SOME VERY NON-KÄHLER MANIFOLDS: THE FRÖLICHER SPECTRAL SEQUENCE CAN BE ARBITRARILY NON DEGENERATE

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ABSTRACT. The Frölicher spectral sequence of a compact complex manifold X measures the difference between Dolbeault cohomology and de Rham cohomology. If X is Kähler then we have the Hodge decomposition and the spectral sequence collapses at the E_1 term. After the Iwasawa manifold few other examples for higher order non-vanishing phenomena were found; in particular no example with $d_n \neq 0$ for $n > 3$ has been described in the literature.

We will give a family of nilmanifolds with left-invariant complex structure X_n such that the n-th differential d_n does not vanish. This answers a question mentioned in the book of Griffith and Harris.

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Introduction. Let X be a compact complex manifold and $\mathcal{A}^{p,q}$ be the sheaf of smooth differential (p,q)-forms. The decomposition of the exterior differential $d = \partial + \bar\partial$ gives rise to a double complex $(\mathcal{A}^{p,q}(X), \partial, \bar\partial)$. The columns of this double complex

$$0 \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \dots$$

calculate the Dolbeault cohomology of X which coincides with the cohomology with values in the sheaf of holomorphic p-form

$$H^q(\Omega^p_X) \cong H^{p,q}(X) = \frac{\ker(\bar\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\bar\partial(\mathcal{A}^{p,q-1}(X))}.$$

The Frölicher spectral sequence, also called Hodge-de Rham spectral sequence, is the spectral sequence $(E_n^{p,q}, d_n)$ associated to this double complex such that $d_0 = \bar\partial$ and it converges to the usual de Rham cohomology of complex valued differential forms (see e.g. [GH78], p. 444).

It was used by Frölicher in [Frö55] to produce relations between the topological and the holomorphic invariants of X.

Since $E_1^{p,q} = H^{p,q}(X)$ the spectral sequence measures the difference between the Dolbeault cohomology groups and the de Rham cohomology groups. Indeed, if X is a Kähler manifold then we have the Hodge-decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

and hence the spectral sequence degenerates at the E_1 term. In this sense non-zero differentials in higher terms of the spectral sequence measure how far a complex manifold is from being a Kähler manifold.

Kodaira showed that for compact complex surfaces d_1 is always zero and for a long time very few manifolds with non-vanishing differential of higher order were known. It was in fact speculated if $E_2 = E_\infty$ holds for every compact complex manifold.

The historically first example with $d_1 \neq 0$ was the Iwasawa manifold: consider the nilpotent complex Lie group

$$G := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{C} \right\}$$

and the discrete cocompat subgroup $\Gamma := G \cap \mathrm{Gl}(3, \mathbb{Z}[i]) \subset G$. Then $X := G/\Gamma$ is a complex parallelisable nilmanifold such that $E_1 \ncong E_2 = E_\infty$. Later Sakane showed that $d_2 = 0$ always holds for such manifolds (see [CFG91]).

Cordero, Fernández and Gray [CFG91] constructed a nilmanifold with left-invariant complex structure such that $E_3 \ncong E_4 = E_\infty$ and together with Ugarte [CFUG99] they showed that for 3-folds several different non-degeneracy phenomena can occur up to $E_2 \ncong E_3 = E_\infty$.

Pittie [Pit89] was able to give a simply-connected example by constructing a left-invariant complex structure on Spin(9) such that $d_2 \neq 0$.

The aim of this short note is to answer the question mentioned in the book of Griffith and Harris [GH78] and repeated by Cordero, Fernández and Gray if we can exhibit manifolds with $d_n \neq 0$ for arbitrary large n.

Theorem 1 — *For every $n \geq 2$ there exist a complex $2n$-dimensional nilmanifold with left-invariant complex structure $X_n \to T_n$, which is a principal holomorphic torus bundles over a complex torus of dimension n with fibre of dimension n, such that the Frölicher spectral sequence does not degenerate at the E_n term, i.e., $d_n \neq 0$.*

Since we do not want to go too far into the general theory of nilmanifolds with left-invariant complex structure which can be found elsewhere (see e.g. [CFUG99, CF01, Sal01, Rol07]) we prefer to give an elementary and explicit construction of the example.

Construction of the example. We will specify a Lie algebra with a complex structure and explain how to obtain from this data a principal holomorphic torus bundle X_n over a complex torus T_n. We then give explicitly a class in $E_n^{0,n-1}$ which is not mapped to zero by d_n using left-invariant differential forms.

Let $n \geq 2$ and $\mathfrak{g}$ be the $4n$-dimensional, real Lie algebra with basis

$$\{x_k, y_k, e_k, f_k\}_{k=1\dots n}$$

and the bracket relations generated by

$$[x_k, x_1] = \begin{cases} 0 & k = 1 \\ e_{k+1} + (-1)^n e_k & 1 < k < n \\ (-1)^n e_n & k = n \end{cases},$$

$$[y_k, y_1] = \begin{cases} 0 & k = 1 \\ e_{k+1} - (-1)^n e_k & 1 < k < n \\ -(-1)^n e_n & k = n \end{cases},$$

(1)

$$[y_k, x_1] = \begin{cases} \frac{1}{2}(f_1 - f_2) & k = 1 \\ (-1)^n f_k - f_{k+1} & 1 < k < n \\ (-1)^n f_n & k = n \end{cases},$$

$$[x_k, y_1] = \begin{cases} \frac{1}{2}(-f_1 + f_2) & k = 1 \\ (-1)^n f_k + f_{k+1} & 1 < k < n \\ (-1)^n f_n & k = n \end{cases}.$$

The centre of $\mathfrak{g}$ is

$$\mathfrak{z} = span_{\mathbb{R}}\{e_1, \dots e_n, f_1, \dots, f_n\}$$

and we get a short exact sequence of Lie algebras

$$0 \to \mathfrak{z} \to \mathfrak{g} \to \mathfrak{h} \to 0 \tag{2}$$

where $\mathfrak{h}$ is also abelian. In other words $\mathfrak{g}$ is a 2-step nilpotent Lie algebra.

Let G, Z and H be associated simply-connected Lie groups. Then (2) induces an exact sequence of Lie groups

$$0 \to Z \to G \xrightarrow{\tilde{\pi}} H \to 0.$$

For nilpotent Lie groups the exponential map is a diffeomorphism and, since the structure constants of the Lie algebra are rational, the images of (some multiple of) the basis vectors of $\mathfrak{g}$ generate a cocompact discrete subgroup $\Gamma \subset G$ ([CG90], Corollary 5.1.8, p. 203).

After taking the quotient by Γ we obtain for every $n \geq 2$ a real principal torus bundle of real dimension $4n$

$$\pi : X_n := \Gamma \backslash G_n \to T_n := \pi(\Gamma)\backslash H$$

over a $2n$-dimensional torus T_n with fibre $(\Gamma \cap Z)\backslash Z$.

Note that we can interpret elements in $\mathfrak{g}^*$ as differential forms on X_n which are left-invariant under the G-action when pulled back to G.

We define an almost complex structure on X_n in the following way: let $J : \mathfrak{g} \to \mathfrak{g}$ be given by

$$\begin{aligned} Jx_k &= y_k & \qquad Je_k &= f_k \\ Jy_k &= -x_k & \qquad Jf_k &= -e_k, \end{aligned}$$

hence $J^2 = -\mathrm{id}_{\mathfrak{g}}$.

Since $\mathfrak{g}$ can be identified with the tangent space at the identity we can extend this map to the tangent bundle of G_n by left translation and it then induces a left-invariant almost complex structure on X_n.

We will check below that J is in fact integrable and makes X_n into a complex manifold. Assuming this for a moment we see that by definition of J the maps in the sequence (2) commute with the action of J and hence are complex linear maps. In particular the differential of π at the identity, and hence everywhere since all structures are left-invariant, is complex linear. Therefore the projection π is holomorphic and $\pi : X_n \to T_n$ becomes a principal holomorphic torus bundle of complex dimension $2n$ with fibre and base complex tori of complex dimension n.

We can also describe X_n in another way: the elements

$$x_2, y_2, \ldots, x_n, y_n, e_1, f_1, \ldots e_n, f_n$$

span an abelian ideal of $\mathfrak{g}$ which is invariant under the action of J and compatible with the definition of Γ. Hence we get a well defined holomorphic torus bundle (which is not a principal bundle)

$$X_n \to E$$

where E is the complex 1-dimensional torus $(\langle x_1, y_1\rangle/\mathbb{Z}x_1 + \mathbb{Z}y_1, J)$.

For sake of completeness we also give a description of $\mathfrak{g}$ as a matrix Lie algebra. Setting

$$z_k := \frac{1}{2}(x_k - iy_k), w_k := \frac{1}{2}(e_k - if_k) \text{ and } \epsilon := (-1)^{n+1}$$

we can describe $\mathfrak{g}$ as the Lie algebra of complex $(n+3)\times(n+3)$ matrices of the form

$$\begin{pmatrix} 0 & z_1 & z_1 & w_1 & w_2 & w_3 & \dots & w_n \\ & 0 & 0 & \frac{1}{2}\bar{z}_1 & 2\epsilon z_2 & 2\epsilon z_2 & \dots & 2\epsilon z_n \\ & & 0 & 0 & \frac{1}{2}\bar{z}_1 & -2\bar{z}_2 & \dots & -2\bar{z}_n \\ \vdots & & & 0 & & \dots & & 0 \\ & & & \vdots & & & & \vdots \\ 0 & & \dots & 0 & & \dots & & 0 \end{pmatrix}$$

The corresponding simply-connected Lie group can be obtained by exponentiation.

We will now study left-invariant differential forms on X_n. The equality

$$d\alpha(a,b) = a(\alpha(b)) - b(\alpha(a)) - \alpha([a,b]) = -\alpha([a,b]) \tag{3}$$

for left-invariant forms and vectorfields enables us to write down the differential of the left-invariant differential forms explicitly.

Let $\{x^k, y^k, e^k, f^k\}$ be the dual basis for $\mathfrak{g}^*$ and

$$\lambda_k := x^k + iy^k, \qquad \omega_k := e^k + if^k$$

in $\mathfrak{g}^*_{\mathbb{C}}$. Then $J\lambda_k = i\lambda_k$, $J\omega_k = i\omega_k$ and hence the space of left-invariant differential forms of type $(1,0)$ is

$$\mathfrak{g}^{*1,0} = span_{\mathbb{C}}\{\lambda_1, \dots, \lambda_n, \omega_1, \dots, \omega_n\} \tag{4}$$

and we have

$$\mathfrak{g}^{*0,1} = span_{\mathbb{C}}\{\bar{\lambda}_1, \dots, \bar{\lambda}_n, \bar{\omega}_1, \dots, \bar{\omega}_n\} = \overline{\mathfrak{g}^{*1,0}},$$
$$\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^{*1,0} \oplus \mathfrak{g}^{*0,1}.$$

It is straightforward to check using (1) and (3) that the differential $d : \mathfrak{g}^*_{\mathbb{C}} \to \Lambda^2\mathfrak{g}^*_{\mathbb{C}}$ is now given by

$$\begin{aligned} d\lambda_k &= 0 \qquad k = 1, \dots, n, \\ d\omega_1 &=, \\ d\omega_k &= \lambda_1 \wedge \bar{\lambda}_{k-1} + (-1)^n \lambda_1 \wedge \lambda_k \qquad k = 2, \dots, n. \end{aligned} \tag{5}$$

and the complex conjugate equations.

We see that the exterior differential decomposes $d = \partial + \bar{\partial}$ where

$$\partial : \mathfrak{g}^{*1,0} \to \Lambda^2\mathfrak{g}^{*1,0} \text{ and } \bar{\partial} : \mathfrak{g}^{*1,0} \to \mathfrak{g}^{*1,0} \otimes \mathfrak{g}^{*0,1}.$$

By left-invariance of J this holds at every point of X_m and it is well known that this is equivalent to the vanishing of the Nijenhuis tensor and hence to the integrability of the almost complex structure J (see e.g. [Huy05], p. 109).

Remark 2 — Note that the datum of (4) and (5) determines uniquely both the complex structure and the Lie algebra. In the construction of examples it is often easier to give $\mathfrak{g}^{*1,0}_{\mathbb{C}}$ and the differential $d : \mathfrak{g}^*_{\mathbb{C}} \to \Lambda^2\mathfrak{g}^*_{\mathbb{C}}$ since we can read off the integrability of the complex structure immediately.

If we extend the differential d to the exterior algebra $\Lambda^*\mathfrak{g}^*$ in the usual way then the Jacobi identity holds if and only if $d^2 = 0$ (see e.g. [Wei94], Chapter 7).

The claim of Theorem 1 follows now directly from

Lemma 3 — *The left-invariant differential form $\beta_0 := \bar{\omega}_1 \wedge \bar{\lambda}_2 \wedge \dots \wedge \bar{\lambda}_{n-1}$ defines a class in $E_n^{0,n-1}$ and*

$$d_n(\beta_0) = [\lambda_1 \wedge \dots \wedge \lambda_n] \neq 0,$$

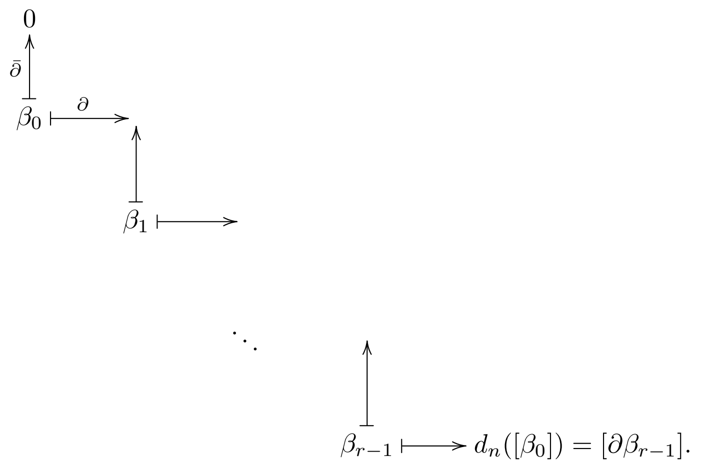
i.e., the map $d_n : E_n^{0,n-1} \to E_n^{n,0}$ is not trivial.

Proof. Since left-invariant differential forms are uniquely determined by their value in one point it suffices to carry out the calculation in $\Lambda^*\mathfrak{g}_{\mathbb{C}}^*$.

Following the exposition in [BT82] (§14, p.161ff) we say that an element $\beta_0 \in E_0^{p,q}$ lives to E_r if it represents a cohomology class in E_r or equivalently if it is a cocycle in $E_0, E_1, \ldots, E_{r-1}$. This is shown to be equivalent to the existence of a *zig-zag* of length r, that is, a collection of elements $\beta_1, \ldots, \beta_{r-1}$ such that

$$\beta_i \in E_0^{p+i,q-i} \qquad \bar\partial\beta_0 = 0 \qquad \partial\beta_{i-1} = \bar\partial\beta_i (i = 1, \ldots r-1).$$

These can be represented as



In this picture we have the first quadrant double complex given by $(E_0^{p,q}, \partial, \bar\partial)$ in mind in which this zig-zag lives.

Furthermore $d_n([\beta_0]) = [\partial\beta_{r-1}]$ is zero in $E_r^{p+n,q-n+1}$ if and only if there exists an element $\beta_r \in E_0^{p+n,q-n}$ such that $\bar\partial\beta_r = \partial\beta_{r-1}$, i.e., we can extend the zigzag by one element.

We will now show that the left-invariant differential form

$$\beta_0 = \bar\omega_1 \wedge \bar\lambda_2 \wedge \cdots \wedge \bar\lambda_{n-1} \in E_0^{0,n-1}$$

admits a zig-zag of length n which cannot be extended.

Since $\bar\partial\bar\omega_1 = 0$ and $d\bar\lambda_k = 0$ we have $\bar\partial\beta_0 = 0$ and calculate

$$\partial\bar\omega_1 = \overline{\bar\partial\omega_1} = \overline{\bar\lambda_1 \wedge \lambda_1} = \lambda_1 \wedge \bar\lambda_1 \quad \Rightarrow \quad \partial\beta_0 = \lambda_1 \wedge \bar\lambda_1 \wedge \cdots \wedge \bar\lambda_{n-1}.$$

Let us define:

$$\begin{aligned}
\beta_1 &:= \omega_2 \wedge \bar\lambda_2 \wedge \cdots \wedge \bar\lambda_{n-1} \\
\beta_i &:= \omega_{i+1} \wedge \bar\lambda_{i+1} \wedge \cdots \wedge \bar\lambda_{n-1} \wedge \lambda_2 \wedge \cdots \wedge \lambda_i \qquad (i = 2, \ldots, n-2) \\
\beta_{n-1} &:= \omega_n \wedge \lambda_2 \wedge \cdots \wedge \lambda_{n-1}
\end{aligned}$$

Then

$$\partial\beta_{i-1} = \bar\partial\beta_i \quad (i = 1, \ldots n-1)$$

and we have the desired zig-zag. We conclude by remarking that

$$d_n([\beta_0]) = [\partial\beta_{n-1}] = [\lambda_1 \wedge \ldots \lambda_n] \neq 0$$

because $E_0^{n,-1} = 0$ and the zig-zag cannot be extended. □

Note that we carried out our calculation inside the subcomplex of left-invariant differential forms

$$(\Lambda^p \mathfrak{g}_n^{*\,1,0} \otimes \Lambda^q \mathfrak{g}_n^{*\,0,1}, \partial, \bar\partial) \hookrightarrow (\mathcal{A}^{p,q}(X), \partial, \bar\partial).$$

This is no coincidence since Cordero, Fernández, Gray and Ugarte [CFGU00] proved that this inclusion induces in fact an isomorphism of spectral sequences after the E_1 term if X is a principal holomorphic torus bundle. This result, namely that the Dolbeault cohomology of a nilmanifold with left-invariant complex structure can be calculated via left-invariant differential forms, was generalised by Console and Fino [CF01] to the cases where the complex structure is rational or generic.

It should not be too difficult to write down similar examples which realise any desired property of the Frölicher spectral sequence using the method described in Remark 2.

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References

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[CF01] S. Console and A. Fino. Dolbeault cohomology of compact nilmanifolds. *Transform. Groups*, 6(2):111–124, 2001.

[CFG91] Luis A. Cordero, Marisa Fernández, and Alfred Gray. The Frölicher spectral sequence for compact nilmanifolds. *Illinois J. Math.*, 35(1):56–67, 1991.

[CFGU00] Luis A. Cordero, Marisa Fernández, Alfred Gray, and Luis Ugarte. Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology. *Trans. Amer. Math. Soc.*, 352(12):5405–5433, 2000.

[CFUG99] L. A. Cordero, M. Fernandez, L. Ugarte, and A. Gray. Frölicher spectral sequence of compact nilmanifolds with nilpotent complex structure. In *New developments in differential geometry, Budapest 1996*, pages 77–102. Kluwer Acad. Publ., Dordrecht, 1999.

[CG90] Lawrence J. Corwin and Frederick P. Greenleaf. *Representations of nilpotent Lie groups and their applications. Part I*, volume 18 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. Basic theory and examples.

[Frö55] Alfred Frölicher. Relations between the cohomology groups of Dolbeault and topological invariants. *Proc. Nat. Acad. Sci. U.S.A.*, 41:641–644, 1955.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[Huy05] Daniel Huybrechts. *Complex geometry*. Universitext. Springer-Verlag, Berlin, 2005.

[Pit89] Harsh V. Pittie. The nondegeneration of the Hodge-de Rham spectral sequence. *Bull. Amer. Math. Soc. (N.S.)*, 20(1):19–22, 1989.

[Rol07] Sönke Rollenske. *Nilmanifolds: Complex structures, geometry and deformations*. PhD thesis, Universität Bayreuth, 2007.

[Sal01] S. M. Salamon. Complex structures on nilpotent Lie algebras. *J. Pure Appl. Algebra*, 157(2-3):311–333, 2001.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

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