On Jordan-Lie Algebras

Sophus Lie (mathematician, 19th cent.)

- LIE algebras   ex. $\mathbb{A}$ assoc. alg.
  $[x,y] = xy - yx$
- LIE groups   ex. $\mathbb{A}^\times$, xy

Pascual Jordan (physicist, 20th cent.)

- JORDAN algebras

ex: $\mathbb{A}$ assoc., $\quad x \bullet y = \frac{1}{2}(xy + yx)$, $\quad x^2 = x \bullet x$

Sym$(\mathcal{H})$ –"–

Herm$(\mathcal{H})$ –"–   (= „observables" in q.m.)

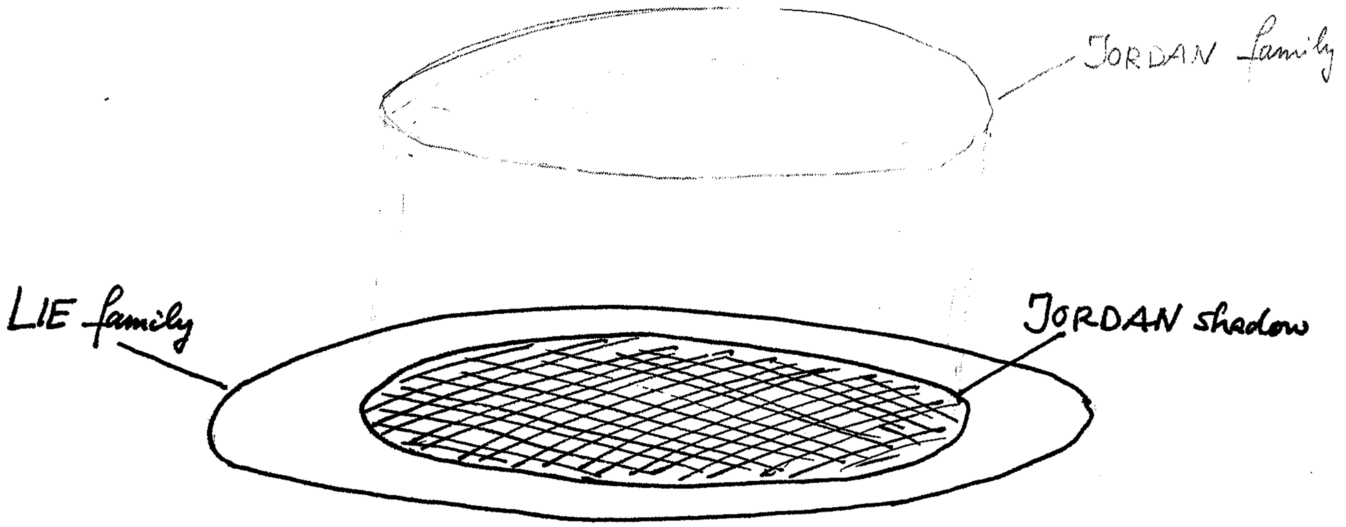
$\{a \in \mathbb{A} \mid a^* = a\}$, $\quad *$ involution

- „JORDAN groups"?

Some algebras are <u>both</u> Jordan- and Lie algebras:
- why are they interesting?
- is there a corresponding geometric object?

THE JORDAN SHADOW ON LIE THEORY

There are several functors from JORDAN to LIE:



Fact: „most" LIE structures lie in the JORDAN shadow:

1.) all „classical" ones

2.) all (pseudo-) Hermitian symmetric spaces
(finite or ∞-dimensional!)

3.) certain parabolic geometries (symmetric R-spaces), e.g.

- — Grassmannians (...)
- — lagrangian -"- (...)
- — projective spaces, $\mathbb{O}\mathbb{P}^2$ included
- — projective quadrics [conformal spheres]

4.) symmetric cones and their analogs

JORDAN algebra V	small group: autom. group. A	medium group: structure group S	big group: conformal group C
$Sym(n,\mathbb{R})$	$O(n)$	$Gl(n,\mathbb{R})$	$Sp(n,\mathbb{R})$
$Herm(n,\mathbb{C})$	$U(n)$	$Gl(n,\mathbb{C})$	$SU(n,n)$
$Herm(n,\mathbb{H})$	$Sp(n)$	$Gl(n,\mathbb{H})$	$SO^*(4n)$
$M(n,n;\mathbb{R})$	$\mathbb{P}Gl(n,\mathbb{R})$	$\mathbb{P}(Gl(n,\mathbb{R})\times Gl(n,\mathbb{R}))$	$\mathbb{P}\,Gl(2n,\mathbb{R})$
$\mathbb{R}^{(p,q)}$	$SO(p,q)$	$SO(p,q)\times\mathbb{R}^\times$	$SO(p+1,q+1)$

$\longrightarrow$ symmetric spaces: S/A „conal space": orbit $S.e\subset V$ (e.g., symmetric cones)

C/S „polarized space" (e.g., „Cayley type")

$\longrightarrow$ parabolic geometries C/P

JORDAN-LIE ALGEBRAS

Some algebras are more special than others: they are both Jordan- and Lie algebras

Examples: – $M(n,n,\mathbb{K})$ with $x \bullet y$ and $[x,y] = xy - yx$

– $\mathbb{A}$ assoc. alg –"– –"–

– $\mathrm{Herm}(n;\mathbb{C})$ with $x \bullet y$ and $[x,y] = i(xy - yx)$

– $\mathrm{Herm}(\mathbb{A},*)$ where $*$ is a $\mathbb{C}$-antilinear involution

Definition. A Jordan-Lie algebra is an algebra V with two products, $x \bullet y$ and $[x,y]$, s.th.

(1) $(V,[\,,\,])$ is a Lie algebra,

(2) $(V,\bullet)$ is a Jordan algebra,

(3) Lie derives Jordan:

$$[x, a \bullet b] = [x,a] \bullet b + a \bullet [x,b]$$

(4) non-associativity of both products is proportional:

$$\boxed{\begin{array}{l} \exists \hbar \in \mathbb{K}: \\ \underset{\text{„associator"}}{(x \bullet y) \bullet z - x \bullet (y \bullet z)} = \hbar\big(\underbrace{[[x,y],z] - [x,[y,z]]}_{=[y,[z,x]]}\big) \end{array}}$$

Examples: – $\mathbb{A}$ assoc.: ok. with $\hbar = -\frac{1}{4}$

– $\mathrm{Herm}(\mathbb{A},*)$: ok. with $\hbar = \frac{1}{4}$

– $\hbar = 0$: Commutative Poisson-algebra

BOHR'S QUESTION:

Which classes of algebras admit a tensor product („composition of systems")?

associative: yes, $\mathbb{A} \otimes \mathbb{A}'$ is associative

Jordan: no, e.g. $\mathrm{Sym}(n,\mathbb{R}) \otimes \mathrm{Sym}(m,\mathbb{R})$ is not a Jordan alg.

Lie: no, e.g. $\square(n) \otimes \square(m)$ is not a Lie algebra

Jordan-Lie (with <u>same</u> $\hbar$): yes, e.g. $\boxed{\mathrm{Herm}(n,\mathbb{C}) \otimes \mathrm{Herm}(m,\mathbb{C}) \cong \mathrm{Herm}(mn,\mathbb{C})}$

<u>Answer</u> (Grgin, Petersen: Comm. math. phys. <u>50</u> (1976)):
Precisely the Jordan-Lie algebras!

<u>easy part</u>: sufficency:

for $\hbar < 0$: let $xy := x \circ y + \sqrt{-\hbar}\,[x,y]$, then using (3) & (4)

$$(xy)z - x(yz) = \ldots = 0,$$

hence V is associative and admits tensor products

for $\hbar > 0$: on $V_{\mathbb{C}}$ let $xy := x \circ y + i\sqrt{\hbar}\,[x,y]$, as above:
$V_{\mathbb{C}}$ is associative, and $V = \mathrm{Herm}(V_{\mathbb{C}}, *)$,
and hence admits tensor products.

<u>Converse</u>: study the Grgin-Petersen paper!

THE GEOMETRIC OBJECT

A. the local (linear) picture

Exple. $\hbar < 0$: $V = M(n,n;\mathbb{R})$: The „conal space" $S.e \subset V$
is a group: $S.e = Gl(n,\mathbb{R}) \subset V$.

$\hbar > 0$: $V = Herm(n;\mathbb{C})$: The „conal space" $S.e \subset V$
is a symmetric cone $\Omega = \{X \mid X \gg 0\} = Gl(n,\mathbb{C})/U(n)$,
the compact dual of Ω is a group: $U(n)$.

B. the global (non-linear) picture

V algebra $\longleftrightarrow$ X space („compact-like")

Lie product $[\,,\,]$ $\longleftrightarrow$ X carries (many)
Lie-group structures:
$\forall a,b \in X$: $\exists\, G_{ab} \subset X$ „open dense"

Jordan product $\bullet$ $\longleftrightarrow$ X has features of a
projective line:
$\forall a,b \in X$: $\exists\, \Omega_{ab} \subset X$ „conal space"

case $\hbar < 0$: the conal structure is of group type:
$$\Omega_{ab} = G_{ab} \text{ as symmetric spaces}$$
case $\hbar > 0$: the conal structure is dual to the group structure:
$$G_{ab} = \text{dual of } \Omega_{ab} \text{ as symmetric space}$$

GEOMETRIC OBJECT: CASE $\hbar < 0$

$V = \mathbb{A}$ „plain" associative algebra

<u>Definition</u>. The <u>projective line over $\mathbb{A}$</u> is the set X of all right-submodules $E \subset \mathbb{A} \oplus \mathbb{A}$, isomorphic to $\mathbb{A}$ and having a complement F that is isomorphic to $\mathbb{A}$:

$$X = \mathbb{A}\mathbb{P}^1 \cong GL(\mathbb{A} \oplus \mathbb{A})/P, \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \dots \right\}$$

<u>Exple 1</u>. $\mathbb{A} = M(n,n;\mathbb{R})$: $\quad X \cong \mathrm{Gras}_n(\mathbb{R}^{2n}) = Gl(2n,\mathbb{R})/P$

<u>Exple 2</u>. $\mathbb{A}$ field or skew-field: $\quad X = \mathbb{A}\mathbb{P}^1$ as usual

$\forall a \in X$: $\quad U_a := X \setminus \{a\}$ affine space $\cong \mathbb{A}$: $\quad$ „$a = \infty$"

$\forall a \neq b$: $\quad U_{ab} := U_a \cap U_b = X \setminus \{a,b\} \cong \mathbb{A}^\times$: $\quad$ „$b = 0$"

$\forall y \in U_{ab}$: $\quad (U_{ab}, y) \cong \mathbb{A}^\times$ is a group: $\quad$ „$y = 1$"

$\Gamma(x,a,y,b,z) := x \cdot_{a,y,b} z :=$ product $x \cdot z$ in (U_{ab}, y)

<u>Thm</u>. (B-Kinyon „Assoc. Geom. I", JoLT <u>20</u> (2010))

- definition of Γ for general $\mathbb{A}$
- explicit formula for Γ
- properties of Γ
- extension of Γ to a „globally defined" map

GEOMETRIC OBJECT: CASE $\hbar > 0$

$V = \mathrm{Herm}(\mathbb{A}, *)$, $\quad *$ $\mathbb{C}$-antilinear involution

<u>Proposition–Definition.</u> There is a unique involution of $\mathbb{A}\mathbb{P}^1$, τ „extending" the involution $*$ of $\mathbb{A}$. The <u>Hermitian projective line</u> over $(\mathbb{A}, *)$ is the fixed point space $\quad Y := (\mathbb{A}\mathbb{P}^1)^\tau$

<u>Exple 1.</u> $\mathbb{A} = M(n,n;\mathbb{C})$, $\quad X^* = \overline{X}^t$

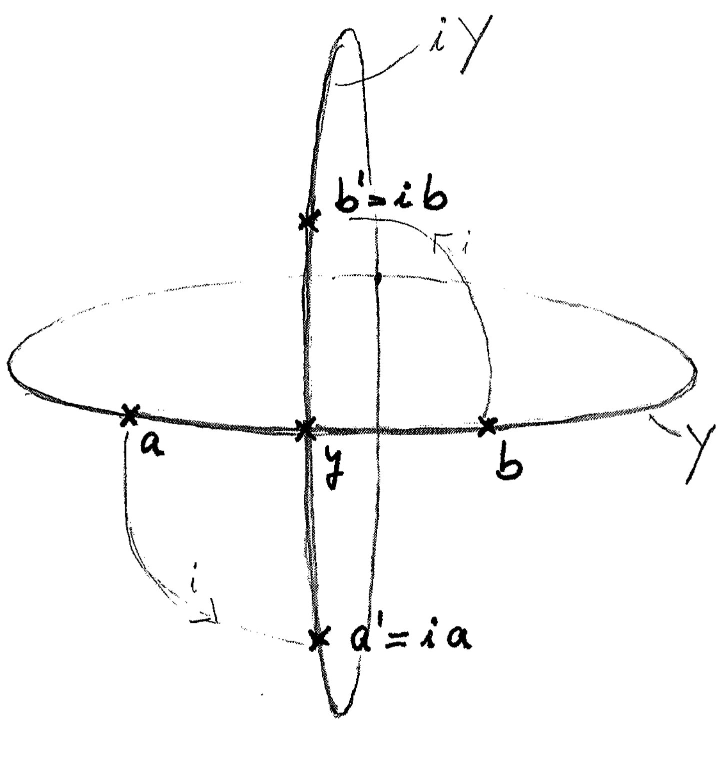
τ = orthocompl. w.r.t. Hermitian form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{C}^{2n}$

$Y = \{ E \in \mathrm{Gras}_n(\mathbb{C}^{2n}) \mid E = E^\perp \} \cong \mathbb{P}U(n,n)/P$

Lagrangian Grassmannian

<u>Exple 2.</u> $\mathbb{A} = \mathbb{C}$, $\quad z^* = \overline{z}$

$Y = \mathbb{R}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^1$



given: $y, a, b \in Y$ $\quad$ pairw. dist.

$c := \frac{a+b}{2}$ in U_y $\quad$ harmonic pt.

$i := i_{y,c}$:= dilation by i in (U_{c}, y)

then $U_{a'b'} \cong \mathbb{C}^* \subset \mathbb{C}\mathbb{P}^1$ group

$$\boxed{\begin{array}{l} G_{ab} := U_{a'b'} \cap Y \text{ is a group} \\ \cong S^1 \cong U(1) \end{array}}$$

(polar dec. of $U_{a'b'}$!)

Observables *and* Generators

"...it has been proposed to model quantum mechanics on Jordan algebras rather than on associative algebras (Jordan, von Neumann, Wigner, 1934). This approach is corroborated by the fact that many physically relevant properties of observables are adequately described by Jordan constructs. However, it is an important feature of quantum mechanics that the physical variables play a dual role, as observables *and* as generators of transformation groups. ... Therefore both the Jordan product and the Lie product of a C^*-algebra are needed for physics, and the decomposition of the associative product into its Jordan part and its Lie part separates two aspects of a physical variable."

(E. Alfsen and F. E. Schultz, *State Spaces of Operator Algebras*, Birkhäuser, Boston 2001, p. vii)