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Products of coalgebras

H. Peter Gumm and Tobias Schröder

Dedicated to Viktor Aleksandrovich Gorbunov

Abstract. We prove that the category of F-coalgebras is complete, that is products and equalizers exist, provided that the type functor F is bounded or preserves mono sources. This generalizes and simplifies a result of Worrell ([Wor98]). We also describe the relationship between the product $\mathcal{A} \times \mathcal{B}$ and the largest bisimulation $\sim_{\mathcal{A},\mathcal{B}}$ between $\mathcal{A}$ and $\mathcal{B}$ and find an example of two finite coalgebras whose product is infinite.

1. Introduction

Only recently has it been discovered that many structures in theoretical computer science, including automata, transition systems, object oriented systems, and lazy data types can be put into a common framework, that of universal coalgebra. Many phenomena that had previously been studied individually in each of those theories are seen to be instances of some general structure theory as provided by the new field of universal coalgebra.

There are many parallels to the situation in universal algebra, when it was found that the basic structure theory of groups, rings, lattices and several related structures can be dealt with uniformly on the level of universal algebras. The advantage is not only an "economy of thought", replacing many individual proofs by the proof of one general theorem, of even greater benefit seems to be the fact that a framework is provided for each new theory – there is no doubt as to which notion of homomorphism, substructure, factor, etc. to settle for.

Coalgebras are not simply obtained by dualizing the established concept of universal algebra. This way one would obtain a coalgebra of type $(n_i)_{i\in I}$ as a family $\alpha_i : A \to (n_i \cdot A)$ of maps from A to the n_i-fold sum of A. Even though such structures have been studied some 30 years ago, they failed to attract much attention, probably due to the lack of interesting applications, let alone interesting mathematical questions.

The most important observation is that a type $(n_i)_{i\in I}$ can simply be understood as an encoding of the functor $F : \mathcal{S}et \to \mathcal{S}et$ on the category of sets, which associates with a set X the disjoint sum of its n_i-fold powers of X:

$$F(X) = \Sigma_{i\in I} X^{n_i}.$$

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A universal algebra $\mathcal{A} = (A, (f_i)_{i\in I})$ of type $(n_i)_{i\in I}$ can be now coded by a single map

$$f : F(A) \to A$$

and a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ between algebras $\mathcal{A} = (A, (f_i)_{i\in I})$ and $\mathcal{B} = (B, (g_i)_{i\in I})$ is just a map $\varphi : A \to B$ making the following diagram commute:

$$\begin{tikzcd} F(A) \arrow[r, "F(\varphi)"] \arrow[d, "f"'] & F(B) \arrow[d, "g"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

This is the more general setting that must be dualized to obtain the proper notion of universal coalgebra. It is surprising that in spite of the high level of generality a rich structure theory of coalgebras can be developed which in many ways parallels the theory of universal algebra up to and including coalgebraic versions of Birkhoff's theorems ([Gum99b]).

The first general introduction into the field is by Jan Rutten ([Rut96]). This work systematically develops the structure theory of coalgebras and at the same time explains a large number of relevant applications. A major insight underlying much of this work was the observation that in most of the relevant examples the type functor F *weakly preserves generalized pullbacks*. Taking this into account, one is led to a richer structure theory, and as a consequence much of the subsequent literature uses this assumption on F.

Nonetheless, there are still functors which do not obey the mentioned assumption of weakly preserving generalized pullbacks. One of those is the *filter functor* $\mathcal{F}$ studied in [Gum98] whose coalgebras include all topological spaces. An introduction to the general theory of coalgebras, without any assumption on the type functor F, has therefore been given in [Gum99a]. It turned out that, in fact, the structure theory can be developed in the very general case, again yielding a dual of Birkhoff's theorem. The structure theoretic results equivalent to various preservation properties of the functor F are analyzed in [GS00].

It is well known (see [Bar93]) that the category $\mathcal{S}et_F$ of all coalgebras of type F has coequalizers and sums, hence arbitrary colimits. In fact it is known (see [Bar93]) and easy to check, that every colimit in $\mathcal{S}et_F$ exists and is constructed exactly as in the underlying category of sets. In category theoretic parlance, the forgetful functor from $\mathcal{S}et_F$ to $\mathcal{S}et$ creates and preserves colimits.

The case is different for limits. Products of coalgebras need not exist, but when they do, their base set will often have to be different from the cartesian product of the base sets. One may be lucky, in that the functor F preserves a certain type of limit. In that case, this very type of limit exists for F-coalgebras and it is constructed as in $\mathcal{S}et$ ([Bar93]). In general though, the functors needed to model most applications of interest fail to preserve arbitrary limits.

Still, J. Worrell was able to show in [Wor98] that $\mathcal{S}et_F$ is complete, that is products and equalizers exist, provided the type functor F weakly preserves pullbacks and F is *bounded*, a term that we shall define later. Worrell's proof uses the theory of monads and some further category theoretic machinery which, when translated into more elementary notions, makes the proof rather long and hard to follow. One purpose of this article is therefore, to give a short and elementary proof of this result and at the same time extend it by removing the assumption that the type functor F should weakly preserve generalized pullbacks.

Doing so, we shall have to redefine the notion of a *bounded functor*. The usual definition requires a cardinal bound on the size of one-generated subcoalgebras. When F does not preserve weak generalized pullbacks, the notion of one-generated subcoalgebra makes no sense, since subcoalgebras need not be closed under intersection. We give a proper definition of a *bounded functor* in section 7 and show that terminal[1] coalgebras, more generally, arbitrarily large cofree coalgebras exist in $\mathcal{S}et_F$ whenever F is bounded.

There is a strong connection between the product $\mathcal{A}\times\mathcal{B}$ of two coalgebras and the largest bisimulation $\sim_{\mathcal{A},\mathcal{B}}$. We characterize precisely, when $\mathcal{A}\times\mathcal{B} \cong \sim_{\mathcal{A},\mathcal{B}}$. Some examples demonstrate the possibilities for products of two coalgebras. In particular, the product $\mathcal{A}\times\mathcal{B}$ of two nontrivial finite coalgebras $\mathcal{A}$ and $\mathcal{B}$ may be empty, isomorphic to $\mathcal{A}$ or even infinite.

During the writing of this article, A. Kurz [Kur99] has also given a direct proof for the existence of products in $\mathcal{S}et_F$, our Theorem 6.3. His proof still uses more category theoretic machinery than we shall require here.

2. Preliminaries

Let $F : \mathcal{S}et \to \mathcal{S}et$ be a functor. An *F-coalgebra*, or *coalgebra of type F*, is a pair $\mathcal{A} = (A, \alpha)$ where $\alpha : A \to F(A)$ is an arbitrary map. A is called the base set and α the *co-operation* or *structure map* of $\mathcal{A}$.

2.1. *Homomorphisms*

Given F-coalgebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$, a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a map $\varphi : A \to B$ which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ F(A) \arrow[r, "F(\varphi)"'] & F(B) \end{tikzcd}$$

[1] Some authors prefer the notion "final".

We shall frequently use without mentioning the following diagram lemmata from [Gum99a]:

LEMMA 2.1. (First Diagram Lemma) *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be F-coalgebras, $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{C}$ homomorphisms. If φ is surjective, then there is a (necessarily unique) homomorphism $\chi : \mathcal{B} \to \mathcal{C}$ with $\chi \circ \varphi = \psi$ iff* $\ker(\varphi) \subseteq ker(\psi)$.

$$\begin{tikzcd} A \arrow[r, two heads, "\varphi"] \arrow[dr, "\psi"'] & B \arrow[d, dashed, "\chi"] \\ & C \end{tikzcd}$$

LEMMA 2.2. (Second Diagram Lemma) *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be F-coalgebras, $\varphi : \mathcal{B} \to \mathcal{A}$ and $\psi : \mathcal{C} \to \mathcal{A}$ homomorphisms. If φ is injective, then there is a (necessarily unique) homomorphism $\chi : \mathcal{C} \to \mathcal{B}$ with $\varphi \circ \chi = \psi$ iff $\psi[C] \subseteq \varphi[B]$.*

$$\begin{tikzcd} A & B \arrow[l, tail, "\varphi"'] \\ & C \arrow[ul, "\psi"] \arrow[u, dashed, "\chi"'] \end{tikzcd}$$

2.2. *The category $\mathcal{S}et_F$*

For a fixed functor F, the class of all F-coalgebras forms a category $\mathcal{S}et_F$. Epimorphisms in $\mathcal{S}et_F$ are just the surjective homomorphisms, however, monomorphisms in $\mathcal{S}et_F$ need not be injective, see [GS00].

It is not hard to prove (see [Bar93]), that $\mathcal{S}et_F$ is co-complete as a category, that is every colimit (sum, coequalizer) exists in $\mathcal{S}et_F$, and it is formed just as in $\mathcal{S}et$, the category of sets. In particular, the sum in $\mathcal{S}et_F$ of a family of coalgebras $\mathcal{A}_i = (A_i, \alpha_i)$ has as base set the disjoint union of the A_i and the injection of each A_i into the disjoint union is a homomorphism.

The corresponding property is not true for limits. In fact, products need not exist in $\mathcal{S}et_F$. However, if F *preserves* a certain type of limit, then this type of limit exists and it is again formed just like in $\mathcal{S}et$.

For instance, a functor F is said to *preserve pullbacks*, if it transforms each pullback diagram into a pullback diagram. Pullbacks in the category of sets have an easy description: Given maps $f : A \to C$ and $g : B \to C$, their pullback in $\mathcal{S}et$ is given as $(pb(f, g), \pi_1, \pi_2)$ where

$$pb(f, g) := \{(a, b) \in A \times B \mid f(a) = g(b)\},$$

and $\pi_1 : pb(f, g) \to A$ and $\pi_2 : pb(f, g) \to B$ are the canonical projections. Therefore, F preserves pullbacks, iff there exists a unique map $\psi : pb(F(f), F(g)) \to F(pb(f, g))$ such that $F(\pi_1)(\psi(u, v)) = u$ and $F(\pi_2)(\psi(u, v)) = v$. Thus we have:

LEMMA 2.3. *F preserves the pullback of maps $f : A \to C$ and $g : B \to C$, iff for every (u, v) with $F(f)(u) = F(g)(v)$ there exists a unique $q \in F(pb(f, g))$ so that $F(\pi_1)(q) = u$ and $F(\pi_2)(q) = v$.*

If we drop the uniqueness requirement, then F is said to *weakly preserve* pullbacks.

2.3. *Subcoalgebras*

If $\mathcal{A} = (A, \alpha)$ is a coalgebra, and U a subset of A, then U is called *closed*, if a coalgebra structure $\mathcal{U} = (U, \delta)$ can be defined on U so that the natural inclusion $\subseteq : U \to A$ is a homomorphism. In this case $\mathcal{U}$ is called a subcoalgebra of $\mathcal{A}$ and we write $\mathcal{U} \leq \mathcal{A}$. A structure map δ as above on a closed set U is easily seen to be unique, therefore closed sets are often called subcoalgebras.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and $\mathcal{U} \leq \mathcal{A}$ then

$$\varphi[U] := \{\varphi(u) \mid u \in U\}$$

is a subcoalgebra of $\mathcal{B}$. In particular, $\varphi[A] \leq \mathcal{B}$.

On the other hand, if $\mathcal{V}$ is a subcoalgebra of $\mathcal{B}$, then

$$\varphi^{-}[V] := \{a \in A \mid \varphi(a) \in V\}$$

need *not* be a subcoalgebra of $\mathcal{A}$, unless F weakly preserves pullbacks!

The union of a family $(U_i)_{i \in I}$ of subcoalgebras of a fixed coalgebra $\mathcal{A}$ is again a subcoalgebra of $\mathcal{A}$, hence for any subset $S \subseteq A$ there is always a largest subcoalgebra of $\mathcal{A}$ contained in S, we denote it by $[S]$ and call it the coalgebra *cogenerated by S*.

2.4. *Bisimulations*

Bisimulations are the structure preserving relations between coalgebras. A bisimulation between coalgebras $\mathcal{A}$ and $\mathcal{B}$ is a binary relation $R \subseteq A \times B$ on which a coalgebra structure $\delta : R \to F(R)$ can be defined so that the projections $\pi_1, \pi_2 : R \to \mathcal{A}$ are homomorphisms:

$$\begin{tikzcd}
A \arrow[d, "\alpha_1"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "\delta"] & B \arrow[d, "\alpha_2"] \\
F(A) & F(R) \arrow[l, "F(\pi_1)"] \arrow[r, "F(\pi_2)"'] & F(B)
\end{tikzcd}$$

Note that for a given bisimulation R there may be different structure maps δ, δ', making the diagram commute. Each of them will be called a *bisimulation structure for R*.

The diagonal $\Delta_A := \{(a, a) \mid a \in A\}$ is always a bisimulation on $\mathcal{A}$, but $A \times A$ in general is not. A union of bisimulations between $\mathcal{A}$ and $\mathcal{B}$ is again a bisimulation between $\mathcal{A}$ and $\mathcal{B}$, hence there is always a largest one, called $\sim_{\mathcal{A},\mathcal{B}}$. Moreover, if $\mathcal{Q}$ is any coalgebra and if $\varphi_1, \varphi_2 : \mathcal{Q} \to \mathcal{A}$ are homomorphisms, then $(\varphi_1, \varphi_2)Q := \{(\varphi_1(q), \varphi_2(q)) \mid q \in Q\}$ is a bisimulation, see [Rut96].

2.5. *An example*

We shall illustrate the above notions with an example which we shall need in a later section. A *nondeterministic transition system* is commonly defined as a set S of states together with a transition relation σ. If $(a, b) \in \sigma$, we say that there is a transition from a to b and illustrate it graphically with an arrow:

$$a \xrightarrow{\sigma} b.$$

Transition systems are commonly used to model nondeterministic systems and they often are equipped with extra structure. Here we only consider the simplest possible case.

A transition system will be modelled as a coalgebra for the powerset functor $\mathcal{P}$ which associates with a set X the set of $\mathcal{P}(X)$ of all subsets of X. Given a map $f : X \to Y$, the functor returns the map $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ which is defined on an arbitrary $U \in \mathcal{P}(X)$ as

$$\mathcal{P}(f)(U) := f[U] := \{f(u) \mid u \in U\}.$$

We record, for later reference, several possible modifications of this functor, such as

- the nonempty powerset functor: $\mathcal{P}_+(X) := \mathcal{P}(X) - \{\emptyset\}$
- the finite powerset functor: $\mathcal{P}_\omega(X) := \{U \subseteq X \mid |U| < \omega\}$.

In each case, the functor acts on maps just like the powerset functor.

Let now (S, σ) be a transition system, we consider it as a $\mathcal{P}$-coalgebra $\mathcal{S} = (S, \alpha)$ by defining

$$\alpha(s) := \{s' \in S \mid (s, s') \in \sigma\}.$$

Then a map φ between transition systems $\mathcal{S} = (S, \sigma)$ and $\mathcal{T} = (T, \tau)$ is easily seen to be a homomorphism iff for all $s, s' \in S$ and all $t \in T$ we have

1. $s \xrightarrow{\sigma} s' \Longrightarrow \varphi(s) \xrightarrow{\tau} \varphi(s')$, and
2. $\varphi(s) \xrightarrow{\tau} t \Longrightarrow \exists s'. s \xrightarrow{\sigma} s'$ and $\varphi(s') = t$.

Similarly, a relation $R \subseteq S \times T$ is easily found to be a bisimulation, if for any $s, s' \in S$ and $t, t' \in T$ we have:

If $(s, t) \in R$ then

1. $s \xrightarrow{\sigma} s' \Longrightarrow \exists t' \in T.t \xrightarrow{\tau} t'$ and $(s', t') \in R$,
2. $t \xrightarrow{\tau} t' \Longrightarrow \exists s' \in S.s \xrightarrow{\sigma} s'$ and $(s', t') \in R$.

3. Congruences

Congruences on coalgebras were originally introduced by Aczel and Mendler in [AM89]. They were later given up in favor of *bisimulations* by Rutten in his treatment [Rut96]. Indeed, whenever F weakly preserves pullbacks, each congruence is a bisimulation. In general, though, this is not the case, and it turns out that we shall need congruences in order to construct, in a later section, the terminal coalgebra.

A *congruence* on a coalgebra $\mathcal{A}$ is defined as the kernel of a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, i.e. as

$$Ker(\varphi) := \{(x, y) \mid \varphi(x) = \varphi(y)\}.$$

If θ is a congruence on $\mathcal{A}$, then there is a canonical coalgebra structure on A/θ, the set of all θ-classes and $\pi_\theta : \mathcal{A} \to \mathcal{A}/\theta$ is a surjective homomorphism with $Ker(\pi_\theta) = \theta$.

Given congruences θ and ϕ, their intersection, in general, will not be a congruence. Fortunately, however, the supremum of a family $(\theta_i)_{i \in I}$ of congruences exists. This is shown in the following lemma:

LEMMA 3.1. *Let $(\theta_i)_{i \in I}$ be a nonempty family of congruences. Then the supremum of the θ_i exists and it is given as the transitive closure of their union, i.e.*

$$\bigvee_{i \in I} \theta_i = \left(\bigcup_{i \in I} \theta_i \right)^* .$$

Proof. $\Phi := (\bigcup_{i \in I} \theta_i)^*$ is the smallest equivalence relation containing all the θ_i. Every congruence θ is the kernel of some homomorphism $\pi_\theta : \mathcal{A} \to \mathcal{A}/\theta$. Form the pushout $(\mathcal{Q}, \psi_i)$ of all homomorphisms $\pi_{\theta_i} : \mathcal{A} \to \mathcal{A}/\theta_i$.

$$\begin{tikzcd}
 & A/\theta_i \arrow[dr, "\psi_i"] & \\
A \arrow[ur, "\pi_{\theta_j}"] \arrow[dr, "\pi_{\theta_j}"'] & & \mathcal{Q} \\
 & A/\theta_j \arrow[ur, "\psi_j"'] &
\end{tikzcd}$$

Since pushouts in $\mathcal{S}et_F$ are formed just as in $\mathcal{S}et$, the kernel of the homomorphism $\psi_i \circ \pi_{\theta_i}$ is just Φ, the congruence generated by all θ_i. $\square$

The supremum over the empty family also exists, it is the diagonal $\Delta_A := \{(a, a) \mid a \in A\}$. For every reflexive relation R, we can therefore define the congruence *cogenerated* by R as the supremum of all congruences contained in R, i.e.

$$Con[R] := \left(\bigcup\{\theta \mid \theta \text{ congruence}, \theta \subseteq R\}\right)^* .$$

We now have:

THEOREM 3.2. *The set of all congruences on a coalgebra $\mathcal{A}$ is a complete lattice. Suprema and infima are given as*

$$\begin{aligned} \bigvee_{i\in I} \theta_i &= \left(\bigcup_{i\in I} \theta_i\right)^* \\ \bigwedge_{i\in I} \theta_i &= Con\left[\bigcap_{i\in I} \theta_i\right]. \end{aligned}$$

The smallest congruence on $\mathcal{A}$ is always the diagonal Δ_A, but the largest congruence, we shall call it $\nabla_{\mathcal{A}}$, will in general be a proper subset of $A \times A$.

The second diagram lemma 2.2 has as consequence that the congruences on $\mathcal{A}/\theta$ correspond uniquely to the congruences on $\mathcal{A}$ which contain θ. In particular, the congruence lattice of $\mathcal{A}/\theta$ is isomorphic to the interval above θ of the congruence lattice of $\mathcal{A}$.

4. Simple and extensional coalgebras

A universal algebra is called *simple*, if it does not have any nontrivial congruence relation. We suggest to use the same definition for coalgebras.

Unfortunately, Rutten ([Rut96]) calls a coalgebra simple, if it does not have any nontrivial bisimulation, that is, if the largest bisimulation is the diagonal. We suggest to call such coalgebras *extensional*. When the type functor weakly preserves pullbacks, both notions agree, but in general they don't, see e.g. [GS00]. The situation of the largest bisimulation $\sim$ being the diagonal can be expressed as a proof rule:

$$\frac{x \sim y}{x = y}.$$

Since bisimilarity represents indistinguishability by observations, we can understand this proof rule as a *principle of extensionality*:

Two elements that can not be distinguished by observations are equal.

With this in mind, we shall call a coalgebra *extensional* if the largest bisimulation is the diagonal.

The largest congruence relation of a simple coalgebra is the diagonal, and conversely, factoring any coalgebra by its largest congruence relation yields a simple factor. Moreover, we have:

LEMMA 4.1. *Every simple coalgebra is extensional.*

Proof. Any bisimulation R on $\mathcal{A}$ yields two homomorphisms $\pi_1, \pi_2 : R \to A$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be the coequalizer of π_1 and π_2. If $\mathcal{A}$ is simple, then $Ker(\varphi)$ is trivial, hence $\pi_1 = \pi_2$, so R is the diagonal. □

Extensional coalgebras can be characterized in the following way:

LEMMA 4.2. *For a coalgebra $\mathcal{A}$ the following are equivalent:*

1. *$\mathcal{A}$ is extensional.*
2. *For every coalgebra $\mathcal{B}$ there is at most one homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$.*

Proof. If $\mathcal{A}$ has a proper bisimulation R, then there are two different homomorphisms $\pi_1, \pi_2 : R \to A$. Conversely, if there are two different homomorphisms $\varphi_1, \varphi_2 : \mathcal{B} \to \mathcal{A}$, then $(\varphi_1, \varphi_2)B$ is a nontrivial bisimulation on $\mathcal{A}$. □

Let us consider the example of automata with a fixed alphabet Σ. Conventionally, a Σ-automaton is given as a triple (A, δ, T) where $\delta : A \times \Sigma \to A$ is the transition function and $T \subseteq A$ is the set of terminal states. This information can conveniently be coded coalgebraically into a single map:

$$\alpha : A \to A^{\Sigma} \times \{true,\ false\}$$

where $\alpha(a) = (\tau_a, b)$ with

$$\begin{aligned} \tau_a(\sigma) &= \delta(a, \sigma), \quad \text{and} \\ b &= (a \in T). \end{aligned}$$

Thus automata are coalgebras of type F, where F is the functor associating with a set X the set

$$F(X) := X^{\Sigma} \times \{true,\ false\}.$$

A map $f : X \to Y$ is transformed into a map $F(f) : F(X) \to F(Y)$ given as

$$F(f)(\gamma, b) := (f \circ \gamma, b).$$

It is not hard to see that a congruence relation on such a coalgebra is just an equivalence relation θ satisfying for every $x, y \in A$ and every $\sigma \in \Sigma$:

$$x\theta y \Longrightarrow (\delta(x, \sigma)\theta\delta(y, \sigma),\ \text{and}\ (x \in T \iff y \in T)).$$

The largest congruence on such a coalgebra is therefore just the *Nerode congruence* and factoring by it yields the corresponding minimal automaton.

5. Limits

We begin with equalizers, that is limits of two parallel arrows. They always exist without any requirement on the functor F. The following theorem is stated in [Wor98] for the case that F weakly preserves pullbacks:

THEOREM 5.1. *Given homomorphisms $\varphi_1, \varphi_2 : \mathcal{A} \to \mathcal{B}$. Let*

$$E := \{a \in A \mid \varphi_1(a) = \varphi_2(a)\},$$

then $[E] \leq \mathcal{A}$ is the equalizer of φ_1 and φ_2.

Proof. For the canonical embedding $\leq : [E] \to \mathcal{A}$ we clearly have that $\varphi_1 \circ \leq = \varphi_2 \circ \leq$. Let $\kappa : \mathcal{Q} \to \mathcal{A}$ be given with $\varphi_1 \circ \kappa = \varphi_2 \circ \kappa$ then $\kappa[Q] \leq \mathcal{A}$, and consequently, $\kappa[Q] \leq [E]$, hence κ uniquely factors through $[E]$.

$$\begin{tikzcd}
[E] \arrow[r, "\leq"] & A \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & B \\
Q \arrow[u, dashed] \arrow[ur, "\kappa"'] & &
\end{tikzcd}$$

□

It is well known that a category is *complete*, i.e. it has all possible limits, provided equalizers and products exist. The next result shows that the product of a family $(\mathcal{A}_i)_{i\in I}$ exists, provided each $\mathcal{A}_i$ can be embedded into some larger $\mathcal{B}_i$ for which the product of the $(\mathcal{B}_i)_{i\in I}$ exists:

THEOREM 5.2. *Assume that the product $\mathcal{B} := \Pi_{i\in I}\mathcal{B}_i$ exists with projections $\eta_i : \mathcal{B} \to \mathcal{B}_i$ for each $i \in I$. If $\mathcal{A}_i \leq_i \mathcal{B}_i$ is a subcoalgebra for each $i \in I$ then the product $\Pi_{i\in I}\mathcal{A}_i$ exists and it is isomorphic to a subcoalgebra of $\mathcal{B}$.*

Proof. Let

$$D := \bigcup\{U \mid \mathcal{U} \leq \mathcal{B}, \forall i \in I.\ \eta_i[U] \subseteq A_i\},$$

then D is a subcoalgebra of $\mathcal{B}$ and for every $i \in I$ the homomorphism $\eta_i \circ \leq : \mathcal{D} \to \mathcal{B}_i$ factors through $\mathcal{A}_i$, that is there is a homomorphism $\nu_i : D \to A_i$ with $\eta_i \circ \leq = \leq_i \circ \nu_i$. We claim that $(D, (\nu_i)_{i\in I})$ is the product of the $(\mathcal{A}_i)_{i\in I}$. Let $\mathcal{Q}$ with homomorphisms

$\mu_i : \mathcal{Q} \to \mathcal{A}_i$ be a competitor for $\mathcal{D}$, then $\mathcal{Q}$ with the morphisms $(\leq_i \circ \mu_i) : Q \to \mathcal{B}_i$ is a competitor for the product of the $\mathcal{B}_i$. Hence there is a unique homomorphism $\psi : Q \to \mathcal{B}$ so that $\eta_i \circ \psi = \leq_i \circ \mu_i$ for all $i \in I$. $\psi[Q]$ is a subcoalgebra of $\mathcal{B}$ and $\eta_i[\psi[Q]] \subseteq A_i$, hence ψ factors through $\mathcal{D}$ as $\psi = \leq \circ \tilde{\psi}$. It follows

$$\begin{aligned} \leq_i \circ \nu_i \circ \tilde{\psi} &= \eta_i \circ \leq \circ \tilde{\psi} \\ &= \eta_i \circ \psi \\ &= \leq_i \circ \mu_i . \end{aligned}$$

Since $\leq_i$ is mono, it follows that $\nu_i \circ \tilde{\psi} = \mu_i$. $\tilde{\psi}$ is unique with this property, for assume that there was another homomorphisms $\hat{\psi}$ with $\nu_i \circ \hat{\psi} = \mu_i$, then it follows that $\eta_i \circ \leq \circ \tilde{\psi} = \eta_i \circ \leq \circ \hat{\psi}$ for all $i \in I$, hence $\leq \circ \tilde{\psi} = \leq \circ \hat{\psi}$ and thus $\tilde{\psi} = \hat{\psi}$, since $\leq$ is left cancellable.

$$\begin{tikzcd} Q \arrow[dr, dashed, "\tilde{\psi}"] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\mu_i"'] & & \\ & D \arrow[r, "\leq"] \arrow[d, "\nu_i"] & \prod_{i\in I} B_i \arrow[d, "\eta_i"] \\ & A_i \arrow[r, "\leq_i"] & B_i \end{tikzcd}$$

□

Until now, we have only considered products (and limits) with respect to the class $\mathcal{S}et_F$. However, limits with respect to subclasses $\mathcal{K}$ of $\mathcal{S}et_F$ are rather easily obtained from limits with respect to $\mathcal{S}et_F$, provided that $\mathcal{K}$ is closed under homomorphic images ($\mathcal{K} = \mathcal{H}(\mathcal{K})$) and sums ($\mathcal{K} = \Sigma(\mathcal{K})$). Such classes are called co-quasivarieties in [Gum99b]. The limit of any diagram D in $\mathcal{K}$ is obtained by first forming the limit $\mathcal{L}$ of D in $\mathcal{S}et_F$ and then taking the largest subcoalgebra $\mathcal{U}$ of $\mathcal{L}$ which is a member of $\mathcal{K}$. In particular:

PROPOSITION 5.3. *If $\mathcal{S}et_F$ is complete, then so is every subclass $\mathcal{K}$ of $\mathcal{S}et_F$, which is closed under sums and homomorphic images.*

6. Cofree Coalgebras

Let X be any set. A coalgebra $\mathcal{S}_X$ with a map $\epsilon_X : \mathcal{S}_X \to X$ is called *cofree over X* if for any F-coalgebra $\mathcal{A}$ and any map $g : A \to X$ there exists exactly one homomorphism $\tilde{g} : \mathcal{A} \to \mathcal{S}_X$ with $\varepsilon_X \circ \tilde{g} = g$.

$$\begin{tikzcd} & X \\ A \arrow[ur, "g"] \arrow[r, dashed, "\tilde{g}"] & \mathcal{S}_X \arrow[u, "\epsilon_X"'] \end{tikzcd}$$

The set X is often thought of as a set of "colors" and g a coloring. Note that, in particular, the cofree coalgebra over the one-element set is the same as the terminal object P in $\mathcal{S}et_F$. The structure map $\alpha_P : P \to F(P)$ on a terminal coalgebra P must be an isomorphism, in particular, P and $F(P)$ must have the same cardinality. This observation is due to Lambek, see [Gum99a] for a proof. This is the reason that terminal coalgebras cannot exist, for instance, for $F = \mathcal{P}$, the powerset functor.

If there is no terminal coalgebra, then there can be no cofree coalgebra $\mathcal{S}_X$, unless $X = \emptyset$. This follows from the following lemma:

LEMMA 6.1. *If $\mathcal{S}_Y$ exists, then $\mathcal{S}_X$ exists for any $X \subseteq Y$, in fact $\mathcal{S}_X \leq \mathcal{S}_Y$.*

Proof. Let ι be the canonical embedding of X into Y and define

$$Q := \bigcup \{\widetilde{\iota \circ g}[A] \mid g : \mathcal{A} \to X, \mathcal{A} \in \mathcal{S}et_F\}.$$

The embedding $\leq : Q \to \mathcal{S}_Y$ composed with $\epsilon_Y : \mathcal{S}_Y \to Y$ factors through X, yielding the required map $\epsilon_X : Q \to X$. It is easy to check that (Q, ϵ_X) is cofree over X.

$$\begin{tikzcd} & X \arrow[r, "\iota"] & Y \\ A \arrow[ur, dotted, "g"] \arrow[r, dotted, "\tilde{g}"'] & Q \arrow[u, dashed, "\epsilon_X"'] \arrow[r, "\leq"] & \mathcal{S}_Y \arrow[u, "\epsilon_Y"'] \end{tikzcd}$$

□

The following proposition is now immediate to check:

PROPOSITION 6.2. *Let $(X_i)_{i \in I}$ be a family of sets and Y any set larger than their cartesian product. If $\mathcal{S}_Y$ exists, then $\Pi_{i \in I} \mathcal{S}_{X_i} \cong \mathcal{S}_{\Pi_{i \in I} X_i}$.*

Thus we get the following criterion for the existence of limits in $\mathcal{S}et_F$:

THEOREM 6.3. *If arbitrarily large cofree coalgebras of type F exist, then $\mathcal{S}et_F$ is complete.*

Proof. Due to theorem 5.1 it is enough to consider products of a family $(\mathcal{A}_i)_{i\in I}$. Each $\mathcal{A}_i$ is isomorphic to a subcoalgebra of a cofree coalgebra $\mathcal{S}_{X_i}$ whenever $|X_i| \geq |A_i|$. By Theorem 5.2 then $\Pi_{i\in I}\mathcal{A}_i$ exists and it is a subcoalgebra of $\mathcal{S}_{\Pi_{i\in I}X_i}$. □

This criterion is almost an equivalent characterization, for if products exist and if there is at least one cofree coalgebra $\mathcal{S}_X$ with $|X| > 1$, then by Lemma 6.1 and Proposition 6.2 cofree coalgebras exist over every color set. That is, we have:

COROLLARY 6.4. *If the cofree coalgebra $\mathcal{S}_X$ over some color set X with $|X| > 1$ exists, then $\mathcal{S}et_F$ is complete if and only if cofree coalgebras exist over arbitrarily large color sets.*

7. Bounded functors

Our criterion for completeness of $\mathcal{S}et_F$ as given in Theorem 6.3 is rather abstract. The question remains, how to check whether arbitrary cofree coalgebras exist, without constructing them. In [Rut96] a criterion for the existence of cofree coalgebras is formulated as a boundedness condition on the type functor: F is called bounded if there is a cardinality bound on the size of all one-generated F-coalgebras.

This definition was made with functors weakly preserving pullbacks in mind, for unless F weakly preserves generalized pullbacks of injective maps, intersections of subcoalgebras do not exist (see [GS00]), so the concept of "generated subcoalgebra" makes no sense. For general functors, therefore, we shall need an appropriate substitute for boundedness:

DEFINITION 7.1. A functor $F : \mathcal{S}et \to \mathcal{S}et$ is called *bounded*, if there is a cardinality κ, so that for every F-coalgebra $\mathcal{A}$ and any $a \in A$ there exists a subcoalgebra $\mathcal{U}_a \leq \mathcal{A}$ of cardinality at most κ with $a \in U_a$.

It is easy to see that this notion of boundedness is equivalent to saying that the category $\mathcal{S}et_F$ has a set of generators. Therefore, the following result could be obtained by invoking the "Special Adjoint Functor Theorem", see e.g. [Lan71], but in the present context a direct proof is much simpler:

THEOREM 7.2. *If F is bounded then the terminal F-coalgebra exists.*

Proof. Let $(\mathcal{G}_i)_{i\in I}$ be a family of coalgebras containing an isomorphic copy of each coalgebra A of cardinality less than or equal to κ, where κ is the bound of F. Let $\mathcal{G} = \Sigma_{i\in I}\mathcal{G}_i$ be the sum of all $\mathcal{G}_i$ and let $\nabla_{\mathcal{G}}$ be the largest congruence relation on $\mathcal{G}$. We claim that $\mathcal{P} := \mathcal{G}/\nabla_{\mathcal{G}}$ is terminal.

Consider any coalgebra $\mathcal{A} \in \mathcal{S}et_F$, we need to find a homomorphism from $\mathcal{A}$ to $\mathcal{P}$. From the sum $\Sigma_{a\in A}\mathcal{U}_a$ we get a surjective homomorphism φ to $\mathcal{A}$. Since each $\mathcal{U}_a$ has an isomorphic copy amongst the $\mathcal{G}_i$, we also have a canonical homomorphism $\psi : \Sigma_{a\in A}\mathcal{U}_a \to \mathcal{G}$.

Form the pushout $(\mathcal{Q}, \hat{\varphi}, \hat{\psi})$ of φ with ψ. Since pushouts of epis are always epi, $\hat{\varphi}$ is an epimorphism and its kernel must be contained in $\nabla_{\mathcal{G}}$. Thus we obtain a unique homomorphism $\chi : \mathcal{Q} \to \mathcal{P}$, and $\chi \circ \hat{\psi}$ is a homomorphism, as required, from $\mathcal{A}$ to $\mathcal{P}$. By construction, $\mathcal{P}$ is simple, hence also extensional by Lemma 4.1, consequently, by Lemma 4.2, the homomorphism from $\mathcal{A}$ to $\mathcal{P}$ is unique.

$$
\begin{tikzcd}
A \arrow[r, "\hat{\psi}"] & Q \arrow[r, "\chi"] & P \\
U_a \arrow[u, hook] \arrow[r] \arrow[rr, bend right] & \sum_{a\in A} U_a \arrow[ul, "\varphi"'] \arrow[r, "\psi"] & \sum_{i\in I} G_i \arrow[ul, "\hat{\varphi}"'] \arrow[u, "\pi_{\nabla_G}"']
\end{tikzcd}
$$

□

Now let X be a fixed set. We consider the functor $X \times F(-)$. An $X \times F(-)$ coalgebra is just a triple (A, α, ϵ) where (A, α) is an F-coalgebra and $\epsilon : A \to X$ is any map.

If F is bounded, then obviously, $X \times F(-)$ is bounded too. Moreover, (A, α, ϵ) is terminal as $X \times F(-)$-coalgebra if and only if (A, α) with coloring $\epsilon : A \to X$ is cofree over X. Hence we obtain from 7.2:

THEOREM 7.3. *If the functor F is bounded, then cofree coalgebras exist over every color set X.*

As a consequence of 6.3, we finally have a usable criterion for completeness of $\mathcal{S}et_F$:

THEOREM 7.4. *If the functor F is bounded, then $\mathcal{S}et_F$ is complete.*

The converse does not hold. In fact, J. Adámek [Adá00] has shown us an example of an unbounded functor for which all cofree coalgebras exist.

8. Products and Bisimulations

Constructing the product $\mathcal{A}_1 \times \mathcal{A}_2$ of two coalgebras can, in practice, be rather cumbersome, for one needs to construct a coalgebra which is cofree over a color set of size $|A_1| \cdot |A_2|$. One would hope to get by with a simpler construction. In general, this seems impossible, but for many functors arising in practice the product allows for an easier description, indeed.

First of all, we note some fundamental relationships between the product $\mathcal{A}_1 \times \mathcal{A}_2$, and $\sim_{\mathcal{A}_1, \mathcal{A}_2}$, the largest bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$, for we have:

LEMMA 8.1. *Let* $\mathcal{A}_1 \times \mathcal{A}_2$ *with projections* η_1 *and* η_2 *be the product in* $\mathcal{S}et_F$ *of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, and let* $\sim_{\mathcal{A}_1,\mathcal{A}_2}$ *be the largest bisimulation between* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*. Then* $(\eta_1, \eta_2)(\mathcal{A}_1 \times \mathcal{A}_2) = \sim_{\mathcal{A}_1,\mathcal{A}_2}$.

Proof. We know that the set $(\eta_1, \eta_2)(\mathcal{A}_1 \times \mathcal{A}_2)$ is a bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$, so it is contained in $\sim_{\mathcal{A}_1,\mathcal{A}_2}$. On the other hand, $\sim_{\mathcal{A}_1,\mathcal{A}_2}$ can be equipped with a coalgebra structure so that the projections $\pi_i : \sim_{\mathcal{A}_1,\mathcal{A}_2} \to \mathcal{A}_i$ become homomorphisms. This way, we have created a competitor for the product, so there is a (unique) homomorphism $\psi : \sim_{\mathcal{A}_1,\mathcal{A}_2} \to \mathcal{A}_1 \times \mathcal{A}_2$ with $\eta_i \circ \psi = \pi_i$. Given $(a, b) \in \sim_{\mathcal{A}_1,\mathcal{A}_2}$, we therefore find $\psi(a, b) \in \mathcal{A}_1 \times \mathcal{A}_2$ with $(\eta_1, \eta_2)(\psi(a, b)) = (a, b)$. $\square$

Let us now fix a bisimulation structure for $\sim_{\mathcal{A}_1,\mathcal{A}_2}$, i.e. a structure map $\delta : \sim_{\mathcal{A}_1,\mathcal{A}_2} \to F(\sim_{\mathcal{A}_1,\mathcal{A}_2})$, for which the projections are homomorphisms. The map $(\eta_1, \eta_2) : \mathcal{A}_1 \times \mathcal{A}_2 \to \sim_{\mathcal{A}_1,\mathcal{A}_2}$ is in general not a homomorphism, but if it is, then it will actually be an isomorphism. To be precise:

LEMMA 8.2. *If the product* $\mathcal{A}_1 \times \mathcal{A}_2$ *with projections* $\eta_i : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_i$ *exists, then the following are equivalent:*

1. $(\eta_1, \eta_2) : \mathcal{A}_1 \times \mathcal{A}_2 \to \sim_{\mathcal{A}_1,\mathcal{A}_2}$ *is a homomorphism.*
2. *There is a homomorphism* $\varphi : \mathcal{A}_1 \times \mathcal{A}_2 \to \sim_{\mathcal{A}_1,\mathcal{A}_2}$ *with* $\pi_i \circ \varphi = \eta_i$.
3. $\sim_{\mathcal{A}_1,\mathcal{A}_2}$ *with the projections* $\pi_i : \sim_{\mathcal{A}_1,\mathcal{A}_2} \to \mathcal{A}_i$ *is the product of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$.

Proof. The equivalence of (1.) and (2.) and the implication (3.) $\to$ (2.) are obvious. Now assuming (1.), we know by Lemma 8.1 that (η_1, η_2) is surjective. To see that it actually must be an isomorphism, observe that there is a homomorphism $\psi : \sim_{\mathcal{A}_1,\mathcal{A}_2} \to \mathcal{A}_1 \times \mathcal{A}_2$ with $\eta_i \circ \psi = \pi_i$. It follows that $\eta_i \circ id_{\mathcal{A}_1 \times \mathcal{A}_2} = \eta_i \circ \psi \circ (\eta_1, \eta_2)$, hence $id_{\mathcal{A}_1 \times \mathcal{A}_2} = \psi \circ (\eta_1, \eta_2)$, so (η_1, η_2) is an isomorphism. $\square$

The above lemma suggests to investigate how one might be able to define a coalgebra structure on $\sim_{\mathcal{A}_1,\mathcal{A}_2}$ so that not only the projections, but also the canonical map (η_1, η_2) become homomorphisms. We certainly need a condition on the functor F.

DEFINITION 8.3. A 2-source (X, f, g) is a pair of maps with common domain X. (X, f, g) is called a *mono source*, if for any two maps $h_1, h_2 : Y \to X$ we have:

$(f \circ h_1 = f \circ h_2)$ and $(g \circ h_1 = g \circ h_2)$ implies $h_1 = h_2$.

Typical mono sources arise from limits, in particular, the cartesian product $X \times Y$ of two sets X and Y together with the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ forms a mono source $(X \times Y, \pi_1, \pi_2)$. A functor is said to *preserve* a mono source (X, f, g), just in case $(F(X), F(f), F(g))$ is again a mono source.

LEMMA 8.4. *For a functor $F : \mathcal{S}et \to \mathcal{S}et$ the following are equivalent:*

1. *F preserves mono sources.*
2. *F preserves mono sources of the form $(Y \times Z, \pi_1, \pi_2)$.*
3. *For any sets Y, Z and any $u, v \in F(Y \times Z)$ we have*

$$F(\pi_1)(u) = F(\pi_1)(v) \text{ and } F(\pi_2)(u) = F(\pi_2)(v) \Longrightarrow u = v.$$

Proof. (1.) $\to$ (2.) and (2.) $\leftrightarrow$ (3.) are obvious. For the direction (2.) $\to$ (1.) observe that maps $f : X \to Y$ and $g : X \to Z$ form a mono source if and only if the canonical map $(f, g) : X \to Y \times Z$ is injective.

$$\begin{tikzcd}
X \arrow[r, "{(f,g)}"] \arrow[d, "f"'] \arrow[dr, "g"] & Y \times Z \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \\
Y & Z
\end{tikzcd}$$

If $X = \emptyset$ then there is nothing to prove, since $\emptyset = \emptyset \times \emptyset$, otherwise $F(f, g) : F(X) \to F(Y \times Z)$ is injective. By assumption, $(F(Y \times Z), F(\pi_1), F(\pi_2))$ is a mono source, therefore, the same is true for $(F(X), F(\pi_1) \circ F(f, g), F(\pi_2) \circ F(f, g))$. But this is nothing else but $(F(X), F(f), F(g))$. □

Using the last criterion of Lemma 8.4, it is easy to check that the following functors preserve mono sources:

- The identity functor,
- the power functor: $X \to X^\Sigma$ for a fixed set Σ,
- constant functors: $X \to C$ for a constant set C.
- If functors F and G preserve mono sources then so do
  - $F \times G$
  - $F + G$.
- If G preserves mono sources and $\nu : F \to G$ is a natural transformation with all ν_A injective, then F preserves mono sources.

In particular, the type functor for deterministic automata with alphabet Σ, that is $F(X) := X^\Sigma \times 2$, preserves mono sources. Also, the functor $(-)^3_2$ which was a rich source of counterexamples in [GS00], preserves mono sources.[2]

[2] On a set X define $(X)^3_2 := \{(x_1, x_2, x_3) \in X^3 \mid |\{x_1, x_2, x_3\}| \leq 2\}$. Maps are extended componentwise. There is an obvious natural transformation ν from $(-)^3_2$ to $(-)^3$ with all components ν_X being injective mappings.

Every bisimulation R between A_1 and A_2 with its projection homomorphisms $\pi_i : R \to A_i$ provides a mono source (R, π_1, π_2). If F preserves mono sources, then $(F(R), F(\pi_1), F(\pi_2))$ is also a mono source, so from the diagram defining a bisimulation (Section 2.4) we read:

LEMMA 8.5. *If F preserves mono sources then every bisimulation R has a unique bisimulation structure.*

We are now able to characterize when the product of two coalgebras is the largest bisimulation:

THEOREM 8.6. *For any functor $F : \mathcal{S}et \to \mathcal{S}et$ the following are equivalent:*

1. $\mathcal{A}_1 \times \mathcal{A}_2 \cong \sim_{\mathcal{A}_1,\mathcal{A}_2}$ *for every $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}et_F$ and for an appropriate bisimulation structure on $\sim_{\mathcal{A}_1,\mathcal{A}_2}$.*
2. *F preserves mono sources.*
3. *Every bisimulation R between coalgebras in $\mathcal{S}et_F$ has a unique bisimulation structure.*
4. $\sim_{\mathcal{A}_1,\mathcal{A}_2}$ *has a unique bisimulation structure for any $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}et_F$.*

Proof. (2.) $\to$ (3.) is Lemma 8.5, (3.) $\to$ (4.) is a specialization and (1.) $\to$ (4.) is straightforward. It is therefore enough to show the implications (4.) $\to$ (2.) and (2.) $\to$ (1.).

We start with the latter, (2.) $\to$ (1.). Let $\sim$ be the largest bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$ and $\delta : \sim \to \mathcal{F}(\sim)$ a structure map on $\sim$ with projection homomorphisms $\pi_i : \sim \to \mathcal{A}_i$. Consider any coalgebra P together with homomorphisms $\eta_i : P \to \mathcal{A}_i$. We have to show that there is a unique homomorphism $\psi : P \to \sim$ with $\pi_i \circ \psi = \eta_i$. Uniqueness is clear, for $(\sim, \pi_1, \pi_2)$ is a mono source. Setting $\psi := (\eta_1, \eta_2)$, we only need to establish that it is a homomorphism.

The following diagram depicts the situation.

$$\begin{tikzcd}
P \arrow[r, "(\eta_1,\eta_2)"'] \arrow[rr, bend left, "\eta_i"] & \sim \arrow[r, "\pi_i"'] \arrow[d, "\delta"] & A_i \\
F(P) \arrow[u, "\alpha_p"] \arrow[r, "F(\eta_1,\eta_2)"] \arrow[rr, bend right, "F(\eta_i)"'] & F(\sim) \arrow[r, "F(\pi_i)"] & F(A_i) \arrow[u, "\alpha_i"']
\end{tikzcd}$$

We have

$$\begin{aligned} F(\pi_i) \circ \delta \circ (\eta_1, \eta_2) &= \alpha_i \circ \eta_i \\ &= F(\eta_i) \circ \alpha_P \\ &= F(\pi_i) \circ F(\eta_1, \eta_2) \circ \alpha_P. \end{aligned}$$

Since F preserves mono sources, we can cancel $F(\pi_i)$, which means that (η_1, η_2) is a homomorphism.

Finally, we prove (4.) $\rightarrow$ (2.). Assume that F does not preserve mono sources, we need to find coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with two different bisimulation structures on $\sim_{\mathcal{A}_1,\mathcal{A}_2}$. Invoking Lemma 8.4, we find sets Y, Z and $u, v \in F(Y \times Z)$ with $u \neq v$ and

$$F(\pi_1)(u) = F(\pi_1)(v) \text{ and } F(\pi_2)(u) = F(\pi_2)(v).$$

On Y we introduce a coalgebra structure $\alpha_Y : Y \rightarrow F(Y)$ by constantly mapping every element of Y to $F(\pi_1)(u)$. Similarly, we define $\alpha_Z : Z \rightarrow F(Z)$ as the constant function with result $F(\pi_2)(u)$. Now the constant maps $\delta_u, \delta_v : Y \times Z \rightarrow F(Y \times Z)$ with result u, resp. v, yield two different bisimulation structures on $Y \times Z$. Clearly, $Y \times Z$ is the largest bisimulation between the given coalgebras on Y, resp. Z. □

Observe that in the proof of this theorem we did not need to assume that the product $\mathcal{A}_1 \times \mathcal{A}_2$ exists. Hence, we have the following corollary:

COROLLARY 8.7. *If F preserves mono sources then the product $\mathcal{A}_1 \times \mathcal{A}_2$ exists for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}et_F$.*

Most of the literature on coalgebras assumes that the type functor F preserves weak pullbacks. The reason is that almost all applications in Computer Science can be modeled with a type functor having this property.

LEMMA 8.8. *F preserves weak pullbacks and mono sources iff F preserves pullbacks.*

Proof. The *only-if*-direction is immediately checked. Let now F preserve pullbacks, then it also preserves weak pullbacks according to [Rut96]. To check that F preserves mono sources, it is enough, by 8.4, to consider a mono source of the form $(X \times Y, \pi_1, \pi_2)$. However, this is the same as the pullback of the constant maps from X, resp. Y, into a one-element set. The hypothesis now tells us, that $(F(X \times Y), F(\pi_1), F(\pi_2))$ is a pullback, i.e. a limit. Clearly, every limit is a mono source. □

COROLLARY 8.9. *Suppose that F preserves weak pullbacks. Then the following are equivalent:*

1. *$\mathcal{A}_1 \times \mathcal{A}_2 \cong \sim_{\mathcal{A}_1,\mathcal{A}_2}$ for all coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$.*
2. *F preserves pullbacks.*
3. *F preserves mono sources.*

In this section we have only considered products of two coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$. We could as well generalize all results to arbitrary products $(A_k)_{k\in\kappa}$ for any cardinal number $\kappa > 0$. We need to replace 2-sources by κ-sources and *bi*simulations by κ-simulations, see [Gum99a]. A mono source, in general, will then be any κ-source $(X, (f_k)_{k\in\kappa})$ so that for any Y and all maps $g, h : Y \to X$ we have

$$(\forall k \in \kappa.\ f_k \circ g = f_k \circ h) \Longrightarrow g = h.$$

In particular we get: If F preserves arbitrary mono sources then $\mathcal{S}et_F$ has non-empty products, and the product of any family $(A_i)_{i\in\kappa} \subseteq \mathcal{S}et_F$ is given by the greatest κ-simulation between the $(A_i)_{i\in\kappa}$. In fact, Trnková has shown in [Trn71] that in this case F is bounded. In order to prove that all cofree coalgebras exist, a weaker condition suffices:

THEOREM 8.10. *If F preserves ω-mono-sources then $\mathcal{S}et_F$ has all cofree coalgebras.*

Proof. For an arbitrary fixed set X one checks that F preserves ω-mono-sources if and only if the functor $X \times F(-)$ preserves ω-mono-sources. Since a coalgebra cofree over X is nothing but a terminal $X \times F(-)$-coalgebra, it suffices to show that $\mathcal{S}et_F$ has a terminal F-coalgebra.

The problem can be further reduced to finding a *weakly terminal* coalgebra, that is an F-coalgebra $\mathcal{L} = (L, \rho)$ so that for each F-coalgebra $\mathcal{A}$ there is a (not necessarily unique) F-homomorphism $\mathcal{A} \to \mathcal{L}$; factorizing by the largest congruence $\nabla_{\mathcal{L}}$, we get that $\mathcal{L}/\nabla_{\mathcal{L}}$ is terminal.

We consider the *terminal sequence of F* ([Bar93], [Wor99]; from the latter reference the idea for the following proof is adapted):

$$1 \xleftarrow{\ !\ } F(1) \xleftarrow{F(!)} F^2(1) \longleftarrow \cdots \longleftarrow F^n(1) \xleftarrow{F^n(!)} F^{n+1}(1) \longleftarrow \cdots$$

Here 1 denotes a one-element set, and for any set X we let $!_X$, or simply $!$, denote the unique map from X to 1.

Let $(L, (\sigma_n : L \to F^n(1))_{n\in\omega})$ be the limit of the above terminal sequence. L is nonempty, since each $F^n(!)$ is surjective.[3]

Now $(F(L), (\tau_n)_{n\in\omega})$ is a competitor for this limit, if we set

$$\tau_0 := !_{F(L)}, \text{ and } \tau_{n+1} := F(\sigma_n).$$

[3] We may assume $F(1) \neq \emptyset$, since otherwise $F(A) = \emptyset$ for each set A and $\mathcal{S}et_F = \{\emptyset\}$, for which the claim of the theorem is trivial.

Therefore, we get a (unique) mediating map $m : F(L) \to L$ with

$$\forall n \in \omega.\ \sigma_{n+1} \circ m \ = \ F(\sigma_n).$$

The assumption on F implies that $(F(L), (F(\sigma_n))_{n\in\omega})$ is a mono source, hence m is injective.

$$\begin{tikzcd}
F(L) \arrow[r, "F(\sigma_n)"] \arrow[d, "m"] & F^{n+1}(1) \arrow[d, "F^n(!)"] \\
L \arrow[r, "\sigma_n"'] \arrow[u, dashed, bend left, "\rho"] \arrow[ur, "\sigma_{n+1}"] & F^n(1)
\end{tikzcd}$$

Choose a left inverse of m, that is any map $\rho : L \to F(L)$ with $\rho \circ m = id_{F(L)}$. We claim that the coalgebra $\mathcal{L} := (L, \rho)$ is weakly terminal.

So let $\mathcal{A} = (A, \alpha)$ be an arbitrary F-coalgebra. We have to construct a homomorphism $\varphi : \mathcal{A} \to \mathcal{L}$. We begin by setting up a family $(\kappa_n)_{n\in\omega}$ of maps from A into the terminal sequence:

$$\kappa_0 \ := !_A, \text{ and } \ \kappa_{n+1} := F(\kappa_n) \circ \alpha.$$

A straightforward induction shows that $F^n(!) \circ \kappa_{n+1} = \kappa_n$, i.e. $(\kappa_n)_{n\in\omega}$ is a cone over the terminal sequence. Hence there is a unique mediating map $\varphi : A \to L$ so that for all $n \in \omega$:

$$\sigma_n \circ \varphi = \kappa_n.$$

$$\begin{tikzcd}
F(L) \arrow[r, "F(\sigma_n)"] \arrow[d, "m"] & F^{n+1}(1) \arrow[d, "F^n(!)"] & F(\mathrm{A}) \arrow[l, "F(\kappa_n)"'] \arrow[ll, bend right, "F(\varphi)"'] \\
L \arrow[r, "\sigma_n"'] \arrow[u, bend left, "\rho"] \arrow[ur, "\sigma_{n+1}"] & F^n(1) & A \arrow[l, "\kappa_n"] \arrow[u, "\alpha"'] \arrow[ul, "\kappa_{n+1}"'] \arrow[ll, bend left, "\varphi"]
\end{tikzcd}$$

For arbitrary $n \in \omega$ we calculate

$$\begin{aligned}
\sigma_{n+1} \circ \varphi &= \kappa_{n+1} \\
&= F(\kappa_n) \circ \alpha
\end{aligned}$$

$$\begin{aligned} &= F(\sigma_n \circ \varphi) \circ \alpha \\ &= F(\sigma_n) \circ F(\varphi) \circ \alpha \\ &= \sigma_{n+1} \circ m \circ F(\varphi) \circ \alpha. \end{aligned}$$

Obviously, we also have

$$\sigma_0 \circ \varphi \;=\; \sigma_0 \circ m \circ F(\varphi) \circ \alpha,$$

so we can cancel the mono source $(L, (\sigma_n)_{n\in\omega})$, yielding

$$\varphi \;=\; m \circ F(\varphi) \circ \alpha$$

and therefore

$$\begin{aligned} \rho \circ \varphi &= \rho \circ m \circ F(\varphi) \circ \alpha \\ &= F(\varphi) \circ \alpha. \end{aligned}$$

This equation says that $\varphi : (A, \alpha) \to (L, \rho)$ is an F-homomorphism. $\square$

9. Examples and counterexamples

Not always are products as well behaved as one might hope. Even when F weakly preserves pullbacks, a hypothesis often used in the literature, we shall be able to find examples of finite coalgebras $\mathcal{A}$ and $\mathcal{B}$ with $|A| \geq 2$ and $|B| \geq 2$ where

- $\mathcal{A} \times \mathcal{B}$ is empty, or
- $\mathcal{A} \times \mathcal{B}$ is infinite, or
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{A}$.

In each case, we find an example within $\mathcal{S}et_{\mathcal{P}_\omega}$, i.e. the class of all coalgebras of $\mathcal{P}_\omega$, the finite powerset functor. $\mathcal{P}_\omega$ is bounded (by ω), hence products of $\mathcal{P}_\omega$-coalgebras exist. A $\mathcal{P}_\omega$-coalgebra $\mathcal{A} = (A, \alpha)$ consists of a set A and a map $\alpha : A \to \mathcal{P}_\omega(A)$. In other words, a $\mathcal{P}_\omega$-coalgebra is just an *image finite* transition system, where from every state there are only finitely many possible transitions into a next state. We shall depict these coalgebras by drawing an arrow for every transition.

EXAMPLE 9.1. Let $\mathcal{A} = \bullet \longrightarrow \bullet$ and $\mathcal{B} = \bullet \longrightarrow \bullet\!\circlearrowleft$. The greatest bisimulation between $\mathcal{A}$ and $\mathcal{B}$ is empty, hence the product of $\mathcal{A}$ and $\mathcal{B}$ must be the empty coalgebra.

EXAMPLE 9.2. In general, if $\nabla_{\mathcal{A}}$ is the largest congruence relation on $\mathcal{A}$, then it is easy to check that $(\mathcal{A}, id_A, \pi_{\nabla_{\mathcal{A}}})$ is the product of $\mathcal{A}$ with $\mathcal{A}/\nabla_{\mathcal{A}}$, thus $\mathcal{A} \times \mathcal{A}/\nabla_{\mathcal{A}} \cong \mathcal{A}$. In particular, for $\mathcal{A} = \bullet \longrightarrow \bullet$, one gets $\nabla_{\mathcal{A}} = \Delta_A$, hence $\mathcal{A} \cong \mathcal{A} \times \mathcal{A}$.

EXAMPLE 9.3. The last example is a transition system whose transition relation is the complete graph on two elements:

$$\mathcal{C}_2 = \circlearrowleft 0 \rightleftarrows 1 \circlearrowright .$$

The rest of this section will be devoted to proving:

PROPOSITION 9.4. *The product $\mathcal{C}_2 \times \mathcal{C}_2$ in $\mathcal{S}et_{\mathcal{P}_\omega}$ is infinite.*

Proof. $\mathcal{C}_2$ is given as a transition system with transition relation $\sigma_{\mathcal{C}} = \{0, 1\}^2$. As a $\mathcal{P}_\omega$-coalgebra we have $\mathcal{C}_2 = (\{0, 1\}, \alpha_C)$ with $\alpha_C(0) = \alpha_C(1) = \{0, 1\}$.

In general, we shall use the arrow notation as introduced in Section 2.5, that is for any $\mathcal{P}_\omega$-coalgebra $\mathcal{A} = (A, \alpha)$ we write

$$a \xrightarrow{\mathcal{A}} a' \iff a' \in \alpha_A(a).$$

We shall drop the label $\mathcal{A}$, if it is clear from the context.

$\mathcal{C}_2$ is special, in that for any map $\varphi : A \to C_2$, the first homomorphism condition from Section 2.5 is automatically satisfied, thus φ is already a homomorphism if for any $a \in A$ and $c' \in C_2$ we have

$$\varphi(a) \xrightarrow{\mathcal{C}_2} c' \Longrightarrow \exists a' \in A.\, a \xrightarrow{\mathcal{A}} a',\ \varphi(a') = c'.$$

In particular, the map $\nu(x) := 1 - x$ is an automorphism of $\mathcal{C}_2$.

On the set $\omega = \{0, 1, 2, \ldots\}$ we define a transition relation by

$$x \xrightarrow{\omega} y \ :\iff\ x > y \text{ or } y \leq 1,$$

that is, by the transitive hull of the following relation:

$$0 \rightleftarrows 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots .$$

Let $\mathcal{A}$ be the coalgebra given by the transition relation $\xrightarrow{\omega}$. Note that C_2 is a subcoalgebra of $\mathcal{A}$, but, more importantly, we have homomorphisms $\psi_1, \psi_2 : \mathcal{A} \to \mathcal{C}_2$ given by $\psi_1(x) := \min\{x, 1\}$ and $\psi_2 := \nu \circ \psi_1$.

Thus, there must be a homomorphism $\psi : \mathcal{A} \to \mathcal{C}_2 \times \mathcal{C}_2$ with $\psi_i = \eta_i \circ \psi$, for $i = 1, 2$. We claim that ψ is injective.

Assuming the contrary, we have a smallest element $z \in \omega$ for which there is some $k > 0$ with $\psi(z) = \psi(z + k)$. Let $p := \psi(k)$, then from $(z + k) \xrightarrow{\mathcal{A}} z$, the first homomorphism condition yields $p \xrightarrow{\mathcal{C}_2} p$. As a result, the second homomorphism condition assures that there exists a z' with $z \xrightarrow{\mathcal{A}} z'$ and $\psi(z') = p$. The choice of z only permits $z \leq z'$, so it follows $z \in \{0, 1\}$. But $\psi_1(0) \neq \psi_1(k)$, and $\psi_2(1) \neq \psi_2(1+k)$ for any $k \neq 0$, contradicting $\psi(z) = \psi(z+k)$. Consequently, ψ is injective, hence the product $\mathcal{C}_2 \times \mathcal{C}_2$ must have at least $|\omega|$ many elements. $\square$

Clearly, the proof goes through with any ordinal κ in place of ω, hence the product of $\mathcal{C}_2 \times \mathcal{C}_2$ in the category $\mathcal{S}et_{\mathcal{P}_\kappa}$ has at least size $|\kappa|$. We conclude, that the product of $\mathcal{C}_2$ with itself does not exist in the category $\mathcal{S}et_{\mathcal{P}}$.

Given any ordinal number κ and defining $x \xrightarrow{\kappa} y :\Longleftrightarrow x > y$ for any $x, y \in \kappa$, one obtains a $\mathcal{P}_\kappa$-coalgebra, which, by an argument similar to the proof of Proposition 9.4, does not possess any proper homomorphic image. This provides a straightforward argument for the fact that the terminal $\mathcal{P}_\kappa$ coalgebra has size at least $|\kappa|$, and that a terminal $\mathcal{P}$-coalgebra cannot exist.

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H. P. Gumm, T. Schröder
Philipps-Universität Marburg
35032 Marburg
Germany
e-mail: {*gumm,tschroed*}*@mathematik.uni-marburg.de*