathfrak{l}$ with $\mathrm{Ad}(L)\mathfrak{p} \subset \mathfrak{p}$
  $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$

  Pull back the inner product g_o on $T_o(M)$ to an inner product on $\mathfrak{p}$, denoted by $< \ , \ >$.
  Note that $< \ , \ >$ is an $\mathrm{Ad}(L)$-invariant inner product on $\mathfrak{p}$

- For $X \in \mathfrak{k}$, we will denote by $X_{\mathfrak{l}}$ (resp. $X_{\mathfrak{p}}$) the $\mathfrak{l}$-component (resp. $\mathfrak{p}$-component) of X.
- A homogeneous Riemannian metric on M is said to be naturally reductive with respect to K, if there exist K and $\mathfrak{p}$ as above such that

$$< [Z, X]_{\mathfrak{p}}, Y > + < X, [Z, Y]_{\mathfrak{p}} >= 0 \quad \text{for} \quad X, Y, Z \in \mathfrak{p}.$$

That is, when we write the Riemannain connection ∇ as, for $X, Y \in \mathfrak{p}$,

$$\nabla_X Y = \frac{1}{2}[X, Y]_{\mathfrak{p}} - U(X, Y),$$

$U(X, Y) = 0$ for any $X, Y \in \mathfrak{p}$, where $U(X, Y)$ is defined by

$$< U(X, Y), Z >= \frac{1}{2}(< [Z, X]_{\mathfrak{p}}, Y > + < X, [Z, Y]_{\mathfrak{p}} >)$$

.

Naturally reductive metrics on a compact Lie group

- D'Atri and Ziller (Memoirs Amer. Math. Soc. 19 (215) (1979)) investigated naturally reductive metrics among the left invariant metrics on compact Lie groups and obtained a complete classification of the metrics in the case of simple Lie groups.
- For a compact semi-simple Lie group G and a closed subgroup H, the group $G \times H$ acts transitively on G by

$$(g, h)y = gyh^{-1} \quad ((g, h) \in G \times H, \ y \in G)$$

  and the Lie group G can be expressed as $(G \times H)/\Delta H$, where $\Delta H = \{(h, h) \mid h \in H\}$.
- Note that the Killing form of a compact semi-simple Lie algebra $\mathfrak{g}$ is negative definite. We set $B = -$ Killing form. Then B is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$.

Naturally reductive metrics on a compact Lie group

- Let $\mathfrak{m}$ be an orthogonal complement of $\mathfrak{h}$ (the Lie algebra of the Lie subgroup H) in $\mathfrak{g}$ with respect to B. Then we have

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}.$$

- Let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$ be the decomposition into ideals of $\mathfrak{h}$, where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\mathfrak{h}_i$ $(i = 1, \cdots, p)$ are simple ideals of $\mathfrak{h}$. Let $A_0|_{\mathfrak{h}_0}$ be an arbitrary metric on $\mathfrak{h}_0$.

Theorem

(D'Atri-Ziller 1979) *Under the notations above, a left invariant metric $<\ ,\ >$ on G of the form*

$$<\ ,\ >= x \cdot B|_{\mathfrak{m}} + A_0|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + \cdots + u_p \cdot B|_{\mathfrak{h}_p} \qquad (x, u_1, \cdots, u_p \in \mathbb{R}_+) \tag{1}$$

is naturally reductive with respect to $G \times H$.
Note that $(G = (G \times H)/\Delta H)$.

Conversely, if a left invariant metric $<\ ,\ >$ on a compact simple Lie group G is naturally reductive, then there exists a closed subgroup H of G and the metric $<\ ,\ >$ is given of the form (1).

Naturally reductive Einstein metrics on a compact Lie group

- D'Atri and Ziller (1979) studied **naturally reductive** Einstein metrics on a compact simple Lie group G in the case which $\mathrm{Ad}(H)$ acts on $\mathfrak{m}$ irreducibly mainly, that is, the left invariant metric determined by
  an irreducible symmetric space of compact type and isotropy irreducible spaces..
- In particular, D'Atri and Ziller found at least the following number of left invariant Einstein metrics:
- $n+1$ on $\mathrm{SU}(2n+2)$, $\mathrm{SU}(2n+3)$, $\mathrm{Sp}(2n)$, $\mathrm{Sp}(2n+1)$,
- $3n-2$ on $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$,
- for exceptional Lie groups, 5 on G_2, 10 on F_4, 14 on E_6, 15 on E_7, 11 on E_8.

- D'Atri and Ziller (1979) asked a following question:

Is there non-naturally reductive left invariant Einstein metrics on a compact Lie group?

Known results on Non-naturally reductive Einstein metrics on compact Lie groups

- **Theorem**[K. Mori, 1996]
  On a compact Lie group $\mathrm{SU}(n)$ $(n \geq 6)$, there exist non-naturally reductive Einstein metrics. (preprint) (Generalized flag manifolds and/or Generalized Wallach spaces)
  For this case, the space of the metrics has been studied from $\mathrm{SU}(2+2+m)/\,\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(m))$ $(m \geq 2)$.
- **Theorem**[Arvanitoyeorgos, Mori and S., 2008] (Geom. Dedicata)
  On a compact simple Lie group G, either $\mathrm{SO}(n)$ $(n \geq 11)$, $\mathrm{Sp}(n)$ ($n \geq 3$), E_6, E_7 or E_8, there exist non-naturally reductive Einstein Einstein metrics. (Generalized flag manifolds)
  For this case, the space of the metrics has been studied from the Generalized flag manifolds G/H with two irreducible summands.

Non-naturally reductive Einstein metrics on compact Lie groups

- **Theorem**[Chen and Liang, 2014] (Ann. Glob. Anal. Geom.)
  On the compact Lie group F_4 there exists a non-naturally reductive Einstein Einstein metric.
  (Generalized Wallach spaces) For this case, the space of the metrics has been studied from the Generalized Wallach space $F_4/\operatorname{SO}(8)$ with fiber $\operatorname{SO}(9)/\operatorname{SO}(8)$
- **Theorem**[Arvanitoyeorgos, S. and Statha]
  (to appear in Geometry, Imaging and Computing vol. 2.2)
  The compact simple Lie groups $\operatorname{SO}(n)$ $(n \geq 7)$ admit left-invariant Einstein metrics which are not naturally reductive.
  (Generalized Wallach spaces)
  For this case, the space of the metrics has been studied from the Generalized Wallach space
  $\operatorname{SO}(3+3+n-6)/\operatorname{SO}(3)\times\operatorname{SO}(3)\times\operatorname{SO}(n-6)$.

- **Theorem**[Arvanitoyeorgos, S. and Statha]
  The compact simple Lie groups $\mathrm{Sp}(n)$ $(n \geq 3)$ admit left-invariant Einstein metrics which are not naturally reductive.
  Current Developments in Differential Geometry and its Related Fields, (Proceedings), 2015.
  (Generalized Wallach spaces) For this case, the space of the metrics has been studied from the Generalized Wallach space $\mathrm{Sp}(n)/\,\mathrm{Sp}(n-2) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$.
- **Theorem**[Chrysikos and S.] (arXiv:1511.03993)
  The compact simple Lie groups G_2, F_4, E_6, E_7 and E_8 admit left-invariant Einstein metrics which are not naturally reductive.
  (Generalized flag manifolds) For this case, the space of the metrics has been studied from the Generalized flag manifolds G/H with the second Betti number $b_2(G/H) = 1$ and three irreducible summands.

- **Theorem**[Arvanitoyeorgos, S. and Statha]
  On a compact Lie group $\mathrm{SU}(n+3)$ $(n \geq 2)$, there exist non-naturally reductive Einstein metrics which are different from K. Mori's results.

  The space of the metrics has been studied from $\mathrm{SU}(1+2+n)/\,\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(n))$ for $n \geq 2$.
  (Generalized flag manifolds and/or Generalized Wallach spaces)
  Note that in this case The Lie algebra of the group $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(n))$ has two dimensional center.

- **Theorem**[Arvanitoyeorgos, S. and Statha]
  On a compact Lie group $\mathrm{SU}(n+3)$ $(n \geq 5)$, there exist non-naturally reductive Einstein metrics which are different from K. Mori's results.

  The space of the metrics has been studied from $\mathrm{SU}(3+n)/(\mathrm{U}(1)\,\mathrm{SO}(3)\,\mathrm{SU}(n))$ for $n \geq 5$, where the Lie subgroup $\mathrm{SO}(3)$ is a natural subgroup of $\mathrm{SU}(3)$ and $\mathrm{SU}(3+n)/(\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(n))$ is a complex Grassmann manifold.

- Suppose that a homogeneous space G/H has the following property: the modules $\mathfrak{p}$ is decomposed as a direct sum of three $\mathrm{Ad}(H)$-invariant irreducible modules pairwise orthogonal with respect to B (negative of Killing form), that is,

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$$

such that

$$[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h} \quad \text{for } \; i \in \{1, 2, 3\}.$$

Homogeneous spaces with this property are called **generalized Wallach spaces**.

- Note that the inclusion $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ implies that $\mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{p}_i$ is a subalgebra of $\mathfrak{g}$ for any i, and the pair $(\mathfrak{k}_i, \mathfrak{h})$ is irreducible symmetric (it could be non-effective). We also see that

$$[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_k$$

for distinct i, j, k. Therefore,

$$[\mathfrak{p}_i \oplus \mathfrak{p}_k, \mathfrak{p}_j \oplus \mathfrak{p}_k] \subset \mathfrak{h} \oplus \mathfrak{p}_i, \quad \{i, j, k\} = \{1, 2, 3\},$$

and all the pairs $(\mathfrak{g}_i, \mathfrak{k}_i)$ are also irreducible symmetric.

Examples of generalized Wallach spaces

- Wallach spaces:

$$\mathrm{SU}(3)/T^2,\ \mathrm{Sp}(3)/(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)),\ F_4/\,\mathrm{Spin}(8).$$

  These spaces are also interesting in that they admit invariant Riemannian metrics of positive sectional curvature.

- Other examples of generalized Wallach spaces are some generalized flag manifolds such as

$$\mathrm{SU}(n_1+n_2+n_3)/S\,(\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\mathrm{U}(n_3)),$$

$$\mathrm{SO}(2n)/(\mathrm{U}(1)\times\mathrm{U}(n-1)),\quad E_6/(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{Spin}(8))$$

  There are two more 3-parameter families of generalized Wallach spaces:

$$\mathrm{SO}(n_1+n_2+n_3)/(\mathrm{SO}(n_1)\times\mathrm{SO}(n_2)\times\mathrm{SO}(n_3)),$$

$$\mathrm{Sp}(n_1+n_2+n_3)/(\mathrm{Sp}(n_1)\times\mathrm{Sp}(n_2)\times\mathrm{Sp}(n_3))$$

- Recently, Yu. Nikonorov has classified all generalized Wallach spaces for compact simple Lie groups. (to appear in Geometriae Dedicata, DOI 10.1007/s10711-015-0119-z and/or in ArXiv: 1411.3131v1 12 Nov 2014)

  There are 15 cases with 5 series for classical groups and 10 exceptional Lie groups.

Summary

- Now we want to summarize the results for left-invariant non-naturally reductive Einstein metrics on compact simple Lie groups.
- For $\mathrm{SU}(n)$ $(n \geq 5)$, $\mathrm{SO}(n)$ ($n \geq 7$), $\mathrm{Sp}(n)$ ($n \geq 3$), E_6, E_7, E_8, F_4 and G_2, there exist non-naturally reductive Einstein metrics. (Recently we obtained more non-naturally reductive Einstein metrics on $\mathrm{SU}(n)$ $(n \geq 5$) which are different from K. Mori's results. In particular, the case for $\mathrm{SU}(5)$ is the first example.)
- For the cases of $\mathrm{SU}(n)$ $(n = 3, 4)$, we still do not know whether there exist non-naturally reductive Einstein metrics or not.
- Also $\mathrm{SO}(5) = \mathrm{Sp}(2)$ (locally) and $\mathrm{SO}(6) = \mathrm{SU}(4)$ (locally) are still open.

- In the following, we assume that $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ is a decomposition into irreducible Ad(H)-modules $\mathfrak{m}_j$ $(j = 1, \cdots, q)$ and that Ad(H)-modules $\mathfrak{m}_j$ are **mutually non-equivalent** and $\underline{\mathbf{dim}\, \mathfrak{h}_0 \leq 1}$.
- We consider the following left invariant metric on G which is Ad(H)-invariant:

$$< \, , \, >= u_0 B|_{\mathfrak{h}_0} + u_1 B|_{\mathfrak{h}_1} + \cdots + u_p B|_{\mathfrak{h}_p} + x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} \quad (2)$$

where $u_0, u_1, \cdots, u_p, x_1, \cdots, x_q \in \mathbb{R}_+$, and the G-invariant Riemannian metric on G/H:

$$(\, , \,) = x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}. \quad (3)$$

- Note that left invariant symmetric covariant 2-tensors on G which are Ad(H)-invariant are the same form as the metrics, and this is also true for G-invariant symmetric covariant 2-tensors on G/H.

- In particular, the Ricci tensor r of a left invariant Riemannian metric $<\ ,\ >$ on G is a left invariant symmetric covariant 2-tensor on G which is Ad(H)-invariant and thus r is of the same form as (2), and Ricci tensor $\bar{r}$ of a G-invariant Riemannian metric on G/H is of the same form as (3).

- For simplicity, we write the decomposition $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ (resp. $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$) as $\mathfrak{g} = \mathfrak{w}_0 \oplus \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_p \oplus \mathfrak{w}_{p+1} \oplus \cdots \oplus \mathfrak{w}_{p+q}$ (resp. $\mathfrak{m} = \mathfrak{w}_{p+1} \oplus \cdots \oplus \mathfrak{w}_{p+q}$).

Ricci tensor of compact Lie groups

- Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{g}$, i.e., $e_\alpha \in \mathfrak{w}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{w}_i$ and $e_\beta \in \mathfrak{w}_j$).
- We put $A^\gamma_{\alpha\beta} = B(\left[e_\alpha, e_\beta\right], e_\gamma)$, so that $\left[e_\alpha, e_\beta\right] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{w}_i,\ e_\beta \in \mathfrak{w}_j,\ e_\gamma \in \mathfrak{w}_k$. Then, $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the B-orthonormal bases chosen for $\mathfrak{w}_i, \mathfrak{w}_j, \mathfrak{w}_k$, and we have

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}. \tag{4}$$

Structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ are introduced by Ziller and Wang .

- For simplicity, we now write a metric of the form (2) on a compact Lie group G as follows:

$$g = y_0 \cdot B|_{\mathfrak{w}_0} + y_1 \cdot B|_{\mathfrak{w}_1} + \cdots + y_p \cdot B|_{\mathfrak{w}_p} + y_{p+1} \cdot B|_{\mathfrak{w}_{p+1}} + \cdots + y_{p+q} \cdot B|_{\mathfrak{w}_{p+q}} \tag{5}$$

and a metric of the form (3) on a compact space G/H as follows:

$$h = w_{p+1} \cdot B|_{\mathfrak{w}_{p+1}} + \cdots + w_{p+q} \cdot B|_{\mathfrak{w}_{p+q}} \tag{6}$$

- Note that the metric of the form (5) is naturally reductive on a compact simple Lie group G with respect to $G \times H$ if and only if $y_{p+1} = \cdots = y_{p+q}$.

Lemma

Let $d_k = \dim \mathfrak{w}_k$*.*
(i) *The components* $r_0, r_1, \cdots, r_{p+q}$ *of Ricci tensor* r *of the metric* g *of the form* (2) *on* G *are given by*

$$r_k = \frac{1}{2y_k} + \frac{1}{4d_k} \sum_{j,i} \frac{y_k}{y_j y_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{y_j}{y_k y_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 0, 1, \cdots, p+q),$$

where the sum is taken over $i, j = 0, 1, \cdots, p+q$*. Moreover, for each* k*, we have* $\sum_{i,j} \begin{bmatrix} j \\ ki \end{bmatrix} = d_k$*.*

- We consider the homogeneous space $G/K = \mathrm{SO}(k_1 + k_2 + k_3)/\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)$, where the embedding of K in G is diagonal. The tangent space $\mathfrak{m}$ of G/K decomposes into three $\mathrm{Ad}(K)$-submodules

$$\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23},$$

where

$$\mathfrak{m}_{12} = \begin{pmatrix} 0 & A_{12} & 0 \\ -{}^tA_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{m}_{13} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ -{}^tA_{13} & 0 & 0 \end{pmatrix}, \mathfrak{m}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & -{}^tA_{23} & 0 \end{pmatrix}$$

and $A_{12} \in M(k_1, k_2), A_{13} \in M(k_1, k_3), A_{23} \in M(k_2, k_3)$ ($M(p, q)$ the set of all $p \times q$ matrices). Note that the irreducible $\mathrm{Ad}(K)$-submodules $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$ and $\mathfrak{m}_{23}$ are mutually non-equivalent.

- For the tangent space $\mathfrak{so}(k_1 + k_2 + k_3)$ of the Lie group $G = \mathrm{SO}(k_1 + k_2 + k_3)$, we consider the decomposition

$$\mathfrak{so}(k_1 + k_2 + k_3) = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3) \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}, \quad (7)$$

where corresponding $\mathrm{Ad}(K)$-submodules are non-equivalent. We write the decomposition (7) as

$$\mathfrak{so}(k_1 + k_2 + k_3) = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}. \quad (8)$$

By taking into account the diffeomorphism

$$G/\{e\} \cong G \times (\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))/\Delta(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3)),$$

left-invariant metrics on G that are also $\mathrm{Ad}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3))$-invariant are given by

$$\begin{aligned} \langle\ ,\ \rangle &= x_1\, B|_{\mathfrak{m}_1} + x_2\, B|_{\mathfrak{m}_2} + x_3\, B|_{\mathfrak{m}_3} \\ &\quad + x_{12}\, B|_{\mathfrak{m}_{12}} + x_{13}\, B|_{\mathfrak{m}_{13}} + x_{23}\, B|_{\mathfrak{m}_{23}} \end{aligned} \quad (9)$$

for $k_1 \geq 2$, $k_2 \geq 2$ and $k_3 \geq 2$.

Now we obtain the following relations:

$$[\mathfrak{m}_1, \mathfrak{m}_1] = \mathfrak{m}_1, \quad [\mathfrak{m}_2, \mathfrak{m}_2] = \mathfrak{m}_2, \quad [\mathfrak{m}_3, \mathfrak{m}_3] = \mathfrak{m}_3,$$

$$[\mathfrak{m}_1, \mathfrak{m}_{12}] = \mathfrak{m}_{12}, \quad [\mathfrak{m}_1, \mathfrak{m}_{13}] = \mathfrak{m}_{13}, \quad [\mathfrak{m}_2, \mathfrak{m}_{12}] = \mathfrak{m}_{12},$$

$$[\mathfrak{m}_2, \mathfrak{m}_{23}] = \mathfrak{m}_{23}, \quad [\mathfrak{m}_3, \mathfrak{m}_{13}] = \mathfrak{m}_{13}, \quad [\mathfrak{m}_3, \mathfrak{m}_{23}] = \mathfrak{m}_{23},$$

$$[\mathfrak{m}_{12}, \mathfrak{m}_{12}] = \mathfrak{m}_1 + \mathfrak{m}_2, \quad [\mathfrak{m}_{13}, \mathfrak{m}_{13}] = \mathfrak{m}_1 + \mathfrak{m}_3, \quad [\mathfrak{m}_{23}, \mathfrak{m}_{23}] = \mathfrak{m}_2 + \mathfrak{m}_3,$$

$$[\mathfrak{m}_{12}, \mathfrak{m}_{23}] = \mathfrak{m}_{13}, \quad [\mathfrak{m}_{13}, \mathfrak{m}_{23}] = \mathfrak{m}_{12}, \quad [\mathfrak{m}_{12}, \mathfrak{m}_{13}] = \mathfrak{m}_{23}.$$

Thus we see that the only non-zero symbols (up to permutation of indices) are

$$\begin{bmatrix} 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 22 \end{bmatrix}, \begin{bmatrix} 3 \\ 33 \end{bmatrix}, \begin{bmatrix} (12) \\ 1(12) \end{bmatrix}, \begin{bmatrix} (13) \\ 1(13) \end{bmatrix}, \begin{bmatrix} (12) \\ 2(12) \end{bmatrix}, \begin{bmatrix} (23) \\ 2(23) \end{bmatrix}, \begin{bmatrix} (13) \\ 3(13) \end{bmatrix}, \begin{bmatrix} (23) \\ 3(23) \end{bmatrix}, \begin{bmatrix} (13) \\ (12)(23) \end{bmatrix},$$

where $\begin{bmatrix} i \\ ii \end{bmatrix}$ is non-zero only for $k_i \geq 3$ $(i = 1, 2, 3)$. (because of $\mathfrak{so}(k_i)$)

Now we have the following (due to A. Arvanitoyeorgos, V.V. Dzhepko and Yu. G. Nikonorov):

Lemma

For $a, b, c = 1, 2, 3$ and $(a-b)(b-c)(c-a) \neq 0$ the following relations hold:

$$\begin{bmatrix} a \\ aa \end{bmatrix} = \frac{k_a(k_a-1)(k_a-2)}{2(n-2)},$$

$$\begin{bmatrix} a \\ (ab)(ab) \end{bmatrix} = \frac{k_a k_b(k_a-1)}{2(n-2)},$$

$$\begin{bmatrix} (ac) \\ (ab)(bc) \end{bmatrix} = \frac{k_a k_b k_c}{2(n-2)}.$$

Lemma

The components of the Ricci tensor r for the left-invariant metric $\langle\ ,\ \rangle$ on G defined by (9), are given as follows

$$
\begin{aligned}
r_1 &= \frac{k_1 - 2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2 \frac{x_1}{{x_{12}}^2} + k_3 \frac{x_1}{{x_{13}}^2}\right),\\
r_2 &= \frac{k_2 - 2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1 \frac{x_2}{{x_{12}}^2} + k_3 \frac{x_2}{{x_{23}}^2}\right),\\
r_3 &= \frac{k_3 - 2}{4(n-2)x_3} + \frac{1}{4(n-2)}\left(k_1 \frac{x_3}{{x_{13}}^2} + k_2 \frac{x_3}{{x_{23}}^2}\right),
\end{aligned}
$$

Lemma

$$
\begin{aligned}
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{12}}^2} + (k_2-1)\frac{x_2}{{x_{12}}^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{13}}^2} + (k_3-1)\frac{x_3}{{x_{13}}^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) \\
&\quad - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2} + (k_3-1)\frac{x_3}{{x_{23}}^2}\right).
\end{aligned}
$$

We consider the system of equations

$$r_1 = r_2, \quad r_2 = r_3, \quad r_3 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23}. \tag{10}$$

Then finding Einstein metrics of the form (9) reduces to finding positive solutions of system (10).

But, in general, it would be difficult to solve the system of equations in general.

If we put $k_1 = k_2 = 2, k_3 = n - 4$, we can not get any non-naturally reductive Einstein metrics as far as we checked several cases for n.

So we put $k_1 = k_2 = 3, k_3 = n - 6$. But it is still difficult to solve in general.

We put $k_1 = k_2 = 3, k_3 = n - 6$ and consider our equations by putting

$$x_{13} = x_{23} = 1, \quad x_2 = x_1.$$

Then the system of equations (10) reduces to the system of equations:

$$\left.\begin{array}{rcl} g_1 & = & n{x_1}^2{x_{12}}^2x_3 - nx_1{x_{12}}^2 - 6{x_1}^2{x_{12}}^2x_3 + 3{x_1}^2x_3 \\ & & -6x_1{x_{12}}^2{x_3}^2 + 8x_1{x_{12}}^2 + {x_{12}}^2x_3 = 0, \\ g_2 & = & -nx_12^3x_3 + n{x_{12}}^2 + 4x_1x_3 + 6{x_{12}}^3x_3 \\ & & +6{x_{12}}^2{x_3}^2 - 8{x_{12}}^2 - 8x_{12}x_3 = 0, \\ g_3 & = & nx_12^3 + n{x_{12}}^2x_3 - 2n{x_{12}}^2 + 2x_1{x_{12}}^2 - 4x_1 \\ & & -3{x_{12}}^3 - 7{x_{12}}^2x_3 + 4{x_{12}}^2 + 8x_{12} = 0. \end{array}\right\} \qquad (11)$$

Theorem

The compact simple Lie groups $\mathrm{SO}(n)$ $(n \geq 9)$ *admit left-invariant Einstein metrics which are not naturally reductive.*

To show our theorem, we consider a polynomial ring $R = \mathbb{Q}[z, x_3, x_1, x_{12}]$ and an ideal I generated by $\{g_1,\ g_2,\ g_3,\ z(\,x_1 - x_{12})\,x_1\,x_{12}\,x_3 - 1\}$ to find non-zero solutions of equations (11) with $x_1 \neq x_{12}$.

We take a lexicographic order $>$ with $z > x_3 > x_1 > x_{12}$ for a monomial ordering on R. Then, by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomials $\{h(x_{12}), p_1(x_{12}, x_1), p_2(x_{12}, x_3)\}$, where $h(x_{12})$ is a polynomial of x_{12} given by

$$
\begin{aligned}
h(x_{12}) = {} & (n-6)^2(n-3)\left(n^2-7n+24\right){x_{12}}^8 \\
& -2(n-6)^2(n-2)\left(n^2-n+6\right){x_{12}}^7 \\
& +(n-6)\left(n^4+26n^3-269n^2+686n-516\right){x_{12}}^6 \\
& -44(n-6)(n-3)(n-2)(n+2){x_{12}}^5 \\
& +\left(14n^4+273n^3-3034n^2+5687n+1164\right){x_{12}}^4 \\
& -2(n-2)\left(157n^2-157n-2778\right){x_{12}}^3 \\
& +\left(49n^3+1658n^2-6539n+836\right){x_{12}}^2 \\
& -728(n-2)(n+5)x_{12}+2704(n-1),
\end{aligned}
\tag{12}
$$

(Here note that the coefficients of the polynomial $h(x_{12})$ are positive for even degree terms and negative for odd degree terms for $n \geq 9$ and that, if the equation $h(x_{12}) = 0$ has real solutions, then these are all positive.)

$p_1(x_{12}, x_1)$ is a polynomial of x_{12} and x_1 given by

$$
\begin{aligned}
&p_1(x_{12}, x_1) = \\
&8(2n-5)\left(n^2-7n+27\right)x_1 \\
&+(n-6)^3(n-3)\left(n^2-7n+24\right){x_{12}}^7 \\
&-2(n-6)^3(n-2)\left(n^2-n+6\right){x_{12}}^6 \\
&+(n-6)^2\left(n^4+19n^3-199n^2+371n-12\right){x_{12}}^5 \\
&-6(n-6)^2(n-2)\left(5n^2-5n-58\right){x_{12}}^4 \\
&+(n-6)\left(7n^4+140n^3-1641n^2+3090n+1248\right){x_{12}}^3 \\
&-104(n-6)^2(n-2)(n+5){x_{12}}^2 \\
&+8\left(48n^3-625n^2+2305n-1719\right)x_{12}
\end{aligned}
\tag{13}
$$

and $p_2(x_{12}, x_3)$ is a polynomial of x_{12} and x_3 given by

$$
\begin{aligned}
&p_2(x_{12}, x_3) = \\
&-(n-6)^2(n-3)\left(n^2 - 7n + 24\right)\left(2n^2 - 14n + 15\right) {x_{12}}^7 \\
&+2(n-6)^2(n-2)\left(n^2 - n + 6\right)\left(2n^2 - 14n + 15\right) {x_{12}}^6 \\
&-(n-6)\left(2n^6 + 25n^5 - 666n^4 + 3955n^3 - 8860n^2 + 7452n - 2124\right) {x_{12}}^5 \\
&+2(n-6)(n-2)\left(31n^4 - 248n^3 + 127n^2 + 2142n - 2448\right) {x_{12}}^4 \\
&-(15n^6 + 194n^5 - 5442n^4 + 33531n^3 - 73361n^2 + 38979n + 18396){x_{12}}^3 \\
&+2(n-2)\left(119n^4 - 952n^3 - 1276n^2 + 21873n - 28098\right) {x_{12}}^2 \\
&-\left(7n^5 + 849n^4 - 11830n^3 + 53569n^2 - 79135n + 24552\right) x_{12} \\
&+52(n-7)(n-1)\left(n^2 - 7n + 27\right) x_3 + 624(n-2)\left(n^2 - 7n + 27\right).
\end{aligned}
\tag{14}
$$

Thus we see that, if there exists a real root $x_{12} = \alpha_{12}$ of $h(x_{12}) = 0$, then there are **a real solution** $x_1 = \alpha_1$ of $p_1(\alpha_{12}, x_1) = 0$ and **a real solution** $x_3 = \alpha_3$ of $p_1(\alpha_{12}, x_3) = 0$.

Now we have $h(0) = 2704(n-1)$ for $n > 1$,
$h(2) = 4(16n^5 - 424n^4 + 4625n^3 - 25470n^2 + 70193n - 77128)$
$= 4(16(n-6)^5$
$+56(n-6)^4 + 209(n-6)^3 + 756(n-6)^2 + 1397(n-6) + 1022) > 0$ for $n \geq 6$ and $h(1) = -2(n-9)(n-1)n^2 < 0$ for $n > 9$. Note that for $n = 9$ $h(6/5) = -1751152/390625 < 0$.

Thus we see that the equation $h(x_{12}) = 0$ has **two positive roots** $x_{12} = \alpha_{12}, \beta_{12}$ with $0 < \alpha_{12} < 1 < \beta_{12} < 2$ <u>for $n > 9$</u>. For $n = 9$ we have roots $x_{12} = 1, \beta_{12}$ with $6/5 < \beta_{12} < 2$.

We also take a lexicographic order $>$ with $z > x_3 > x_{12} > x_1$ for a monomial ordering on R. Then we see that a Gröbner basis for the ideal I contains the polynomial $h_1(x_1)$ of x_1, and moreover, take a lexicographic order $>$ with $z > x_1 > x_{12} > x_3$ for a monomial ordering on R. Then we see that a Gröbner basis for the ideal I contains the polynomial $h_3(x_3)$ of x_3.

Now we see that the polynomial $h_1(x_1)$ of x_1 and the polynomial $h_3(x_3)$ of x_3 have the same property as $h(x_{12})$, that is, for $n \geq 9$, the coefficients of the polynomials $h_1(x_1)$ and $h_3(x_3)$ are positive for even degree terms and negative for odd degree terms and that, if the equations $h_1(x_1) = 0$ and $h_3(x_3) = 0$ have real solutions, then **these are all positive**.

We see that there exist at least two positive solutions of the system for $n \geq 10$ of the form

$$\langle\ ,\ \rangle = \alpha_1 B|_{\mathfrak{m}_1} + \alpha B|_{\mathfrak{m}_2} + \gamma B|_{\mathfrak{m}_3} + \beta B|_{\mathfrak{m}_{12}} + B|_{\mathfrak{m}_{13}} + B|_{\mathfrak{m}_{23}}$$

(α, β are different, $\beta \neq 1$).
We can see that these metrics are not naturally reductive by

Lemma

If a left invariant metric $\langle\ ,\ \rangle$ of the form (9) *on $\mathrm{SO}(n)$ is naturally reductive with respect to $\mathrm{SO}(n) \times L$ for some closed subgroup L of $\mathrm{SO}(n)$, then one of the following holds:*

1) $x_1 = x_2 = x_{12}, x_{13} = x_{23}$ 2) $x_2 = x_3 = x_{23}, x_{12} = x_{13}$
3) $x_1 = x_3 = x_{13}, x_{12} = x_{23}$, 4) $x_{12} = x_{13} = x_{23}$.

Conversely, if one of the conditions 1), 2), 3), 4) *is satisfied, then the metric $\langle\ ,\ \rangle$ of the form* (9) *is naturally reductive with respect to $\mathrm{SO}(n) \times L$ for some closed subgroup L of $\mathrm{SO}(n)$.*

- For $G = \mathrm{SO}(7), \mathrm{SO}(8))$, we consider separately and we see that $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3))$-invariant Einstein metrics on $\mathrm{SO}(7)$, that is, $k_1 = 3, k_2 = 3, k_3 = 1$ and
  $\mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathrm{SO}(2))$-invariant Einstein metrics on $\mathrm{SO}(8)$, that is, $k_1 = 3, k_2 = 3, k_3 = 2$
  Both cases we can show there are non-naturally reductive Einstein metrics.

Generalized flag manifolds

- A generalized flag manifold M is an adjoint orbit of a compact connected semi-simple Lie group G, and is a homogeneous space of the form $M = G/C(S)$, where $C(S)$ is the centralizer of a torus S in G.
- Generalized flag manifolds exhaust compact simply connected homogeneous Kähler manifolds.
- A generalized flag manifold admits a finite number of G-invariant complex structures. For each G-invariant complex structure there is a compatible Kähler-Einstein metric.
- Generalized flag manifolds can be classified by use of painted Dynkin diagrams.
- Generalized flag manifolds are also referred to as Kähler C-spaces.

Examples of Generalized flag manifolds

- Set $G = SU(n+1)$, $K = S(U(n) \times U(1))$. Then G/K is a complex projective space $\mathbb{C}P^n$.
- Set $G = SU(n+m)$, $K = S(U(n) \times U(m))$. Then G/K is a Grassmann manifold $G_{m+n,n}(\mathbb{C})$.
- Set $G = SU(n+m+\ell)$, $K = S(U(n) \times U(m) \times U(\ell))$. Then G/K is a generalized flag manifold.
- Set $G = Sp(n+1)$, $K = Sp(n) \times U(1)$. Then G/K is a a complex projective space $\mathbb{C}P^{2n-1}$.

- Let G be a compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{h}$ a maximal abelian subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and $\mathfrak{h}$ respectively.
- We identify an element of the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ with an element of $\mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}$ by the duality defined by the Killing form of $\mathfrak{g}^{\mathbb{C}}$. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental system of Δ and $\{\Lambda_1, \cdots, \Lambda_l\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π, that is

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad (1 \leq i, j \leq \ell).$$

- Let Π_0 be a subset of Π and $\Pi - \Pi_0 = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ $(1 \leq \alpha_{i_1} < \cdots < \alpha_{i_r} \leq \ell)$. We put $[\Pi_0] = \Delta \cap \{\Pi_0\}_{\mathbb{Z}}$, where $\{\Pi_0\}_{\mathbb{Z}}$ denotes the subspace of $\mathfrak{h}_0$ generated by Π_0.

Structures of generalized flag manifolds

- Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}^{\mathbb{C}}.$$

For a subset Π_0 of Π, we define a parabolic subalgebra $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{u} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in [\Pi_0] \cup \Delta^+} \mathfrak{g}_{\alpha}^{\mathbb{C}},$$

where Δ^+ is the set of all positive roots relative to Π.

- Note that the nilradical $\mathfrak{n}$ of $\mathfrak{u}$ is given by

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+ - [\Pi_0]} \mathfrak{g}_{\alpha}^{\mathbb{C}}.$$

We put $\Delta_{\mathfrak{m}}^+ = \Delta^+ - [\Pi_0]$.

- Let $G^{\mathbb{C}}$ be a simply connected complex semi-simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$ and U the parabolic subgroup of $G^{\mathbb{C}}$ generated by $\mathfrak{u}$. Then the complex homogeneous manifold $G^{\mathbb{C}}/U$ is compact simply connected and G acts transitively on $G^{\mathbb{C}}/U$. Note also that $K = G \cap U$ is a connected closed subgroup of G, $G^{\mathbb{C}}/U = G/K$ as C^{∞}-manifolds, and $G^{\mathbb{C}}/U$ admits a G-invariant Kähler metric.
  Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{k}^{\mathbb{C}}$ the complexification of $\mathfrak{k}$. Then we have a direct decomposition

$$\mathfrak{u} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{n}, \qquad \mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in [\Pi_0]} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

- We put $\mathfrak{t} = \left\{ H \in \mathfrak{h}_0 \mid (H,\ \Pi_0) = (0) \right\}$. Then $\{\Lambda_{i_1}, \cdots, \Lambda_{i_r}\}$ is a basis of $\mathfrak{t}$. Put $\mathfrak{s} = \sqrt{-1}\mathfrak{t}$. Then the Lie algebra $\mathfrak{k}$ is given by $\mathfrak{k} = \mathfrak{z}(\mathfrak{s})$ (the Lie algebra of centralizer of a torus S in G).

$\mathfrak{t}$-roots of generalized flag manifolds

- We consider the restriction map

$$\kappa : \mathfrak{h}_0^* \to \mathfrak{t}^* \quad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

  and set $\Delta_{\mathfrak{t}} = \kappa(\Delta)$. The elements of $\Delta_{\mathfrak{t}}$ are called $\mathfrak{t}$-roots. (The notion of $\mathfrak{t}$-roots is introduced by Alekseevky and Perelomov around 1985 to study invariant Kähler-Einstein metrics of generalized flag manifolds.)
- There exists a 1-1 correspondence between $\mathfrak{t}$-roots ξ and irreducible submodules $\mathfrak{m}_\xi$ of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$ that is given by

$$\Delta_{\mathfrak{t}} \ni \xi \mapsto \mathfrak{m}_\xi = \sum_{\kappa(\alpha)=\xi} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

- Thus we have a decomposition of the $\mathrm{Ad}_G(K)$-module $\mathfrak{m}^{\mathbb{C}}$:

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in \Delta_{\mathfrak{t}}} \mathfrak{m}_\xi.$$

Decomposition associated to generalized flag manifolds

- Denote by $\Delta_{\mathfrak{t}}^{+}$ the set of all positive $\mathfrak{t}$-roots, that is, the restricton of the system Δ^{+}. Then $\mathfrak{n} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} \mathfrak{m}_{\xi}$.
- Denote by τ the complex conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$ (note that τ interchanges $\mathfrak{g}_{\alpha}^{\mathbb{C}}$ and $\mathfrak{g}_{-\alpha}^{\mathbb{C}}$) and by $\mathfrak{v}^{\tau}$ the set of fixed points of τ in a (complex) vector subspace $\mathfrak{v}$ of $\mathfrak{g}^{\mathbb{C}}$. Thus we have a decomposition of $\mathrm{Ad}_G(K)$-module $\mathfrak{m}$ into irreducible submodules:
$$\mathfrak{m} = \sum_{\xi \in \Delta_{\mathfrak{t}}^{+}} \left(\mathfrak{m}_{\xi} + \mathfrak{m}_{-\xi} \right)^{\tau}.$$

- For integers $j_1, \cdots, j_r$ with $(j_1, \cdots, j_r) \neq (0, \cdots, 0)$, we put $\Delta(j_1, \cdots, j_r) = \left\{ \sum_{j=1}^{\ell} m_j\alpha_j \in \Delta^+ \;\middle|\; m_{i_1} = j_1, \cdots, m_{i_r} = j_r \right\}$.
  There exists a natural 1-1 correspondence between $\Delta_{\mathfrak{t}}^+$ and the set $\{\Delta(j_1, \cdots, j_r) \neq \emptyset\}$
- For a generalized flag manifold G/K, we have a decomposition of $\mathfrak{m}$ into mutually non-equivalent irreducible $\mathrm{Ad}_G(H)$-modules :

$$\mathfrak{m} = \sum_{\xi \in \Delta_{\mathfrak{t}}^+} \left(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi}\right)^\tau = \sum_{j_1, \cdots, j_r} \mathfrak{m}(j_1, \cdots, j_r).$$

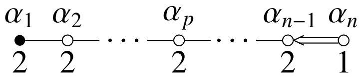
Thus a G-invariant metric g on G/K can be written as

$$g = \sum_{\xi \in \Delta_{\mathfrak{t}}^+} x_\xi B|_{(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi})^\tau} = \sum_{j_1, \cdots, j_r} x_{j_1 \cdots j_r} B|_{\mathfrak{m}(j_1, \cdots, j_r)} \tag{15}$$

for positive real numbers x_ξ, $x_{j_1 \cdots j_r}$.

$\mathfrak{t}$-roots and decompositions

- From now on we assume that the Lie group G is simple. We denote by q the number of elements of $\Delta_{\mathfrak{t}}^{+}$ for a generalized flag manifold G/K, that is, the number of irreducible components of $\mathrm{Ad}_G(K)$-module $\mathfrak{m}$.
- If $q = 1$, then $\Delta_{\mathfrak{t}}^{+} = \{\xi\}$ and G/K is an irreducible Hermitian symmetric space with the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.
- If $q = 2$, then we see that $r = b_2(G/K) = 1$ and $\mathfrak{m} = \mathfrak{m}(1) \oplus \mathfrak{m}(2) == \mathfrak{m}_1 \oplus \mathfrak{m}_2$, that is, $\Delta_{\mathfrak{t}}^{+} = \{\xi, 2\xi\}$. We say this case that $\mathfrak{t}$-roots system is of type $A_1(2)$.
- Example. $\mathbb{C}P^{2n-1} = Sp(n)/(Sp(n-1) \times U(1))$



- To show a result of Arvanitoyeorgos, Mori and S., 2008, we have used generalized flag manifolds G/H with a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ as irreducible Ad(H)-modules. That is, we have used the following pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h})$:
$(\mathfrak{g},\ \mathfrak{h}) = (\mathfrak{so}(n),\ \mathfrak{su}(3) \oplus \mathfrak{so}(n-6) \oplus \mathbb{R}) \quad (n \geq 11)$,
$(\mathfrak{g},\ \mathfrak{h}) = (\mathfrak{sp}(n),\ \mathfrak{su}(2) \oplus \mathfrak{sp}(n-2) \oplus \mathbb{R}) \quad (n \geq 3)$,
$(\mathfrak{g},\ \mathfrak{h}) = (E_6,\ \mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R})$,
$(\mathfrak{g},\ \mathfrak{h}) = (E_7,\ \mathfrak{su}(2) \oplus \mathfrak{so}(10) \oplus \mathbb{R})$
- Note that Lie algebra $\mathfrak{g}$ can be decomposed into irreducible Ad(H)-modules as

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \ (= \mathfrak{w}_0 \oplus \mathfrak{w}_1 \oplus \mathfrak{w}_2 \oplus \mathfrak{w}_3 \oplus \mathfrak{w}_4),$$

where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\dim \mathfrak{h}_0 = 1$.

- Since we have $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h} + \mathfrak{m}_2$, $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h}$, $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$ and $[\mathfrak{h}_2, \mathfrak{m}_2] = (0)$, we see that $\begin{bmatrix} k \\ ij \end{bmatrix} = 0$, except $\begin{bmatrix} 3 \\ 03 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 04 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 13 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 14 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 22 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 23 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 33 \end{bmatrix}$.

Now we can give the components $r_0, r_1, \cdots, r_4$ of the Ricci tensor r of the metric

$$< \ , \ >= u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + u_2 \cdot B|_{\mathfrak{h}_2} + x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2} \quad (16)$$

by using structures constants $\begin{bmatrix} k \\ ij \end{bmatrix}$:

$$
\begin{cases}
r_0 &= \dfrac{u_0}{4\,{x_1}^2}\begin{bmatrix}0\\33\end{bmatrix} + \dfrac{u_0}{4\,{x_2}^2}\begin{bmatrix}0\\44\end{bmatrix}\\
r_1 &= \dfrac{1}{4\,d_1\,u_1}\begin{bmatrix}1\\11\end{bmatrix} + \dfrac{u_1}{4\,d_1\,{x_1}^2}\begin{bmatrix}1\\33\end{bmatrix} + \dfrac{u_1}{4\,d_1\,{x_2}^2}\begin{bmatrix}1\\44\end{bmatrix}\\
r_2 &= \dfrac{1}{4\,d_2\,u_2}\begin{bmatrix}2\\22\end{bmatrix} + \dfrac{u_2}{4\,d_2\,{x_1}^2}\begin{bmatrix}2\\33\end{bmatrix}\\
r_3 &= \dfrac{1}{2x_1} - \dfrac{x_2}{2\,d_3\,{x_1}^2}\begin{bmatrix}4\\33\end{bmatrix} - \dfrac{1}{2\,d_3\,{x_1}^2}\left(u_0\begin{bmatrix}0\\33\end{bmatrix} + u_1\begin{bmatrix}1\\33\end{bmatrix} + u_2\begin{bmatrix}2\\33\end{bmatrix}\right)\\
r_4 &= \dfrac{1}{x_2}\left(\dfrac{1}{2} - \dfrac{1}{2\,d_4}\begin{bmatrix}3\\43\end{bmatrix}\right) + \dfrac{x_2}{4\,d_4\,{x_1}^2}\begin{bmatrix}4\\33\end{bmatrix} - \dfrac{1}{2\,d_4\,{x_2}^2}\left(u_0\begin{bmatrix}0\\44\end{bmatrix} + u_1\begin{bmatrix}1\\44\end{bmatrix}\right).
\end{cases}
$$

Now we can determine structures constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ explicitly.

- In the decomposition $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, we set $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}_2$ and $\mathfrak{k}_1 = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_2$. Then $\mathfrak{k}$ and $\mathfrak{k}_1$ are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1$.
- Note that the pair $(\mathfrak{g}, \mathfrak{k})$ is a pair of irreducilbe symmetric space of compact type and the irreducible decomposition of $\mathfrak{g}$ as a Ad(K)-module is given by $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1$.
- Now a left invariant metric $<< \, , \, >>$ on the compact Lie group G which is Ad(K)-invariant, is given by

$$<< \, , \, >> = w_1 \cdot B|_{\mathfrak{k}_1} + w_2 \cdot B|_{\mathfrak{h}_2} + w_3 \cdot B|_{\mathfrak{m}_1}.$$

- Note that the Ad(K)-invariant metric can be regarded as a special case of the metric which is Ad(H)-invariant (that is, the metric obtained by setting $w_1 = u_0 = u_1 = x_2$, $w_2 = u_2$, $w_3 = x_1$ in (7)).

- Now by comparing the Ricci tensors of the Ad(K)-invariant metrics and Ad(H)-invariant metrics, we get the following:

$$\begin{bmatrix} 0 \\ 33 \end{bmatrix} = \frac{d_3}{(d_3 + 4d_4)} \qquad \begin{bmatrix} 0 \\ 44 \end{bmatrix} = \frac{4d_4}{(d_3 + 4d_4)}$$

$$\begin{bmatrix} 1 \\ 11 \end{bmatrix} = \frac{2d_4(2d_1 + 2 - d_4)}{(d_3 + 4d_4)} \qquad \begin{bmatrix} 1 \\ 33 \end{bmatrix} = \frac{d_1 d_3}{(d_3 + 4d_4)}$$

$$\begin{bmatrix} 1 \\ 44 \end{bmatrix} = \frac{2d_4(d_4 - 2)}{(d_3 + 4d_4)} \qquad \begin{bmatrix} 2 \\ 22 \end{bmatrix} = d_2 - \frac{d_3(d_3 + 2d_4 - 2d_1 - 2)}{2(d_3 + 4d_4)}$$

$$\begin{bmatrix} 2 \\ 33 \end{bmatrix} = \frac{d_3(d_3 + 2d_4 - 2d_1 - 2)}{2(d_3 + 4d_4)} \qquad \begin{bmatrix} 4 \\ 33 \end{bmatrix} = \frac{d_3 d_4}{(d_3 + 4d_4)}.$$

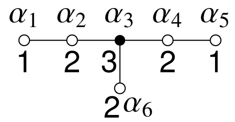
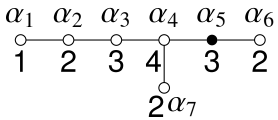
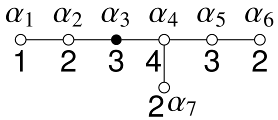
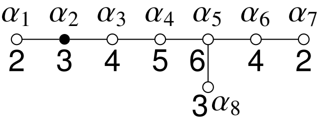
- Now we have that the $\mathrm{Ad}(H)$-invariant metric is Eistein iff there exist positive numbers $\{u_0, u_1, u_2, x_1, x_2, e\}$ which satisfies the system of equations

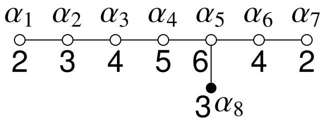
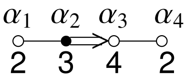
$$r_0 = e, \quad r_1 = e, \quad r_2 = e, \quad r_3 = e, \quad r_4 = e.$$

- We normalize the system of equations by setting $x_1 = 1$ and then we can solve the system of equations. Each of u_0, u_1, u_2, e can be expressed by a rational polynomial of x_2 and the equation for x_2 is a polynomials of degree 16. We see that the polynomial equation for x_2 has a solution with $x_2^0 > 1$ and also we can see that the corresponding values u_0^0, u_1^0, u_2^0, e^0 iare positive and thus we have an Einstein metric on G.

- To see that the metric is not naturally reductive, we have the following:
  If a Ad(H)-invariant metric
  $< \ , \ >= u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + u_2 \cdot B|_{\mathfrak{h}_2} + x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2}$ is naturally reductive for $G \times L$ (where L is a closed subgroup of G), then one of the followng holds:
  1) $x_1 = x_2$, 2) $u_0 = u_1 = x_2$, 3) $u_0 = u_1 = u_2 = x_1 = x_2$, (that is, Ad(G)-invariant metric).
- We can work for E_8 by the similar method.
- **Theorem**[Arvanitoyeorgos, Mori and S., 2008]
  On a compact simple Lie group G, either $\mathrm{SO}(n)$ $(n \geq 11)$, $\mathrm{Sp}(n)$ ($n \geq 3$), E_6, E_7 or E_8, there exist non-naturally reductive Einstein Einstein metrics.

- To show a result of Chrysikos and S. in 2015, we have used generalized flag manifolds G/H with a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ as irreducible $\mathrm{Ad}(H)$-modules.
- We say the case of $r = b_2(G/K) = 1$ and $q = 3$ that $\mathfrak{t}$-roots system is of type $A_1(3)$, that is, $\Delta_{\mathfrak{t}}^{+} = \{\xi, 2\xi, 3\xi\}$. There are 7 cases and the Lie group G is always exceptional, that is, E_6, E_7, E_8, F_4 and G_2 (for E_7, E_8, there are 2 cases.)
- We can compute $\begin{bmatrix} i \\ jk \end{bmatrix}$ by dividing into cases.

E_6	
E_7	
E_7	
E_8	

E_8	
F_4	
G_2