THE MULTIPLICITY CONJECTURE FOR BARYCENTRIC SUBDIVISIONS

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Abstract. For a simplicial complex Δ we study the effect of barycentric subdivision on ring theoretic invariants of its Stanley-Reisner ring. In particular, for Stanley-Reisner rings of barycentric subdivisions we verify a conjecture by Huneke and Herzog & Srinivasan, that relates the multiplicity of a standard graded k-algebra to the product of the maximal and minimal shifts in its minimal free resolution up to the height. On the way to proving the conjecture we develop new and list well known results on behavior of dimension, Hilbert series, multiplicity, local cohomology, depth and regularity when passing from the Stanley-Reisner ring of Δ to the one of its barycentric subdivision.

1. Introduction

For a simplicial complex Δ on ground set Ω its Stanley-Reisner ideal I_Δ is the ideal in $S = k[x_\omega \mid \omega \in \Omega]$ generated by the monomials $\underline{\mathbf{x}}_A := \prod_{\omega \in A} x_\omega$ for $A \subseteq \Omega$ and $A \notin \Delta$. Many combinatorial invariants of Δ are encoded in ring-theoretic invariants of its Stanley-Reisner ring $k[\Delta] := S/I_\Delta$. Here we are interested in the behavior of these invariants when passing from $k[\Delta]$ to $k[\mathrm{sd}(\Delta)]$, where $\mathrm{sd}(\Delta)$ denotes the barycentric subdivision of Δ. Recall, that $\mathrm{sd}(\Delta)$ is the simplicial complex on ground set $\mathring{\Delta} := \Delta \setminus \{\emptyset\}$ whose simplices are flags $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_i$ of elements $A_j \in \mathring{\Delta}$, $0 \leq j \leq i$. Note, that throughout the paper we assume that if Δ is a simplicial complex on ground set Ω then $\{\omega\} \in \Delta$ for all $\omega \in \Omega$. In particular, I_Δ will not contain any variable.

Our main result is the verification of the multiplicity conjecture by Huneke and Herzog & Srinivasan [18] for $k[\mathrm{sd}(\Delta)]$. In recent years, this conjecture has attracted attention from commutative algebra and combinatorics (see for example [10], [13], [14], [15], [19], [21], [22], [23], [24], [25]). Here we provide another link to combinatorics and add a large class of rings for which the conjecture holds.

In general, for a standard graded k-algebra $A = T/I$, where $T = k[x_1, \ldots, x_n]$, the conjecture relates the multiplicity of A and the shifts in the minimal free resolution of A up to its height. More precisely, let

$$0 \to \bigoplus_{j \geq 0} T(-j)^{\beta_{r,j}} \to \cdots \to \bigoplus_{j \geq 0} T(-j)^{\beta_{1,j}} \to T \to A \to 0$$

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be the minimal free resolution of A as a T-module. Let $e(A)$ denote the multiplicity of A, $h = \text{height}(I)$ be the height or codimension of I and set $M_i = \max\{ j \mid j \geq 0 \text{ and } \beta_{i,j} \neq 0\}$ respectively $m_i = \min\{ j \mid j \geq 0 \text{ and } \beta_{i,j} \neq 0\}$. Then the conjecture states:

Conjecture 1.1 (Multiplicity Conjecture)**.**

$$e(A) \leq \frac{1}{h!} \prod_{i=1}^{h} M_i.$$

If A is Cohen-Macaulay then

$$e(A) \geq \frac{1}{h!} \prod_{i=1}^{h} m_i.$$

If A is Cohen-Macaulay then equality holds if and only if A has a pure resolution over T.

We refer the reader for more background in commutative algebra to the books by Eisenbud [9] and Bruns & Herzog [8]. We also note that additionally it has been conjectured the the equality case can only appear for Cohen-Macaulay rings. We were not able to verify this extension in our setting.

Thus our main result states:

Theorem 1.2. *Let Δ be a simplicial complex. Then the Multiplicity Conjecture holds for $k[\text{sd}(\Delta)]$.*

In independent work Novik and Swartz [24] have verified the multiplicity conjecture for some important classes of Cohen-Macaulay simplicial complexes. The classes of simplicial complexes treated by Novik and Swartz and our Theorem 1.2 overlap in a small fraction of either class. The lower bound in Theorem 1.2 is also a consequence of parallel work by Michael Goff [12].

For the proof of Theorem 1.2 we need to study the behavior of a few ring theoretic invariants when passing from $k[\Delta]$ to $k[\text{sd}(\Delta)]$. We take this as an opportunity to list in Section 2 the relation of the most important ring theoretic invariants of $k[\Delta]$ and $k[\text{sd}(\Delta)]$.

The proof of Theorem 1.2 will then be given in Section 4 and will rely on the Hochster formula for the Betti numbers of the minimal free resolution of a Stanley-Reisner ring $k[\Delta]$.

2. Invariants for Barycentric Subdivisions

2.1. **Basic Definitions.** Before we can discuss the behavior of ring theoretic invariants when passing from $k[\Delta]$ to $k[\text{sd}(\Delta)]$, we need to introduce some basic notation about simplicial complexes.

For the formulation of the results and proofs, we adopt the following standard notation for simplicial complexes (see [2] for more details). For $F \in \Delta$ we denote by ∂F the simplicial complex of all $G \subsetneq F$ that lie in the boundary of the simplex F. We call an element F of Δ a face of Δ. An inclusionwise maximal face is called facet. For a face F its dimension is given as $\dim F = |F| - 1$ and the dimension of Δ is the maximum dimension of one of its faces. The vector $\mathfrak{f}^\Delta := (f_{-1}^\Delta, \ldots, f_{\dim \Delta}^\Delta)$ where f_i^Δ is the number of i-dimensional faces of Δ is called the f-vector of Δ. The vector $\mathfrak{h}^\Delta := (h_0^\Delta, \ldots, h_{\dim \Delta+1}^\Delta)$ defined by

$$\sum_{i=0}^{\dim \Delta+1} h_i^\Delta t^{\dim \Delta - i+1} = \sum_{i=0}^{\dim \Delta+1} f_{i-1}^\Delta (t-1)^{\dim \Delta+1-i}$$

is called the h-vector of Δ. For a face $F \in \Delta$ we write $\mathrm{lk}_\Delta(F) := \{G \in \Delta \mid F \cap G = \emptyset,\ F \cup G \in \Delta\}$ for the link of F in Δ. By $\widetilde{H}_i(\Delta; k)$ we denote the i-th reduced simplicial homology group with coefficients in k. Also we use $[n]$ to denote for a natural number n the set $\{1, \ldots, n\}$.

2.2. **Krull Dimension.** Since $\dim \Delta = \dim \mathrm{sd}(\Delta)$ it follows that

$$\dim k[\Delta] = \dim \Delta + 1 = \dim \mathrm{sd}(\Delta) + 1 = \dim k[\mathrm{sd}(\Delta)].$$

2.3. **Hilbert Series.**

Proposition 2.1. [5, Theorem 2.2] *Let Δ be a $(d-1)$-dimensional simplicial complex. Then:*

$$\begin{aligned} \mathrm{Hilb}\,(k[\Delta], t) &= \frac{h_0^\Delta + h_1^\Delta t + \cdots + h_d^\Delta t^d}{(1-t)^d} \\ \mathrm{Hilb}\,(k[\mathrm{sd}(\Delta)], t) &= \frac{h_0^{\mathrm{sd}(\Delta)} + h_1^{\mathrm{sd}(\Delta)} t + \cdots + h_d^{\mathrm{sd}(\Delta)} t^d}{(1-t)^d} \\ &= \frac{\sum_{j=0}^{d} \Big(\sum_{i=0}^{d} h_i^\Delta A(d+1, j, i+1) \Big) t^j}{(1-t)^d} \end{aligned}$$

where $A(d+1, j, i+1)$ denotes the number of permutations $\sigma \in S_{d+1}$ such that $\sigma(1) = i+1$ and $\mathrm{des}(\sigma) := \#\{l \in [d] \mid \sigma(l) > \sigma(l+1)\} = j$.

2.4. **Local Cohomology.** We denote by $H^i(k[\Delta]) = H^i_{\mathfrak{m}}(k[\Delta])$ the i-th local cohomology module of $k[\Delta]$ with respect to $\mathfrak{m} = (x_1, \ldots, x_n)$ where $n = f_0^\Delta$ (for more background see [6]). We recall the $\mathbb{Z}$-graded version of a theorem of Hochster for the Hilbert series of the i-th local cohomology of $k[\Delta]$.

Proposition 2.2 (see Theorem 5.3.8 in [8]). *Let Δ be a simplicial complex. Then the $\mathbb{Z}$-graded Hilbert series of the i-th local cohomology module of $k[\Delta]$ is given by*

$$\mathrm{Hilb}\,\left(H^i\left(k[\Delta]\right), t\right) = \sum_{F \in \Delta} \dim_k \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right) \cdot \left(\frac{1}{t-1}\right)^{|F|}.$$

We also need the following simple lemma about links in barycentric subdivisions, whose verification is left to the reader.

Lemma 2.3. *Let Δ be a simplicial complex,* $\mathrm{sd}(\Delta)$ *its barycentric subdivision. Then for a face F of Δ and a flag $F_1 \subsetneq \ldots \subsetneq F_r := F$ of* $\mathrm{sd}(\Delta)$ *it holds that*

$$\widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right) = \widetilde{H}_{i-r-1}\left(\mathrm{lk}_{\mathrm{sd}(\Delta)}\left(F_1 \subsetneq \ldots \subsetneq F_r\right); k\right).$$

Proposition 2.4. *Let Δ be a $(d-1)$-dimensional simplicial complex. Then the $\mathbb{Z}$-graded Hilbert series of the i-th local cohomology module of $k[\mathrm{sd}(\Delta)]$ is given by*

$$\mathrm{Hilb}\left(H^i(k[\mathrm{sd}(\Delta)]), t\right) = \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right) +$$

$$\sum_{m=1}^{d} \sum_{\substack{F\in\Delta \\ |F|=m}} \left(\frac{\displaystyle\sum_{k=0}^{m-1} \Big| \{\sigma \in S_m \mid \mathrm{des}(\sigma) = k\} \Big| \cdot t^k}{(t-1)^m} \right) \cdot \dim_k \widetilde{H}_{i-m-1}\left(\mathrm{lk}_\Delta F; k\right).$$

Proof. For $F \in \mathring{\Delta}$ we set

$$\mathrm{sd}(\Delta)[F] := \{F_1 \subsetneq \ldots \subsetneq F_r \in \mathrm{sd}(\Delta) \mid F_r = F,\ r \geq 1\}.$$

By Proposition 2.2 and Lemma 2.3 it holds that

$$\begin{aligned}
&\mathrm{Hilb}\left(H^i\left(k[\mathrm{sd}(\Delta)]\right), t\right) \\
=\ & \sum_{\sigma\in\mathrm{sd}(\Delta)} \dim_k \widetilde{H}_{i-|\sigma|-1}\left(\mathrm{lk}_{\mathrm{sd}(\Delta)}\left(\sigma\right); k\right) \left(\frac{1}{t-1}\right)^{|\sigma|} \\
=\ & \sum_{F\in\Delta} \sum_{r=1}^{|F|} \sum_{\substack{\sigma\in\mathrm{sd}(\Delta)[F] \\ |\sigma|=r}} \dim_k \widetilde{H}_{i-|\sigma|-1}\left(\mathrm{lk}_{\mathrm{sd}(\Delta)}\left(\sigma\right); k\right) \left(\frac{1}{t-1}\right)^{r} \\
&+ \dim_k \widetilde{H}_{i-|\emptyset|-1}\left(\mathrm{lk}_{\mathrm{sd}(\Delta)}\left(\emptyset\right); k\right) \cdot \left(\frac{1}{t-1}\right)^{|\emptyset|} \\
=\ & \sum_{F\in\Delta} \sum_{r=1}^{|F|} \sum_{\substack{\sigma\in\mathrm{sd}(\Delta)[F] \\ |\sigma|=r}} \dim_k \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right) \cdot \left(\frac{1}{t-1}\right)^{r} \\
&+ \dim_k \widetilde{H}_{i-1}\left(\mathrm{sd}(\Delta); k\right) \\
=\ & \sum_{F\in\Delta} \sum_{r=1}^{|F|} \Big| \{\sigma \in \mathrm{sd}(\Delta)[F] \mid |\sigma| = r\} \Big| \cdot \frac{\dim_k \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right)}{(t-1)^r} \\
&+ \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \sum_{F\in\Delta}\sum_{r=1}^{|F|} f_{r-2}^{\mathrm{sd}(\partial F)} \cdot \frac{\dim_k \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right)}{(t-1)^r} \\
& + \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right) \\
= \quad & \sum_{F\in\Delta}\sum_{r=1}^{|F|} f_{r-2}^{\mathrm{sd}(\partial F)} \cdot \dim_k \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right) \cdot \left(\frac{1}{t-1}\right)^r \\
& + \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right).
\end{aligned}
$$

Since $f_{r-2}^{\mathrm{sd}(\partial F)} = r! \cdot S\left(|F|, r\right)$, where $S\left(m, r\right)$ denotes the Stirling number of the second kind (see [26]), it follows that

$$
\begin{aligned}
& \mathrm{Hilb}\ \left(H^i\left(k[\mathrm{sd}(\Delta)]\right), t\right) \\
= \quad & \sum_{F\in\Delta}\sum_{r=1}^{|F|} r! \cdot S\left(|F|, r\right) \cdot \dim_k \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right) \cdot \frac{1}{(t-1)^r} \\
& + \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right) \\
= \quad & \sum_{F\in\Delta}\left(\sum_{r=1}^{|F|} r! \cdot S\left(|F|, r\right) \cdot \frac{1}{(t-1)^r}\right) \cdot \dim_k \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta F; k\right) \\
& + \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right) \\
= \quad & \sum_{m=1}^{\dim\Delta+1} \sum_{\substack{F\in\Delta\\|F|=m}} \left(\frac{\sum_{r=1}^{m}\left(r! S\left(m, r\right)\sum_{k=0}^{m-r}\binom{m-r}{k} t^k (-1)^{m-r-k}\right)}{(t-1)^m}\right) \cdot \\
& \qquad \cdot\ \dim_k \widetilde{H}_{i-m-1}\left(\mathrm{lk}_\Delta F; k\right) \\
& + \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right) \\
= \quad & \sum_{m=1}^{\dim\Delta+1} \sum_{\substack{F\in\Delta\\|F|=m}} \left(\frac{\sum_{k=0}^{m-1}\left(\sum_{r=1}^{m-k} r! S\left(m, r\right)\binom{m-r}{k}(-1)^{m-r-k}\right) t^k}{(t-1)^m}\right) \cdot \\
& \qquad \cdot\ \dim_k \widetilde{H}_{i-m-1}\left(\mathrm{lk}_\Delta F; k\right) \\
& + \dim_k \widetilde{H}_{i-1}\left(\Delta; k\right) \\
\overset{(*)}{=} \quad & \sum_{m=1}^{\dim\Delta+1} \sum_{\substack{F\in\Delta\\|F|=m}} \left(\frac{\sum_{k=0}^{m-1} \left|\{\sigma\in S_m \mid \mathrm{des}(\sigma)=k\}\right| \cdot t^k}{(t-1)^m}\right) \cdot
\end{aligned}
$$

$$\cdot \dim_k \widetilde{H}_{i-m-1}(\mathrm{lk}_\Delta F; k)$$
$$+ \dim_k \widetilde{H}_{i-1}(\Delta; k).$$

All manipulations are straight forward, except for (*) which uses a well known formula for the Eulerian numbers (see [4, Corollary 1.18]). □

2.5. **Depth.**

Corollary 2.5. *Let Δ be a simplicial complex. Then*

$$\mathrm{depth}(k[\Delta]) = \mathrm{depth}(k[\mathrm{sd}(\Delta)]).$$

Proof. By a theorem of Grothendieck (see [6, Theorem 6.2.7]) the depth of $k[\Delta]$ is given by

$$\mathrm{depth}(k[\Delta]) = \min\left\{i \mid H^i(k[\Delta]) \neq 0\right\}.$$

By Proposition 2.2 for the depth of $k[\Delta]$ we get

$$\begin{aligned}
\mathrm{depth}(k[\Delta]) &= \min\left\{i \mid H^i(k[\Delta]) \neq 0\right\} \\
&= \min\left\{i \mid \mathrm{Hilb}\,(H^i(k[\Delta]), t) \neq 0\right\} \\
&= \min\left\{i \mid \exists F \in \Delta : \dim_k \widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F; k) \neq 0\right\}.
\end{aligned}$$

Analogously, $\mathrm{depth}(k[sd(\Delta)]) =$

$$\min\left\{i \mid \exists \sigma \in \mathrm{sd}(\Delta) : \dim_k \widetilde{H}_{i-|\sigma|-1}\left(lk_{\mathrm{sd}(\Delta)}(\sigma); k\right) \neq 0\right\}.$$

By Lemma 2.3 for $\sigma = F_1 \subsetneq \cdots \subsetneq F_r := F$ we have

$$\widetilde{H}_{m-|\sigma|-1}\left(\mathrm{lk}_{\mathrm{sd}(\Delta)}(\sigma); k\right) = \widetilde{H}_{m-|F|-1}(\mathrm{lk}_\Delta F; k),$$

which implies the assertion. □

2.6. **Projective Dimension.** We denote by $\mathrm{pdim}(k[\Delta])$ the projective dimension of $k[\Delta]$. For a simplicial complex Δ over ground set $[n]$ and $f_0^\Delta = n$ we obtain using The Auslander-Buchsbaum formula (see [9, Theorem 19.9]):

$$\begin{aligned}
\mathrm{pdim}(k[\Delta]) &= f_0^\Delta - \mathrm{depth}(k[\Delta]) \\
&= \sum_{i\geq 0} f_i^\Delta - \sum_{i\geq 1} f_i^\Delta - \mathrm{depth}(k[\mathrm{sd}(\Delta)]) \\
&= f_0^{\mathrm{sd}(\Delta)} - \mathrm{depth}(k[\mathrm{sd}(\Delta)]) - \sum_{i\geq 1} f_i^\Delta \\
&= \mathrm{pdim}(k[\mathrm{sd}(\Delta)]) - \sum_{i\geq 1} f_i^\Delta.
\end{aligned}$$

2.7. **Regularity.**

Proposition 2.6. *Let Δ be a simplicial complex on vertex set $[n]$.*

$$\mathrm{reg}(k[\Delta]) \le \mathrm{reg}(k[\mathrm{sd}(\Delta)]) = \begin{cases} \dim\Delta & \text{if} \quad \widetilde{H}_{\dim\Delta}(\Delta;k) = 0 \\ \dim\Delta + 1 & \text{if} \quad \widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0 \end{cases}$$

Moreover, if $\widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0$ then

$$\mathrm{reg}(k[\Delta]) = \mathrm{reg}(k[\mathrm{sd}(\Delta)]) = \dim\Delta + 1.$$

Proof. We use the following characterization of regularity [6]

$$\mathrm{reg}(k[\Delta]) = \max\{i + j \mid H^i(k[\Delta])_j \neq 0\}.$$

By Proposition 2.2 we have that

$$\begin{aligned}
\mathrm{Hilb}\left(H^i(k[\Delta]), t\right) &= \sum_{l\in\mathbb{Z}} \dim_k H^i(k[\Delta])_l \cdot t^l \\
&= \sum_{F\in\Delta} \dim_k \widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F;k) \cdot \left(\frac{1}{t-1}\right)^{|F|} \\
&= \sum_{F\in\Delta} \dim_k \widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F;k) \cdot \left(\sum_{n\ge1}\left(\frac{1}{t}\right)^n\right)^{|F|} \\
&= \sum_{F\in\Delta} \dim_k \widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F;k) \cdot \sum_{a\in(\mathbb{Z}_-\setminus\{0\})^{|F|}} t^{|a|}.
\end{aligned}$$

Here $|a| = a_1 + \ldots + a_{|F|}$ and $\mathbb{Z}_- = \{0, -1, -2, \ldots\}$. We conclude

$$\begin{aligned}
& H^i(k[\Delta])_l \neq 0 \\
\iff\ & \dim_k H^i(k[\Delta])_l \neq 0 \\
\iff\ & \exists a \in \mathbb{Z}_-^n,\ |a| = l \text{ and } \exists\ F\in\Delta \text{ such that } \Big|\{s \mid a_s < 0\}\Big| = |F| \\
& \text{and } \dim_k \widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F;k) \neq 0
\end{aligned}$$

As usual, for a $\mathbb{Z}$-graded module $M = \bigoplus_{m\in\mathbb{Z}} M_m$, we write $\mathrm{end}(M)$ for $\sup\{m\in\mathbb{Z} \mid M_m \neq 0\}$. The above directly yields $\mathrm{end}(H^i(k[\Delta])) \le 0$ and choosing $a \in \{0,-1\}^n$ we see that $\mathrm{end}(H^i(k[\Delta])) \ge -(\dim\Delta+1)$ if $H^i(k[\Delta]) \neq 0$.
More precisely, if $H^i(k[\Delta]) \neq 0$ then $\mathrm{end}(H^i(k[\Delta]))$ is given by

$$\begin{aligned}
\mathrm{end}(H^i(k[\Delta])) &= \sup\left\{m\in\mathbb{Z} \mid H^i(k[\Delta])_m \neq 0\right\} \\
&= \sup\left\{0 \ge m \ge -(\dim\Delta+1) \mid H^i(k[\Delta])_m \neq 0\right\} \\
&= \sup\left\{0 \ge m \ge -(\dim\Delta+1) \ \Big|{}_{\widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F;k)\neq 0}^{\exists F\in\Delta,\ |F|=-m:}\right\} \\
&= \sup\left\{-|F| \mid F\in\Delta :\ \widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F;k) \neq 0\right\}
\end{aligned}$$

$$= -\inf\left\{|F| \mid F \in \Delta : \widetilde{H}_{i-|F|-1}(\mathrm{lk}_\Delta F; k) \neq 0\right\}.$$

We now show $\mathrm{reg}(k[\Delta]) \le \mathrm{reg}(k[\mathrm{sd}(\Delta)])$.
From the previous consideration and $|F_1 \subsetneq \ldots \subsetneq F_t| \le |F_t|$ it follows that

$$\begin{aligned}
\mathrm{end}(H^i(k[\mathrm{sd}(\Delta)])) &= -\inf\left\{ t \,\Big|\, {}^{\exists F_1 \subsetneq \ldots \subsetneq F_t \in \mathrm{sd}(\Delta):}_{\widetilde{H}_{i-t-1}(\mathrm{lk}_{\mathrm{sd}(\Delta)}(F_1 \subsetneq \ldots \subsetneq F_t);k)\neq 0}\right\}\\
&= -\inf\left\{ t \,\Big|\, {}^{\exists F_1 \subsetneq \ldots \subsetneq F_t \in \mathrm{sd}(\Delta):}_{\widetilde{H}_{i-|F_t|-1}(\mathrm{lk}_\Delta F_t;k)\neq 0}\right\}\\
&\ge -\inf\left\{ |F_t| \,\Big|\, {}^{\exists F_1 \subsetneq \ldots \subsetneq F_t \in \mathrm{sd}(\Delta):}_{\widetilde{H}_{i-|F_t|-1}(\mathrm{lk}_\Delta F_t;k)\neq 0}\right\}\\
&= -\inf\left\{ |F_t| \,\Big|\, F_t \in \Delta : \widetilde{H}_{i-|F_t|-1}(\mathrm{lk}_\Delta F_t; k) \neq 0\right\}\\
&= \mathrm{end}(H^i(k[\Delta])).
\end{aligned}$$

Thus $\mathrm{end}(H^i(k[\Delta])) + i \le \mathrm{end}(H^i(k[\mathrm{sd}(\Delta)])) + i$ for all i. It follows that $\mathrm{reg}(k[\Delta]) \le \mathrm{reg}(k[\mathrm{sd}(\Delta)])$, as desired.

We claim $\mathrm{reg}(k[\mathrm{sd}(\Delta)]) = \begin{cases} \dim\Delta & \text{if } \widetilde{H}_{\dim\Delta}(\Delta;k) = 0 \\ \dim\Delta + 1 & \text{if } \widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0 \end{cases}$

Case 1: $\widetilde{H}_{\dim\Delta}(\Delta;k) = 0$.
By Grothendieck's Non-Vanishing Theorem (see [6, 6.1.4]) we have that $H^{\dim k[\Delta]}(k[\Delta]) = H^{\dim\Delta+1}(k[\Delta]) \neq 0$. Along with the above consideration we conclude that there exists a face $F \in \Delta$ such that $\widetilde{H}_{\dim\Delta+1-|F|-1}(\mathrm{lk}_\Delta F; k) = \widetilde{H}_{\dim\Delta-|F|}(\mathrm{lk}_\Delta F; k) \neq 0$.
Since $\widetilde{H}_{\dim\Delta}(\Delta;k) = 0$ this face cannot be the empty face.
Therefore, we can consider F as a one-element flag in $\mathrm{sd}(\Delta)$. From Lemma 2.3 we deduce

$$\widetilde{H}_{(\dim\Delta+1)-\underbrace{|F|}_{=1 \text{ in } \mathrm{sd}(\Delta)}-1}\left(\mathrm{lk}_{\mathrm{sd}(\Delta)}F;k\right) = \widetilde{H}_{\dim\Delta-\underbrace{|F|}_{\text{in } \Delta}}(\mathrm{lk}_\Delta F;k) \neq 0.$$

Thus

$$\begin{aligned}
&\mathrm{end}(H^{\dim\Delta+1}(k[\mathrm{sd}(\Delta)]))\\
&= -\inf\left\{|F_1 \subsetneq \ldots \subsetneq F_t| \,\Big|\, {}^{F_1 \subsetneq \ldots \subsetneq F_t \in \mathrm{sd}(\Delta):}_{\widetilde{H}_{\dim\Delta-|F_1 \subsetneq \ldots \subsetneq F_t|}(\mathrm{lk}_{\mathrm{sd}(\Delta)}(F_1 \subsetneq \ldots \subsetneq F_t);k)\neq 0}\right\}\\
&\ge -\underbrace{|F|}_{=1 \text{ in } \mathrm{sd}(\Delta)} = -1. \qquad (2.1)
\end{aligned}$$

This implies

$$\begin{aligned}
\mathrm{reg}(k[\mathrm{sd}(\Delta)]) &\ge \mathrm{end}(H^{\dim\Delta+1}(k[\mathrm{sd}(\Delta)])) + \dim\Delta + 1\\
&\ge -1 + \dim\Delta + 1 = \dim\Delta.
\end{aligned}$$

We also know that $\mathrm{end}(H^i(k[\mathrm{sd}(\Delta)])) \le 0$. Hence

$$\mathrm{end}(H^i(k[\mathrm{sd}(\Delta)])) + i \le i < \dim\Delta + 1,\ 0 \le i \le \dim\Delta.$$

From Inequality (2.1) we deduce $0 \geq \operatorname{end}(H^{\dim\Delta+1}(k[\operatorname{sd}(\Delta)])) \geq -1$. Since

$$\widetilde{H}_{\dim\Delta-|\emptyset|}\left(\operatorname{lk}_{\operatorname{sd}(\Delta)}(\emptyset);k\right) = \widetilde{H}_{\dim\Delta}(\operatorname{sd}(\Delta);k) = \widetilde{H}_{\dim\Delta}(\Delta;k) = 0$$

we conclude that $\operatorname{end}(H^{\dim\Delta+1}(k[\operatorname{sd}(\Delta)])) \neq 0$. Therefore,

$$\operatorname{end}(H^{\dim\Delta+1}(k[\operatorname{sd}(\Delta)])) = -1.$$

Thus $\operatorname{end}(H^{\dim\Delta+1}(k[\operatorname{sd}(\Delta)]))+\dim\Delta+1 = \dim\Delta$. This finally proves the claim for this case.

Case 2: $\widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0$.

From the fact that $\operatorname{end}(H^i(k[\Delta]) \leq 0$ it follows that $\operatorname{end}(H^i(k[\Delta])) + i \leq \dim\Delta$, $0 \leq i \leq \dim\Delta$.

By $\widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0$ we get

$$\begin{aligned} 0 &\geq \operatorname{end}(H^{\dim\Delta+1}(k[\Delta])) \\ &= -\inf\left\{|F| \;\middle|\; F\in\Delta:\ \widetilde{H}_{\dim\Delta+1-|F|}(\operatorname{lk}_\Delta(F);k) \neq 0\right\} \geq |\emptyset| = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{reg}(k[\Delta]) &= \sup\left\{\operatorname{end}(H^i(k[\Delta]))+i \mid 0\leq i\leq \dim\Delta+1\right\} \\ &= \sup\left(\left\{\underbrace{\operatorname{end}(H^i(k[\Delta]))+i}_{\leq\dim\Delta} \mid 0\leq i\leq\dim\Delta\right\}\right. \\ &\qquad \left.\cup\left\{\underbrace{\operatorname{end}(H^{\dim\Delta+1}(k[\Delta]))+\dim\Delta+1}_{=\dim\Delta+1}\right\}\right) \\ &= \dim\Delta+1. \end{aligned}$$

By $\dim\Delta = \dim\operatorname{sd}(\Delta)$ and $\widetilde{H}_{\dim\Delta}(\Delta) = \widetilde{H}_{\dim\Delta}(\operatorname{sd}(\Delta))$ the above also applies to $\operatorname{sd}(\Delta)$ and $\operatorname{reg}(k[\operatorname{sd}(\Delta)]) = \dim\Delta+1$ follows. ☐

2.8. Height and Multiplicity.

Proposition 2.7. *Let Δ be a simplicial complex with f-vector $\mathfrak{f}^\Delta = \left(f_0^\Delta,\ldots,f_{\dim\Delta}^\Delta\right)$. Then*

$$\begin{aligned} \operatorname{height}(I_{\operatorname{sd}(\Delta)}) &= \sum_{l=0}^{\dim\Delta}(f_l^\Delta - 1) \text{ and} \\ e(k[\operatorname{sd}(\Delta)]) &= (\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta. \end{aligned}$$

Proof. From a result by Herzog & Srinivasan ([7, 2. Antichains]) one deduces that $\operatorname{height}(I_\Delta) = f_0^\Delta - (\dim\Delta+1)$ and $e(k[\Delta]) = f_{\dim\Delta}^\Delta$. Thus

$$\begin{aligned} \operatorname{height}(I_{\operatorname{sd}(\Delta)}) &= f_0^{\operatorname{sd}(\Delta)} - (\dim\operatorname{sd}(\Delta)+1) \\ &= \sum_{l=0}^{\dim\Delta} f_l^\Delta - \dim\Delta - 1 \end{aligned}$$

$$= \sum_{l=0}^{\dim \Delta} (f_l^\Delta - 1).$$

A simple counting argument shows that

$$\begin{aligned} e(k[\mathrm{sd}(\Delta)]) &= f_{\dim \Delta}^{\mathrm{sd}(\Delta)} \\ &= (\dim \Delta + 1)! \cdot f_{\dim \Delta}^{\Delta}. \end{aligned}$$

□

2.9. **Cohen-Macaulay-ness.** A simplicial complex is Cohen-Macaulay over a field k if $k[\Delta]$ is a Cohen-Macaulay ring (see [8, Chapter 5] for background on Cohen-Macaulay simplicial complexes). It is a well known fact from geometric combinatorics that Cohen-Macaulay-ness over a field k of a simplicial complex depends on its topological realization only (see [2]). Since Δ and $\mathrm{sd}(\Delta)$ have homeomorphic geometric realizations it follows that Δ is Cohen-Macaulay over k if and only if $\mathrm{sd}(\Delta)$ is.

2.10. **Koszulness.** The minimal nonfaces of $\mathrm{sd}(\Delta)$ are of cardinality two – the pairs of faces of Δ that are incomparable. Therefore, the Stanley-Reisner ideal of $\mathrm{sd}(\Delta)$ is generated by (squarefree) monomials of degree 2 and hence by a result of Fröberg (see [11] for the result and background on Koszul algebras) it is Koszul.

2.11. **Golod-ness.** We have already seen in Section 2.10 that $I_{\mathrm{sd}(\Delta)}$ is generated by squarefree monomials of degree two. By a result of Berglund & Jöllenbeck [1, Theorem 7.4] we know that in this situation $k[\mathrm{sd}(\Delta)]$ is Golod if and only if the 1-skeleton of $\mathrm{sd}(\Delta)$ is a chordal graph; i.e. any cycle of length ≥ 4 has chord. We refer the reader to [16] for background on Golod-ness. Now assume the 1-skeleton of Δ has a chordless cycle of length $\ell \geq 3$ – here we regard triangles as chordless cycles. Then after barycentric subdivision this chordless cycle turns into a chordless cycle of length $2\ell \geq 6$. Hence, $\mathrm{sd}(\Delta)$ cannot be Golod. So assume the 1-skeleton of Δ has no chordless cycle of length ≥ 3. Then $\dim \Delta \leq 1$ and Δ is a graph. Having no chordless cycle of length ≥ 3 then implies that Δ has no cycle and hence is a forest. Now the barycentric subdivision of a forest is a forest and hence has no cycles which implies that $\mathrm{sd}(\Delta)$ is Golod. Thus:

Proposition 2.8. *Let Δ be a simplicial complex. Then* $\mathrm{sd}(\Delta)$ *is Golod if and only if Δ is a forest.*

3. Auxiliary Lemmas and Inequalities

The basic result which allows us to verify the Multiplicity Conjecture for barycentric subdivisions is the following classical theorem by Hochster which expresses the Betti numbers of $k[\Delta]$ in terms of homology groups of restrictions of Δ. For a simplicial complex on ground set Ω the restriction Δ_W of Δ to a subset $W \subseteq \Omega$ is the simplicial complex $\Delta_W := \{F \in \Delta \mid F \subseteq W\}$.

Proposition 3.1 ([17]). *Let Δ be a simplicial complex on vertex set $[n]$ and let β_{ij} be the graded Betti-numbers of $k[\Delta]$. Then for $i,j \in \mathbb{N}$*

$$\beta_{ij} = \sum_{\substack{W\subseteq[n]\\|W|=j}} \dim_k \widetilde{H}_{|W|-i-1}\left(\Delta_W;k\right).$$

The following corollary is a direct consequence of Proposition 3.1.

Corollary 3.2. *Let Δ be a simplicial complex on vertex set $[n]$ and β_{ij} the graded Betti numbers of $k[\Delta]$. Then it holds that*

$$\beta_{ij} \neq 0 \iff \exists W \subseteq [n],\ |W| = j \text{ such that } \widetilde{H}_{j-i-1}\left(\Delta_W;k\right) \neq 0.$$

Lemma 3.3. *Let Δ be a simplicial complex and β_{ij} the graded Betti numbers of $k[\mathrm{sd}(\Delta)]$. Then*

(i) For $1 < m < \dim\Delta$ and $2^{m+1} - 2 - m \leq i < 2^{m+2} - 2 - (m+1)$ we have $\beta_{i,i+m} \neq 0$.
(ii) If $1 < \dim\Delta$ then $\mathrm{pdim}(k[\mathrm{sd}(\Delta)]) \geq 4$. Equivalently, for $1 \leq i \leq 4$ there exist $k_i \geq 1$ such that $\beta_{i,i+k_i} \neq 0$.

Proof. Clearly, a simplicial complex Δ contains at least one $(\dim\Delta)$-simplex. Thus $\sum_{l=0}^{\dim\Delta} f_l^\Delta \geq 2^{\dim\Delta+1} - 1$. From Section 2.6 and by $\dim k[\Delta] = \dim(\Delta) + 1$ we deduce

$$\begin{aligned}
\mathrm{pdim}(k[\mathrm{sd}(\Delta)]) &= \mathrm{pdim}(k[\Delta]) + \sum_{i\geq 1} f_i^\Delta \\
&= f_0^\Delta - \mathrm{depth}(k[\Delta]) + \sum_{i\geq 1} f_i^\Delta \\
&\geq \sum_{l=0}^{\dim\Delta} f_l^\Delta - \dim\Delta - 1
\end{aligned}$$

In particular, if $\dim(\Delta) > 1$ then $\mathrm{pdim}(k[\mathrm{sd}(\Delta)]) \geq 4$, which proves (ii).

The same inequality shows that if $\dim\Delta \geq m+1$ then $\mathrm{pdim}(k[\mathrm{sd}(\Delta)]) \geq 2^{m+2} - 2-(m+1)$. Thus for $1 < m < \dim\Delta$ and for $2^{m+1}-2-m \leq i < 2^{m+2}-2-(m+1)$ there exist $j \geq 1$ such that $\beta_{i,i+j} \neq 0$.

Let $F \in \Delta$ with $\dim F = m$. The boundary ∂F of F is homeomorphic to an $(m-1)$-sphere. Hence $\mathrm{sd}(\partial F) \cong \mathrm{sd}(\Delta)_{\mathring{\partial F}} \cong \mathbb{S}^{m-1}$. Thus

$$k = \widetilde{H}_{m-1}\left(\mathrm{sd}(\Delta)_{\mathring{\partial F}};k\right) = \widetilde{H}_{|\mathring{\partial F}|-(|\mathring{\partial F}|-m)-1}\left(\mathrm{sd}(\Delta)_{\mathring{\partial F}};k\right).$$

Using Corollary 3.2 we conclude

$$\beta_{|\mathring{\partial F}|,|\mathring{\partial F}|-m} \overset{|\mathring{\partial F}|=2^{m+1}-2}{=} \beta_{2^{m+1}-2-m,2^{m+1}-2} \neq 0.$$

It now remains to show that $\beta_{i,i+m} \neq 0$ for $1 < m < \dim \Delta$ and $2^{m+1} - 2 - m < i < 2^{m+2} - 2 - (m+1)$. Since $\dim \Delta \geq m+1$ there exists an $(m+1)$-dimensional face $G \in \Delta$. Choose $v \in G$ and set $F = G \setminus \{v\}$. Then $\mathrm{sd}(\Delta)_{\mathring{\partial F}} \cong \mathbb{S}^{m-1}$ and $\mathrm{sd}(\Delta)_{\mathring{\partial F} \cup H} \simeq \mathbb{S}^{m-1}$ for all $H \subseteq \mathring{\Delta}_G \setminus \{G, F, v\}$ and $H \cap \partial F = \emptyset$. The last assertion holds because when restricting $\mathrm{sd}(\Delta)$ to $\mathring{\partial F} \cup H$ we are only adding simplices to $\mathrm{sd}(\Delta)_{\mathring{\partial F}}$ which can be contracted to $\mathrm{sd}(\Delta)_{\mathring{\partial F}}$. Moreover, since we do not add v, F and G the complex $\mathrm{sd}(\Delta)_{\mathring{\partial F} \cup H}$ still contains the cycle induced by ∂F. We conclude

$$\begin{aligned}
\widetilde{H}_{m-1}\left(\mathrm{sd}(\Delta)_{\mathring{\partial F} \cup H}; k\right) &= \widetilde{H}_{|H|+|\mathring{\partial F}|-(|H|+|\mathring{\partial F}|-m)-1}\left(\mathrm{sd}(\Delta)_{\mathring{\partial F} \cup H}; k\right) \\
&= \widetilde{H}_{|H|+|\mathring{\partial F}|-(|H|+|\mathring{\partial F}|-m)-1}\left(\mathbb{S}^{m-1}; k\right) \\
&= \widetilde{H}_{m-1}\left(\mathbb{S}^{m-1}; k\right) \neq 0.
\end{aligned}$$

By Corollary 3.2 we obtain $\beta_{|H|+|\mathring{\partial F}|-m, |H|+|\mathring{\partial F}|} \neq 0$.
From

$$2^{m+1} - 2 \overset{H=\emptyset}{\leq} |H| + |\mathring{\partial F}| \overset{H \cup \mathring{\partial F} = \mathring{\Delta}_G \setminus \{G,F,v\}}{\leq} 2^{m+2} - 4$$

it follows that $\beta_{j-m,j} \neq 0$ for $2^{m+1} - 2 \leq j \leq 2^{m+2} - 4$.
Thus $\beta_{i,i+m} \neq 0$ for $2^{m+1} - 2 - m \leq i \leq 2^{m+2} - 4 - m < 2^{m+2} - 2 - (m+1)$, as desired. ☐

Lemma 3.4. *Let Δ be a simplicial complex such that $\widetilde{H}_{\dim \Delta}(\Delta; k) = 0$ and let β_{ij} be the graded Betti numbers of $k[\mathrm{sd}(\Delta)]$. Then*

$$\beta_{i,i+\dim \Delta} \neq 0 \quad \text{for } 2^{\dim \Delta + 1} - 2 - \dim \Delta \leq i \leq \sum_{j=0}^{\dim \Delta} (f_j^\Delta - 1).$$

Proof. Let $F \in \Delta$ with $\dim F = \dim \Delta$. We show that $\widetilde{H}_{\dim \Delta - 1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta} \setminus \{F\}}; k\right) \neq 0$. By an elementary homotopy $\mathrm{sd}(\Delta)_{\mathring{\Delta} \setminus \{F\}} \simeq \mathrm{sd}(\Delta) \setminus \{F\}$. Now consider the long exact sequence in homology of the pair $(\mathrm{sd}(\Delta), \mathrm{sd}(\Delta) \setminus \{F\})$.

$$\to \underbrace{\widetilde{H}_{\dim \Delta}(\mathrm{sd}(\Delta); k)}_{=0 \text{ by assumption}} \overset{q_{\dim \Delta}}{\to} \widetilde{H}_{\dim \Delta}(\mathrm{sd}(\Delta), \mathrm{sd}(\Delta) \setminus \{F\}; k) \overset{\partial}{\to}$$

$$\widetilde{H}_{\dim \Delta - 1}(\mathrm{sd}(\Delta) \setminus \{F\}; k) \overset{i}{\to} \widetilde{H}_{\dim \Delta - 1}(\mathrm{sd}(\Delta); k) \overset{q_{\dim \Delta - 1}}{\to}$$

$$\widetilde{H}_{\dim \Delta - 1}(\mathrm{sd}(\Delta), \mathrm{sd}(\Delta) \setminus \{F\}; k) \to \dots$$

Since $(\mathrm{sd}(\Delta), \mathrm{sd}(\Delta) \setminus F)$ is a good pair we have

$$\widetilde{H}_{\dim \Delta}(\mathrm{sd}(\Delta), \mathrm{sd}(\Delta) \setminus \{F\}; k) \cong \widetilde{H}_{\dim \Delta}(\mathrm{sd}(\Delta)/(\mathrm{sd}(\Delta) \setminus \{F\}); k)$$

$$\cong \widetilde{H}_{\dim \Delta}\left(\mathbb{S}^{\dim \Delta}; k\right) = k.$$

The same argument shows $\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta),\mathrm{sd}(\Delta)\setminus\{F\};k\right)=0$. Along with the above sequence being exact this implies

$$\mathrm{Im}\ i=\mathrm{Ker}\ q_{\dim\Delta-1}=\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta);k\right).$$

Thus

$$\begin{aligned}\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta);k\right) &\cong \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)\setminus\ \{F\};k\right)/\mathrm{Ker}\ i\\ &= \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)\setminus\{F\};k\right)/\mathrm{Im}\ \partial.\end{aligned}$$

Since the above sequence is exact it holds that $0=\mathrm{Im}\ q_{\dim\Delta}=\mathrm{Ker}\ \partial$. This yields

$$\mathrm{Im}\ \partial\cong\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta),\mathrm{sd}(\Delta)\setminus\{F\};k\right)\cong k.$$

Thus

$$\begin{aligned}\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)\setminus\{F\};k\right) &\cong \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta);k\right)\oplus\mathrm{Im}\ \partial\\ &\cong \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta);k\right)\oplus k\neq 0.\end{aligned}$$

It follows that $\beta_{\sum_{j=0}^{\dim\Delta}(f_j^\Delta-1),\sum_{j=0}^{\dim\Delta}f_j^\Delta-1}=\beta_{|\mathring{\Delta}\setminus\{F\}|-\dim\Delta,|\mathring{\Delta}\setminus\{F\}|}\neq 0$. Our next aim is to prove that $\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})};k\right)\neq 0$ for $A\subseteq\Delta\setminus\Delta_F$. First using induction on the cardinality of A we show, that $\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A};k\right)=0$.
For $|A|=0$ this is our assumption on the homology of Δ. Assume $|A|\geq 1$. Let $\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A};k\right)=0$ for all $A\subseteq\Delta\setminus\Delta_F$ with $|A|=m$ and let $B\subseteq\Delta\setminus\Delta_F$ with $|B|=m+1$. Consider $A:=B\setminus\{v\}$ for some $v\in B$. By the induction hypothesis we have $\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A};k\right)=0$. Consider the exact homology sequence of the $\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A},\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}\right)$:

$$\begin{aligned}\cdots\widetilde{H}_{\dim\Delta+1}&\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A},\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})};k\right)\xrightarrow{\partial}\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})};k\right)\\ &\xrightarrow{i}\underbrace{\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A};k\right)}_{=0\text{ by induction hypothesis}}\to\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A},\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})};k\right)\cdots\end{aligned}$$

Since $\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}$ and $\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})}$ are $(\dim\Delta)$-dimensional CW-complexes it follows that $\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}/\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})}$ is a $(\dim\Delta)$-dimensional CW-complex. In particular, the complex has no cells in dimension $\dim\Delta+1$. Thus

$$\begin{aligned}&\widetilde{H}_{\dim\Delta+1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A},\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})};k\right)\\ =\ &\widetilde{H}_{\dim\Delta+1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}/\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})};k\right)=0.\end{aligned}$$

Exactness of the preceding sequence implies the desired fact

$$\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{v\})};k\right)=\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus B};k\right)=0.$$

Consider the long exact sequence of the pair $\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}, \mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}\right)$ for an arbitrary $A \subseteq \Delta \setminus \Delta_F$.

$$\begin{aligned}
\ldots\ & \widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}; k\right) \to \widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}, \mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}; k\right) \\
\to\ & \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}; k\right) \to \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}; k\right) \\
\to\ & \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}, \mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}; k\right) \to \ldots
\end{aligned}$$

Since $\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}, \mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}\right)$ is a good pair it holds that

$$\begin{aligned}
& \widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}, \mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}; k\right) \\
\cong\ & \widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}/\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}; k\right) \\
\cong\ & \widetilde{H}_{\dim\Delta}\left(\mathbb{S}^{\dim\Delta}; k\right) = k
\end{aligned}$$

The same argument shows $\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}, \mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}; k\right) = 0$.
Analogous to the case $A = \emptyset$ we deduce from the above long exact sequence

$$\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(A\cup\{F\})}; k\right) = \widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus A}; k\right) \oplus k \neq 0$$

By Proposition 3.1 it follows that

$$\beta_{|\mathring{\Delta}\setminus(A\cup\{F\})|-\dim\Delta, |\mathring{\Delta}\setminus(A\cup\{F\})|} = \beta_{\sum_{l=0}^{\dim\Delta} f_l^\Delta - 1 - |A| - \dim\Delta, \sum_{l=0}^{\dim\Delta} f_l^\Delta - 1 - |A|} \neq 0.$$

Since $A \subseteq \Delta \setminus \Delta_F$ we have $0 \leq |A| \leq \sum_{l=0}^{\dim\Delta} f_l^\Delta - (2^{\dim\Delta+1} - 1)$. Therefore $\beta_{i,i+\dim\Delta} \neq 0$ for $2^{\dim\Delta+1} - 2 - \dim\Delta \leq i \leq \sum_{l=0}^{\dim\Delta}(f_l^\Delta - 1)$. $\square$

Lemma 3.5. *Let Δ be a simplicial complex such that $\widetilde{H}_{\dim\Delta}(\Delta; k) \neq 0$ and let β_{ij} be the graded Betti numbers of $k[\mathrm{sd}(\Delta)]$. Then $\beta_{i,i+\dim\Delta} \neq 0$ or $\beta_{i,i+\dim\Delta+1} \neq 0$ for*

$$2^{\dim\Delta+1} - 2 - \dim\Delta \leq i \leq \sum_{j=0}^{\dim\Delta}(f_j^\Delta - 1).$$

Proof. The assumption yields

$$\begin{aligned}
\widetilde{H}_{\dim\Delta}(\Delta; k) &= \widetilde{H}_{\dim\Delta}(\mathrm{sd}(\Delta); k) \\
&= \widetilde{H}_{\sum_{l=0}^{\dim\Delta} f_l^\Delta - (\sum_{l=0}^{\dim\Delta} f_l^\Delta - \dim\Delta - 1) - 1}(\mathrm{sd}(\Delta)_{\mathring{\Delta}}; k) \neq 0.
\end{aligned}$$

By Corollary 3.2 it follows that $\beta_{\sum_{l=0}^{\dim\Delta}(f_l^\Delta - 1), \sum_{l=0}^{\dim\Delta} f_l^\Delta} \neq 0$ which proves the assertion for $i = \sum_{j=0}^{\dim\Delta}(f_j^\Delta - 1)$.
Now assume $i < \sum_{j=0}^{\dim\Delta}(f_j^\Delta - 1)$. We successively remove vertices of Δ from

$\mathrm{sd}(\Delta)$ until the homology in dimension $\dim\Delta$ vanishes. Let $v_1,\dots,v_r$ be vertices of Δ such that $\widetilde{H}_{\dim\Delta}(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_j\}\}};k)\neq 0$ for $1\leq j\leq r-1$ and $\widetilde{H}_{\dim\Delta}(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_r\}\}};k)=0$. Therefore, by Corollary 3.2,

$$\begin{aligned}&\beta_{|\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_j\}\}|-\dim\Delta-1,|\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_j\}\}|}\\ =\ &\beta_{\sum_{l=0}^{\dim\Delta}f_l^{\Delta}-j-\dim\Delta-1,\sum_{l=0}^{\dim\Delta}f_l^{\Delta}-j}\\ \neq\ &0 \text{ for } 0\leq j\leq r-1.\end{aligned}$$

Consider the complexes $\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_{r-1}\}\}}$ and $\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_r\}\}}$. By construction it holds that $\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots\{v_r\}\}};k\right)=0$ and $\widetilde{H}_{\dim\Delta}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots\{v_{r-1}\}\}};k\right)\neq 0$. Successively applying the fact that for a simplicial complex Δ on ground set Ω and a vertex v of Δ we get that $\mathrm{sd}(\Delta_{\Omega\setminus\{v\}})\simeq\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v\}\}}$. It follows that $\mathrm{sd}(\Delta_{\Omega\setminus\{v_1,\dots,v_{r-1}\}})\simeq\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_{r-1}\}\}}$ and $\mathrm{sd}(\Delta_{\Omega\setminus\{v_1,\dots,v_r\}})\simeq\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_r\}\}}$. Since the homology of a simplicial complex is invariant under barycentric subdivision this implies

$$\widetilde{H}_{\dim\Delta}\left(\Delta_{\Omega\setminus\{v_1,\dots,v_{r-1}\}};k\right)\neq 0 \text{ and } \widetilde{H}_{\dim\Delta}\left(\Delta_{\Omega\setminus\{v_1,\dots,v_r\}};k\right)=0.$$

Therefore, $\Delta_{\Omega\setminus\{v_1,\dots,v_{r-1}\}}$ contains a homology cycle in dimension $\dim\Delta$. We obtain the complex $\Delta_{\Omega\setminus\{v_1,\dots,v_r\}}$ from $\Delta_{\Omega\setminus\{v_1,\dots,v_{r-1}\}}$ by removing v_r and all faces containing v_r. Since there is a homology cycle in dimension $\dim\Delta$ the maximal dimensional faces of $\Delta_{\Omega\setminus\{v_1,\dots,v_{r-1}\}}$ cannot have a vertex in common. Hence, there is at least one $(\dim\Delta)$-dimensional face in $\Delta_{\Omega\setminus\{v_1,\dots,v_r\}}$. Thus we have $\dim(\Delta_{\Omega\setminus\{v_1,\dots,v_r\}})=\dim\Delta$.

Choose $F\in\Delta_{\Omega\setminus\{v_1,\dots,v_r\}}$ with $\dim F=\dim\Delta$. It follows by $\mathrm{sd}(\Delta_{\Omega\setminus\{v_1,\dots,v_r\}})\subseteq\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_r\}\}}$ that $\mathrm{sd}(\partial F)\subseteq\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\{\{v_1\},\dots,\{v_r\}\}}$ and $\mathrm{sd}(\partial F)$ is homeomorphic to a $(\dim\Delta-1)$-sphere.

The same arguments as in proof of Lemma 3.4 show that

$$\widetilde{H}_{\dim\Delta-1}\left(\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus(\{\{v_1\},\dots,\{v_r\}\}\cup A\cup\{F\})};k\right)\neq 0$$

for $A\subseteq\Delta\setminus(\Delta_F\cup\{\{v_1\},\dots,\{v_r\}\})$. By Corollary 3.2 and

$$|\mathring{\Delta}\setminus(\{\{v_1\},\dots,\{v_r\}\}\cup A\cup\{F\})|=\sum_{l=0}^{\dim\Delta}f_l^{\Delta}-r-|A|-1$$

we deduce that

$$\beta_{\sum_{l=0}^{\dim\Delta}(f_l^{\Delta}-1)-r-|A|,\sum_{l=0}^{\dim\Delta}f_l^{\Delta}-r-|A|-1}\neq 0.$$

Since

$$0\leq|A|\overset{A=\Delta\setminus(\Delta_F\cup\{v_1,\dots,v_r\})}{\leq}\sum_{l=0}^{\dim\Delta}f_l^{\Delta}-r-2^{\dim\Delta+1}+1$$

it follows that $\beta_{i,i+\dim\Delta}\neq 0$ for $2^{\dim\Delta+1}-\dim\Delta-2\leq i\leq\sum_{l=0}^{\dim\Delta}(f_l^{\Delta}-1)-r$ what finally completes the proof. $\square$

The following lemma is a simple consequence of the characterization [3, Theorem 1] of pairs of the vector $(\dim \widetilde{H}_i(\Delta; k))_{0 \leq i \leq \dim \Delta}$ encoding the Betti numbers of Δ and the f-vector $(f_i^\Delta)_{-1 \leq i \leq \dim \Delta}$ of Δ. We leave the verification to the reader.

Lemma 3.6. *Let Δ be a d-dimensional simplicial complex such that $\widetilde{H}_d(\Delta; k) = 0$. Then $f_{d-1}^\Delta \geq f_d^\Delta + d$.*

The following lemmas include simple but crucial inequalities that will be used in the derivation of the main theorem. Their proofs are straightforward and left to the reader.

Lemma 3.7. *For $d \geq 1$*

$$\frac{\prod_{l=2}^{d+1}(2^{d+1} - l)}{(d+1)! \cdot \prod_{m=2}^{d}(2^{m+1} - 3)} \geq \begin{cases} 1 & \text{if } 1 \leq d \leq 3 \\ 2 & \text{if } d \geq 4 \end{cases}$$

Lemma 3.8. *For $n \geq 11$*

$$(n+1)! \leq 2^{\frac{n^2}{2} - \frac{5}{2}n}.$$

Lemma 3.9. *For $n \in \mathbb{N}$ and $k \geq 2$*

$$\frac{\prod_{l=0}^{n-1}(2^{n+1} + 2k - 4 + l)}{(n+1)! \cdot k \cdot \prod_{m=2}^{n}(2^{m+1} - 3)} \geq 1.$$

Lemma 3.10. *For $d \geq 4$ it holds that*

$$d \cdot \prod_{l=0}^{d-2}(2^{d+2} - d - 6 - l) \geq (d+1)! \cdot \prod_{l=2}^{d}(2^{l+1} - 3).$$

4. Proof of the Theorem 1.2

Before we proceed to the proof of the main theorem, we consider Δ with small dimension. If $\dim \Delta = 0$ then I_Δ is generated by all squarefree monomials of degree 2. It is well known that the resolution of this ideal is linear. Hence the Multiplicity Conjecture holds (see e.g. [18]). For the cases $\dim \Delta = 1, 2$ the conjecture was settled in [24, Theorem 4.3] except for the equality statement. For dimensions 3 and 4 the result follows from [24] in case the complex is Gorenstein.

Proof of Theorem 1.2. Upper Bound: By the argumentation above we may assume that $\dim(\Delta) = \dim(\mathrm{sd}(\Delta)) \geq 1$. We set $F^\Delta := \sum_{l=0}^{\dim \Delta}(f_l^\Delta - 1)$. By Proposition 2.7 we have to show that

$$(\dim \Delta + 1)! \cdot f_{\dim \Delta}^\Delta \leq \frac{1}{(F^\Delta)!} \cdot \prod_{i=1}^{F^\Delta} M_i.$$

First, we consider the case $\widetilde{H}_{\dim \Delta}(\Delta; k) = 0$. From Lemmas 3.3 and 3.4 we deduce that $M_i \geq m + i$ for $2^{m+1} - 2 - m \leq i < 2^{m+2} - 2 - (m+1)$ and

$1 \leq m < \dim\Delta$ and $M_i \geq i + \dim\Delta$ for $2^{\dim\Delta+1} - 2 - \dim\Delta \leq i \leq F^\Delta$. Therefore:

$$\begin{aligned}
&\prod_{i=1}^{F^\Delta} M_i \\
=\ & \prod_{m=1}^{\dim\Delta-1}\left(\prod_{i=2^{m+1}-2-m}^{2^{m+2}-2-(m+1)-1} M_i\right) \cdot \prod_{i=2^{\dim\Delta+1}-2-\dim\Delta}^{F^\Delta} M_i \\
\geq\ & \prod_{m=1}^{\dim\Delta-1}\left(\prod_{i=2^{m+1}-2-m}^{2^{m+2}-m-4} (m+i)\right) \cdot \prod_{i=2^{\dim\Delta+1}-2-\dim\Delta}^{F^\Delta} (i+\dim\Delta) \\
=\ & \left(\prod_{m=1}^{\dim\Delta-1} \frac{(2^{m+2}-m-4+m)!}{(2^{m+1}-2-m+m-1)!}\right) \\
& \cdot \frac{\big(F^\Delta + \dim\Delta\big)!}{(2^{\dim\Delta+1}-2-\dim\Delta+\dim\Delta-1)!} \\
=\ & \left(\prod_{m=1}^{\dim\Delta-1} \frac{(2^{m+2}-4)!}{(2^{m+1}-3)!}\right) \cdot \frac{(F^\Delta+\dim\Delta)!}{(2^{\dim\Delta+1}-3)!} \\
=\ & \frac{\prod_{m=2}^{\dim\Delta}(2^{m+1}-4)!}{\prod_{m=1}^{\dim\Delta}(2^{m+1}-3)!} \cdot (F^\Delta+\dim\Delta)! \\
=\ & \frac{\prod_{m=2}^{\dim\Delta}(2^{m+1}-4)!}{\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)!} \cdot \big(F^\Delta+\dim\Delta\big)! \\
=\ & \prod_{m=2}^{\dim\Delta} \frac{1}{2^{m+1}-3} \cdot \big(F^\Delta+\dim\Delta\big)!.
\end{aligned}$$

It follows

$$\begin{aligned}
& \frac{1}{(F^\Delta)! \cdot (\dim\Delta+1)! \cdot f^\Delta_{\dim\Delta}} \cdot \prod_{i=1}^{F^\Delta} M_i \\
\geq\ & \frac{1}{(F^\Delta)! \cdot (\dim\Delta+1)! \cdot f^\Delta_{\dim\Delta}} \cdot \prod_{m=2}^{\dim\Delta} \frac{1}{2^{m+1}-3} \cdot \big(F^\Delta+\dim\Delta\big)! \\
=\ & \frac{\prod_{m=1}^{\dim\Delta}\big(F^\Delta+\dim\Delta+1-m\big)}{(\dim\Delta+1)! \cdot f^\Delta_{\dim\Delta} \cdot \prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}.
\end{aligned}$$

Along with Lemma 3.6 this yields

$$\frac{1}{(F^\Delta)! \cdot (\dim\Delta+1)! \cdot f^\Delta_{\dim\Delta}} \cdot \prod_{i=1}^{F^\Delta} M_i$$

$$\geq \quad \frac{\prod_{m=1}^{\dim\Delta}\left(\sum_{l=0}^{\dim\Delta-2} f_l^\Delta + 2f_{\dim\Delta}^\Delta + \dim\Delta - m\right)}{(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}$$
$$= \quad \frac{\prod_{m=0}^{\dim\Delta-1}\left(\sum_{l=0}^{\dim\Delta-2} f_l^\Delta + 2f_{\dim\Delta}^\Delta + m\right)}{(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}.$$

Assume that $f_{\dim\Delta}^\Delta = 1$. Then $f_i^\Delta \geq \binom{\dim\Delta+1}{i+1}$, $0 \leq i \leq \dim\Delta$. This implies

$$\frac{1}{(F^\Delta)!\cdot(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta}\cdot\prod_{i=1}^{F^\Delta} M_i$$
$$\geq \quad \frac{\prod_{m=0}^{\dim\Delta-1}\left(\sum_{l=0}^{\dim\Delta-2}\binom{\dim\Delta+1}{l+1} + 2 + m\right)}{(\dim\Delta+1)!\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}$$
$$= \quad \frac{\prod_{m=0}^{\dim\Delta-1}\left(2^{\dim\Delta+1} - (1+\binom{\dim\Delta+1}{\dim\Delta}+1) + 2 + m\right)}{(\dim\Delta+1)!\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}$$
$$= \quad \frac{\prod_{m=0}^{\dim\Delta-1}\left(2^{\dim\Delta+1} - \dim\Delta - 1 + m\right)}{(\dim\Delta+1)!\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}$$
$$= \quad \frac{\prod_{m=2}^{\dim\Delta+1}\left(2^{\dim\Delta+1} - m\right)}{(\dim\Delta+1)!\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}.$$

Since by Lemma 3.7 the latter expression is greater or equal than 1 this shows the claim in case $\widetilde{H}_{\dim\Delta}(\Delta;k) = 0$ and $f_{\dim\Delta}^\Delta = 1$.
Let $f_{\dim\Delta}^\Delta > 1$. Clearly, in this case $f_i^\Delta \geq \binom{\dim\Delta+1}{i+1} + 1$. Therefore

$$\frac{1}{(F^\Delta)!\cdot(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta}\cdot\prod_{i=1}^{F^\Delta} M_i$$
$$\geq \quad \frac{\prod_{m=0}^{\dim\Delta-1}\left(\sum_{l=0}^{\dim\Delta-2}(\binom{\dim\Delta+1}{l+1}+1) + 2f_{\dim\Delta}^\Delta + m\right)}{(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}$$
$$= \quad \frac{\prod_{m=0}^{\dim\Delta-1}\left(\sum_{l=0}^{\dim\Delta-2}\binom{\dim\Delta+1}{l+1} + \dim\Delta - 1 + 2f_{\dim\Delta}^\Delta + m\right)}{(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}$$
$$= \quad \frac{\prod_{m=0}^{\dim\Delta-1}\left(2^{\dim\Delta+1} - (1+(\dim\Delta+1)+1) + \dim\Delta - 1 + 2f_{\dim\Delta}^\Delta + m\right)}{(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}$$
$$= \quad \frac{\prod_{m=0}^{\dim\Delta-1}(2^{\dim\Delta+1} - 4 + 2f_{\dim\Delta}^\Delta + m)}{(\dim\Delta+1)!\cdot f_{\dim\Delta}^\Delta\cdot\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)} \geq 1.$$

The last inequality holds by Lemma 3.9. This proves the Multiplicity Conjecture if $\widetilde{H}_{\dim\Delta}(\Delta;k) = 0$ and $f_{\dim\Delta}^\Delta > 1$.

Let $\widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0$. If Δ has dimension 1 or 2 the claim follows from [24, Theorem 4.3]. Let $\dim\Delta \geq 3$. By Lemma 3.3 and 3.5 it holds that $M_i \geq m+i$ for $1 \leq m < \dim\Delta$ and $2^{m+1}-2-m \leq i < 2^{m+2}-2-(m+1)$ and $M_i \geq i+\dim\Delta$ for $2^{\dim\Delta+1}-2-\dim\Delta \leq i \leq F^\Delta-1$ and $M_{F^\Delta} \geq F^\Delta+\dim\Delta+1 = \sum_{l=0}^{\dim\Delta} f_l^\Delta$. Therefore, the same calculation as in the first part of the proof yields

$$\prod_{i=1}^{F^\Delta} M_i \geq \prod_{m=2}^{\dim\Delta} \frac{1}{2^{m+1}-3} \cdot \frac{\big(F^\Delta+\dim\Delta+1\big)!}{F^\Delta+\dim\Delta}.$$

Thus it suffices to show that

$$\frac{\big(F^\Delta+\dim\Delta+1\big)!}{(\dim\Delta+1)!f_{\dim\Delta}^\Delta(F^\Delta)!\,(F^\Delta+\dim\Delta)\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)} \geq 1.$$

We have that

$$\begin{aligned} &\frac{\big(F^\Delta+\dim\Delta+1\big)!}{(\dim\Delta+1)!f_{\dim\Delta}^\Delta(F^\Delta)!\,(F^\Delta+\dim\Delta)\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)} \\ = &\frac{\prod_{m=2}^{\dim\Delta}\big(F^\Delta+\dim\Delta+1-m\big)\cdot(F^\Delta+\dim\Delta+1)}{(\dim\Delta+1)!f_{\dim\Delta}^\Delta\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}. \end{aligned}$$

Since $\widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0$ it holds that $f_i^\Delta \geq f_i^{\partial(\Delta_{\dim\Delta+1})} = \binom{\dim\Delta+2}{i+1}$, where $\partial(\Delta_{\dim\Delta+1})$ denotes the boundary of the $(\dim\Delta+1)$-simplex. It follows that

$$\begin{aligned} \sum_{l=0}^{\dim\Delta-1} f_l^\Delta &\geq \sum_{l=0}^{\dim\Delta-1}\binom{\dim\Delta+2}{l+1} \\ &= 2^{\dim\Delta+2}-(1+\binom{\dim\Delta+2}{\dim\Delta+1}+1) \\ &= 2^{\dim\Delta+2}-\dim\Delta-4. \qquad (4.1) \end{aligned}$$

We conclude that

$$\begin{aligned} &\frac{\big(F^\Delta+\dim\Delta+1\big)!}{(\dim\Delta+1)!f_{\dim\Delta}^\Delta(F^\Delta)!\,(F^\Delta+\dim\Delta)\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)} \\ \geq &\prod_{m=2}^{\dim\Delta}\big(2^{\dim\Delta+2}-\dim\Delta-4+f_{\dim\Delta}^\Delta-m\big) \\ &\cdot\frac{(2^{\dim\Delta+2}-\dim\Delta-4+f_{\dim\Delta}^\Delta)}{(\dim\Delta+1)!f_{\dim\Delta}^\Delta\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)} \\ = &\prod_{m=6+\dim\Delta}^{4+2\dim\Delta}\big(2^{\dim\Delta+2}-m+f_{\dim\Delta}^\Delta\big) \\ &\cdot\frac{\big(2^{\dim\Delta+2}-\dim\Delta-4+f_{\dim\Delta}^\Delta\big)}{(\dim\Delta+1)!f_{\dim\Delta}^\Delta\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\dim\Delta \cdot f^{\Delta}_{\dim\Delta} \cdot \prod_{m=6+\dim\Delta}^{4+2\dim\Delta}(2^{\dim\Delta+2}-m)}{(\dim\Delta+1)! f^{\Delta}_{\dim\Delta}\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}\\
&= \frac{\dim\Delta \cdot \prod_{m=6+\dim\Delta}^{4+2\dim\Delta}(2^{\dim\Delta+2}-m)}{(\dim\Delta+1)!\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}\\
&= \frac{\dim\Delta \cdot \prod_{m=0}^{\dim\Delta-2}(2^{\dim\Delta+2}-\dim\Delta-6-m)}{(\dim\Delta+1)!\prod_{m=2}^{\dim\Delta}(2^{m+1}-3)}\\
&\geq 1,
\end{aligned}$$

where the last inequality holds by Lemma 3.10 for $\dim\Delta \geq 4$.
It remains to show the assertion for $\dim\Delta = 3$.
By Equation (4.1) we have that

$$\begin{aligned}
&\frac{(\sum_{l=0}^{3} f_l^{\Delta}-3)(\sum_{l=0}^{3} f_l^{\Delta}-2)(\sum_{l=0}^{3} f_l^{\Delta})}{4!\cdot f_3^{\Delta}\cdot\prod_{m=2}^{3}(2^{m+1}-3)}\\
\geq\ &\frac{(2^5-10+f_3^{\Delta})(2^5-9+f_3^{\Delta})(2^5-7+f_3^{\Delta})}{24\cdot f_3^{\Delta}\cdot 5\cdot 13}\\
=\ &\frac{(22+f_3^{\Delta})(23+f_3^{\Delta})(25+f_3^{\Delta})}{1560 f_3^{\Delta}}\\
=\ &\frac{(f_3^{\Delta})^3+70(f_3^{\Delta})^2+1631 f_3^{\Delta}+12650}{1560 f_3^{\Delta}}\\
\geq\ &\frac{1631 f_3^{\Delta}}{1560 f_3^{\Delta}} \geq 1
\end{aligned}$$

This finally concludes the proof of the upper bound in the Multiplicity Conjecture.

Cohen-Macaulay Case and Lower Bound: By Proposition 2.7 we have to show that

$$\frac{1}{(F^{\Delta})!}\cdot\prod_{i=1}^{F^{\Delta}} m_i \leq (\dim\Delta+1)!\cdot f^{\Delta}_{\dim\Delta}.$$

$\triangleright$ Case 1: First, we consider simplicial complexes Δ for which $\dim\Delta \geq 7$ and there exists a face $F\in\Delta$ of dimension $\dim\Delta-1$ that is contained in a unique facet G. Then, the restricted complex $\mathrm{sd}(\Delta)_{\mathring{\Delta}_G\setminus(\{G\}\cup\partial\mathring{F})}$ consists of at least two connected components. Therefore, $\mathrm{sd}(\Delta)_W$ is disconnected if $|W|\geq 2$, $F\in W$ and $W\subseteq\mathring{\Delta}\setminus(\{G\}\cup\partial\mathring{F})$. This implies $\widetilde{H}_0(\mathrm{sd}(\Delta)_W;k)\neq 0$ for $|W|\geq 2$, $F\in W$ and $W\subseteq\mathring{\Delta}\setminus(\{G\}\cup\partial\mathring{F})$. From Corollary 3.2 we deduce $\beta_{|W|-1,|W|}\neq 0$ for $2\leq |W|\leq \sum_{j=0}^{\dim\Delta} f_j^{\Delta}-(2^{\dim\Delta}-2)-1$, i.e. $\beta_{i,i+1}\neq 0$ for $1\leq |W|\leq \sum_{j=0}^{\dim\Delta} f_j^{\Delta}-2^{\dim\Delta}$. Thus $m_i\leq i+1$ for $1\leq |W|\leq\sum_{j=0}^{\dim\Delta} f_j^{\Delta}-2^{\dim\Delta}$. From Section 2.7 we know that $\mathrm{reg}(k[\mathrm{sd}(\Delta)])\leq \dim\Delta+1$. Hence, $m_i\leq i+\dim\Delta+1$

for $\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+1 \leq i \leq F^\Delta$. This implies

$$\begin{aligned}
&\prod_{i=1}^{F^\Delta} m_i \cdot \frac{1}{(F^\Delta)!} \\
&= \prod_{i=1}^{\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}} m_i \cdot \prod_{i=\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+1}^{F^\Delta} m_i \cdot \frac{1}{(F^\Delta)!} \\
&\leq \frac{1}{(F^\Delta)!} \cdot \prod_{i=1}^{\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}} (i+1) \cdot \prod_{i=\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+1}^{F^\Delta} (i+\dim\Delta+1) \\
&= \frac{1}{(F^\Delta)!} \cdot (\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+1)! \cdot \frac{(F^\Delta+\dim\Delta+1)!}{(\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+\dim\Delta+1)!} \\
&= \frac{\prod_{i=1}^{\dim\Delta+1}(F^\Delta+i)}{\prod_{i=1}^{\dim\Delta}(\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+1+i)}.
\end{aligned}$$

The claim now follows if we show that

$$\begin{aligned}
&\prod_{i=1}^{\dim\Delta+1} (F^\Delta+i) \\
&\leq (\dim\Delta+1)! \cdot f_{\dim\Delta}^\Delta \cdot \prod_{i=1}^{\dim\Delta} (\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+1+i).
\end{aligned} \tag{4.2}$$

For $d \geq 8$ it holds that $d! \geq 2^{2d-1}$. From

$$\begin{aligned}
2^{\dim\Delta+1} \cdot f_{\dim\Delta}^\Delta &\geq \sum_{j=0}^{\dim\Delta} f_j^\Delta \\
&= F^\Delta+\dim\Delta+1
\end{aligned}$$

we conclude that

$$\begin{aligned}
(\dim\Delta+1)! \cdot f_{\dim\Delta}^\Delta &\geq 2^{2\cdot\dim\Delta+1} \cdot f_{\dim\Delta}^\Delta \\
&= 2^{\dim\Delta} \cdot 2^{\dim\Delta+1} \cdot f_{\dim\Delta}^\Delta \\
&\geq 2^{\dim\Delta} \cdot (F^\Delta+\dim\Delta+1)
\end{aligned}$$

for $\dim\Delta \geq 7$.
Therefore, it suffices to show that

$$\prod_{i=1}^{\dim\Delta+1} (F^\Delta+i) \leq 2^{\dim\Delta} \cdot (F^\Delta+\dim\Delta+1) \cdot \prod_{i=1}^{\dim\Delta} (\sum_{j=0}^{\dim\Delta} f_j^\Delta - 2^{\dim\Delta}+1+i),$$

i.e.

$$\prod_{i=1}^{\dim\Delta}(F^{\Delta}+i) \leq \prod_{i=1}^{\dim\Delta} 2\cdot(\sum_{j=0}^{\dim\Delta} f_j^{\Delta} - 2^{\dim\Delta}+1+i).$$

By $\sum_{j=-1}^{\dim\Delta} f_j^{\Delta} \geq 2^{\dim\Delta+1}$ it follows that

$$\begin{aligned} & F^{\Delta}+2\dim\Delta+2+i \\ = & \sum_{j=-1}^{\dim\Delta} f_j^{\Delta}+\dim\Delta+1+i \\ \geq & 2^{\dim\Delta+1} \end{aligned}$$

for $1 \leq i \leq \dim\Delta$. This implies

$$\begin{aligned} & F^{\Delta}+i \\ \leq & 2\cdot(F^{\Delta}+\dim\Delta+1-2^{\dim\Delta}+1+i) \\ = & 2\cdot(\sum_{j=0}^{\dim\Delta} f_j^{\Delta}-2^{\dim\Delta}+1+i) \end{aligned}$$

for $1 \leq i \leq \dim\Delta$. This concludes the proof in Case 1.

$\triangleright$ Case 2: Now, we consider simplicial complexes Δ such that $\dim\Delta \geq 6$ and every $(\dim\Delta-1)$-dimensional face lies in at least two facets. Let $F \in \Delta$ be a face of dimension $\dim\Delta$. The restriction $\mathrm{sd}(\Delta)_{\mathring{\Delta}\setminus\partial\mathring{F}}$ consists of two connected components. In particular, every restriction of the form $\mathrm{sd}(\Delta)_W$ where $W \subseteq \mathring{\Delta}\setminus\partial\mathring{F}$, $F \in W$ and $|W| \geq 2$, is disconnected, i.e. $\widetilde{H}_0(\mathrm{sd}(\Delta)_W;k) \neq 0$. From Corollary 3.2 we deduce that $\beta_{|W|-1,|W|} \neq 0$. Therefore, $\beta_{i,i+1} \neq 0$ for $2 \leq i+1 \leq \sum_{j=0}^{\dim\Delta} f_j^{\Delta} - (2^{\dim\Delta+1}-2)$. This implies $m_i \leq i+1$ for $1 \leq i \leq \sum_{j=0}^{\dim\Delta} f_j^{\Delta} - 2^{\dim\Delta+1}+1$. As in Case 1, Section 2.7 implies that $m_i \leq i+\dim\Delta+1$ for $\sum_{j=0}^{\dim\Delta} f_j^{\Delta} - 2^{\dim\Delta+1}+2 \leq i \leq F^{\Delta}$. Therefore,

$$\begin{aligned} & \frac{1}{(F^{\Delta})!}\cdot\prod_{i=1}^{F^{\Delta}} m_i \\ = & \frac{1}{(F^{\Delta})!}\cdot\prod_{i=1}^{\sum_{j=0}^{\dim\Delta} f_j^{\Delta}-2^{\dim\Delta+1}+1} m_i \cdot \prod_{i=\sum_{j=0}^{\dim\Delta} f_j^{\Delta}-2^{\dim\Delta+1}+2}^{F^{\Delta}} m_i \\ \leq & \frac{1}{(F^{\Delta})!}\cdot\prod_{i=1}^{\sum_{j=0}^{\dim\Delta} f_j^{\Delta}-2^{\dim\Delta+1}+1} (i+1) \cdot \prod_{i=\sum_{j=0}^{\dim\Delta} f_j^{\Delta}-2^{\dim\Delta+1}+2}^{F^{\Delta}} (i+\dim\Delta+1) \\ = & \frac{(F^{\Delta}+\dim\Delta+1-2^{\dim\Delta+1}+2)!}{(F^{\Delta})!}\cdot\frac{(F^{\Delta}+\dim\Delta+1)!}{(F^{\Delta}+2\dim\Delta+2-2^{\dim\Delta+1}+1)!} \end{aligned}$$

$$= \frac{\prod_{i=1}^{\dim\Delta+1}(F^\Delta+i)}{\prod_{i=1}^{\dim\Delta}(F^\Delta+\dim\Delta+1-2^{\dim\Delta+1}+2+i)}.$$

Thus, it suffices to show that

$$\prod_{i=1}^{\dim\Delta+1}(F^\Delta+i) \le (\dim\Delta+1)!\cdot f^\Delta_{\dim\Delta}\cdot\prod_{i=1}^{\dim\Delta}(F^\Delta+\dim\Delta+3-2^{\dim\Delta+1}+i).$$

Since Δ is pure, every i-dimensional face is contained in a $(\dim\Delta-1)$-dimensional face and then by assumption in at least two facets of Δ. This implies $f_i^\Delta \le \binom{\dim\Delta+1}{i+1}\cdot f^\Delta_{\dim\Delta}\cdot\frac{1}{2}$ for $0\le i\le\dim\Delta-1$. Hence

$$\begin{aligned}\sum_{i=0}^{\dim\Delta} f_i^\Delta &\le \sum_{i=0}^{\dim\Delta-1}\binom{\dim\Delta+1}{i+1}\cdot f^\Delta_{\dim\Delta}\cdot\frac{1}{2}+f^\Delta_{\dim\Delta}\\ &= (2^{\dim\Delta+1}-2)\cdot f^\Delta_{\dim\Delta}\cdot\frac{1}{2}+f^\Delta_{\dim\Delta}\\ &= 2^{\dim\Delta}\cdot f^\Delta_{\dim\Delta}, \qquad (4.3)\end{aligned}$$

i.e. $2^{\dim\Delta}\cdot f^\Delta_{\dim\Delta}\ge F^\Delta+\dim\Delta+1$.

Using $d!\ge 2^{2d-2}$ for $d\ge 7$ we obtain that $(\dim\Delta+1)!\cdot f^\Delta_{\dim\Delta}\ge 2^{\dim\Delta}\cdot(F^\Delta+\dim\Delta+1)$ for $\dim\Delta\ge 6$. Therefore, it suffices to show that

$$\prod_{i=1}^{\dim\Delta+1}(F^\Delta+i)\le 2^{\dim\Delta}\cdot(F^\Delta+\dim\Delta+1)\cdot\prod_{i=1}^{\dim\Delta}(F^\Delta+\dim\Delta+3-2^{\dim\Delta+1}+i),$$

i.e.

$$\prod_{i=1}^{\dim\Delta}(F^\Delta+i)\le\prod_{i=1}^{\dim\Delta}2\cdot(F^\Delta+\dim\Delta+3-2^{\dim\Delta+1}+i)$$

for $\dim\Delta\ge 6$.

For a simplicial complex with the property that every face of dimension $\dim\Delta-1$ is contained in at least two facets each entry of its f-vector is bounded from below by the same entry in the f-vector of the boundary of the $(\dim\Delta+1)$-simplex. Hence,

$$\sum_{j=-1}^{\dim\Delta} f_j^\Delta \ge 2^{\dim\Delta+2}-1. \qquad (4.4)$$

This implies

$$\begin{aligned}&F^\Delta+2\cdot\dim\Delta+6+i\\ =\ &\sum_{j=-1}^{\dim\Delta} f_j^\Delta+\dim\Delta+4+i\ge 2^{\dim\Delta+2}\end{aligned}$$

for $1\le i\le\dim\Delta$. We deduce that

$$F^\Delta+i\le 2\cdot(F^\Delta+\dim\Delta+3-2^{\dim\Delta+1}+i)$$

for $1 \leq i \leq \dim \Delta$. This proves the lower bound conjecture in Case 2.

▷ Small Dimensions: We now prove the lower bound of the Multiplicity Conjecture for the dimensions not covered in Case 1 and Case 2.
Since Δ is a Cohen-Macaulay complex there exists an ordering $F_1, \ldots, F_{f^{\Delta}_{\dim \Delta}}$ of the facets of Δ such that $\dim \overline{(F_1 \cup \ldots \cup F_i)} \cap F_{i+1} = \dim \Delta - 1$. This implies

$$\begin{aligned} & \sum_{j=0}^{\dim \Delta} f_j^{\Delta} \\ \leq\ & (2^{\dim \Delta+1} - 1) + (f^{\Delta}_{\dim \Delta} - 1) \cdot (2^{\dim \Delta+1} - 1 - (2^{\dim \Delta} - 1)) \\ =\ & 2^{\dim \Delta} - 1 + 2^{\dim \Delta} \cdot f^{\Delta}_{\dim \Delta}. \end{aligned} \tag{4.5}$$

By the arguments preceding the proof we may assume $\dim \Delta \geq 3$.
$\dim\Delta = 3$. In the situation of Case 1 we have to show that

$$(F^{\Delta} + 1) \cdots (F^{\Delta} + 4) \leq 4! \cdot f_3^{\Delta} \cdot (F^{\Delta} + 4 - 2^3 + 1 + 1) \cdot (F^{\Delta} - 1) \cdot F^{\Delta}.$$

Inequality (4.5) yields $F^{\Delta} + 4 \leq 8 \cdot f_3^{\Delta} + 7$. If Δ is the 3-simplex, then $\mathrm{sd}(\Delta)$ is Gorenstein an the result follows by the [24]. If Δ is not the 3-simplex it follows from the above considerations that it suffices to show that

$$(F^{\Delta} + 1) \cdot (F^{\Delta} + 2) \cdot (F^{\Delta} + 3) \leq 2 \cdot (F^{\Delta} - 2) \cdot (F^{\Delta} - 1) \cdot F^{\Delta}$$

which is equivalent to $(F^{\Delta})^3 - 12 \cdot (F^{\Delta})^2 - 7 \cdot F^{\Delta} - 6 \geq 0$. Since Δ is not the 3-simplex we can deduce $F^{\Delta} \geq 2^4 - 1 + (2^4 - 1 - (2^3 - 1)) - 4 = 19$. Since

$$\begin{aligned} 6 + 7 \cdot F^{\Delta} + 12 \cdot (F^{\Delta})^2 &\leq 8 \cdot F^{\Delta} + 12 \cdot (F^{\Delta})^2 \\ &\leq (F^{\Delta})^2 + 12 \cdot (F^{\Delta})^2 \\ &= 13 \cdot (F^{\Delta})^2 \leq (F^{\Delta})^3 \end{aligned}$$

for $F^{\Delta} \geq 19$ the claim follows.
Now we turn to the situation of Case 2. If Δ is the boundary of the 4-simplex then the complex and its barycentric subdivision are Gorenstein. Hence the result follows from [24]. Assume Δ is not the boundary of the 4-simplex. Then Δ must have at least one additional 3-simplex (i.e, $f_3^{\Delta} \geq 5$). From the Kruskal-Katona theorem we infer that $f_2^{\Delta}, f_1^{\Delta} \geq 12$ and $f_0^{\Delta} \geq 6$. This implies $F^{\Delta} \geq 32$. We have to show that

$$(F^{\Delta} + 1) \cdot \ldots \cdot (F^{\Delta} + 4) \leq 4! \cdot f_3^{\Delta} \cdot (F^{\Delta} + 4 - 2^4 + 2 + 1) \cdot (F^{\Delta} - 8) \cdot (F^{\Delta} - 7).$$

By Inequality (4.3) it holds that $F^{\Delta} + 4 \leq 8 \cdot f_3^{\Delta}$. Hence, it suffices to show that $(F^{\Delta} + 1) \cdot (F^{\Delta} + 2) \cdot (F^{\Delta} + 3) \leq 3 \cdot (F^{\Delta} - 9) \cdot (F^{\Delta} - 8) \cdot (F^{\Delta} - 7)$. The inequality is satisfied for $F^{\Delta} \geq 31$ and we are done.
$\dim\Delta = 4$. In the situation Case 1 the desired inequality is the following

$$(F^{\Delta}+1) \cdot \ldots \cdot (F^{\Delta}+5) \leq 5! \cdot f_4^{\Delta} \cdot (F^{\Delta}+5-2^4+1+1) \cdot (F^{\Delta}-8) \cdot (F^{\Delta}-7) \cdot (F^{\Delta}-6).$$

Inequality (4.5) yields $F^\Delta + 5 \leq 16 \cdot f_4^\Delta + 15 \leq 24 \cdot f_4^\Delta$ if $f_4^\Delta \geq 2$, i.e. Δ is not the 4-simplex. It then suffices to show

$$(F^\Delta+1)\cdot(F^\Delta+2)\cdot(F^\Delta+3)\cdot(F^\Delta+4) \leq 5\cdot f_4^\Delta\cdot(F^\Delta-9)\cdot(F^\Delta-8)\cdot(F^\Delta-7)\cdot(F^\Delta-6),$$

which is equivalent to

$$4\cdot(F^\Delta)^4 - 160\cdot(F^\Delta)^3 + 1640\cdot(F^\Delta)^2 - 8300\cdot F^\Delta + 15096 \geq 0.$$

From $F^\Delta \geq 2^5 - 1 + 2^4 - 5 = 42$ for $f_4^\Delta \geq 2$ we deduce that $4 \cdot F^\Delta \geq 160$ and $1640 \cdot F^\Delta \geq 8300$. This finally implies the claim. If Δ is the 4-simplex then again the result follows from [24]. Now assume the situation of Case 2. If Δ is the boundary of the 5-simplex then the assertion follows by Gorenstein-ness from [24]. If Δ is not the boundary of the 5-simplex then $f_4^\Delta \geq 7$. Here Kruskal-Katona theorem implies $f_3^\Delta, f_1^\Delta \geq 19, f_2^\Delta \geq 26, f_0^\Delta \geq 7$. Thus $F^\Delta \geq 74$. We we have to show that

$$(F^\Delta+1)\cdot\ldots\cdot(F^\Delta+5) \leq 5\cdot f_4^\Delta\cdot(F^\Delta+5-2^5+2+1)\cdot(F^\Delta-23)\cdot\ldots\cdot(F^\Delta-21).$$

By Inequality (4.3) it holds that $F^\Delta + 5 \leq 16 \cdot f_4^\Delta$. It thus suffices to show that

$$(F^\Delta+1)\cdot\ldots\cdot(F^\Delta+4) \leq \frac{15}{2}\cdot(F^\Delta-24)\cdot(F^\Delta-23)\cdot(F^\Delta-22)\cdot(F^\Delta-21).$$

The latter inequality is true for $F^\Delta \geq 61$. Hence we are done.
dim$\Delta = 5$. In the situation of Case 1 we have to show

$$(F^\Delta+1)\cdot\ldots\cdot(F^\Delta+6) \leq 6!\cdot f_5^\Delta\cdot(F^\Delta+6-2^5+1+1)\cdot(F^\Delta-23)\cdot\ldots\cdot(F^\Delta-20).$$

Inequality (4.5) implies $F^\Delta + 6 \leq 32 \cdot f_5^\Delta + 31 \leq 48 \cdot f_5^\Delta$ if $f_5^\Delta \geq 2$, i.e. Δ is not the 5-simplex. It then suffices to show that

$$(F^\Delta+1)\cdot\ldots\cdot(F^\Delta+5) \leq 15\cdot(F^\Delta-24)\cdot\ldots\cdot(F^\Delta-20).$$

The latter inequality is true for $F^\Delta \geq 57$. By $\dim\Delta = 5$ it follows that $F^\Delta \geq 2^6 - 1 - 6 = 57$ which implies the assertion.
In the situation of Case 2 we have to show that

$$(F^\Delta+1)\cdot\ldots\cdot(F^\Delta+6) \leq 6!\cdot f_5^\Delta\cdot(F^\Delta+6-2^6+2+1)\cdot(F^\Delta-54)\cdot\ldots\cdot(F^\Delta-51).$$

By Inequality (4.3) we know that $F^\Delta + 6 \leq 2^5 \cdot f_5^\Delta$. Therefore, it suffices to show that

$$(F^\Delta+1)\cdot\ldots\cdot(F^\Delta+5) \leq 22,5\cdot(F^\Delta-55)\cdot\ldots\cdot(F^\Delta-51).$$

The above inequality is satisfied for $F^\Delta \geq 118$. Since by Inequality (4.4) $F^\Delta \geq 2^7 - 1 - 1 - 6 = 120$ this implies the claim. This concludes the proof of the lower bound part of the Multiplicity Conjecture if every $(\dim\Delta - 1)$-dimensional face of Δ lies in at least two facets of Δ.

dim$\Delta = 6$. We only need to consider the situation of Case 1. We have to show that

$$(F^\Delta+1)\cdot\ldots\cdot(F^\Delta+7) \leq 7!\cdot f_6^\Delta\cdot(F^\Delta+7-2^6+1+1)\cdot(F^\Delta-54)\cdot\ldots\cdot(F^\Delta-50).$$

From Inequality (4.5) we deduce that $F^\Delta + 7 \leq 2^6 - 1 + 2^6 \cdot f_6^\Delta = 64 \cdot f_6^\Delta + 63 \leq 96 \cdot f_6^\Delta$ if $f_6^\Delta \geq 2$, i.e. Δ is not the 6-simplex. It then suffices to show that

$$(F^\Delta + 1) \cdot \ldots \cdot (F^\Delta + 6) \leq 52,5 \cdot (F^\Delta - 55) \cdot \ldots \cdot (F^\Delta - 50).$$

This finally concludes the proof of the lower bound of the Multiplicity Conjecture since the latter inequality is satisfied for $F^\Delta \geq 113$ and since from $\dim \Delta = 6$ we deduce that $F^\Delta \geq 2^7 - 1 - 7 = 120$.

Cohen-Macaulay Case and Equality: It remains to study the equality case when $k[\mathrm{sd}(\Delta)]$ is Cohen-Macaulay. By Section 2.9 we know that $k[\mathrm{sd}(\Delta)]$ is Cohen-Macaulay if and only if $k[\Delta]$ is. Assume $\dim(\Delta) \geq 2$. From the fact that Δ is Cohen-Macaulay we infer that either Δ is the 2-simplex or Δ contains two 2-dimensional faces that intersect along a 1-dimensional face. If Δ is a 2-simplex then by inspection one sees that the resolution of $k[\mathrm{sd}(\Delta)]$ is pure – indeed linear – and satisfies the Multiplicity Conjecture with equality. Hence after relabeling we may assume that Δ contains the face $\{1, 2, 3\}$ and the face $\{1, 2, 4\}$. We show that in this case the minimal free resolution of $k[\mathrm{sd}(\Delta)]$ is never pure and the inequality always strict. The restriction of $\mathrm{sd}(\Delta)$ to the vertices $\{1\}$, $\{1, 2, 3\}$, $\{2\}$ and $\{1, 2, 4\}$ yields a 4-gon which shows by Corollary 3.2 that $\beta_{2,4} \neq 0$. On the other hand the restriction of $\mathrm{sd}(\Delta)$ to the vertices $\{1\}$, $\{2\}$, $\{3\}$ yields three isolated points which then again by Corollary 3.2 shows that $\beta_{2,3} \neq 0$. Thus the minimal free resolution can never be pure. But our reasoning also implies that $M_2 \geq 4$. Since our estimates only use $M_2 \geq 3$ this then shows that the inequality is strict. The case $\dim(\mathrm{sd}(\Delta)) = 0$ was already covered by the arguments preceding the whole proof. Hence it remains to consider $\dim(\Delta) = \dim(\mathrm{sd}(\Delta)) = 1$. By Theorem 4.3 from [24] it follows that equality implies pureness of the minimal free resolution for $k[\mathrm{sd}(\Delta)]$. Assume the minimal free resolution of $k[\mathrm{sd}(\Delta)]$ is pure. It is possible to prove by elementary reasoning that equality holde in this situation. But more generally it is well known by a result of Huneke and Miller [20] that equality holds for any standrard graded Cohen-Macaulay k which has a pure resolution.

□

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