

POSITIVITY OF RELATIVE CANONICAL BUNDLES AND APPLICATIONS

GEORG SCHUMACHER

Dedicated to the memory of Eckart Viehweg

ABSTRACT. Given a family $f : \mathcal{X} \rightarrow S$ of canonically polarized manifolds, the unique Kähler-Einstein metrics on the fibers induce a hermitian metric on the relative canonical bundle $\mathcal{K}_{\mathcal{X}/S}$. We use a global elliptic equation to show that this metric is strictly positive on \mathcal{X} , unless the family is infinitesimally trivial.

For degenerating families we show that the curvature form on the total space can be extended as a (semi-)positive closed current. By fiber integration it follows that the generalized Weil-Petersson form on the base possesses an extension as a positive current. We prove an extension theorem for hermitian line bundles, whose curvature forms have this property. This theorem can be applied to a determinant line bundle associated to the relative canonical bundle on the total space. As an application the quasi-projectivity of the moduli space \mathcal{M}_{can} of canonically polarized varieties follows.

The direct images $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$, $m > 0$, carry natural hermitian metrics. We prove an explicit formula for the curvature tensor of these direct images. We apply it to the morphisms $S^p\mathcal{T}_S \rightarrow R^pf_*\Lambda^p\mathcal{T}_{\mathcal{X}/S}$ that are induced by the Kodaira-Spencer map and obtain a differential geometric proof for hyperbolicity properties of \mathcal{M}_{can} .

CONTENTS

1. Introduction	2
2. Fiber integration and Lie derivatives	7
2.1. Definition of fiber integrals and basic properties	7
2.2. Direct images and differential forms	10
3. Estimates for resolvent and heat kernel	11
4. Positivity of $K_{\mathcal{X}/S}$	13
5. Fiber integrals and Quillen metrics	17

2010 *Mathematics Subject Classification.* 32L10, 14D20, 32Q20.

Key words and phrases. Relative canonical bundle; Families of Kähler-Einstein manifolds; Curvature of direct image sheaves; Moduli spaces.

6. An extension theorem for hermitian line bundles and positive currents	19
7. Degenerating families of canonically polarized varieties	27
8. Extension of the Weil-Petersson form for canonically polarized varieties to the compactified Hilbert scheme	30
9. Moduli of canonically polarized manifolds	31
10. Curvature of $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ – Statement of the theorem and Applications	33
10.1. Statement of the theorem	33
10.2. Hyperbolicity conjecture of Shafarevich	38
10.3. Finsler metric on the moduli stack	40
11. Computation of the curvature	41
11.1. Setup	42
11.2. Cup Product	42
11.3. Lie derivatives	43
References	52

1. INTRODUCTION

For any holomorphic family $f : \mathcal{X} \rightarrow S$ of canonically polarized, complex manifolds, the unique Kähler-Einstein metrics on the fibers define an intrinsic metric on the relative canonical bundle $\mathcal{K}_{\mathcal{X}/S}$, whose curvature form has at least as many positive eigenvalues as the dimension of the fibers indicates. The construction is functorial in the sense of compatibility with base changes.

Main Theorem. *Let $\mathcal{X} \rightarrow S$ be a holomorphic family of canonically polarized, compact, complex manifolds, which is nowhere infinitesimally trivial. Then the curvature of the hermitian metric on $\mathcal{K}_{\mathcal{X}/S}$ that is induced by the Kähler-Einstein metrics on the fibers is strictly positive.*

Actually the first variation of the metric tensor in a family of compact Kähler-Einstein manifolds contains the information about the corresponding deformation, more precisely, it contains the harmonic representatives of the Kodaira-Spencer classes. The positivity of the hermitian metric will be measured in terms of a certain global function. Essential is an elliptic equation on the fibers, which relates this function to the pointwise norm of the harmonic Kodaira-Spencer forms. The strict positivity of the corresponding (fiberwise) operator $(\square + id)^{-1}$, where \square is the complex Laplacian, can be seen in a direct way (cf. [SCH4]). For families of compact Riemann surfaces this operator had been considered in the context of automorphic forms by Wolpert [WO].

Later in higher dimensions a similar equation arose in the work of Siu [SIU2] for families of canonical polarized manifolds.

Here we reduce estimates for the positivity of the curvature of $\mathcal{K}_{\mathcal{X}/S}$ on \mathcal{X} to estimates of the resolvent kernel of the above integral operator, whose positivity was already shown by Yosida in [YO]. Finally estimates for the resolvent kernel follow from the estimates for the heat kernel, which were achieved by Cheeger and Yau in [C-Y].

The positivity of the relative canonical bundle and the methods involved are closely related to different questions. We will treat the construction of a positive line bundle on the moduli space of canonically polarized manifolds proving its quasi-projectivity, and the question about its hyperbolicity.

Theorem I. *Let $(\mathcal{K}_{\mathcal{X}/\mathcal{H}}, h)$ be the relative canonical bundle on the total space over the Hilbert scheme, equipped with the hermitian metric that is induced by the Kähler-Einstein metrics on the fibers. Then the curvature form $\omega_{\mathcal{X}}$ extends to the total space $\overline{\mathcal{X}}$ over the compact Hilbert scheme $\overline{\mathcal{H}}$ as a positive, closed current $\omega_{\overline{\mathcal{X}}}$ with at most analytic singularities.*

The proof depends on Yau's C^0 -estimates [Y] that he used for the construction of Kähler-Einstein metrics.

We consider a certain determinant line bundle of the relative canonical bundle, which is defined in a functorial way on the base of any family of canonically polarized varieties. In particular, it exists on the open part of the Hilbert scheme. It carries a Quillen metric, and its curvature form is the generalized Weil-Petersson form. The latter is equal to the fiber integral $\int_{\mathcal{X}/\mathcal{H}} c_1(\mathcal{K}_{\mathcal{X}/\mathcal{H}}, h)^{n+1}$, where n is the fiber dimension. These quantities descend to the moduli space of canonically polarized manifolds. Again we see that the curvature form extends as a closed, (semi-)positive current to a compactification of the moduli space, which is known to exist as a Moishezon space according to Artin's theorem.

Based upon Siu's decomposition theorem for positive closed currents we prove the following theorem.

Theorem II. *Let Y be a normal space and $Y' \subset Y$ the complement of a closed analytic, nowhere dense subset. Let L' be a holomorphic line bundle on Y' together with a hermitian metric h' of semi-positive curvature, which may be singular. Assume that the curvature current can be extended to Y as a positive, closed current ω . Then there exists a holomorphic line bundle (L, h) with a singular hermitian metric of semi-positive curvature, whose restriction to Y' is isomorphic to*

(L', h') . The metric h can be chosen with at most analytic singularities, if ω has this property.

This proves the extension of the determinant line bundle and the Quillen metric to a compactification of the normalization. Finally the complex structure at points of the compactifying divisor has to be changed.

Theorem III. *The generalized Weil-Petersson form on the moduli stack of canonically polarized varieties is strictly positive. A multiple is the Chern form of a determinant line bundle, equipped with a Quillen metric. This line bundle extends to a compactification of the moduli space, and the Quillen metric extends as a (semi-)positive singular hermitian metric with at most analytic singularities (in the orbifold sense).*

These facts imply a short proof for the quasi-projectivity of the moduli space of canonically polarized manifolds (cf. Section 9). It was pointed out by Eckart Viehweg in [V2] that the extension of the Weil-Petersson current is not automatic. This issue is closely related to the extendability of the determinant line bundle, a question emphasized by Kollar [KO2] – these problems are addressed in our manuscript.

We note that our approach uses the functoriality of the construction of a determinant line bundle, equipped with a Quillen metric, which was generalized in [F-S] to smooth families over singular base spaces. Once this generalization is established, when dealing with singular fibers, we can use desingularizations of the base spaces, allowing for non-reduced fibers, and prove extension theorems for line bundles and curvature currents.

Responding to a remark by Robert Berman we mention that the theory of non-pluripolar products of globally defined currents of Boucksom, Eyssidieux, Guedj, and Zeriahi from [BEGZ] applies to the Weil-Petersson volume of a component \mathcal{M} of the moduli space of canonically polarized manifolds:

Corollary 1. *Let ω^{WP} be the Weil-Petersson form on the (open) moduli space \mathcal{M} , which is of class C^∞ in the orbifold sense. Then*

$$\int_{\mathcal{M}} (\omega^{WP})^{\dim \mathcal{M}} < \infty.$$

The *second application* concerns the direct image sheaves

$$R^{n-p} f_* \Omega_{X/S}^p (\mathcal{K}_{X/S}^{\otimes m}).$$

These are equipped with a natural hermitian metric that is induced by the L^2 -inner product of harmonic tensors on the fibers of f . In the second part of this article we will give an explicit formula for the curvature tensor. Estimates will be related to the above discussion of the resolvent kernel.

A main motivation of our approach is the study of the curvature of the classical Weil-Petersson metric, in particular the computation of the curvature tensor by Wolpert [WO] and Tromba [TR]. It immediately implies the hyperbolicity of the classical Teichmüller space. For families of higher dimensional manifolds with metrics of constant Ricci curvature the generalized Weil-Petersson metric was explicitly computed by Siu in [SIU2]. In [SCH2] a formula in terms of elliptic operators and harmonic Kodaira-Spencer tensors was derived. However, the curvature of the generalized Weil-Petersson metric seems not to satisfy any negativity condition. This difficulty was overcome in the work of Viehweg and Zuo in [V-Z1]. Their approach to hyperbolicity makes use of a period map.

On the other hand our results are motivated by Berndtsson's result on the Nakano-positivity for certain direct images. In [SCH4] we showed that the positivity of $f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes 2}$ together with the curvature of the generalized Weil-Petersson metric is sufficient to imply a hyperbolicity result for moduli of canonically polarized complex manifolds in dimension two.

For ample K_X , and a fixed number $m \geq 1$, e.g. $m = 1$, the cohomology groups $H^{n-p}(X, \Omega_X^p(K_X^{\otimes m}))$ are critical with respect to the Kodaira-Nakano vanishing theorem. The understanding of this situation is the other main motivation. We will consider the relative case. Let $A_{i\bar{j}}^\alpha(z, s)\partial_\alpha dz^{\bar{\beta}}$ be a harmonic Kodaira-Spencer form. Then for $s \in S$ the cup product together with the contraction defines a mapping

$$\begin{aligned} A_{i\bar{j}}^\alpha \partial_\alpha dz^{\bar{\beta}} \cup \lrcorner : \mathcal{A}^{0,n-p}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^p(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})) &\rightarrow \mathcal{A}^{0,n-p+1}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^{p-1}(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})) \\ A_{j\bar{\alpha}}^{\bar{\beta}} \partial_{\bar{\beta}} dz^\alpha \cup \lrcorner : \mathcal{A}^{0,n-p}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^p(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})) &\rightarrow \mathcal{A}^{0,n-p-1}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^{p+1}(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})). \end{aligned}$$

We will apply the above products to harmonic $(0, n-p)$ -forms. In general the results are not harmonic. We denote the pointwise L^2 inner product by a dot.

The computation of the curvature tensor yields the following result. For necessary assumptions cf. Section 10.

Theorem IV. *The curvature tensor for $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ is given by*

$$\begin{aligned}
 R_{i\bar{j}}^{\bar{\ell}k}(s) &= m \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) \cdot (\psi^k \cdot \bar{\psi}^{\bar{\ell}}) g dV \\
 (1) \quad &+ m \int_{\mathcal{X}_s} (\square + m)^{-1} (A_i \cup \psi^k) \cdot (A_{\bar{j}} \cup \bar{\psi}^{\bar{\ell}}) g dV \\
 &+ m \int_{\mathcal{X}_s} (\square - m)^{-1} (A_i \cup \bar{\psi}^{\bar{\ell}}) \cdot (A_{\bar{j}} \cup \psi^k) g dV.
 \end{aligned}$$

The potentially negative third term is identically zero for $p = n$, i.e. for $f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes(m+1)}$. From our Main Theorem, with Theorem IV we immediately get a fact which also follows from the Main Theorem with [B, Theorem 1.2]¹:

Corollary 2. *The locally free sheaf $f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes(m+1)}$ is Nakano-positive.*

An estimate is given in Corollary 6. It contains the following inequality.

$$R(A, \bar{A}, \psi, \bar{\psi}) \geq P_n(d(\mathcal{X}_s)) \cdot \|A\|^2 \cdot \|\psi\|^2,$$

where the coefficient $P_n(d(\mathcal{X}_s)) > 0$ is an explicit function depending on the dimension and the diameter $d(\mathcal{X}_s)$ of the fibers.

For one dimensional fibers and $m = 1$, we are considering the L^2 -inner product on the sheaf $f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes 2}$ of quadratic holomorphic differentials for the Teichmüller space of Riemann surfaces of genus larger than one, which is dual to the classical Weil-Petersson metric: According to Wolpert [WO] the sectional curvature is negative, and the holomorphic sectional curvature is bounded from above by a negative constant. A stronger curvature property, which is related to strong rigidity, was shown in [SCH1].

The strongest result on curvature by Liu, Sun, and Yau [L-S-Y] now follows from Corollary 5 (cf. also Section 10).

Corollary 3 ([L-S-Y]). *The Weil-Petersson metric on the Teichmüller space of Riemann surfaces of genus $p > 1$ is dual Nakano negative.*

By Serre duality Theorem IV yields the following version, which contains the curvature formula for the generalized Weil-Petersson metric for $p = 1$. Again a tangent vector of the base is identified with a harmonic Kodaira-Spencer form A_i , and ν_k stands for a section of the relevant sheaf:

¹Very recently Berndtsson considered the curvature of $f_*(\mathcal{K}_{\mathcal{X}/S} \otimes \mathcal{L})$ in [B1].

Theorem V. *The curvature for $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}$ equals*

$$\begin{aligned}
 R_{i\bar{j}k\bar{\ell}}(s) = & - \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) \cdot (\nu_k \cdot \nu_{\bar{\ell}}) g dV \\
 (2) \quad & - \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \wedge \nu_{\bar{\ell}}) \cdot (A_{\bar{j}} \wedge \nu_k) g dV \\
 & - \int_{\mathcal{X}_s} (\square - 1)^{-1} (A_i \wedge \nu_k) \cdot (A_{\bar{j}} \wedge \nu_{\bar{\ell}}) g dV.
 \end{aligned}$$

Now an upper semi-continuous Finsler metric of negative holomorphic curvature on any relatively compact subspace of the moduli stack of canonically polarized varieties can be constructed so that any such space is hyperbolic with respect to the orbifold structure.

We get immediately the following fact related to Shafarevich's hyperbolicity conjecture for higher dimensions, which was solved by Migliorini [M], Kovács [KV1, KV2, KV3], Bedulev-Viehweg [B-V], and Viehweg-Zuo [V-Z1, V-Z2].

Application. *Let $\mathcal{X} \rightarrow C$ be a non-isotrivial holomorphic family of canonically polarized manifolds over a compact curve. Then $g(C) > 1$.*

Similar results hold for families of polarized Ricci-flat manifolds. These will appear elsewhere.²

2. FIBER INTEGRATION AND LIE DERIVATIVES

2.1. Definition of fiber integrals and basic properties. We denote by $\{\mathcal{X}_s\}_{s \in S}$ a holomorphic family of compact complex manifolds \mathcal{X}_s of dimension $n > 0$ parameterized by a reduced complex space S . By definition, it is given by a proper holomorphic submersion $f : \mathcal{X} \rightarrow S$, such that the \mathcal{X}_s are the fibers $f^{-1}(s)$ for $s \in S$. In case of a smooth space S , if η is a differential form of class C^∞ of degree $2n + r$ the fiber integral

$$\int_{\mathcal{X}/S} \eta$$

is a differential form of degree r on S . It can be defined as follows: Fix a point $s_0 \in S$ and denote by $X = \mathcal{X}_{s_0}$ the fiber. Let $U \subset S$ be an open neighborhood of s_0 such that there exists a C^∞ trivialization of

²Dissertation in progress.

the family:

$$\begin{array}{ccc} f^{-1}U & \xleftarrow[\sim]{\Phi} & X \times U \\ f|f^{-1}U \downarrow & \swarrow pr & \\ U & & \end{array}$$

Let $z = (z^1, \dots, z^n)$ and $s = (s^1, \dots, s^k)$ be local (holomorphic) coordinates on X and S resp.

The pull-back $\Phi^*\eta$ possesses a summand η' , which is of the form $\sum \eta_k(z, s) dV_z \wedge d\sigma^{k_1} \wedge \dots \wedge d\sigma^{k_r}$, where the σ^k run through the real and complex parts of s^j , and where dV_z denotes the relative Euclidean volume element. Now

$$\int_{\mathcal{X}/S} \eta := \int_{X \times S/S} \Phi^*\eta := \sum_{k=(k_1, \dots, k_r)} \left(\int_{\mathcal{X}_s} \eta_k(z, s) dV_z \right) d\sigma^{k_1} \wedge \dots \wedge d\sigma^{k_r}.$$

The definition is independent of the choice of coordinates and differentiable trivializations. The fiber integral coincides with the push-forward of the corresponding current. Hence, if η is a differentiable form of type $(n+r, n+s)$, then the fiber integral is of type (r, s) .

Singular base spaces are treated as follows: Using deformation theory, we can assume that $S \subset W$ is a closed subspace of some open set $W \subset \mathbb{C}^N$, and that an almost complex structure is defined on $X \times S$ so that \mathcal{X} is contained in the integrable locus. Then by definition, a differential form of class C^∞ on \mathcal{X} will be given on the whole ambient space $X \times W$ (with a type decomposition defined on \mathcal{X}).

Fiber integration commutes with taking exterior derivatives:

$$(3) \quad d \int_{\mathcal{X}/S} \eta = \int_{\mathcal{X}/S} d\eta,$$

and since it preserves the type (or to be seen explicitly in local holomorphic coordinates), the same equation holds true for ∂ and $\bar{\partial}$ instead of d .

A Kähler form $\omega_{\mathcal{X}}$ on a singular space, by definition is a form that possesses locally a $\partial\bar{\partial}$ -potential, which is the restriction of a C^∞ function on a smooth ambient space. It follows from the above facts that given a Kähler form $\omega_{\mathcal{X}}$ on the total space, the fiber integral

$$(4) \quad \int_{\mathcal{X}/S} \omega_{\mathcal{X}}^{n+1}$$

is a Kähler form on the base space S , which possesses locally a smooth $\partial\bar{\partial}$ -potential, even if the base space of the smooth family is singular.

For the actual computation of exterior derivatives of fiber integrals (3), in particular of functions, given by integrals of (n, n) -forms on

the fibers, the above definition seems to be less suitable. Instead the problem is reduced to derivatives of the form

$$(5) \quad \frac{\partial}{\partial s^i} \int_{\mathcal{X}_s} \eta,$$

where only the vertical components of η contribute to the integral. Here and later we will always use the summation convention.

Lemma 1. *Let*

$$w_i = \left(\frac{\partial}{\partial s^i} + b_i^\alpha(z, s) \frac{\partial}{\partial z^\alpha} + c_i^{\bar{\beta}}(z, s) \frac{\partial}{\partial z^{\bar{\beta}}} \right) \Big|_s$$

be differentiable vector fields, whose projections to S equal $\frac{\partial}{\partial s^i}$. Then

$$\frac{\partial}{\partial s^i} \int_{\mathcal{X}_s} \eta = \int_{\mathcal{X}_s} L_{w_i}(\eta),$$

where L_{w_i} denotes the Lie derivative.

Concerning singular base spaces, it is obviously sufficient that the above equation is given on the first infinitesimal neighborhood of s in S .

Proof of Lemma 1. Because of linearity, one may consider the real and imaginary parts of $\partial/\partial s^i$ and w_i resp. separately.

Let $\partial/\partial t$ stand for $Re(\partial/\partial s^i)$, and let $\Phi_t : X \rightarrow \mathcal{X}_t$ be the one parameter family of diffeomorphisms generated by $Re(w_i)$. Then

$$\frac{d}{dt} \int_{\mathcal{X}_s} \eta = \int_X \frac{d}{dt} \Phi_t^* \eta = \int_X L_{Re(w_i)}(\eta).$$

In general the vector fields $Re(w_i)$ and $Im(w_i)$ need not commute. \square

Remark 1. Taking Lie derivatives of differential forms or tensors is in general not type-preserving.

In our applications, the form η will typically consist of inner products of differential forms with values in hermitian vector bundles, whose factors need to be treated separately. This will be achieved by Lie derivatives. In this context, we will have to use covariant derivatives with respect to the given hermitian vector bundle on the total space and to the Kähler metrics on the fibers. k

2.2. Direct images and differential forms. Let (\mathcal{E}, h) be a hermitian, holomorphic vector bundle on \mathcal{X} , whose direct image $R^q f_* \mathcal{E}$ is *locally free*. Furthermore we assume that for all $s \in S$ the cohomology $H^{q+1}(\mathcal{X}_s, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_s})$ *vanishes*. Then the statement of the Grothendieck-Grauert comparison theorem holds for $R^q f_* \mathcal{E}$, in particular $R^q f_* \mathcal{E} \otimes_{\mathcal{O}_S} \mathbb{C}(s)$ can be identified with $H^q(\mathcal{X}_s, \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}_s})$. Since \mathcal{E} is S -flat, by Grauert's theorem, for $R^q f_* \mathcal{E}$ to be locally free, it is sufficient to assume that $\dim_{\mathbb{C}} H^q(\mathcal{X}_s, \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}_s})$ is constant as long as the resp. base space is reduced. Reducedness will always be assumed unless stated differently.

For simplicity, we assume at this point that the base space S is smooth. Locally, after replacing S by a neighborhood of any given point, we can represent sections of the q -th direct image sheaf in terms of Dolbeault cohomology by $\bar{\partial}$ -closed $(0, q)$ -forms with values in \mathcal{E} . On the other hand, fiberwise, we have harmonic representatives of cohomology classes with respect to the Kähler form and hermitian metric on the fibers. The following fact will be essential. Under the above assumptions we have:

Lemma 2. *Let $\tilde{\psi} \in R^q f_* \mathcal{E}(S)$ be a section. Let $\psi_s \in \mathcal{A}^{0,q}(\mathcal{X}_s, \mathcal{E}_s)$ be the harmonic representatives of the cohomology classes $\tilde{\psi}|_{\mathcal{X}_s}$.*

Then locally with respect to S there exists a $\bar{\partial}$ -closed form $\psi \in \mathcal{A}^{0,q}(\mathcal{X}, \mathcal{E})$, which represents $\tilde{\psi}$, and whose restrictions to the fibers \mathcal{X}_s equal ψ_s .

Proof. For the sake of completeness we give the simple argument. The harmonic representatives ψ_s define a relative form, which we denote by $\psi_{\mathcal{X}/S}$. Let $\Phi \in \mathcal{A}^{0,q}(\mathcal{X}, \mathcal{E})$ represent $\tilde{\psi}$. Denote by $\Phi_{\mathcal{X}/S}$ the induced relative form. Then there exists a relative $(0, q-1)$ -form $\chi_{\mathcal{X}/S}$ on \mathcal{X} , whose exterior derivative in fiber direction $\bar{\partial}_{\mathcal{X}/S}(\chi_{\mathcal{X}/S})$ satisfies

$$\psi_{\mathcal{X}/S} = \Phi_{\mathcal{X}/S} + \bar{\partial}_{\mathcal{X}/S}(\chi_{\mathcal{X}/S}).$$

Let $\{U_i\}$ be a covering of \mathcal{X} , which possesses a partition of unity $\{\rho_i\}$ such that all $\chi_{\mathcal{X}/S}|_{U_i}$ can be extended to $(0, q-1)$ -forms χ_i on U_i . Then with $\chi = \sum \rho_i \chi_i$ we set $\psi = \Phi + \bar{\partial}\chi$. \square

The relative Serre duality can be treated in terms of such differential forms. Let $\mathcal{E}^\vee = \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{O}_{\mathcal{X}})$. Then (under the above assumptions)

$$R^p f_* \mathcal{E} \otimes_{\mathcal{O}_S} R^{n-p} f_* (\mathcal{E}^\vee \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{K}_{\mathcal{X}/S}) \rightarrow \mathcal{O}_S$$

is given by the fiber integral of the wedge product of $\bar{\partial}$ -closed differential forms in the sense of Lemma 2. By (3) (for the operator $\bar{\partial}$), the result is a $\bar{\partial}$ -closed 0-form i.e. a holomorphic function.

3. ESTIMATES FOR RESOLVENT AND HEAT KERNEL

Let (X, ω_X) be a compact Kähler manifold. The Laplace operator for differentiable functions is given by $\square = \partial\bar{\partial}^* + \bar{\partial}^*\partial$, where the adjoint $\bar{\partial}^*$ is the formal adjoint operator. The Laplacian is self-adjoint with non-negative eigenvalues.

Proposition 1. *Let (X, ω_X) be a Kähler-Einstein manifold of constant Ricci curvature -1 with volume element $g dV$. Then there exists a strictly positive function $P_n(d(X))$, depending on the dimension n of X and the diameter $d(X)$ with the following property:*

If χ is a non-negative continuous function and ϕ a solution of

$$(6) \quad (1 + \square)\phi = \chi,$$

then

$$(7) \quad \phi(z) \geq P_n(d(X)) \cdot \int_X \chi g dV$$

for all $z \in X$.

The minimum principle immediately shows that $\phi \geq 0$. In [SCH4] we gave an elementary proof that $\chi \geq 0$, implies that ϕ is strictly positive everywhere or identically zero for real analytic functions χ , which was sufficient for our application. Bo Berndtsson pointed out to the author the existence of a refined minimum principle yielding the same fact without the assumption of analyticity.

The explicit estimates claimed in the above Proposition are based on the theory of resolvent kernels.

We summarize some known facts. Let $\{\psi_\nu\}$ be an orthonormal basis of the space of square integrable real valued functions consisting of eigenfunctions of the Laplacian with eigenvalues λ_ν . Then the resolvent kernel for the above equation is given by

$$P(z, w) = \sum_\nu \frac{1}{1 + \lambda_\nu} \psi_\nu(z) \psi_\nu(w),$$

and the solution ϕ of (6) equals

$$\phi(z) := (\square + id)^{-1}(\chi)(z) = \int_X P(z, w) \chi(w) g(w) dV_w.$$

So the integral kernel $P(z, w)$ must be non-negative for all z and w .

In a similar way we denote by $P(t, z, w)$ the integral kernel for the heat operator

$$\frac{d}{dt} + \square$$

so that the solution of the heat equation

$$\left(\frac{d}{dt} + \square\right)\phi = 0$$

with initial function $\chi(z)$ for $t = 0$ is given by

$$\int_X P(t, z, w) \chi(w) g(w) dV_w.$$

Here $P(t, z, w) = \sum_{\nu} e^{-t\lambda_{\nu}} \psi_{\nu}(z) \psi_{\nu}(w)$. Now, since the eigenvalues of the Laplacian are non-negative, the equation $\int_0^{\infty} e^{-t(\lambda+1)} dt = 1/(1+\lambda)$ implies the following Lemma, (cf. also [C-Y, (3.13)]).

Lemma 3. *Let $P(z, w)$ be the integral kernel of the resolvent operator and denote by $P(t, z, w)$ the heat kernel. Then*

$$(8) \quad P(z, w) = \int_0^{\infty} e^{-t} P(t, z, w) dt.$$

We now apply the lower estimates for the heat kernel by Cheeger and Yau [C-Y] to the resolvent kernel. Assuming constant negative Ricci curvature -1 , we use the estimates in the form of [ST, (4.3) Corollary].

$$(9) \quad P(t, z, w) \geq Q_n(t, r(z, w)) := \frac{1}{(2\pi t)^n} e^{-\frac{r^2(z, w)}{t}} e^{-\frac{2n-1}{4}t},$$

Where $r = r(z, w)$ denotes the geodesic distance (and $n = \dim X$).

Let

$$(10) \quad P_n(r) = \int_0^{\infty} e^{-t} Q_n(t, r) dt > 0.$$

Using Lemma 3, equation (9), and the monotonicity of P_n we get

$$(11) \quad P(z, w) \geq P_n(r(z, w)) \geq P_n(d(X)),$$

where $d(X)$ denotes the diameter of X . This shows Proposition 1. \square

We mention that $\lim_{r \rightarrow \infty} P_n(r) = 0$.

Conversely, if we require that for all functions χ the solutions ϕ of (6) satisfy an estimate of the form

$$\phi(z) \geq P \cdot \int_X \chi g dV,$$

then $P \leq \inf P(z, w)$ follows immediately.

We mention that symbolic integration of (9) with (10) yields the explicit estimate

$$P_n(r) \geq \frac{1}{(2\pi)^n} \frac{(2n+3)^{\frac{n-1}{2}}}{2^{n-2}} \frac{1}{r^{n-1}} \text{BesselK}(n-1, \sqrt{2n+3}r).$$

4. POSITIVITY OF $K_{\mathcal{X}/S}$

Let X be a canonically polarized manifold of dimension n , equipped with a Kähler-Einstein metric ω_X . In terms of local holomorphic coordinates (z^1, \dots, z^n) we write

$$\omega_X = \sqrt{-1} g_{\alpha\bar{\beta}}(z) dz^\alpha \wedge d\bar{z}^\beta$$

so that the Kähler-Einstein equation reads

$$(12) \quad \omega_X = -\text{Ric}(\omega_X), \text{ i.e. } \omega_X = \sqrt{-1} \partial \bar{\partial} \log g(z),$$

where $g := \det g_{\alpha\bar{\beta}}$. We consider g as a hermitian metric on the anti-canonical bundle K_X^{-1} .

For any holomorphic family of compact, canonically polarized manifolds $f : \mathcal{X} \rightarrow S$ of dimension n with fibers \mathcal{X}_s for $s \in S$ the Kähler-Einstein forms $\omega_{\mathcal{X}_s}$ depend differentiably on the parameter s . The resulting relative Kähler form will be denoted by

$$\omega_{\mathcal{X}/S} = \sqrt{-1} g_{\alpha\bar{\beta}}(z, s) dz^\alpha \wedge d\bar{z}^\beta.$$

The corresponding hermitian metric on the relative anti-canonical bundle is given by $g = \det g_{\alpha\bar{\beta}}(z, s)$. We consider the real $(1, 1)$ -form

$$\omega_{\mathcal{X}} = \sqrt{-1} \partial \bar{\partial} \log g(z, s)$$

on the total space \mathcal{X} . We will discuss the question, whether $\omega_{\mathcal{X}}$ is a Kähler form on the total space.

The Kähler-Einstein equation (12) implies that

$$\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s}$$

for all $s \in S$. In particular $\omega_{\mathcal{X}}$, restricted to any fiber, is positive definite. Our result is the following statement (cf. Main Theorem).

Theorem 1. *Let $\mathcal{X} \rightarrow S$ be a holomorphic family of canonically polarized, compact, complex manifolds. Then the curvature form $\omega_{\mathcal{X}}$ of the hermitian metric on $K_{\mathcal{X}/S}$ induced by the Kähler-Einstein metrics on the fibers is semi-positive and strictly positive on all fibers. It is strictly positive in horizontal directions, for which the family is not infinitesimally trivial.*

More precisely, in terms of the generalized Weil-Petersson form ω^{WP} :

$$(13) \quad \omega_{\mathcal{X}}^{n+1} \geq P_n(d(\mathcal{X}_s)) \cdot f^* \omega^{WP} \wedge \omega_{\mathcal{X}}^n,$$

where the right-hand side only depends on the relative volume form $\omega_{\mathcal{X}/S}^n$.

(The form ω^{WP} will be given in Definition 2. We denote by $\omega_{\mathcal{X}}^n$ etc. the n -fold exterior power divided by $n!$, cf. also the remark on notation in Section 11.)

Both the statement of the theorem and the methods are valid for smooth, proper families over singular (even non-reduced) complex spaces (for the necessary theory cf. [F-S, §12]).

It is sufficient to prove the theorem for base spaces of dimension one assuming $S \subset \mathbb{C}$. (In order to treat singular base spaces, the claim can be reduced to the case where the base is a double point $(0, \mathbb{C}[s]/(s^2))$. The arguments below will still be meaningful and can be applied literally.)

We denote the Kodaira-Spencer map for the family $f : \mathcal{X} \rightarrow S$ at a given point $s_0 \in S$ by

$$\rho_{s_0} : T_{s_0}S \rightarrow H^1(X, \mathcal{T}_X)$$

where $X = \mathcal{X}_{s_0}$. The family is called *effectively parameterized* at s_0 , if ρ_{s_0} is injective. The Kodaira-Spencer map is induced by the edge homomorphism for the short exact sequence

$$0 \rightarrow \mathcal{T}_{\mathcal{X}/S} \rightarrow \mathcal{T}_{\mathcal{X}} \rightarrow f^*\mathcal{T}_S \rightarrow 0.$$

If $v \in T_{s_0}S$ is a tangent vector, say $v = \frac{\partial}{\partial s}|_{s_0}$, and if $\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}$ is any lift of class C^∞ to \mathcal{X} along X , then

$$\bar{\partial} \left(\frac{\partial}{\partial s} + b^\alpha(z) \frac{\partial}{\partial z^\alpha} \right) = \frac{\partial b^\alpha(z)}{\partial z^{\bar{\beta}}} \frac{\partial}{\partial z^\alpha} dz^{\bar{\beta}}$$

is a $\bar{\partial}$ -closed form on X , which represents $\rho_{s_0}(\partial/\partial s)$.

We will use the semi-colon notation as well as raising and lowering of indices for covariant derivatives with respect to the *Kähler-Einstein metrics on the fibers*. The s -direction will be indicated by the index s . In this sense the coefficients of $\omega_{\mathcal{X}}$ will be denoted by $g_{s\bar{s}}$, $g_{\alpha\bar{s}}$, $g_{\alpha\bar{\beta}}$ etc.

Next, we define *canonical lifts* of tangent vectors of S as differentiable vector fields on \mathcal{X} along the fibers of f in the sense of Siu [SIU2]. By definition these satisfy the property that the induced representatives of the Kodaira-Spencer class are *harmonic*.

Since the form $\omega_{\mathcal{X}}$ is positive, when restricted to fibers, *horizontal lifts* of tangent vectors with respect to the pointwise sesquilinear form $\langle -, - \rangle_{\omega_{\mathcal{X}}}$ are well-defined (cf. also [SCH2]).

Lemma 4. *The horizontal lift of $\partial/\partial s$ equals*

$$v = \partial_s + a_s^\alpha \partial_\alpha,$$

where

$$a_s^\alpha = -g^{\bar{\beta}\alpha} g_{s\bar{\beta}}.$$

Proposition 2. *The horizontal lift of a tangent vector of the base induces the harmonic representative of its Kodaira-Spencer class.*

Proof. The Kodaira-Spencer form of the tangent vector ∂/∂_s is $\bar{\partial}v|_{\mathcal{X}_s} = a_{s;\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}$. We consider the tensor

$$A_{s\bar{\beta}}^\alpha := a_{s;\bar{\beta}}^\alpha|_{\mathcal{X}_{s_0}}$$

on X . Then $\bar{\partial}^*(A_{s\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}})$ is given by

$$\begin{aligned} g^{\bar{\beta}\gamma} A_{s\bar{\beta};\gamma}^\alpha &= -g^{\bar{\beta}\gamma} g^{\bar{\delta}\alpha} g_{s\bar{\delta};\bar{\beta}\gamma} = -g^{\bar{\beta}\gamma} g^{\bar{\delta}\alpha} g_{s\bar{\beta};\bar{\delta}\gamma} = -g^{\bar{\beta}\gamma} g^{\bar{\delta}\alpha} \left(g_{s\bar{\beta};\gamma\bar{\delta}} - g_{s\bar{\tau}} R_{\bar{\beta}\bar{\delta}\gamma}^{\bar{\tau}} \right) \\ &= -g^{\bar{\delta}\alpha} \left((\partial \log g / \partial s)_{;\bar{\delta}} + g_{s\bar{\tau}} R_{\bar{\delta}}^{\bar{\tau}} \right) = 0 \end{aligned}$$

because of the Kähler-Einstein property. \square

It follows immediately from the proposition that the harmonic Kodaira-Spencer forms induce symmetric tensors. This fact reflects the close relationship between the Kodaira-Spencer tensors and the Kähler-Einstein metrics:

Corollary 4. *Let $A_{s\bar{\beta}\bar{\delta}} = g_{\alpha\bar{\beta}} A_{s\bar{\delta}}^\alpha$. Then*

$$(14) \quad A_{s\bar{\beta}\bar{\delta}} = A_{s\bar{\delta}\bar{\beta}}.$$

Next, we introduce a *global* function $\varphi(z, s)$, which is by definition the *pointwise inner product of the canonical lift v of $\partial/\partial s$ at $s \in S$ with itself taken with respect to $\omega_{\mathcal{X}}$* .

Definition 1.

$$(15) \quad \varphi(z, s) := \langle \partial_s + a_s^\alpha \partial_\alpha, \partial_s + a_s^\beta \partial_\beta \rangle_{\omega_{\mathcal{X}}}$$

Since $\omega_{\mathcal{X}}$ is not known to be positive definite in all directions, φ is not known to be non-negative at this point.

Lemma 5.

$$(16) \quad \varphi = g_{s\bar{s}} - g_{\alpha\bar{s}} g_{s\bar{\beta}} g^{\bar{\beta}\alpha}$$

Proof. The proof follows from Lemma 4 and

$$\varphi = g_{s\bar{s}} + g_{s\bar{\beta}} a_{\bar{s}}^{\bar{\beta}} + a_s^\alpha g_{\alpha\bar{s}} + a_s^\alpha a_{\bar{s}}^{\bar{\beta}} g_{\alpha\bar{\beta}}.$$

\square

Denote by $\omega_{\mathcal{X}}^{n+1}$ the $(n+1)$ -fold exterior product, divided by $(n+1)!$ and by dV the Euclidean volume element in fiber direction. Then the global real function φ satisfies the following property:

Lemma 6.

$$\omega_{\mathcal{X}}^{n+1} = \varphi \cdot g dV \sqrt{-1} ds \wedge \bar{ds}.$$

Proof. Compute the following $(n+1) \times (n+1)$ -determinant

$$\det \begin{pmatrix} g_{s\bar{s}} & g_{s\bar{\beta}} \\ g_{\alpha\bar{s}} & g_{\alpha\bar{\beta}} \end{pmatrix},$$

where $\alpha, \beta = 1, \dots, n$. □

So far we are looking at *local* computations, which essentially only involve derivatives of certain tensors. The only *global ingredient* is the fact that we are given global solutions of the Kähler-Einstein equation.

The essential quantity is the differentiable function φ on \mathcal{X} . Restricted to any fiber it associates the yet to be proven positivity of the hermitian metric on the relative canonical bundle with the canonical lift of tangent vectors, which is related to the harmonic Kodaira-Spencer forms.

We use the Laplacian operators $\square_{g,s}$ with non-negative eigenvalues on the fibers \mathcal{X}_s so that for a real valued function χ the Laplacian equals $\square_{g,s}\chi = -g^{\bar{\beta}\alpha}\chi_{;\alpha\bar{\beta}}$.

Proposition 3. *The following elliptic equation holds fiberwise:*

$$(17) \quad (\square_{g,s} + \text{id})\varphi(z, s) = \|A_s(z, s)\|^2,$$

where

$$A_s = A_{s\bar{\beta}}^{\alpha} \frac{\partial}{\partial z^{\alpha}} dz^{\bar{\beta}}.$$

Proof. In order to prove such an elliptic equation for φ on the fibers, we need to eliminate the second order derivatives with respect to the base parameter. This is achieved by the left hand side of (17). First,

$$\begin{aligned} g^{\bar{\delta}\gamma} g_{s\bar{s};\gamma\bar{\delta}} &= g^{\bar{\delta}\gamma} \partial_s \partial_{\bar{s}} g_{\gamma\bar{\delta}} \\ &= \partial_s (g^{\bar{\delta}\gamma} \partial_{\bar{s}} g_{\gamma\bar{\delta}}) - a_s^{\gamma;\bar{\delta}} \partial_{\bar{s}} g_{\gamma\bar{\delta}} \\ &= \partial_s \partial_{\bar{s}} \log g + a_s^{\gamma;\bar{\delta}} a_{\bar{s}\gamma;\bar{\delta}} \\ &= g_{s\bar{s}} + a_{s;\gamma}^{\sigma} a_{\bar{s}\sigma;\bar{\delta}} g^{\bar{\delta}\gamma}. \end{aligned}$$

Next

$$(a_s^{\sigma} a_{\bar{s}\sigma})_{;\gamma\bar{\delta}} g^{\bar{\delta}\gamma} = (a_s^{\sigma}{}_{;\gamma\bar{\delta}} a_{\bar{s}\sigma} + A_{s\bar{\delta}}^{\sigma} A_{\bar{s}\sigma\gamma} + a_{s;\gamma}^{\sigma} a_{\bar{s}\sigma;\bar{\delta}} + a_s^{\sigma} A_{\bar{s}\sigma\gamma;\bar{\delta}}) g^{\bar{\delta}\gamma}.$$

The last term vanishes because of the harmonicity of A_s , and

$$\begin{aligned} a_{s;\gamma\bar{\delta}}^{\sigma} g^{\bar{\delta}\gamma} &= A_{s\bar{\delta};\gamma}^{\sigma} g^{\bar{\delta}\gamma} + a_s^{\lambda} R_{\lambda\gamma\bar{\delta}}^{\sigma} g^{\bar{\delta}\gamma} \\ &= 0 - a_s^{\lambda} R_{\lambda}^{\sigma} \\ &= a_s^{\sigma}. \end{aligned}$$

□

Proof of Theorem 1. It is sufficient to show that the function φ from Definition 1 is strictly positive, since $\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s}$ is known to be positive definite. The positivity of φ follows from (17) in Proposition 3 together with Proposition 1. □

Definition 2. *The Weil-Petersson hermitian product on $T_s S$ is given by the L^2 -inner product of harmonic Kodaira-Spencer forms:*

$$(18) \quad \left\| \frac{\partial}{\partial s} \right\|_{WP}^2 := \int_{\mathcal{X}_s} A_{s\bar{\beta}}^\alpha A_{s\bar{\gamma}}^{\bar{\delta}} g_{\alpha\bar{\delta}} g^{\bar{\beta}\gamma} g dV = \int_{\mathcal{X}_s} A_{s\bar{\beta}}^\alpha A_{s\bar{\alpha}}^{\bar{\beta}} g dV$$

If the tangent vectors $\partial/\partial s^i \in T_s S$ are part of a basis, we denote by $G_{ij}^{WP}(s)$ the inner product $\langle \partial/\partial s^i|_s, \partial/\partial s^j|_s \rangle_{WP}$, and set

$$\omega^{WP} := \sqrt{-1} G_{ij}^{WP} ds^i \wedge ds^{\bar{j}}.$$

Observe that the generalized Weil-Petersson form is equal to a fiber integral (cf. [F-S, Theorem 7.8]). The above approach yields a simple proof of this fact, which also implies the Kähler property of ω^{WP} :

Proposition 4.

$$(19) \quad \omega^{WP} = \int_{\mathcal{X}/S} \omega_{\mathcal{X}}^{n+1}.$$

The *proof* follows from Lemma 6 and Proposition 3.

5. FIBER INTEGRALS AND QUILLEN METRICS

In this section we summarize the methods how to produce a positive line bundle on the base of a holomorphic family from [F-S, §10]. Let $f : \mathcal{X} \rightarrow S$ be a proper, smooth holomorphic map of reduced complex spaces and $\omega_{\mathcal{X}/S}$ a closed real $(1, 1)$ -form on \mathcal{X} , whose restrictions to the fibers are Kähler forms. Let (\mathcal{E}, h) be a hermitian vector bundle on \mathcal{X} . We denote the determinant line bundle of \mathcal{E} in the derived category by

$$\lambda(\mathcal{E}) = \det f_!(\mathcal{E}).$$

The main result of Bismut, Gillet and Soulé from [BGS] states the existence of a Quillen metric h^Q on the determinant line bundle such that the following equality holds for its Chern form on the base S and the component in degree two of a fiber integral:

$$(20) \quad c_1(\lambda(\mathcal{E}), h^Q) = \left[\int_{\mathcal{X}/S} td(\mathcal{X}/S, \omega_{\mathcal{X}/S}) ch(\mathcal{E}, h) \right]_2$$

Here ch and td stand for the Chern and Todd character forms.

The formula is applied to a virtual bundle of degree zero (cf. [F-S, Remark 10.1] for the notion of virtual bundles). It follows immediately from the definition that the determinant line bundle of a virtual vector bundle is well-defined (as a line bundle).

Let (\mathcal{L}, h) be a hermitian line bundle, and set $\mathcal{E} = (\mathcal{L} - \mathcal{L}^{-1})^{n+1}$. The difference is taken in the Grothendieck group, and the product is the tensor product. Since the term of degree zero in the Chern character $\text{ch}(\mathcal{L} - \mathcal{L}^{-1})$ is equal to the (virtual) rank, which is zero, and the first term is $2c_1(\mathcal{L})$, we conclude that $\text{ch}(\mathcal{E})$ is equal to

$$(21) \quad 2^{n+1}c_1(\mathcal{L})^{n+1}$$

plus higher degree terms. Furthermore, the hermitian structure on \mathcal{L} provides \mathcal{E} with a natural Chern character form.

Hence the only contribution of the Todd character form in (20) is the constant 1 resulting in the following equality

$$(22) \quad c_1(\lambda(\mathcal{E}), h^Q) = 2^{n+1} \int_{\mathcal{X}/S} c_1(\mathcal{L}, h)^{n+1}.$$

Following [F-S, Theorem 11.10] we apply this construction to families of canonically polarized varieties. Let $f : \mathcal{X} \rightarrow S$ be any (smooth) family of canonically polarized manifolds over a reduced complex space. The generalized Weil-Petersson form ω_S^{WP} on S was proven to be equal to a certain fiber integral. We will use the notion \simeq for equality up to a numerical factor.

We set $\mathcal{L} = \mathcal{K}_{\mathcal{X}/S}$ in (20). Equation (22) yields

$$(23) \quad c_1(\lambda(\mathcal{E}), h^Q) \simeq \int_{\mathcal{X}/S} \omega_{\mathcal{X}}^{n+1},$$

where $\omega_{\mathcal{X}} = c_1(\mathcal{K}_{\mathcal{X}/S}, h)$, with h induced by the Kähler-Einstein volume forms on the fibers.

On the other hand Lemma 6 together with Proposition 3 implied that the fiber integral (23) is equal to the generalized Weil-Petersson form:

$$\omega_S^{WP}(s) = \int_{Z_s} A_{i\bar{\beta}}^{\alpha} A_{\bar{j}\gamma}^{\bar{\delta}} g_{\alpha\bar{\delta}} g^{\bar{\beta}\gamma} g \, dV \sqrt{-1} ds^i \wedge ds^{\bar{j}},$$

where the forms $A_{i\bar{\beta}}^{\alpha} \partial_{\alpha} dz^{\bar{\beta}}$ are the harmonic representatives of the Kodaira-Spencer classes $\rho_s(\partial/\partial s_i|_s)$, i.e.

$$(24) \quad \omega_S^{WP} \simeq \int_{\mathcal{X}/S} c_1(\mathcal{K}_{\mathcal{X}/S}, h)^{n+1}.$$

Now

$$(25) \quad c_1(\det f_!((\mathcal{K}_{\mathcal{X}/S} - \mathcal{K}_{\mathcal{X}/S}^{-1})^{n+1}), h^Q) \simeq \omega_S^{WP}.$$

We consider the situation of Hilbert schemes of canonically polarized varieties.

After fixing the Hilbert polynomial and a multiple m of the canonical bundles in the family that yields very ampleness, we consider the universal embedded family over the Hilbert scheme

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{i} & \mathbb{P}_N \times \mathcal{H} \\ & \searrow f & \downarrow pr \\ & & \mathcal{H}. \end{array}$$

In this sense we modify the determinant line bundle and consider

$$\lambda = \det f_!((\mathcal{K}_{\mathcal{X}/\mathcal{H}}^{\otimes m} - (\mathcal{K}_{\mathcal{X}/\mathcal{H}}^{-1})^{\otimes m})^{n+1}),$$

which only yields an extra factor m^{n+1} in front of the Weil-Petersson form ω^{WP} on \mathcal{H} .

We point out, how singularities of base spaces (and smooth maps) were treated in [F-S, Theorem 10.1 and §12]: Given a (local) deformation of a canonically polarized variety, equipped with a Kähler-Einstein metric over a reduced singular base, the latter is embedded into a smooth ambient space. The deformation is computed in terms of certain elliptic operators, which are meaningful for all neighboring points. The integrability condition for the respective almost complex structures, i.e. the vanishing of the Nijenhuis torsion tensor, determines the singular base space. Now the implicit function theorem yields Kähler-Einstein metrics on neighboring fibers – the solutions also exist (without being too significant) for points of the base, where the almost complex structure is not integrable. This procedure yields a potential for the relative Kähler-Einstein forms that comes from a differentiable function on the smooth ambient space. By fiber integration the Weil-Petersson form is being computed; again, it possesses a $\partial\bar{\partial}$ -potential, which is the restriction of a C^∞ -function on the smooth ambient space.

6. AN EXTENSION THEOREM FOR HERMITIAN LINE BUNDLES AND POSITIVE CURRENTS

Let \mathcal{L} be a holomorphic line bundle on a reduced, complex space \mathcal{Z} . Then a semi-positive, singular hermitian metric h on \mathcal{L} is defined by the

³Also the element $(\mathcal{K}_{\mathcal{X}/S} - \mathcal{O}_{\mathcal{X}})^{n+1}$ of the relative Grothendieck group can be taken instead.

property that the locally defined function $-\log h$ is plurisubharmonic (and locally integrable), when pulled back to the normalization of the space. By definition, a *positive current* takes non-negative values on semi-positive differential forms.

Definition 3. *Let χ be a plurisubharmonic function on an open subset of \mathbb{C}^n . We say that χ has at most analytic singularities, if locally*

$$\chi \geq \gamma \log \sum_{\nu=1}^k |f_\nu|^2 + \text{const.}$$

holds, for holomorphic functions f_ν and some $\gamma > 0$. In this situation, we say that a (locally defined) positive, singular hermitian metric of the form

$$h = e^{-\chi}$$

has at most analytic singularities. This property will be also assigned to a locally $\partial\bar{\partial}$ -exact, positive current of the form

$$\omega = \sqrt{-1} \partial\bar{\partial}\chi.$$

For any positive closed $(1, 1)$ -current T on a complex manifold Y the Lelong number at a point x is denoted by $\nu(T, x)$, and for any $c > 0$ we have the associated sets $E_c(T) = \{x; \nu(T, x) \geq c\}$. According to [SIU3, Main Theorem] these are closed analytic sets.

We will use in an essential way Siu's decomposition formula for positive, closed currents on complex manifolds. We state it for $(1, 1)$ -currents.

Theorem ([SIU3]). *Let ω be a closed positive $(1, 1)$ -current. Then ω can be written as a series of closed positive currents*

$$(26) \quad \omega = \sum_{k=0}^{\infty} \mu_k [Z_k] + R,$$

where the $[Z_k]$ are currents of integration over irreducible analytic sets of codimension one, and R is a closed positive current with the property that $\dim E_c(R) < \dim Y - 1$ for every $c > 0$. This decomposition is locally and globally unique: the sets Z_k are precisely components of codimension one occurring in the sublevel sets $E_c(\omega)$, and $\mu_k = \min_{x \in Z_k} \nu(\omega; x)$ is the generic Lelong number of ω along Z_k .

We want to prove an extension theorem for hermitian line bundles, whose curvature forms ω extend as positive currents. The idea is to treat the non-integer part

$$\sum_{k=0}^{\infty} (\mu_k - \lfloor \mu_k \rfloor) [Z_k].$$

of the decompositions in (26).

Theorem 2. *Let Y be a normal complex space and $Y' = Y \setminus A$ the complement of a nowhere dense, closed, analytic subset. Let L' be a holomorphic line bundle together with a hermitian metric h' of semi-positive curvature, which also may be singular. Assume that the curvature current ω' of (L', h') possesses an extension ω to Y as a closed, positive current. Then there exists a holomorphic line bundle (L, h) with a singular, positive hermitian metric, whose restriction to Y' is isomorphic to (L', h') . If ω has at most analytic singularities, then h can be chosen with this property.*

We first note the following special case:

Proposition 5. *The theorem holds, if Y is a complex manifold and A is a simple normal crossings divisor.*

Proof. We will argue in an elementary way. We first assume that $A \subset Y$ is a smooth, connected hypersurface. Let $\{U_j\}$ be an open covering of Y such that the set $A \cap U_i$ consists of the zeroes of a holomorphic function z_i on U_i . Since all holomorphic line bundles on the product of a polydisk and a punctured disk are trivial, we can choose the sets $\{U_i\}$ such that the line bundle L' extends to such U_j as a holomorphic line bundle. So L' possesses nowhere vanishing sections over $U'_j = U_j \setminus A$. Hence L' is given by a cocycle $g'_{ij} \in \mathcal{O}_Y^*(U'_{ij})$, where $U'_{ij} = U_i \cap U_j$ and $U'_{ij} = U_{ij} \cap Y'$. If necessary, we will replace $\{U_i\}$ by a finer covering.

We will first show the existence of plurisubharmonic functions ψ_i on U_i and holomorphic functions φ'_i on U'_i such that

$$h'_i \cdot |\varphi'_i|^2 = e^{-\psi_i}|U'_i|$$

where the $\psi_i - \psi_j$ are pluriharmonic functions on U_{ij} :

Let

$$\omega|_{U_i} = \sqrt{-1} \partial \bar{\partial} (\psi_i^0)$$

for some plurisubharmonic functions ψ_i^0 on U_i . Now

$$\log(e^{\psi_i^0} h'_i)$$

is pluriharmonic on U'_i . For a suitable number $\beta_i \in \mathbb{R}$ and some holomorphic function f'_i on U'_i we have

$$(27) \quad \log(e^{\psi_i^0} h'_i) + \beta_i \log |z_i| = f'_i + \overline{f'_i}.$$

We write

$$\beta_i = \gamma_i + 2k_i$$

for $0 \leq \gamma_i < 2$ and some integer k_i . We set

$$(28) \quad \psi_i = \psi_i^0 + \gamma_i \log |z_i|.$$

These functions are clearly plurisubharmonic, and $\gamma_i \log |z_i|$ contributes as an analytic singularity to ψ_i^0 . Set

$$\varphi'_i = z_i^{-k_i} e^{f'_i} \in \mathcal{O}^*(U'_i).$$

We use the functions φ'_i to change the bundle coordinates of L' with respect to U'_i . In these bundle coordinates the hermitian metric h' on L' is given by

$$\tilde{h}'_i = h'_i \cdot |\varphi'_i|^{-2}$$

and the transformed transition functions are

$$\tilde{g}'_{ij} = \varphi'_i \cdot g'_{ij} \cdot (\varphi'_j)^{-1}.$$

Now

$$(29) \quad \tilde{h}'_i = |z_i|^{-\gamma_i} e^{-\psi_i^0} |U'_i = e^{-\psi_i} |U'_i,$$

and

$$(30) \quad |\tilde{g}'_{ij}|^2 = \tilde{h}'_j (\tilde{h}'_i)^{-1} = |z_j|^{\gamma_i - \gamma_j} \left| \frac{z_i}{z_j} \right|^{\gamma_i} \cdot e^{\psi_i^0 - \psi_j^0}.$$

Since the function z_i/z_j is holomorphic and nowhere vanishing on U_{ij} , and since the function $\psi_i^0 - \psi_j^0$ is pluriharmonic on U_{ij} , the function

$$\left| \frac{z_i}{z_j} \right|^{\gamma_i} \cdot e^{\psi_i^0 - \psi_j^0}$$

is of class C^∞ on U_{ij} with no zeroes. Now $-2 < \gamma_i - \gamma_j < 2$, and \tilde{g}'_{ij} is holomorphic on U'_{ij} . So we have

$$(31) \quad \gamma_i = \gamma_j =: \gamma_A$$

(whenever $A \cap U_{ij} \neq \emptyset$). Accordingly (30) reads

$$(32) \quad |\tilde{g}'_{ij}|^2 = \tilde{h}'_j (\tilde{h}'_i)^{-1} = \left| \frac{z_i}{z_j} \right|^{\gamma_i} \cdot e^{\psi_i^0 - \psi_j^0},$$

and the transition functions \tilde{g}'_{ij} can be extended holomorphically to all of U_{ij} . So a line bundle L exists.

The functions ψ_i are plurisubharmonic, and the quantity

$$\hat{\omega} = \sqrt{-1} \partial \bar{\partial} \psi_i = \omega + \pi \gamma_A [A]$$

is a well-defined positive current on Y because of (28) and (31). Its restriction to Y' equals $\omega|_{Y'}$.

We define

$$\tilde{h}_i = e^{-\psi_i} = e^{-\psi_i^0} |z_i|^{-\gamma_i}$$

on U_i . It defines a positive, singular, hermitian metric on L . This shows the theorem in the special case.

Observe that the numbers $\beta_j = \gamma_j + 2k_j$ only depend on the smooth hypersurface A . We will denote these by β_A etc.

For the general case of the proposition, we assume for simplicity first that the divisor A consists of two components A_1 and A_2 , which intersect transversally. Let $Y^j = Y \setminus A_j$ for $j = 1, 2$. We know from the first part that there exist two line bundle extensions (Y^1, h^1) and (Y^2, h^2) resp. of (L', h') to Y^1 and Y^2 resp.

Claim. The hermitian line bundles (L^j, h^j) define the extension of (L', h') as a holomorphic line bundle on Y with a singular hermitian metric with positive curvature equal to

$$\omega + \pi \gamma_{A_1} [A_1] + \pi \gamma_{A_2} [A_2],$$

where $0 \leq \gamma_{A_j} < 2$ are given by the first part of the proof.

We prove the *Claim*. Let $p \in A_1 \cap A_2$. For simplicity we may assume that $\dim Y = 2$. Let $U(p) = \Delta \times \Delta = \{(z_1, z_2)\}$ be a neighborhood of p , where $\Delta \subset \mathbb{C}$ denotes the unit disk. Assume that $A_j = V(z_j)$, $j = 1, 2$. Now the metric h^1 is defined on $\Delta^* \times \Delta$ and h^2 is defined on $\Delta \times \Delta^*$, where $\Delta^* = \Delta \setminus \{0\}$. Since any holomorphic line bundle on $\Delta^* \times \Delta$ is trivial, we may use the spaces $\Delta^* \times \Delta$ and $\Delta \times \Delta^*$ as coordinate neighborhoods for the definition of the line bundles L^1 and L^2 resp. Since holomorphic line bundles on $\Delta \times \Delta^*$ and $\Delta^* \times \Delta$ extend to $\Delta \times \Delta$, we find a nowhere vanishing function $\kappa \in \mathcal{O}_{\mathbb{C}^2}^*(\Delta^* \times \Delta^*)$ such that

$$(33) \quad h_1 = |\kappa|^2 h_2.$$

With the above methods it is easy to show that any such function satisfies

$$(34) \quad \kappa(z_1, z_2) = z_1^{m_1} z_2^{m_2} e^\chi$$

for a holomorphic function $\chi \in \mathcal{O}_{\mathbb{C}^2}(\Delta^* \times \Delta^*)$ and $m_j \in \mathbb{Z}$. (In order to see this, write $\log |\kappa|^2 = \sigma_1 \log |z_1| + \sigma_2 \log |z_2| + \phi + \bar{\phi}$, with $\phi \in \mathcal{O}_{\mathbb{C}^2}(\Delta^* \times \Delta^*)$. Then $|\kappa e^{-\phi} z_1^{-q_1} z_2^{-q_2}|^2 = |z_1|^{\tau_1} |z_2|^{\tau_2}$ with $\sigma_j = \tau_j + 2q_j$, $0 \leq \tau_j < 2$, $q_j \in \mathbb{Z}$. Now $\kappa e^{-\phi} z_1^{-q_1} z_2^{-q_2}$ possesses a holomorphic extension to $\Delta \times \Delta$, and $\tau_1 = \tau_2 = 0$.)

We use the arguments and result of the first part, and again the fact that the homotopy group $\pi_1(\Delta^n \setminus V(z_1 \cdots z_k))$ for any $k \leq n$ is abelian. We find

$$(35) \quad -\log h^1 = \psi^0 + \beta_1 \log |z_1| + \gamma_{A_2} \log |z_2| + f_1 + \bar{f}_1$$

$$(36) \quad -\log h^2 = \psi^0 + \gamma_{A_1} \log |z_1| + \beta_2 \log |z_2| + f_2 + \bar{f}_2,$$

where $\beta_j \in \mathbb{R}$, and $f_j \in \mathcal{O}_{\mathbb{C}^2}(\Delta^* \times \Delta^*)$. The numbers $0 \leq \gamma_{A_j} < 2$ are already determined. Let

$$\beta_j = \gamma_j + 2\ell_j \text{ with } 0 \leq \gamma_j < 2 \text{ and } \ell_j \in \mathbb{Z}.$$

Now (33), (34), (35), and (36) imply that

$$\gamma_j = \gamma_{A_j}, \ell_1 = -m_1, \ell_2 = m_2,$$

and

$$e^{f_2 - f_1 - \chi}$$

possesses a holomorphic extension to $\Delta \times \Delta$. Now, like in the first part, the functions $z_1^{\ell_1 - m_1}$ and $z_2^{\ell_2 + m_2}$, together with the function $e^{f_2 - f_1 - \chi}$ can be used as coordinate transformations for the line bundles L_1 and L_2 on $\Delta^* \times \Delta$ and $\Delta \times \Delta^*$ resp. This shows the claim.

The case of a general, simple normal crossings divisor follows in an analogous way. \square

The proof of Proposition 5 implies the following fact (introduce extra auxiliary local smooth divisors).

Proposition 6. *Let $n = \dim Y$. Then the statement of the above proposition still holds, when $A \subset Y$ is an analytic set with smooth irreducible components and transverse intersections such that no more than n components meet at any point. The curvature current of the extended singular hermitian metric differs from the given current only by an added sum*

$$(37) \quad \pi \sum_j \gamma_j [A_j], \text{ where } 0 \leq \gamma_j < 2.$$

If the given line bundle already possesses an extension into a component of A , together with an extension of the singular hermitian metric such that ω is the curvature current, then the above construction reproduces these without any additional currents of integration.

Proof of the Theorem. We first mention that because of Proposition 5 and Proposition 6 we may assume that $A \subset Y$ is of codimension at least two: Namely we extend both the line bundle and the singular hermitian metric into the locus Y'' , where A is of codimension one and Y is smooth. The original current ω is being changed in this way by adding currents of integration of the form (37), where the A_j are components of A of codimension one. These currents can obviously be extended from Y'' to Y .

The remaining case should also be seen relating to the extension theorem of Shiffman [SHI] for positive line bundles on normal spaces.

We first take a desingularization $\tau : \tilde{Y} \rightarrow Y$ and consider $\tilde{A} = \tau^{-1}(A)$. Next we choose a modification $\mu : Z \rightarrow \tilde{Y}$ that defines an embedded resolution of singularities of $\tilde{A} \subset \tilde{Y}$. In particular $\mu^{-1}(\tilde{A}) = B \cup E$ is a transversal union, where the proper transform B of \tilde{A} is the desingularization and the normal crossings divisor E is the exceptional locus of μ . (The divisorial component of B together with E is a simple normal crossings divisor).

We pull back the line bundle L' to $Z \setminus \mu^{-1}(\tilde{A})$ together with the given data. We apply Proposition 5 and Proposition 6, and obtain an extension $\tilde{\mathcal{L}}$ of the line bundle and of the singular hermitian metric $h_{\tilde{\mathcal{L}}}$ of positive curvature. At those places, where an extended line bundle with a singular hermitian metric already exists, the construction yields the pull-back. The determinant line bundle $\det((\tau \circ \mu)_! \tilde{\mathcal{L}})$ defines an extension of L' . Observe that the original line bundle over Y' is reproduced together with the singular hermitian metric. Because of the normality of Y an extension into an analytic set of codimension two or more is unique, if it exists, and the singular metric of positive curvature can be extended. \square

If Y is just a reduced complex space, we still have the following statement.

Proposition 7. *Let Y be a reduced complex space, and $A \subset Y$ a closed analytic subset. Let \mathcal{L} be an invertible sheaf on $Y \setminus A$, which possesses a holomorphic extension to the normalization of Y as an invertible sheaf. Then there exists a reduced complex space Z together with a finite map $Z \rightarrow Y$, which is an isomorphism over $Y \setminus A$ such that \mathcal{L} possesses an extension as an invertible sheaf to Z .*

Proof. Denote by $\nu : \hat{Y} \rightarrow Y$ the normalization of Y . The presheaf

$$U \mapsto \{\sigma \in (\nu_* \mathcal{O}_{\hat{Y}})(U); \sigma|_{U \setminus A} \in \mathcal{O}_Y(U \setminus A)\}$$

defines a coherent \mathcal{O}_Y -module, the so-called *gap sheaf*

$$\mathcal{O}_Y[A]_{\nu_* \mathcal{O}_{\hat{Y}}}$$

on Y (cf. [SIU1, Proposition 2]). It carries the structure of an \mathcal{O}_Y -algebra. According to Houzel [HOU, Prop. 5 and Prop. 2] it follows that it is an \mathcal{O}_Y -algebra of finite presentation, and hence its analytic spectrum provides a complex space Z over Y (cf. also Forster [FO, Satz 1]). \square

Finally, we have to deal with the question of a *global* extension of positive currents. Siu's Thullen-Remmert-Stein type theorem [SIU3,

Theorem 1] gives an answer for currents on open sets in a complex number space \mathbb{C}^N . An extension for $(1, 1)$ -currents exists and is uniquely determined by one local extension into each irreducible hypersurface component of the critical set. In a global situation we need an extra argument, namely the fact that (under some assumption), one can single out an extension, which is unique and does not depend upon the choice of some local extension.

Proposition 8. *Let A be a closed analytic subset of a normal complex space Y , and let $Y' = Y \setminus A$.*

Let ω' be a closed, positive current on Y' , with vanishing Lelong numbers. Assume that for any point of A there exists an open neighborhood $U \subset Y$ such that $\omega'|_{U \cap Y'}$ can be extended to U as a closed positive current. Then ω' can be extended to all of Y as a positive current.

If the local extensions of ω' have at most analytic singularities, then also the constructed global extension has at most analytic singularities.

Proof. We first assume that Y is smooth and that A is a simple normal crossings divisor. Let ω_U be a (positive) extension of $\omega'|_{U \cap Y'}$.

We apply (26):

$$\omega_U = \sum_{k=0}^{\infty} \mu_k[Z_k] + R.$$

Since the Lelong numbers of ω' vanish everywhere, the sets Z_k must be contained in A so that the positive residual current R is the null extension of $\omega'|_{U \cap Y'}$. We now take the currents of the form R as local extensions. If $W \subset Y$ is open, then the difference of any two such extensions is a current of order zero, which is supported on $A \cap W$. By [DE1, Corollary III (2.14)] it has to be a current of integration supported on $A \cap W$, so it must be equal to zero.

In the case, where Y is not necessarily smooth but normal, we consider a desingularization of the pair (Y, A) like in the proof of Theorem 2 and use the local notation. By [SIU3, Theorem 1] we have unique local extensions of the pull-back of ω' into the locus of codimension greater or equal to two. The Lelong numbers are still equal to zero, and the previous argument applies so that the pull-back of ω' extends to Z as a positive current $\tilde{\omega}$. The push-forward of $\tilde{\omega}$ solves the problem. Since $\tilde{\omega}$ differs from the given local extensions (pulled back to Z) only by (locally defined) currents of integration with support in $\mu^{-1}\tau^{-1}(A)$ the push-forward of $\tilde{\omega}$ again has at most analytic singularities, if the local extensions of ω' have this property. \square

7. DEGENERATING FAMILIES OF CANONICALLY POLARIZED VARIETIES

In this section we want to show that in a degenerating family the curvature of the relative canonical bundle can be extended as a positive closed current.

Given a canonically polarized manifold X of dimension n together with an m -canonical embedding $\Phi = \Phi_{mK_X} : X \hookrightarrow \mathbb{P}_N$, the Fubini-Study metric h_{FS} on the hyperplane section bundle $\mathcal{O}_{\mathbb{P}_N}(1)$ defines a volume form

$$(38) \quad \Omega_X^0 = \left(\sum_{i=0}^N |\Phi_i(z)|^2 \right)^{1/m}$$

on the manifold X , such that

$$\omega_X^0 := -\text{Ric}(\Omega_X^0) = \frac{1}{m} \omega^{FS}|_X,$$

where ω^{FS} denotes the Fubini-Study form on \mathbb{P}_N . According to Yau's theorem, ω_X^0 can be deformed into a Kähler-Einstein metric $\omega_X = \omega_X^0 + \sqrt{-1}\partial\bar{\partial}u$. It solves the equation (12), which is equivalent to

$$(39) \quad \omega_X^n = (\omega_X^0 + \sqrt{-1}\partial\bar{\partial}u)^n = e^u \Omega_X^0.$$

We will need the C^0 -estimates for the (uniquely determined) C^∞ -function u .

The deviation of ω_X^0 from being Kähler-Einstein is given by the function

$$(40) \quad F = \log \frac{\Omega_X^0}{(\omega_X^0)^n}$$

so that (39) is equivalent to

$$(41) \quad \omega_X^n = (\omega_X^0 + \sqrt{-1}\partial\bar{\partial}u)^n = e^{u+F} (\omega_X^0)^n.$$

We will use the C^0 -estimate for u from [CH-Y, Proposition 4.1] (cf. [AU, K2]).

C^0 -estimate. *Let \square^0 denote the complex Laplacian on functions with respect to ω_X^0 (with non-negative eigenvalues). Then*

$$(42) \quad u + F \leq -\square^0(u).$$

In particular the function u is bounded from above by $\sup(-F)$.

Now we come to the *relative situation*. Let $\bar{S} \subset \mathbb{C}^q = \{(s^1, \dots, s^q)\}$ be a polydisk around the origin, and let $S = \bar{S} \setminus V(s^1 \cdot \dots \cdot s^r)$ for some

$r \leq q$. Let

$$(43) \quad \begin{array}{ccc} \overline{\mathcal{X}} & \xrightarrow{\Phi} & \mathbb{P}_N \times \overline{S} \\ & \searrow f & \downarrow \text{pr}_2 \\ & & \overline{S} \end{array}$$

be a proper, flat family, which is smooth over S , and Φ an embedding. We set $\mathcal{L} = \Phi^* \mathcal{O}_{\mathbb{P}_N \times \overline{S}}(1)$.

We assume that $\mathcal{L}|_{\mathcal{X}} \simeq \mathcal{K}_{\mathcal{X}/S}^{\otimes m}$. Again for $s \in S$ we equip the fibers \mathcal{X}_s with Kähler-Einstein forms, and we study the induced relative Kähler-Einstein volume form. It defines a hermitian metric on the relative canonical bundle $\mathcal{K}_{\mathcal{X}/S}$, whose curvature form was denoted by $\omega_{\mathcal{X}}$ (cf. Theorem 1).

The following Proposition plays a key role.

Proposition 9. *Under the above assumptions, the form $\omega_{\mathcal{X}}$ extends to $\overline{\mathcal{X}}$ as a closed, real, positive $(1, 1)$ -current.*

Proof. Like in the absolute case, the Fubini-Study hermitian metric h_{FS} on \mathcal{L} defines a relative volume form

$$\Omega_{\mathcal{X}/S}^0 = h_{FS}^{-1/m} |_{\mathcal{X}}.$$

In a similar way $\omega_{\mathcal{X}/S}^0$ is defined:

$$\omega_{\mathcal{X}/S}^0 := -\text{Ric}_{\mathcal{X}/S}(\Omega_{\mathcal{X}/S}^0).$$

We have, restricted to any fiber of $s \in S$,

$$\omega_{\mathcal{X}_s}^0 = \frac{1}{m} \omega^{FS}|_{\mathcal{X}_s}.$$

We denote by u and F the functions with parameter s , which were defined for the absolute case in (39) and (40) above. Let $\sigma(s) = s^1 \cdot \dots \cdot s^r$. Singular fibers occur only, where σ vanishes.

Now both the initial relative volume form $\Omega_{\mathcal{X}/S}^0$, and the relative volume form $(\omega_{\mathcal{X}/S}^0)^n$ that is induced by the relative Fubini-Study form, are given in terms of polynomials. On S we consider the function

$$\sup \left(\frac{(\omega_{\mathcal{X}/S}^0)^n}{\Omega_{\mathcal{X}/S}^0} \Big|_{\mathcal{X}_s} \right).$$

It follows immediately that for some positive exponent k we have

$$|\sigma(s)|^{2k} \sup \left(\frac{(\omega_{\mathcal{X}/S}^0)^n}{\Omega_{\mathcal{X}/S}^0} \Big|_{\mathcal{X}_s} \right) \leq c$$

for $s \in S$, and some real constant c , i.e.

$$|\sigma(s)|^{2k} \sup\{e^{-F(z)}; f(z) = s\} \leq c.$$

(Here \bar{S} may have to be replaced by a smaller neighborhood of $0 \in \bar{S}$.)

We denote by $\omega_{\mathcal{X}/S}$ the relative Kähler-Einstein form. Again

$$\omega_{\mathcal{X}/S}^n = e^u \Omega_{\mathcal{X}/S}^0,$$

and the fiberwise C^0 -estimate for u yields

$$(44) \quad |\sigma(s)|^{2k} \cdot e^u \leq c$$

on \mathcal{X} .

Now the global form on \mathcal{X} constructed in Theorem 1 is

$$\begin{aligned} \omega_{\mathcal{X}} &= \sqrt{-1} \partial \bar{\partial} \log(\omega_{\mathcal{X}/S}^n) \\ &= \sqrt{-1} \partial \bar{\partial} \log(e^u \cdot \Omega_{\mathcal{X}/S}^0) \\ &= \sqrt{-1} \partial \bar{\partial} \log(e^u \cdot h_{FS}^{-1/m} | \mathcal{X}) \\ &= \sqrt{-1} \partial \bar{\partial} \log(|\sigma(s)|^{2k} \cdot e^u \cdot h_{FS}^{-1/m} | \mathcal{X}). \end{aligned}$$

Observe that $\log h_{FS}$ is of class C^∞ on $\bar{\mathcal{X}}$.

We know from Theorem 1 that $\log(|\sigma(s)|^{2k} \cdot e^u \cdot h_{FS}^{-1/m})$ is plurisubharmonic on \mathcal{X} , and by (44) it is bounded from above, hence it possesses a plurisubharmonic extension to $\bar{\mathcal{X}}$ [G-R]. \square

Now we apply Proposition 9 to families over Hilbert schemes. We fix the Hilbert polynomial and consider the Hilbert scheme of embedded flat proper morphisms. Denote by ${}^n\bar{\mathcal{H}}$ the normalization and by ${}^n\mathcal{H} \subset {}^n\bar{\mathcal{H}}$ the locus of smooth fibers.

$$(45) \quad \begin{array}{ccc} \bar{\mathcal{X}} & \xrightarrow{\Phi} & \mathbb{P}_N \times {}^n\bar{\mathcal{H}} \\ & \searrow f & \downarrow \text{pr}_2 \\ & & {}^n\bar{\mathcal{H}} \end{array}$$

Over ${}^n\mathcal{H}$ the fibers are canonically polarized, i.e. $\Phi^* \mathcal{O}_{\mathbb{P}_N \times {}^n\bar{\mathcal{H}}}(1)|_{\mathcal{X}} \simeq \mathcal{K}_{\mathcal{X}/{}^n\mathcal{H}}^{\otimes m}$.

Theorem 3. *Let $(\mathcal{K}_{\mathcal{X}/\mathcal{H}}, h)$ be the relative canonical bundle on the total space over the Hilbert scheme, where the hermitian metric h is induced by the Kähler-Einstein metrics on the fibers. Then the curvature form extends to the total space $\bar{\mathcal{X}}$ over the compact Hilbert scheme $\bar{\mathcal{H}}$ as a positive, closed current $\omega_{\bar{\mathcal{X}}}^{KE}$ with at most analytic singularities.*

Proof. We observe that $\overline{\mathcal{H}}$ may be desingularized (such that the preimage of $\overline{\mathcal{H}} \setminus \mathcal{H}$ is a simple normal crossings divisor): Since $\omega_{\mathcal{X}}$ is already defined on the (possibly singular) space \mathcal{X} , it will not be affected by the process of taking a pull-back and push-forward again. After a desingularization we apply Proposition 9 together with Proposition 8 and take the push-forward of the resulting current. \square

8. EXTENSION OF THE WEIL-PETERSSON FORM FOR CANONICALLY POLARIZED VARIETIES TO THE COMPACTIFIED HILBERT SCHEME

Now we consider extensions of the Weil-Petersson form to the Hilbert scheme $\overline{\mathcal{H}}$. We have the situation of (45).

We want to apply Theorem 3 and consider the fiber integral analogous to (24) for the map $\overline{\mathcal{X}} \rightarrow \overline{\mathcal{H}}$.

$$(46) \quad \omega_{\mathcal{H}}^{WP} = \int_{\mathcal{X}/\mathcal{H}} (\omega_{\mathcal{X}})^{n+1}.$$

Theorem 4. *The Weil-Petersson form $\omega_{\mathcal{H}}^{WP}$ on \mathcal{H} given by the fiber integral (46) can be extended to $\overline{\mathcal{H}}$ as a positive d -closed $(1, 1)$ -current.*

Proof. The statement is about the pull-back of the Weil-Petersson form to a normalization, so again we assume normality of the base space. We desingularize the pair $({}^n\overline{\mathcal{H}}, {}^n\overline{\mathcal{H}} \setminus {}^n\mathcal{H})$. We can apply Proposition 8 and restrict ourselves to the local situation (43). As we want to avoid wedge products of currents, we use the proof of Proposition 9. On \mathcal{X} we have

$$\omega_{\mathcal{X}} := 2\pi c_1(\mathcal{K}_{\mathcal{X}/S}, h) = \sqrt{-1} \partial \bar{\partial} \log(e^u \Omega_{\mathcal{X}/S}^0) = \omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} u,$$

where $\omega_{\mathcal{X}}^0$ comes from the Fubini-Study form pulled back to \mathcal{X} .

We observe that

$$(\omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} u)^{n+1} = (\omega_{\mathcal{X}}^0)^{n+1} + \sum_{j=0}^n \sqrt{-1} \partial \bar{\partial} u (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} u)^{n-j}.$$

The first term $(\omega_{\mathcal{X}}^0)^{n+1}$ can be treated directly. The form $\omega_{\overline{\mathcal{X}}}^0 = (1/m) \omega^{FS}|_{\overline{\mathcal{X}}}$ is the restriction of a positive C^∞ form on the total space – it was shown in [VA1, Lemme 3.4] that the fiber integral

$$\int_{\overline{\mathcal{X}}/\overline{S}} (\omega_{\overline{\mathcal{X}}}^0)^{n+1}$$

exists and defines a d -closed real $(1, 1)$ -current (positive in the sense of currents) on \overline{S} , which possesses a *continuous $\partial \bar{\partial}$ -potential* on \overline{S} . (This fact also follows from the main theorem of [Yo].)

For the second term we use the results of Section 7: Again we may replace u by $\tilde{u} = u + 2k \log |\sigma(s)|$, which does not change the value of the fiber integral on the interior.

Near any point of $\bar{S} \setminus S$ the potentials \tilde{u} are bounded from above uniformly with respect to the parameter of the base. We avoid taking wedge products of currents and consider

$$\begin{aligned} \int_{\mathcal{X}/S} \sqrt{-1} \partial \bar{\partial} \left(\tilde{u} \cdot (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} \tilde{u})^{n-j} \right) = \\ \sqrt{-1} \partial \bar{\partial} \left(\int_{\mathcal{X}/S} \tilde{u} \cdot (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} \tilde{u})^{n-j} \right). \end{aligned}$$

Now the $\partial \bar{\partial}$ -potential is given by integrals over the fibers:

$$\int_{\mathcal{X}_s} \tilde{u} \cdot (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} \tilde{u})^{n-j}$$

The functions \tilde{u} are known to be uniformly bounded from above, whereas the integrals

$$\int_{\mathcal{X}_s} (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} \tilde{u})^{n-j}$$

are constant as functions of s . This shows the boundedness from above of the potential for the singular Weil-Petersson metrics for families of Kähler-Einstein manifolds of negative curvature. Finally, we invoke again Proposition 8. \square

9. MODULI OF CANONICALLY POLARIZED MANIFOLDS

In this section we give a short analytic/differential geometric proof of the quasi-projectivity of moduli spaces of canonically polarized manifolds depending upon the variation of the Kähler-Einstein metrics on such manifolds.

Theorem 5. *Let \mathcal{M} be a component of the moduli space of canonically polarized manifolds. Then there exists a compactification $\overline{\mathcal{M}}$ together with a holomorphic line bundle $\bar{\lambda}$ equipped with a singular hermitian metric \bar{h}^Q with the following properties:*

- (i) *The restriction of \bar{h}^Q to \mathcal{M} is a hermitian metric of class C^∞ in the orbifold sense, whose curvature is strictly positive.*
- (ii) *The curvature form of \bar{h}^Q on $\overline{\mathcal{M}}$ is a (semi-)positive current.*
- (iii) *The metric \bar{h}^Q has at most analytic singularities.*

Proof. We know from Artin's theorem [AR] that \mathcal{M} possesses a compactification, which is a complex Moishezon space.

Construction of an hermitian line bundle $\lambda_{\mathcal{M}}$ on \mathcal{M} . We fix a number m so that for all polarized varieties belonging to \mathcal{M} the m -th power of the canonical line bundle determines an embedding into a projective space (Matsusaka's Big Theorem).

For any smooth family $f : \mathcal{X} \rightarrow S$, where S denotes a reduced complex space, in Section 5 we constructed the determinant line bundle

$$\lambda_S := \det f_!((\mathcal{K}_{\mathcal{X}/S}^{\otimes m} - (\mathcal{K}_{\mathcal{X}/S}^{\otimes m})^{-1})^{n+1})$$

together with a Quillen metric h_S^Q , whose curvature form is equal to the generalized Weil-Petersson form ω_S^{WP} up to a numerical factor. The construction is functorial so that the line bundle, together with the Quillen metric and the Weil-Petersson form descend to a line bundle $\lambda_{\mathcal{M}}$ and a current $\omega_{\mathcal{M}}^{WP}$ on the moduli space \mathcal{M} in the orbifold sense. We also know that the order of the automorphism groups of the fibers is bounded on \mathcal{M} so that a finite power of the above determinant line bundle actually descends as a line bundle to \mathcal{M} . The form $\omega_{\mathcal{M}}^{WP}$ can be interpreted as of class C^∞ in the orbifold sense or as a positive current with vanishing Lelong numbers on \mathcal{M} , since it has local, continuous $\partial\bar{\partial}$ -potentials.

Compactifications. We denote the canonical map by $\nu : \mathcal{H} \rightarrow \mathcal{M}$ and chose a compactification $\overline{\mathcal{M}}$. Since $\overline{\mathcal{M}}$ is Moishezon, we can eliminate the indeterminacy set of ν : Let $\tau : \widetilde{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$ be a modification such that ν extends to a map $\mu : \widetilde{\mathcal{H}} \rightarrow \overline{\mathcal{M}}$:

$$\begin{array}{ccc} \widetilde{\mathcal{H}} & & \\ \tau \downarrow & \searrow \mu & \\ \overline{\mathcal{H}} & \dashrightarrow & \overline{\mathcal{M}} \\ \cup & & \cup \\ \mathcal{H} & \xrightarrow{\nu} & \mathcal{M} \end{array}$$

Extension of the Weil-Petersson current to $\overline{\mathcal{M}}$. The construction of the Weil-Petersson form on the Hilbert scheme \mathcal{H} is known to be functorial, it descends to the (open) moduli space \mathcal{M} in the orbifold sense. In Theorem 4 we constructed an extension to $\overline{\mathcal{H}}$ as a positive, closed current. Here we use the theorem to construct the extension $\omega_{\overline{\mathcal{H}}}^{WP}$ of $\omega_{\mathcal{H}}^{WP}$ to $\overline{\mathcal{H}}$. We claim that restricted to $\mu^{-1}(\mathcal{M})$ this current is equal to the pull-back of the (smooth) Weil-Petersson form $\mu^*\omega_{\mathcal{M}}^{WP}$ as a current. This follows from the construction in Proposition 8, where the extension was defined as a null-extension, and as such it is uniquely

determined. Now the push forward of $\omega_{\overline{\mathcal{H}}}^{WP}$ to $\overline{\mathcal{M}}$ under μ yields an extension of $\omega_{\mathcal{M}}^{WP}$. Over points of $\overline{\mathcal{M}} \setminus \mathcal{M}$ the extension of the Weil-Petersson form cannot be controlled.

Reduction to the normalization. By Proposition 7 it is sufficient consider normalizations.

Extension of the determinant line bundle. We apply Theorem 2. \square

Now the criterion [S-T, Theorem 6] is applicable, which yields the quasi-projectivity of the moduli space [V, V1, V2, KO1].

Finally, we address the finiteness of the volume of the moduli space stated in Corollary 1.

We know from Theorem 4 together with Proposition 8 that we have extensions of ω^{WP} resp. to the compactified space $\overline{\mathcal{M}}$ as a closed, positive current. This current defines the Chern class of a certain line bundle on the compactified moduli space. Although the maximum exterior product of this current need not exist as a current, the theory of non-pluripolar products of globally defined currents of Boucksom, Eyssidieux, Guedj, and Zeriahi from [BEGZ] applies. (Observe that both the orbifold space structure and the singularities do not present any problem.) As a result we get that the integral of the differentiable volume form over the interior \mathcal{M} is finite.

10. CURVATURE OF $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ – STATEMENT OF THE THEOREM AND APPLICATIONS

10.1. Statement of the theorem. We consider an effectively parameterized family $\mathcal{X} \rightarrow S$ of canonically polarized manifolds, equipped with Kähler-Einstein metrics of constant Ricci curvature -1 . For any $m > 0$ the direct image sheaves $f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes(m+1)} = f_*\Omega_{\mathcal{X}/S}^n(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ are locally free. For values of p other than n we assume local freeness of

$$R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m}),$$

i.e. $\dim_{\mathbb{C}} H^{n-p}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^p(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})) = \text{const.}$ In particular the base change property (cohomological flatness) holds. The assumptions of Section 2.2 are satisfied so that we can apply Lemma 2. If necessary, we replace S by a (Stein) open subset, such that the direct image is actually free, and denote by $\{\psi^1, \dots, \psi^r\} \subset R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})(S)$ a basis of the corresponding free \mathcal{O}_S -module. At a given point $s \in S$ we denote by $\{(\partial/\partial s_i)|_s; i = 1, \dots, M\}$ a basis of the complex tangent space $T_s S$ of S over \mathbb{C} , where the s_i are holomorphic coordinate functions of a minimal smooth ambient space $U \subset \mathbb{C}^M$.

Let $A_{i\bar{j}}^\alpha(z, s)\partial_\alpha dz^{\bar{\beta}}$ be a harmonic Kodaira-Spencer form. Then for $s \in S$ the cup product together with the contraction defines

$$(47) \quad A_{i\bar{j}}^\alpha \partial_\alpha dz^{\bar{\beta}} \cup \cdot : \mathcal{A}^{0, n-p}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^p(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})) \rightarrow \mathcal{A}^{0, n-p+1}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^{p-1}(\mathcal{K}_{\mathcal{X}_s}^{\otimes m}))$$

$$(48) \quad A_{j\bar{\alpha}}^{\bar{\beta}} \partial_{\bar{\beta}} dz^\alpha \cup \cdot : \mathcal{A}^{0, n-p}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^p(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})) \rightarrow \mathcal{A}^{0, n-p-1}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^{p+1}(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})),$$

where $p > 0$ in (47) and $p < n$ in (48).

We will apply the above products to harmonic $(0, n-p)$ -forms. In general the results are not harmonic. We use the notation $\psi^{\bar{\ell}} := \overline{\psi^\ell}$ for sections ψ^k (and a notation of similar type for tensors on the fibers):

Theorem 6. *The curvature tensor for $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ is given by*

$$(49) \quad \begin{aligned} R_{i\bar{j}}^{\bar{\ell}k}(s) &= m \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) \cdot (\psi^k \cdot \psi^{\bar{\ell}}) g dV \\ &\quad + m \int_{\mathcal{X}_s} (\square + m)^{-1} (A_i \cup \psi^k) \cdot (A_{\bar{j}} \cup \psi^{\bar{\ell}}) g dV \\ &\quad + m \int_{\mathcal{X}_s} (\square - m)^{-1} (A_i \cup \psi^{\bar{\ell}}) \cdot (A_{\bar{j}} \cup \psi^k) g dV. \end{aligned}$$

The only contribution in (49), which may be negative, originates from the harmonic parts in the third term. It equals

$$- \int_{\mathcal{X}_s} H(A_i \cup \psi^{\bar{\ell}}) \overline{H(A_{\bar{j}} \cup \psi^k)} g dV.$$

Concerning the third term, the theorem contains the fact that the positive eigenvalues of the Laplacian are larger than m . (For $p = 0$ the second term in (49) is equal to zero, and for $p = n$ the third one does not occur.) One can verify that in case $p = 0$ and $m = 1$ the right-hand side of (49) is identically zero as expected.

Theorem 6 will be proved in Section 11.

The pointwise estimate (11) of the resolvent kernel (cf. also Proposition 1) translates into an estimate of the curvature.

Proposition 10. *Let $f : \mathcal{X} \rightarrow S$ be a family of canonically polarized manifolds, and $s \in S$. Let a tangent vector of S at s be given by a harmonic Kodaira-Spencer form A and let ψ be a harmonic $(p, n-p)$ -form on \mathcal{X}_s with values in the m -canonical bundle. Then*

$$(50) \quad R(A, \bar{A}, \psi, \bar{\psi}) \geq P_n(d(\mathcal{X}_s)) \cdot \|A\|^2 \cdot \|\psi\|^2 + \|H(A \cup \psi)\|^2 - \|H(A \cup \bar{\psi})\|^2.$$

Proof of the Proposition. We apply (49). The first term involves the Laplacian on functions. It has already been treated. Concerning the second and third term, we use the fact that on $\mathcal{A}^{0, n-p\pm 1}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^{p\mp 1}(\mathcal{K}_{\mathcal{X}_s}^{\otimes m}))$

according to Lemma 13 of Section 11 the equation $\square_{\partial} = \square_{\bar{\partial}}$ holds and that moreover all non-zero eigenvalues are larger than m by the Claim in the proof of Theorem 6. \square

For $p = n$ we obtain the following result.

Corollary 5. *For $f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes(m+1)}$ the curvature equals*

$$(51) \quad \begin{aligned} R_{i\bar{j}}^{\bar{\ell}k}(s) &= m \int_{\mathcal{X}_s} (\square + m)^{-1} (A_i \cup \psi^k) \cdot (A_{\bar{j}} \cup \psi^{\bar{\ell}}) g dV \\ &\quad + m \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) \cdot (\psi^k \cdot \psi^{\bar{\ell}}) g dV. \end{aligned}$$

The operator $(\square + m)^{-1}$ in the first term of (51) is positive on the respective tensors, the second term yields an estimate: Let

$$(52) \quad H^{\bar{\ell}k} = \int_{\mathcal{X}_s} \psi^k \cdot \psi^{\bar{\ell}} g dV.$$

Corollary 6. *Let $s \in S$ be any point. Let $\xi_k^i \in \mathbb{C}$. Then*

$$(53) \quad R_{i\bar{j}}^{\bar{\ell}k}(s) \xi_k^i \bar{\xi}_{\bar{\ell}}^{\bar{j}} \geq m \cdot P_n(d(\mathcal{X}_s)) \cdot G_{i\bar{j}}^{WP} \cdot H^{\bar{\ell}k} \cdot \xi_k^i \bar{\xi}_{\bar{\ell}}^{\bar{j}}.$$

In particular the curvature is strictly Nakano-positive with the above estimate.

Next, we set $m = 1$ and take a dual basis $\{\nu_i\} \subset R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}(S)$ of the $\{\psi^k\}$ and normal coordinates at a given point $s_0 \in S$. (Again local freeness is assumed.)

We have mappings dual to (47) and (48):

$$(54) \quad A_{i\bar{\beta}}^{\alpha} \partial_{\alpha} dz^{\bar{\beta}} \wedge \lrcorner : \mathcal{A}^{0,p}(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s}) \rightarrow \mathcal{A}^{0,p+1}(\mathcal{X}_s, \Lambda^{p+1} \mathcal{T}_{\mathcal{X}_s})$$

$$(55) \quad A_{j\alpha}^{\bar{\beta}} \partial_{\bar{\beta}} dz^{\alpha} \wedge \lrcorner : \mathcal{A}^{0,p}(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s}) \rightarrow \mathcal{A}^{0,p-1}(\mathcal{X}_s, \Lambda^{p-1} \mathcal{T}_{\mathcal{X}_s}).$$

Here, in (54) the wedge product stands for an exterior product, whereas in (55) the wedge product denotes a contraction with both indices of A_i . Again for $p = n$ in (54), and for $p = 0$ in (55) the value of the wedge product is zero.

Observing that the role of conjugate and non-conjugate tensors is being interchanged, the curvature can be computed from Theorem 6. Again because of S -flatness of $\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S})$ and $\Lambda^p \mathcal{T}_{\mathcal{X}/S}$ resp. it is sufficient to require that $\dim_{\mathbb{C}} H^{n-p}(\mathcal{X}_s, \Omega_{\mathcal{X}_s}^p(\mathcal{K}_{\mathcal{X}_s}))$ is constant, which is equivalent to $\dim_{\mathbb{C}} H^p(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s})$ being constant.

Theorem 7. *The curvature of $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}$ equals*

$$\begin{aligned}
 R_{i\bar{j}k\bar{\ell}}(s) = & - \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) \cdot (\nu_k \cdot \nu_{\bar{\ell}}) g dV \\
 & - \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \wedge \nu_{\bar{\ell}}) \cdot (A_{\bar{j}} \wedge \nu_k) g dV \\
 & - \int_{\mathcal{X}_s} (\square - 1)^{-1} (A_i \wedge \nu_k) \cdot (A_{\bar{j}} \wedge \nu_{\bar{\ell}}) g dV.
 \end{aligned}
 \tag{56}$$

The only possible positive contribution arises from

$$\int_{\mathcal{X}_s} H(A_i \wedge \nu_k) H(A_{\bar{j}} \wedge \nu_{\bar{\ell}}) g dV.$$

We observe that in Theorem 7 for $n = 1$ and $p = 1$ the third term in (56) is not present, and we have the formula for the classical Weil-Petersson metric on Teichmüller space of Riemann surfaces of genus larger than one [TR, WO].

For $p = 1$ we obtain the curvature for the generalized Weil-Petersson metric from [SCH2], (cf. [SIU2]). Again we can estimate the curvature like in Proposition 10.

For $p = 0$, one can verify that the curvature tensor (56) is identically zero as expected.

The following case is of interest.

Proposition 11. *Let $f : \mathcal{X} \rightarrow S$ be a family of canonically polarized manifolds and $s \in S$. Let tangent vectors of S at s be given by harmonic Kodaira-Spencer forms A, A_1, \dots, A_p on \mathcal{X}_s . Let R denote the curvature tensor for $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}$. Then we have in terms of the Weil-Petersson norms:*

$$\begin{aligned}
 & R(A, \bar{A}, H(A_1 \wedge \dots \wedge A_p), \overline{H(A_1 \wedge \dots \wedge A_p)}) \leq \\
 & -P_n(\mathcal{X}_s) \cdot \|A\|^2 \cdot \|H(A_1 \wedge \dots \wedge A_p)\|^2 + \|H(A \wedge A_1 \wedge \dots \wedge A_p)\|^2.
 \end{aligned}
 \tag{57}$$

Proof. Since the A and A_i are $\bar{\partial}$ -closed forms, we have $H(A \wedge H(A_1 \wedge \dots \wedge A_p)) = H(A \wedge A_1 \wedge \dots \wedge A_p)$. \square

Next, we define higher Kodaira-Spencer maps defined on the symmetric powers of the tangent bundle of the base. Let $S^p(R^1 f_* \mathcal{T}_{\mathcal{X}/S})$ denote the p -th symmetric tensor power for $p > 0$. Then the natural morphism

$$S^p(R^1 f_* \mathcal{T}_{\mathcal{X}/S}) \rightarrow R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}$$

together with the Kodaira-Spencer morphism $\rho_S : \mathcal{T}_S \rightarrow R^1 f_* \mathcal{T}_{\mathcal{X}/S}$ induces the Kodaira-Spencer morphism of order p

$$(58) \quad \rho_S^p : S^p \mathcal{T}_S \rightarrow R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}.$$

After tensorizing with $\mathbb{C}(s)$ we have fiberwise the maps

$$\rho_{S,s}^p : S^p T_{S,s} \rightarrow H^p(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s})$$

that send a symmetric power

$$\frac{\partial}{\partial s^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial s^{i_p}}$$

to the class of

$$A_{i_1} \wedge \dots \wedge A_{i_p} := A_{i_1 \bar{\beta}_1}^{\alpha_1} \partial_{\alpha_1} dz^{\bar{\beta}_1} \wedge \dots \wedge A_{i_p \bar{\beta}_p}^{\alpha_p} \partial_{\alpha_p} dz^{\bar{\beta}_p}.$$

Definition 4. *Let the tangent vector $\partial/\partial s \in T_{S,s}$ correspond to the harmonic Kodaira-Spencer tensor A_s . Then the generalized Weil-Petersson function of degree p on the tangent space is*

$$(59) \quad \begin{aligned} \|\partial/\partial s\|_p^{WP} &:= \|A_s\|_p := \|H(\underbrace{A_s \wedge \dots \wedge A_s}_p)\|^{1/p} \\ &:= \left(\int_{\mathcal{X}_s} H(A_s \wedge \dots \wedge A_s) \cdot \overline{H(A_s \wedge \dots \wedge A_s)} g dV \right)^{1/2p} \end{aligned}$$

Obviously $\|\alpha \cdot A_s\|_p = |\alpha| \cdot \|A_s\|_p$ for all $\alpha \in \mathbb{C}$. The triangle inequality is not being claimed.

From now on we assume that S is a locally irreducible (reduced) space.

Next, let $C \rightarrow S$ be a smooth analytic curve, whose image in S again is a curve, and let $\mathcal{X}_C \rightarrow C$ be the pull-back of the given family over S . All $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}_C/C}/\text{torsion}$ are locally free, maybe zero. If on some open subset of C the value $\rho_{C,s}^p(\partial/\partial s|_s) \in H^p(\mathcal{X}_s, \Lambda^p \mathcal{T}_{\mathcal{X}_s})$ is different from zero, where $\partial/\partial s \neq 0$ denotes a tangent vector of C at s , then $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}_C/C}/\text{torsion}$ is not zero, and on a complement of a discrete set of C the sheaf $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}_C/C}$ is locally free.

Given a family over a smooth analytic curve $C \rightarrow S$, the p -th Weil-Petersson function of tangent vectors defines a hermitian (pseudo) metric on the curve, which we denote by G_p . On the complement C' of a discrete subset of C we can estimate the curvature.

Lemma 7. *For any analytic curve $C \rightarrow S$ the curvature K_{G_p} of G_p , at points $s \in C'$ with $G_p(s) \neq 0$ satisfies the following inequality, where*

the tensors A_s represent Kodaira-Spencer classes of tangent vectors of C' at s with $A_s^p \neq 0$.

$$(60) \quad K_{G_p}(s) \leq \frac{1}{p} \left(-P_n(d(\mathcal{X}_s)) \frac{\|A_s\|_1^2}{\|A_s\|_p^2} + \frac{\|A_s\|_{p+1}^{2p+2}}{\|A_s\|_p^{2p+2}} \right).$$

The second summand vanishes identically for $p = n$.

Proof. Let $A^p = H(A_s \wedge \dots \wedge A_s)$ be the harmonic projection of the p -fold exterior product of A_s . Then the curvature tensor for $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/C}$ satisfies

$$(61) \quad \begin{aligned} R(\partial_s, \partial_{\bar{s}}, A^p, \overline{A^p}) &\geq -\frac{\partial^2 \log(G_p^p)}{\partial s \partial \bar{s}} \cdot G_p^p \cdot \|A^p\|^2 = \\ &-p \frac{\partial^2 \log(G_p)}{\partial s \partial \bar{s}} \cdot \|A\|_p^{2p} = p \cdot G_p \cdot K_{G_p} \cdot \|A\|_p^{2p}. \end{aligned}$$

Here $R(\partial_s, \partial_{\bar{s}}, _, _)$ is the curvature form applied to the tangent vectors $\partial/\partial s$ and $\partial/\partial \bar{s}$ resp. With respect to G_p , we identify $G_p = \|\partial/\partial s\|_p^2 = \|A_s\|_p^2$ (cf. (4)) so that

$$R(A_s, \overline{A_s}, A^p, \overline{A^p}) \geq p \cdot K_{G_p} \cdot \|A\|_p^{2p+2}.$$

Now the estimate of Proposition 11 implies

$$K_{G_p} \leq \frac{1}{p} \left(-P_n(d(\mathcal{X}_s)) \|A\|_1^2 \|A\|_p^{2p} + \|A\|_{p+1}^{2(p+1)} \right) / \|A\|_p^{2(p+1)}.$$

□

10.2. Hyperbolicity conjecture of Shafarevich. In this section we describe a short proof of the following special case of Shafarevich's hyperbolicity conjecture for canonically polarized varieties [B-V, KE-KV, KO-KV, KV1, KV2, KO1, M, V-Z1, V-Z2].

Application. *If a compact smooth curve C parameterizes a non-isotrivial family of canonically polarized manifolds, its genus must be greater than one.*

For the proof, we will show that one of the metrics G_p on C has negative curvature.

Let $f : \mathcal{X} \rightarrow C$ denote a non-isotrivial family of canonically polarized varieties over a smooth compact curve. Again we denote by A_s the harmonic Kodaira-Spencer form that represents the Kodaira-Spencer class of a tangent vector $0 \neq \partial/\partial s \in T_{C,s}$, and by A_s^q the harmonic representative of the q -fold wedge product. Let p_0 be the maximum number p for which $A_s^p \neq 0$ on some open subset, i.e. on the complement of a finite subset of C .

Proposition 12. *Under the above assumptions, the locally defined functions $\log G_{p_0}$ are subharmonic, and there exists a number $c > 0$ such that for all s in the complement of a finite set of C and all $A_s \neq 0$*

$$(62) \quad K_{G_{p_0}}(s) \leq -\frac{1}{p_0} P_n(d(\mathcal{X}_s)) \frac{\|A_s\|_1^2}{\|A_s\|_{p_0}^2} \leq -c.$$

Proof. We chose the complement C' of a finite set in C so that for all $s \in C'$ we have $A_s^{p_0} \neq 0$ and also $A_s^{p_0+1} = 0$ for $s \in C'$. On C' by Lemma 7 we have the first inequality in (62). We claim that there exist a number $c' > 0$ such that

$$(63) \quad \|A_s\|_{p_0}/\|A_s\|_1 \leq c' \text{ for all } A_s \neq 0 \text{ and all } s \in C'.$$

Since C is compact, we need to show boundedness only near points $s_0 \in C \setminus C'$: Let a point s_0 be given by $s = 0$, where s is a local holomorphic coordinate on C . Using a differentiable local trivialization of the family, we find representatives B_s of the Kodaira-Spencer classes (which depend in a C^∞ way on the parameter). The p_0 -fold wedge products $B_s \wedge \dots \wedge B_s$ have bounded norm so that also the norms of $A_s^{p_0} = H(B_s \wedge \dots \wedge B_s)$ are bounded. Since the given family is not isotrivial, the sheaf $R^1 f_* \mathcal{T}_{\mathcal{X}/C}/\text{torsion}$ is locally free and not zero. We need to consider only the case that $A_s \rightarrow 0$ for $s \rightarrow 0$. In this case we find some power $k > 0$ so that $s^{-k} A_s$ converges towards some non-zero element from $(R^1 f_* \mathcal{T}_{\mathcal{X}/C}(C)/\text{torsion}) \otimes \mathbb{C}(s_0)$. So $\inf_{s \rightarrow 0} \|s^{-k} A_s\|_1 > 0$. At the same time the norms $\|s^{-k} A_s\|_{p_0}$ stay bounded from above. The factors $|s^{-k}|$ cancel out in $\|s^{-k} A_s\|_{p_0}/\|s^{-k} A_s\|_1$, which shows (63) implying (62). The first inequality in (62) implies that the (locally defined function) $\log G_{p_0}$ is subharmonic on C' , the second implies that it is bounded on C so that $\log G_{p_0}$ is subharmonic on C (being defined in terms of local coordinate systems). \square

In [DE, 3.2] Demailly gives a proof of the Ahlfors lemma (cf. also [G-RZ]) for singular hermitian metrics of negative curvature in the context of currents using an approximation argument. We will need the following special case:

Proposition 13. *Let $\gamma = \gamma(s)\sqrt{-1}ds \wedge \overline{ds}$, $\gamma(s) \geq 0$ be given on an open disk $\Delta_R = \{|s| < R\}$, where $\log \gamma(s)$ is a subharmonic function such that $\sqrt{-1}\partial\bar{\partial}(\log \gamma) \geq A\gamma$ in the sense of currents for some $A > 0$. Let ρ denote the Poincaré metric on Δ_R . Then $\gamma \leq \rho/A$ holds.*

Proof of the Application. Let $\varphi : \Delta_R \rightarrow C$ be a non-constant holomorphic map and $\gamma = \varphi^*(G_{p_0})$. Then the assumptions of Proposition 13 are satisfied by Proposition 12. So the curve C is hyperbolic. \square

Proposition 14. *Any relatively compact open subspace of the moduli space of canonically polarized manifolds is Kobayashi hyperbolic in the orbifolds sense.*

Proof. For any canonically polarized manifold we consider a local universal deformation given by a holomorphic family $f : \mathcal{X} \rightarrow S$. The moduli space possesses an open covering by quotients of the form S/Γ , where Γ is a finite group. Over a relatively compact subspace of the moduli space the diameter of the canonically polarized manifolds \mathcal{X}_s will be bounded. Since S itself will be singular we consider the tangent cone of S , which consists fiberwise of 1-jets. These are the tangent vectors, which are induced by local analytic curves C through the given point $s_0 \in S$. (cf. [K1, Chapter 2.3]). Now the estimates of Proposition 12 hold also for locally defined analytic curves. One can show that numbers $c > 0$ satisfying the second inequality from the statement of the Proposition can be chosen uniformly over any given relatively compact subset of the moduli space, only depending upon p_0 so that the smallest of these numbers yields Kobayashi hyperbolicity. \square

10.3. Finsler metric on the moduli stack. We indicate, how to construct a Finsler metric of negative holomorphic curvature on the moduli stack. Different notions of a Finsler metric are common. We do not assume the triangle inequality/convexity. Such metrics are also called *pseudo-metrics* (cf. [K1]).

Definition 5. *Let Z be a reduced complex space and let $T_c Z$ be the fiber bundle consisting of the tangent cones of 1-jets. An upper semi-continuous function*

$$F : T_c Z \rightarrow [0, \infty)$$

is called Finsler pseudo-metric (or pseudo-length function), if

$$F(av) = |a|F(v) \text{ for all } a \in \mathbb{C}, v \in TZ.$$

The triangle inequality on the fibers is not required for the definition of the holomorphic (or holomorphic sectional) curvature: The holomorphic curvature of a Finsler metric at a certain point p in the direction of a tangent vector v is the supremum of the curvatures of the pull-back of the given Finsler metric to a holomorphic disk through p and tangent to v (cf. [A-P]). (For a hermitian metric, the holomorphic curvature is known to be equal to the holomorphic sectional curvature.) The functions G_p define Finsler (pseudo) metrics. Furthermore, any convex sum $G = \sum_j a_j G_j$, $a_j > 0$ is upper semi-continuous and has the property that $\log G$ restricted to a curve is subharmonic.

Lemma 8 (cf. [SCH3, Lemma 3]). *Let C be a complex curve and G_j a collection of pseudo-metrics of bounded curvature, whose sum has no common zero. Then the curvatures K satisfy the following equation.*

$$(64) \quad K_{\sum_{j=1}^k G_j} \leq \sum_{j=1}^k \frac{G_j^2}{(\sum_{i=1}^k G_i)^2} K_{G_j}.$$

Like in the proof of Proposition 12 one shows that $\|A\|_{p+1}/\|A\|_p$ can be uniformly bounded with respect to S for $\|A\|_p \neq 0$. Using Lemma 8 and Lemma 7 a convex sum $G = \sum_p \alpha_p G_p$, $\alpha_p > 0$ with negative holomorphic curvature is constructed. The metric G on curves defines an (upper semi-continuous) Finsler metric that descends to the moduli space in the orbifold sense.

Proposition 15. *On any relatively compact subset of the moduli space of canonically polarized manifolds there exists a Finsler orbifold metric, whose holomorphic curvature is bounded from above by a negative constant.*

11. COMPUTATION OF THE CURVATURE

The components of the metric tensor for $R^{n-p} f_* \Omega_{X/S}^p (\mathcal{K}_{X/S}^{\otimes m})$ on the base space S are the integrals of inner products of the harmonic representatives of cohomology classes. We know from Lemma 2 that these are the restrictions of certain $\bar{\partial}$ -closed differential forms on the total space. When we compute derivatives with respect to the base of these fiber integrals, we will apply Lie derivatives with respect to lifts of tangent vectors in the sense of Section 2.1. Taking *horizontal* lifts simplifies the computations. The Lie derivatives of these pointwise inner products can be broken up, and Lie derivatives of these differential forms with values in the m -canonical bundle have to be taken. The derivatives are covariant derivatives with respect to the hermitian structure on this line bundle. Since we are dealing with alternating forms we may use covariant derivatives also with respect to the Kähler structure on the fibers, which further simplifies the computations.

Again, we will use the semi-colon notation for covariant derivatives and use a $|$ -symbol for ordinary derivatives, if necessary. Greek indices are being used for fiber coordinates, Latin indices indicate the base direction. Dealing with alternating forms, for instance of degree (p, q) , extra coefficients of the form $1/p!q!$ are sometimes customary; these play a role, when the coefficients of an alternating form are turned into skew-symmetric tensors by taking the average. However, for the sake of a halfway simple notation, we follow the better part of the literature and leave these to the reader.

11.1. Setup. As above, we denote by $f : \mathcal{X} \rightarrow S$ a smooth family of canonically polarized manifolds and we pick up the notation from Section 4. The fiber coordinates were denoted by z^α and the coordinates of the base by s^i . We set $\partial_i = \partial/\partial s^i$, $\partial_\alpha = \partial/\partial z^\alpha$.

Again we have *horizontal lifts of tangent vectors and coordinate vector fields on the base*

$$v_i = \partial_i + a_i^\alpha \partial_\alpha.$$

As above we have the corresponding harmonic representatives

$$A_i = A_{i\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}$$

of the Kodaira-Spencer classes $\rho(\partial_i|_{s_0})$.

For the computation of the curvature it is sufficient to treat the case where $\dim S = 1$. We set $s = s_1$ and $v_s = v_1$ etc. In this case we write s and \bar{s} for the indices 1 and $\bar{1}$ so that

$$v_s = \partial_s + a_s^\alpha \partial_\alpha$$

etc.

Sections of $R^{n-p} f_* \Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ will be denoted by letters like ψ .

$$\begin{aligned} \psi|_{\mathcal{X}_s} &= \psi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \bar{\beta}_n} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge dz^{\bar{\beta}_{p+1}} \wedge \dots \wedge dz^{\bar{\beta}_n} \\ &= \psi_{A_p \bar{B}_{n-p}} dz^{A_p} \wedge dz^{\bar{B}_{n-p}} \end{aligned}$$

where $A_p = (\alpha_1, \dots, \alpha_p)$ and $\bar{B}_{n-p} = (\bar{\beta}_{p+1}, \dots, \bar{\beta}_n)$. The further component of ψ is

$$\psi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \bar{\beta}_{n-1}, \bar{s}} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge dz^{\bar{\beta}_{p+1}} \wedge \dots \wedge dz^{\bar{\beta}_{n-1}} \wedge \bar{ds}.$$

Now Lemma 2 implies

$$(65) \quad \psi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \bar{\beta}_n | \bar{s}} = \sum_{j=p+1}^n (-1)^{n-j} \psi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \widehat{\bar{\beta}_j}, \dots, \bar{\beta}_n, \bar{s} | \bar{\beta}_j}.$$

Since these are the coefficients of alternating forms, on the right-hand side, we may also take the covariant derivatives with respect to the given structure on the fibers

$$\psi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \widehat{\bar{\beta}_j}, \dots, \bar{\beta}_n, \bar{s}; \bar{\beta}_j}.$$

11.2. Cup Product. We define the cup product of a differential form with values in the relative holomorphic tangent bundle and an (line bundle valued) differential form now in terms of local coordinates.

Definition 6. *Let*

$$\mu = \mu_{\alpha_1, \dots, \alpha_p, \bar{\beta}_1, \dots, \bar{\beta}_q}^\sigma \partial_\sigma dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge dz^{\bar{\beta}_1} \wedge \dots \wedge dz^{\bar{\beta}_q},$$

and

$$\nu = \nu_{\gamma_1, \dots, \gamma_a, \bar{\delta}_1, \dots, \bar{\delta}_b} dz^{\gamma_1} \wedge \dots \wedge dz^{\gamma_a} \wedge dz^{\bar{\delta}_1} \wedge \dots \wedge dz^{\bar{\delta}_b}$$

Then

$$(66) \quad \mu \cup \nu := \mu_{\alpha_1, \dots, \alpha_p, \bar{\beta}_1, \dots, \bar{\beta}_q}^\sigma \nu_{\sigma\gamma_2, \dots, \gamma_a, \bar{\delta}_1, \dots, \bar{\delta}_b} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge dz^{\bar{\beta}_1} \wedge \dots \wedge dz^{\bar{\beta}_q} \wedge dz^{\gamma_2} \wedge \dots \wedge dz^{\gamma_a} \wedge dz^{\bar{\delta}_1} \wedge \dots \wedge dz^{\bar{\delta}_b}$$

11.3. Lie derivatives. Let again the base be smooth, $\dim S = 1$ with local coordinate s . Then the induced metric on $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ is given by (52), where the pointwise inner product equals

$$\psi^k \cdot \psi^{\bar{\ell}} g dV = (\sqrt{-1})^n (-1)^{n(n-p)} \frac{1}{g^m} \psi^k \wedge \psi^{\bar{\ell}},$$

and where $1/g^m$ stands for the hermitian metric on the m -canonical bundle on the fibers.

Lemma 9.

$$\frac{\partial}{\partial s} H^{\bar{\ell}k} = \int_{\mathcal{X}_s} L_v(\psi^k \cdot \psi^{\bar{\ell}}) g dV = \langle L_v \psi^k, \psi^{\bar{\ell}} \rangle + \langle \psi^k, L_{\bar{v}} \psi^{\bar{\ell}} \rangle,$$

where L_v denotes the Lie derivative with respect to the canonical lift v of the coordinate vector field $\partial/\partial s$.

Proof. Taking the Lie derivative is not type-preserving. We need the $(1, 1)$ -component: $L_v(g_{\alpha\bar{\beta}}) = [\partial_s + a_s^\alpha \partial_\alpha, g_{\alpha\bar{\beta}}]_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}|s} + a_s^\gamma g_{\alpha\bar{\beta};\gamma} + a_{s;\alpha}^\gamma g_{\gamma\bar{\beta}} - a_{s\bar{\beta};\alpha} + a_{s;\alpha}^\gamma g_{\gamma\bar{\beta}} = 0$. So $L_v(\det(g_{\alpha\bar{\beta}})) = 0$. \square

We have the type decomposition for $\psi = \psi^k$ or $\psi = \psi^{\bar{\ell}}$

$$(67) \quad L_v \psi = L_v \psi' + L_v \psi'',$$

$$\begin{aligned}
(68) \quad L_v \psi' &= [\partial_s + a_s^\alpha \partial_\alpha, \psi_{A_p \bar{B}_{n-p}} dz^{A_p} \wedge dz^{\bar{B}_{n-q}}]_{(p, n-p)} \\
&= (\psi_{;s} + a_s^\alpha \psi_{;\alpha} + \sum_{j=1}^p a_{s;\alpha_j}^\alpha \psi_{\alpha_1, \dots, \underset{j}{\alpha}, \dots, \alpha_p \bar{B}_{n-p}}) dz^{A_p} \wedge dz^{\bar{B}_{n-p}} \\
(69) \quad L_v \psi'' &= [\partial_s + a_s^\alpha \partial_\alpha, \psi_{A_p \bar{B}_{n-p}} dz^{A_p} \wedge dz^{\bar{B}_{n-q}}]_{(p-1, n-p+1)} \\
&= \sum_{j=1}^p A_{s\bar{\beta}_j}^\alpha \psi_{\alpha_1, \dots, \underset{j}{\alpha}, \dots, \alpha_p \bar{B}_{n-p}} \\
&\quad dz^{\alpha_1} \wedge \dots \wedge dz^{\bar{\beta}_p} \wedge \dots \wedge dz^{\alpha_p} \wedge dz^{\bar{\beta}_{p+1}} \wedge \dots \wedge dz^{\bar{\beta}_n}
\end{aligned}$$
$$\begin{aligned}
(70) \quad L_{\bar{v}}\psi' &= [\partial_{\bar{s}} + a_{\bar{s}}^{\bar{\beta}} \partial_{\bar{\beta}}, \psi_{A_p \bar{B}_{n-p}} dz^{A_p} \wedge dz^{\bar{B}_{n-q}}]_{(p,n-p)} \\
&= (\psi_{;\bar{s}} + a_{\bar{s}}^{\bar{\beta}} \psi_{;\bar{\beta}} + \sum_{j=1}^p a_{\bar{s};\bar{\beta}_j}^{\bar{\beta}} \psi_{A_p \bar{\beta}_{p+1}, \dots, \underset{j}{\bar{\beta}}, \dots, \bar{\beta}_n}) dz^{A_p} \wedge dz^{\bar{B}_{n-p}} \\
(71) \quad L_{\bar{v}}\psi'' &= [\partial_{\bar{s}} + a_{\bar{s}}^{\bar{\beta}} \partial_{\bar{\beta}}, \psi_{A_p \bar{B}_{n-p}} dz^{A_p} \wedge dz^{\bar{B}_{n-q}}]_{(p+1, n-p-1)} \\
&= \sum_{j=p+1}^n A_{\bar{s}\alpha_{p+1}}^{\bar{\beta}} \psi_{\alpha_1, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \underset{j}{\bar{\beta}}, \dots, \bar{\beta}_n} \\
&\quad dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge dz^{\bar{\beta}_1} \wedge \dots \wedge dz^{\alpha_{p+1}} \wedge \dots \wedge dz^{\bar{\beta}_n}
\end{aligned}$$
$$\begin{aligned} (72) \quad L_v \psi'' &= A_s \cup \psi \\ (73) \quad L_{\bar{v}} \psi'' &= (-1)^p A_{\bar{s}} \cup \psi \end{aligned}$$
$$\begin{aligned}
L_v \psi'' &= \\
&= \sum_{j=1}^p A_{s\bar{\beta}_p}^{\alpha} \psi_{\alpha_1, \dots, \widehat{\alpha}_j, \dots, \alpha_p, \alpha, \bar{B}_{n-p}} dz^{\alpha_1} \wedge \dots \wedge \widehat{dz^{\alpha_j}} \wedge \dots \wedge dz^{\alpha_p} \wedge dz^{\bar{\beta}_p} \wedge \dots \wedge dz^{\bar{\beta}_n} \\
&= (-1)^{p-1} \sum_{j=1}^p A_{s\bar{\beta}_p}^{\alpha} \psi_{\alpha, \alpha_1, \dots, \alpha_{p-1}, \bar{\beta}_{p+1}, \dots, \bar{\beta}_n} dz^{\alpha_1} \wedge \dots \wedge \dots \wedge dz^{\alpha_{p-1}} \wedge dz^{\bar{\beta}_p} \wedge \dots \wedge dz^{\bar{\beta}_n}.
\end{aligned}$$
☐

Proof of (73). The claim follows in a similar way from (71). \square

The situation is not quite symmetric because of Lemma 2, which implies that the contraction of the global $(0, n-p)$ -form ψ with values in $\Omega_{\mathcal{X}/S}^p(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})$ is well-defined. Like in Definition 6 we have a cup product on the total space (restricted to the fibers).

$$\begin{aligned} \bar{v} \cup \psi &= (\partial_{\bar{s}} + a_{\bar{s}}^{\bar{\beta}} \partial_{\bar{\beta}}) \cup \psi \\ &= \psi_{A_p, \bar{s}, \bar{\beta}_{p+1}, \dots, \bar{\beta}_{n-1}} + a_{\bar{s}}^{\bar{\beta}} \psi_{A_p, \bar{\beta}, \bar{\beta}_{p+1}, \dots, \bar{\beta}_{n-1}} \end{aligned}$$

Lemma 11.

$$(74) \quad L_{\bar{v}} \psi' = (-1)^p \bar{\partial}(\bar{v} \cup \psi).$$

Proof. The proof follows from the fact that, according to Lemma 2, ψ is given by a $\bar{\partial}$ -closed $(0, n-p)$ -form on the total space \mathcal{X} with values in a certain holomorphic vector bundle. \square

We will need that the forms ψ on the fibers are *also harmonic with respect to ∂* (which was defined as the connection of the line bundle $\mathcal{K}_{\mathcal{X}/S}^{\otimes m}$). The curvature of $(\mathcal{K}_{\mathcal{X}/S}, g^{-1})$ being equal to $-\omega_{\mathcal{X}}$ implies the following Lemma.

Lemma 12.

$$(75) \quad \sqrt{-1}[\bar{\partial}, \partial] = -mL_{\mathcal{X}},$$

where $L_{\mathcal{X}}$ denotes the multiplication with $\omega_{\mathcal{X}}$. The analogous formula holds on all fibers \mathcal{X}_s .

Now:

Lemma 13. *The following equation holds on $\mathcal{A}^{(p,q)}(\mathcal{K}_{\mathcal{X}_s}^{\otimes m})$.*

$$(76) \quad \square_{\partial} = \square_{\bar{\partial}} + m \cdot (n-p-q) \cdot id.$$

In particular, the harmonic forms $\psi \in \mathcal{A}^{(p,n-p)}(\mathcal{K}_{\mathcal{X}_s})$ are also harmonic with respect to ∂ .

Proof. We use the formulas

$$\sqrt{-1} \bar{\partial}^* = [\Lambda, \partial] \quad \text{and} \quad -\sqrt{-1} \partial^* = [\Lambda, \bar{\partial}],$$

where Λ denotes the adjoint operator to L . Then

$$\square_{\partial} - \square_{\bar{\partial}} = [\Lambda, \sqrt{-1}(\partial \bar{\partial} + \bar{\partial} \partial)] = [\Lambda, m \cdot \omega_{\mathcal{X}}] = m \cdot (n-p-q) \cdot id.$$

\square

Now we compute the curvature in the following way. Because of (74)

$$\langle \psi^k, L_{\bar{v}}(\psi^\ell)' \rangle = 0$$

holds for all $s \in S$ so that by Lemma 9

$$\begin{aligned} \frac{\partial}{\partial s} H^{\bar{\ell}k} &= \langle L_v \psi^k, \psi^\ell \rangle + \langle \psi^k, L_{\bar{v}} \psi^\ell \rangle = \langle (L_v \psi^k)', \psi^\ell \rangle + \langle \psi^k, (L_{\bar{v}} \psi^\ell)' \rangle \\ &= \langle (L_v \psi^k)', \psi^\ell \rangle. \end{aligned}$$

Later in the computation we will use normal coordinates (of the second kind) at a given point $s_0 \in S$. The condition $(\partial/\partial s)H^{\bar{\ell}k}|_{s_0} = 0$ for all k, ℓ means that for $s = s_0$ the harmonic projection

$$(77) \quad H((L_v \psi^k)') = 0$$

vanishes for all k .

In order to compute the second order derivative of $H^{\bar{\ell}k}$ we begin with

$$(78) \quad \frac{\partial}{\partial s} H^{\bar{\ell}k} = \langle L_v \psi^k, \psi^\ell \rangle.$$

which contains both $(L_v \psi^k)'$ and $(L_v \psi^k)''$. Now

$$\begin{aligned} \partial_{\bar{s}} \partial_s \langle \psi^k, \psi^\ell \rangle &= \langle L_{\bar{v}} L_v \psi^k, \psi^\ell \rangle + \langle L_v \psi^k, L_v \psi^\ell \rangle \\ &= \langle L_{[\bar{v}, v]} \psi^k, \psi^\ell \rangle + \langle L_v L_{\bar{v}} \psi^k, \psi^\ell \rangle + \langle L_v \psi^k, L_v \psi^\ell \rangle \\ &= \langle L_{[\bar{v}, v]} \psi^k, \psi^\ell \rangle + \partial_s \langle L_{\bar{v}} \psi^k, \psi^\ell \rangle - \langle L_{\bar{v}} \psi^k, L_{\bar{v}} \psi^\ell \rangle + \langle L_v \psi^k, L_v \psi^\ell \rangle \end{aligned}$$

We just saw that $\langle L_{\bar{v}} \psi^k, \psi^\ell \rangle \equiv 0$. Hence for all $s \in S$

$$(79) \quad \partial_{\bar{s}} \partial_s \langle \psi^k, \psi^\ell \rangle = \langle L_{[\bar{v}, v]} \psi^k, \psi^\ell \rangle - \langle L_{\bar{v}} \psi^k, L_{\bar{v}} \psi^\ell \rangle + \langle L_v \psi^k, L_v \psi^\ell \rangle$$

Since we are computing Lie-derivatives of n -forms (with values in some line bundle) we obtain

$$\langle L_v \psi^k, L_v \psi^\ell \rangle = \langle (L_v \psi^k)', (L_v \psi^\ell)' \rangle - \langle (L_v \psi^k)'', (L_v \psi^\ell)'' \rangle,$$

and

$$\langle L_{\bar{v}} \psi^k, L_{\bar{v}} \psi^\ell \rangle = \langle (L_{\bar{v}} \psi^k)', (L_{\bar{v}} \psi^\ell)' \rangle - \langle (L_{\bar{v}} \psi^k)'', (L_{\bar{v}} \psi^\ell)'' \rangle.$$

Lemma 14. *Restricted to the fibers \mathcal{X}_s the following equation holds for $L_{[\bar{v}, v]}$ applied to $\mathcal{K}_{\mathcal{X}/S}^{\otimes m}$ -valued functions and differential forms resp.*

$$(80) \quad L_{[\bar{v}, v]} = \left[-\varphi^\alpha \partial_\alpha + \varphi^{\bar{\beta}} \partial_{\bar{\beta}}, \lrcorner \right] - m \cdot \varphi \cdot id$$

Proof. We first compute the vector field $[\bar{v}, v]$ on the fibers:

$$\begin{aligned} [\bar{v}, v] &= [\partial_{\bar{s}} + a_{\bar{s}}^{\bar{\beta}} \partial_{\bar{\beta}}, \partial_s + a_s^\alpha \partial_\alpha] \\ &= \left(\partial_{\bar{s}}(a_s^\alpha) + a_{\bar{s}}^{\bar{\beta}} a_{s|\bar{\beta}}^\alpha \right) \partial_\alpha - \left(\partial_s(a_{\bar{s}}^{\bar{\beta}}) + a_s^\alpha a_{s|\alpha}^{\bar{\beta}} \right) \partial_{\bar{\beta}}. \end{aligned}$$

Now

$$\begin{aligned}\partial_{\bar{s}}(a_s^\alpha) &= -\partial_{\bar{s}}(g^{\bar{\beta}\alpha}g_{s\bar{\beta}}) = g^{\bar{\beta}\sigma}g_{\sigma\bar{s}}\bar{g}^{\bar{\tau}\alpha}g_{s\bar{\beta}} - g^{\bar{\beta}\alpha}g_{s\bar{\beta}}\bar{s} \\ &= g^{\bar{\beta}\sigma}a_{\bar{s}\sigma;\bar{\tau}}\bar{g}^{\bar{\tau}\alpha}a_{s\bar{\beta}} - g^{\bar{\beta}\alpha}g_{s\bar{s};\bar{\beta}}.\end{aligned}$$

Equation (16) implies that the coefficient of ∂_α is $-\varphi^\alpha$. In the same way the coefficient of $\partial_{\bar{\beta}}$ is computed.

Next, we compute the contribution of the connection on $\mathcal{K}_{\mathcal{X}/S}^{\otimes m}$ which we denote by $[\bar{v}, v]_{\mathcal{K}_{\mathcal{X}/S}^{\otimes m}}$. We use (75):

$$\begin{aligned}[\partial_{\bar{s}} + a_{\bar{s}}^{\bar{\beta}}\partial_{\bar{\beta}}, \partial_s + a_s^\alpha\partial_\alpha]_{\mathcal{K}_{\mathcal{X}/S}^{\otimes m}} \\ = -m \left(g_{s\bar{s}} + a_{\bar{s}}^{\bar{\beta}}g_{s\bar{\beta}} + a_s^\alpha g_{\alpha\bar{s}} + a_{\bar{s}}^{\bar{\beta}}a_s^\alpha g_{\alpha\bar{\beta}} \right) = -m\varphi.\end{aligned}$$

□

Lemma 15.

(81)

$$\langle L_{[\bar{v}, v]} \psi^k, \psi^\ell \rangle = -m \langle \varphi \psi^k, \psi^\ell \rangle = -m \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_s \cdot A_{\bar{s}}) \psi^k \bar{\psi}^\ell g \, dV.$$

Proof. The ∂ -closedness of the ψ^k can be read as

$$\psi_{;\alpha}^k = \sum_{j=1}^p \psi_{\alpha_1, \dots, \underset{j}{\alpha}, \dots, \alpha_p \bar{B}_{n-p}; \alpha_j}.$$

Hence

$$\begin{aligned}[\varphi^{;\alpha}\partial_\alpha, \psi_{A_p \bar{B}_{n-p}}^k]' &= \varphi^{;\alpha}\psi_{;\alpha}^k + \sum_{j=1}^p \varphi^{;\alpha}_{;\alpha_j} \psi_{\alpha_1, \dots, \underset{j}{\alpha}, \dots, \alpha_p \bar{B}_{n-p}}^k \\ &= \sum_{j=1}^p (\varphi^{;\alpha} \psi_{\alpha_1, \dots, \underset{j}{\alpha}, \dots, \alpha_p \bar{B}_{n-p}}^k)_{;\alpha_j} \\ &= \partial(\varphi^{;\alpha}\partial_\alpha \cup \psi^k).\end{aligned}$$

Now

$$\begin{aligned}\langle [\varphi^{;\alpha}\partial_\alpha, \psi_{A_p \bar{B}_{n-p}}^k], \psi^\ell \rangle &= \langle [\varphi^{;\alpha}\partial_\alpha, \psi_{A_p \bar{B}_{n-p}}^k]', \psi^\ell \rangle \\ &= \langle \partial(\varphi^{;\alpha}\partial_\alpha \cup \psi^k), \psi^\ell \rangle = \langle \varphi^{;\alpha}\partial_\alpha \cup \psi^k, \partial^* \psi^\ell \rangle = 0.\end{aligned}$$

In the same way we get

$$\langle [\varphi^{;\bar{\beta}}\partial_{\bar{\beta}}, \psi_{A_p \bar{B}_{n-p}}^k], \psi^\ell \rangle = 0,$$

and, according to Lemma 14, we are left with the desired term. □

Proposition 16. *In view of (47) and (48) we have*

$$(82) \quad \bar{\partial}(L_v \psi^k)' = \partial(A_s \cup \psi^k)$$

$$(83) \quad \bar{\partial}^*(L_v \psi^k)' = 0$$

$$(84) \quad \partial^*(A_s \cup \psi^k) = 0$$

$$(85) \quad \bar{\partial}^*(L_{\bar{v}} \psi^k)' = \partial^*(A_{\bar{s}} \cup \psi^k)$$

$$(86) \quad \bar{\partial}(L_{\bar{v}} \psi^k)' = 0$$

$$(87) \quad \bar{\partial}^*(A_{\bar{s}} \cup \psi^k) = 0$$

The proof of the above proposition is the technical part of this article and will be given at the end.

Proof of Theorem 6. Again, we may set $i = j = s$ and use normal coordinates at a given point $s_0 \in S$.

We continue with (79) and apply (81). Let G_{∂} and $G_{\bar{\partial}}$ denote the Green's operators on the spaces of differentiable $\mathcal{K}_{\mathcal{X}_s}$ -valued (p, q) -forms on the fibers with respect to \square_{∂} and $\square_{\bar{\partial}}$ resp. We know from Lemma 13 that for $p + q = n$ the Green's operators G_{∂} and $G_{\bar{\partial}}$ coincide.

We compute $\langle (L_v \psi^k)', L_v \psi^{\ell}' \rangle$: Since the harmonic projection $H((L_v \psi^k)') = 0$ vanishes for $s = s_0$, we have

$$(L_v \psi^k)' = G_{\bar{\partial}} \square_{\bar{\partial}} (L_v \psi^k)' = G_{\bar{\partial}} \bar{\partial}^* \bar{\partial} (L_v \psi^k)' = \bar{\partial}^* G_{\bar{\partial}} \partial (A_s \cup \psi^k)$$

by (83) and (82). The form $\bar{\partial}(L_v \psi^k)' = \partial(A_s \cup \psi^k)$ is of type $(p, n-p+1)$ so that by Lemma 13 on the space of such forms $G_{\bar{\partial}} = (\square_{\partial} + m)^{-1}$ holds.

Now

$$\begin{aligned} \langle (L_v \psi^k)', (L_v \psi^{\ell})' \rangle &= \langle \bar{\partial}^* G_{\bar{\partial}} \partial (A_s \cup \psi^k), (L_v \psi^{\ell})' \rangle \\ &= \langle G_{\bar{\partial}} \partial (A_s \cup \psi^k), \partial (A_s \cup \psi^{\ell}) \rangle = \langle (\square_{\partial} + m)^{-1} \partial (A_s \cup \psi^k), \partial (A_s \cup \psi^{\ell}) \rangle \\ &= \langle \partial^* (\square_{\partial} + m)^{-1} \partial (A_s \cup \psi^k), A_s \cup \psi^{\ell} \rangle. \end{aligned}$$

Because of (84)

$$\begin{aligned} \langle (L_v \psi^k)', (L_v \psi^{\ell})' \rangle &= \langle (\square_{\partial} + m)^{-1} \square_{\partial} (A_s \cup \psi^k), A_s \cup \psi^{\ell} \rangle \\ &= \langle A_s \cup \psi^k, A_s \cup \psi^{\ell} \rangle - m \langle (\square_{\partial} + m)^{-1} (A_s \cup \psi^k), A_s \cup \psi^{\ell} \rangle. \end{aligned}$$

(For $(p-1, n-p+1)$ -forms, we write $\square = \square_{\partial} = \square_{\bar{\partial}}$.) Altogether we have

$$(88) \quad \langle L_v \psi^k, L_v \psi^{\ell} \rangle|_{s_0} = -m \int_{\mathcal{X}_s} (\square + m)^{-1} (A_s \cup \psi^k) \cdot (A_{\bar{s}} \cup \psi^{\bar{\ell}}) g dV.$$

Finally we need to compute $\langle L_{\bar{v}} \psi^k, L_{\bar{v}} \psi^{\ell} \rangle$.

By equation (73) we have $\langle (L_{\bar{v}}\psi^k)'', (L_{\bar{v}}\psi^\ell)'' \rangle = \langle A_{\bar{s}} \cup \psi^k, A_{\bar{s}} \cup \psi^\ell \rangle$. Now Lemma 11 implies that the harmonic projections of the $(L_{\bar{v}}\psi^k)'$ vanish for all parameters s . So

$$\begin{aligned} \langle (L_{\bar{v}}\psi^k)', (L_{\bar{v}}\psi^\ell)' \rangle &= \langle G_{\bar{\partial}} \square_{\bar{\partial}} (L_{\bar{v}}\psi^k)', (L_{\bar{v}}\psi^\ell)' \rangle \\ &\stackrel{(86)}{=} \langle (G_{\bar{\partial}} \bar{\partial} \bar{\partial}^* L_{\bar{v}}\psi^k)', (L_{\bar{v}}\psi^\ell)' \rangle = \langle (G_{\bar{\partial}} \bar{\partial}^* L_{\bar{v}}\psi^k)', \bar{\partial}^* (L_{\bar{v}}\psi^\ell)' \rangle \\ &\stackrel{(85)}{=} \langle G_{\bar{\partial}} \bar{\partial}^* (A_{\bar{s}} \cup \psi^k), \bar{\partial}^* (A_{\bar{s}} \cup \psi^\ell) \rangle. \end{aligned}$$

Now the $(p, n - p - 1)$ -form $\bar{\partial}^* (L_{\bar{v}}\psi^k)' = \bar{\partial}^* (A_{\bar{s}} \cup \psi^k)$ is orthogonal to both the spaces of $\bar{\partial}$ - and ∂ -harmonic forms. On these, by Lemma 13 we have

$$\square_{\bar{\partial}} = \square_{\partial} - m \cdot id.$$

We see that all eigenvalues of \square_{∂} are larger or equal to m for $(p, n - p - 1)$ -forms.

Claim. *Let $\sum_{\nu} \lambda_{\nu} \rho_{\nu}$ be the eigenfunction decomposition of $A_{\bar{s}} \cup \psi^k$. Then all $\lambda_{\nu} > m$ or $\lambda_0 = 0$. In particular $(\square - m)^{-1} (A_{\bar{s}} \cup \psi^k)$ exists.*

In order to verify the claim, we consider $\bar{\partial}^* (A_{\bar{s}} \cup \psi^k) = \sum_{\nu} \bar{\partial}^* (\rho_{\nu})$ with

$$\square_{\partial} \bar{\partial}^* (\rho_{\nu}) = \lambda_{\nu} \bar{\partial}^* (\rho_{\nu}) = \square_{\bar{\partial}} \bar{\partial}^* (\rho_{\nu}) + m \cdot \bar{\partial}^* (\rho_{\nu}).$$

This fact implies that $\sum_{\nu} \bar{\partial}^* (\rho_{\nu})$ is the eigenfunction expansion with respect to $\square_{\bar{\partial}}$ and eigenvalues $\lambda_{\nu} - m \geq 0$ of $\bar{\partial}^* (A_{\bar{s}} \cup \psi^k) = \bar{\partial}^* (L_{\bar{v}}\psi^k)$. The latter is orthogonal to the space of $\bar{\partial}$ -harmonic functions so that $\lambda_{\nu} - m = 0$ does not occur. (The harmonic part of $A_{\bar{v}} \cup \psi^k$ may be present though.) This shows the claim.

Now

$$G_{\bar{\partial}} \bar{\partial}^* (A_{\bar{s}} \cup \psi^k) = (\square_{\partial} - m)^{-1} \bar{\partial}^* (A_{\bar{s}} \cup \psi^k)$$

so that (87) implies

$$\begin{aligned} \langle (L_{\bar{v}}\psi^k)', (L_{\bar{v}}\psi^\ell)' \rangle &= \langle (\square_{\partial} - m)^{-1} \square_{\partial} (A_{\bar{s}} \cup \psi^k), A_{\bar{s}} \cup \psi^\ell \rangle \\ &= \langle A_{\bar{s}} \cup \psi^k, A_{\bar{s}} \cup \psi^\ell \rangle + m \cdot \langle (\square_{\partial} - m)^{-1} (A_{\bar{s}} \cup \psi^k), A_{\bar{s}} \cup \psi^\ell \rangle. \end{aligned}$$

Now (73) yields the final equation (again with $\square_{\bar{\partial}} = \square_{\partial} = \square$ for $(p + 1, n - p - 1)$ -forms)

$$(89) \quad \langle L_{\bar{v}}\psi^k, L_{\bar{v}}\psi^\ell \rangle = m \int_{\mathcal{X}_s} (\square - m)^{-1} (A_{\bar{s}} \cup \psi^k) \cdot (A_s \cup \psi^{\bar{\ell}}) g dV.$$

The main theorem follows from (81), (88), (79), and (89). \square

Proof of Proposition 16. We verify (82): We will need various identities. For simplicity, we drop the superscript k . The tensors below are

meant to be coefficients of alternating forms on the fibers, i.e. skew-symmetrized.

$$(90) \quad \psi_{;s\bar{\beta}_{n+1}} = \psi_{;\bar{\beta}_{n+1}s} - m \cdot g_{s\bar{\beta}_{n+1}} \psi = m \cdot a_{s\bar{\beta}_{n+1}} \psi$$

$$(91) \quad \psi_{;\alpha\bar{\beta}_{n+1}} = \psi_{;\bar{\beta}_{n+1}\alpha} - m \cdot g_{\alpha\bar{\beta}_{n+1}} \psi \\ - \sum_{j=1}^p \psi_{\alpha_1, \dots, \underset{j}{\sigma}, \dots, \alpha_p, \bar{B}_{n-p}} R_{\alpha_j \alpha \bar{\beta}_{n+1}}^\sigma - \sum_{j=p+1}^n \psi_{A_p \bar{\beta}_{p+1}, \dots, \underset{j}{\bar{\tau}}, \dots, \bar{\beta}_n} R_{\bar{\beta}_j \alpha \bar{\beta}_{n+1}}^{\bar{\tau}}$$

$$(92) \quad a_{s;\alpha_j \bar{\beta}_{n+1}}^\alpha = A_{s\bar{\beta}_{n+1};\alpha_j}^\alpha + a_s^\sigma R_{\sigma \alpha_j \bar{\beta}_{n+1}}^\alpha$$

Now, starting from (68) we get, using (90), (91), and (92),

$$\bar{\partial} L_v \psi' = \left(\psi_{;s\bar{\beta}_{n+1}} + A_{s\bar{\beta}_{n+1}}^\alpha \psi_{;\alpha} + a_s^\alpha \psi_{;\alpha \bar{\beta}_{n+1}} + \sum_{j=1}^p a_{s;\alpha_j \bar{\beta}_{n+1}}^\alpha \psi_{\alpha_1, \dots, \alpha_p, \bar{B}_{n-p}} \right. \\ \left. + \sum_{j=1}^p a_{s;\alpha_j}^\alpha \psi_{\alpha_1, \dots, \alpha_p, \bar{B}_{n-p}; \bar{\beta}_{n+1}} \right) dz^{\bar{\beta}_{n+1}} \wedge dz^{A_p} \wedge dz^{\bar{B}_{n-p}} \\ = \left(A_{s\bar{\beta}_{n+1}}^\alpha \psi_{;\alpha} + \sum_{j=1}^p A_{s\bar{\beta}_{n+1};\alpha_j}^\alpha \psi_{\alpha_1, \dots, \alpha_p, \bar{B}_{n-p}} \right) dz^{\bar{\beta}_{n+1}} \wedge dz^{A_p} \wedge dz^{\bar{B}_{n-p}}.$$

Because of the fiberwise ∂ -closedness of ψ this equals

$$\sum_{j=1}^p \left(A_{s\bar{\beta}_{n+1}}^\alpha \psi_{\alpha_1, \dots, \underset{j}{\alpha}, \dots, \alpha_p, \bar{B}_{n-p}} \right)_{;\alpha_j} dz^{\bar{\beta}_{n+1}} \wedge dz^{A_p} \wedge dz^{\bar{B}_{n-p}} \\ = (-1)^n \sum_{j=1}^p \left(A_{s\bar{\beta}_{n+1}}^\alpha \psi_{\alpha, \alpha_2, \dots, \alpha_p, \bar{B}_{n-p}} \right)_{;\alpha_1} dz^{\alpha_1} \wedge dz^{A_{p-1}} \wedge dz^{\bar{\beta}_1} \wedge \dots \wedge dz^{\bar{\beta}_{n+1}} \\ = \partial \left((-1)^n A_{s\bar{\beta}_{n+1}}^\alpha \psi_{\alpha, \alpha_2, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \bar{\beta}_n} dz^{A_{p-1}} \wedge dz^{\bar{B}_{n+1}} \right) = \partial(A_s \cup \psi).$$

This shows (82).

Next, we prove (83). We begin with (70). We first note

$$\partial_s(\Gamma_{\alpha\gamma}^\sigma) = -a_{s;\alpha\gamma}^\sigma$$

which follows in a straightforward way. Now this equation implies

$$\partial_s(\psi_{;\gamma}) = \psi_{;s\gamma} - \sum_{j=1}^p a_{s;\alpha_j \gamma}^\sigma \psi_{\alpha_1, \dots, \underset{j}{\sigma}, \dots, \alpha_p, \bar{B}_{n-p}}$$

so that (with $g^{\bar{\beta}_n \gamma} \psi_{;\gamma} = 0$ and $\partial_s g^{\bar{\beta}_n \gamma} = g^{\bar{\beta}_n \sigma} a_{s;\sigma}^\gamma$)

$$(93) \quad g^{\bar{\beta}_n \gamma} \psi_{;s\gamma} = -\psi_{;\gamma} g^{\bar{\beta}_n \sigma} a_{s;\sigma}^\gamma + \sum_{j=1}^p g^{\bar{\beta}_n \gamma} a_{s;\alpha_j \gamma}^\sigma \psi_{\alpha_1, \dots, \underset{j}{\sigma}, \dots, \alpha_p \bar{B}_{n-p}}$$

follows. Next, since fiberwise ψ is $\bar{\partial}^*$ -closed,

$$(94) \quad g^{\bar{\beta}_n \gamma} (a_s^\alpha \psi_{;\alpha})_{;\gamma} = g^{\bar{\beta}_n \gamma} a_{s;\gamma}^\alpha \psi_{;\alpha},$$

and with the same argument

$$(95) \quad g^{\bar{\beta}_n \gamma} \left(\sum_{j=1}^p a_{s;\alpha_j}^\sigma \psi_{\alpha_1, \dots, \underset{j}{\sigma}, \dots, \alpha_p \bar{B}_{n-p}} \right)_{;\gamma} = g^{\bar{\beta}_n \gamma} \sum_{j=1}^p a_{s;\alpha_j \gamma}^\sigma \psi_{\alpha_1, \dots, \underset{j}{\sigma}, \dots, \alpha_p \bar{B}_{n-p}}.$$

Now $\bar{\partial}^*(L_v \psi') = 0$ follows from (93), (94), and (95).

We come to the ∂^* -closedness (84) of $A_s \cup \psi$. We need to show that

$$(A_{s\bar{\beta}_{n+1}}^\alpha \psi_{\alpha, \alpha_2, \dots, \alpha_p, \bar{\beta}_{p+1}, \dots, \beta_n})_{;\bar{\delta}} g^{\bar{\delta} \alpha_p}$$

vanishes. Since $\partial^* \psi = 0$ the above quantity equals

$$A_{s\bar{\beta}_{n+1};\bar{\delta}}^\alpha \psi_{\alpha, \alpha_2, \dots, \alpha_p, \bar{B}_{n-p}} g^{\bar{\delta} \alpha_p}.$$

Because of the $\bar{\partial}$ -closedness of A_s this equals

$$(A_s^{\alpha \alpha_p})_{;\bar{\beta}_{n+1}} \psi_{\alpha, \alpha_2, \dots, \alpha_p, \bar{B}_{n-p}}.$$

However,

$$A_s^{\alpha \alpha_p} = A_s^{\alpha_p \alpha}$$

whereas ψ is skew-symmetric so that also this contribution vanishes.

The proof of (85), (86), and (87) is similar, we remark that (86) also follows from Lemma 11. \square

Acknowledgements. This work was begun during a visit to Harvard University. The author would like to thank Professor Yum-Tong Siu for his cordial hospitality and many discussions about the hyperbolicity and curvature. His thanks also go to Robert Berman, Bo Berndtsson, Indranil Biswas, Jeff Cheeger, Jean-Pierre Demailly, Gordon Heier, Stefan Kebekus, Janos Kollár, Sándor Kovács, Mihai Paun, Thomas Peternell, and Stefano Trapani for discussions and remarks. Also, he would like to thank the referees for comments and remarks.

He would have liked to thank Eckart Viehweg also in writing for discussing details of the analytic proof of the quasi-projectivity of \mathcal{M}_{can} – it is too late now.

REFERENCES

- [A-P] Abate, M., Patrizio, G.: Holomorphic curvature of Finsler metrics and complex geodesics. *J. Geom. Anal.* **6**, 341–363 (1996).
- [AR] Artin, M.: Algebraization of formal moduli, *Global Analysis, Papers in Honor of K. Kodaira* 21–71 (1969), and Algebraization of formal moduli II, Existence of modifications. *Ann. Math.* **91**, 88–135 (1970).
- [AU] Aubin, T.: Equation du type de Monge-Ampère sur les variétés Kähleriennes compactes, *C. R. Acad. Sci. Paris* **283**, 119–121 (1976) / *Bull. Sci. Math.* **102**, 63–95 (1978).
- [B-V] Bedulev E., Viehweg, E.: Shafarevich conjecture for surfaces of general type over function fields. *Invent. Math.* **139**, 603–615 (2000).
- [B] Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations. *Ann. Math* **169**, 531–560 (2009).
- [B1] Berndtsson, B.: Strict and nonstrict positivity of direct image bundles, preprint, personal communication.
- [B-P] Berndtsson, B., Paun, M.: Bergman kernels and the pseudoeffectivity of relative canonical bundles. *Duke Math. J.* **145**, 341–378 (2008).
- [BI] Bismut, J.-M.: Métriques de Quillen et dégénérescence de variétés kähleriennes. *C. R. Acad. Sci. Paris Sér. I. Math.* **319**, 1287–1291 (1994).
- [BGS] Bismut, J.-M.; Gillet, H.; Soulé, Ch.: Analytic torsion and holomorphic determinant bundles I, II, III. *Comm. Math. Phys.* **115**, 49–78, 79–126, 301–351 (1988).
- [BO] Bombieri, E.: Algebraic values of meromorphic maps, *Invent. Math.* **10**, 267–287 (1970) and Addendum, *Invent. Math.* **11** 163–166 (1970).
- [BEGZ] Boucksom, S., Eyssidieux, P., Guedj, V., Zeriahi, A.: Monge-Ampère equations in big cohomology classes. *Acta Math.* **205**, 199–262 (2010).
- [C-Y] Cheeger, J., Yau, S.-T.: A lower bound for the heat kernel. *Commun. Pure Appl. Math.* **34**, 465–480 (1981).
- [CH-Y] Cheng, S.Y., Yau, S.T.: On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation. *Comm. Pure Appl. Math.* **33**, 507–544 (1980).
- [DE] Demailly, J.P.: Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. *Lecture notes Santa Cruz* (1995). Electronically published.
- [DE1] Demailly, J.P.: *Complex Analytic and Differential Geometry*, Grenoble 1997.
- [FO] Forster, O.: Zur Theorie der Steinschen Algebren und Moduln. *Math. Z.* **97**, 376–405 (1967).
- [F-S] Fujiki, A., Schumacher, G.: The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics. *Publ. Res. Inst. Math. Sci.* **26**, 101–183 (1990).
- [G-R] Grauert, H., Remmert, R.: Plurisubharmonische Funktionen in komplexen Räumen. *Math. Z.* **65**, 175–194 (1956).
- [G-RZ] Grauert, H., Reckziegel, H.: Hermitesche Metriken und normale Familien holomorpher Abbildungen. *Math. Z.* **89**, 108–125 (1965).
- [HOU] Houzel, Ch.: Géométrie analytique locale, II. Théorie des morphismes finis. *Séminaire Cartan*, 13e année, 1969/61, no **19**.

- [KE-KV] Kebekus, S., Kovács, S.: Families of canonically polarized varieties over surfaces. *Invent. Math.* **172**, 657–682 (2008).
- [K1] Kobayashi, Sh.: *Hyperbolic complex spaces*. Grundlehren der Mathematischen Wissenschaften. **318**. Berlin: Springer, (1998).
- [K2] Kobayashi, R.: Kähler-Einstein metric on an open algebraic manifold. *Osaka J. Math.* **21**, 399–418 (1984).
- [KO1] Kollár, J.: Projectivity of complete moduli, *J. Differ. Geom.* **32** (1990), 235–268.
- [KO2] Kollár, J.: Non-quasi-projective moduli spaces. *Ann. Math.* **164** (2006), 1077–1096.
- [KO-KV] Kollár, J., Kovács, S.: Log canonical singularities are Du Bois. *J. Am. Math. Soc.* **23**, 791–813 (2010).
- [K-M] Kollár, J., Mori, S.: *Birational Geometry of Algebraic Varieties*. Cambridge University Press 1998. Translated from Souyuuiri Kikagaku published by Iwanami Shoten, Publishers, Tokyo, 1998.
- [KV1] Kovács, S.: Smooth families over rational and elliptic curves. *J. Alg. Geom.* **5**, 369–385 (1996).
- [KV2] Kovács, S.: On the minimal number of singular fibres in a family of surfaces of general type. *J. Reine Angew. Math.* **487**, 171–177 (1997).
- [KV3] Kovács, Algebraic hyperbolicity of fine moduli spaces, *J. Alg. Geom.* **9**, (2000) 165–174.
- [M] Migliorini, L.: A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial. *J. Alg. Geom.* **4**, 353–361 (1995).
- [M-T] Mourougane, Ch., Takayama, S.: Hodge metrics and the curvature of higher direct images. *Ann. Sci. Éc. Norm. Supér. (4)* **41**, 905–924 (2008).
- [L-S-Y] Liu, K., Sun, X., Yau, S.T.: Good geometry on the curve moduli. *Publ. Res. Inst. Math. Sci.* **44**, 699–724 (2008).
- [SCH1] Schumacher, G.: Harmonic maps of the moduli space of compact Riemann surfaces. *Math. Ann.* **275**, 455–466 (1986).
- [SCH2] Schumacher, G.: The curvature of the Petersson-Weil metric on the moduli space of Kähler-Einstein manifolds. Ancona, V. (ed.) et al., *Complex analysis and geometry*. New York: Plenum Press. The University Series in Mathematics. 339–354 (1993).
- [SCH3] Schumacher, G.: Asymptotics of Kähler-Einstein metrics on quasi-projective manifolds and an extension theorem on holomorphic maps. *Math. Ann.* **311**, 631–645 (1998).
- [SCH4] Schumacher, G.: Positivity of relative canonical bundles for families of canonically polarized manifolds, arXiv:0808.3259v2
- [SCH5] Schumacher, G.: Curvature of higher direct images and applications, arXiv:1002.4858
- [S-T] Schumacher, G., Tsuji, H.: Quasi-projectivity of moduli spaces of polarized varieties. *Ann. Math.* **159**, 597–639 (2004).
- [SHI] Shiffman, B.: Extension of positive line bundles and meromorphic maps. *Invent. math.* **15**, 332–347 (1972).
- [SIU1] Siu, Y.-T.: Absolute gap-sheaves and extensions of coherent analytic sheaves. *Trans. Am. Math. Soc.* **141**, 361–376 (1969).

- [SIU2] Siu, Y.-T.: Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class. Contributions to several complex variables, Hon. W. Stoll, Proc. Conf. Complex Analysis, Notre Dame/Indiana 1984, Aspects Math. E9, 261–298 (1986).
- [SIU3] Siu, Y.-T. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. Invent. Math. **27**, 53–156 (1974).
- [SK] Skoda, H.: Sous-ensembles analytiques d'ordre fini ou infini dans \mathbb{C}^n . Bull. Soc. Math. France **100**, 353–408 (1972).
- [ST] Sturm, K.Th.: Heat kernel bounds on manifolds. Math. Ann. **292** 149–162 (1992).
- [TR] Tromba A.J.: On a natural affine connection on the space of almost complex structures and the curvature of the Teichmüller space with respect to its Weil-Petersson metric. Manuscr. Math. **56** (1986) 475–497.
- [VA1] Varouchas, J.: Stabilité de la classe des variétés Kähleriennes par certains morphismes propres. Invent. Math. **77**, 117–127 (1984).
- [VA2] Varouchas, J.: Kähler spaces and proper open morphisms. Math. Ann. **283**, 13–52 (1989).
- [V] Viehweg, E.: Weak positivity and stability of certain Hilbert points I,II,III, Invent. Math. **96**, 639–669 (1989). Invent. Math. **101** 191–223 (1990), Invent. Math. **101** 521–543 (1990).
- [V1] Viehweg, E.: Quasi-projective moduli for polarized manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. **30**. Berlin: Springer-Verlag, 1995.
- [V2] Viehweg, E.: Compactifications of smooth families and of moduli spaces of polarized manifolds. Ann. of Math. **172**, 809–910 (2010).
- [V-Z1] Viehweg, E.; Zuo, K.: On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds. Duke Math. J. **118**, 103–150 (2003).
- [V-Z2] Viehweg, E.; Zuo, K.: On the isotriviality of families of projective manifolds over curves, J. Algebr. Geom. **10**, 781–799 (2001).
- [WO] Wolpert, S.: Chern forms and the Riemann tensor for the moduli space of curves. Invent. Math. **85**, 119–145 (1986).
- [Y] Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, Commun. Pure Appl. Math. **31**, 339–411 (1978).
- [Yo] Yoshikawa, K.-I.: On the singularity of Quillen metrics. Math. Ann. **337** (2007), 61–89.
- [YO] Yosida, K.: On the existence of the resolvent kernel for elliptic differential operator in a compact Riemann space. Nagoya Math. J. **4** (1952) 63–72.

FACHBEREICH MATHEMATIK UND INFORMATIK, PHILIPPS-UNIVERSITÄT MARBURG, LAHNBERGE, HANS-MEERWEIN-STRASSE, D-35032 MARBURG, GERMANY
E-mail address: `schumac@mathematik.uni-marburg.de`