ON THE DIAMETER OF PLANE CURVES.

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Recently, differential geometric properties of embedded projective varieties have gained increasing interest. In this note, we consider plane algebraic curves equipped with the Fubini-Study metric from ${}_2(\ )$ and give an estimate for the diameter in terms of the degree, initiated by a paper of F. A. Bogomolov [2]. In particular this paper implied that contrary to a general belief the diameter is not bounded from above – a result, which was later extended by N. A'Campo [1]. The curvature had already been explicitly computed by L. Ness [5]. Her results show the existence of areas of negative curvature and that the curvature is not bounded from below in the family of all embedded algebraic curves of a fixed degree. Using curvature when proving an estimate for the diameter requires a careful consideration of these areas. Bogomolov pointed out that the best estimate to expect is logarithmic, since Gromov's Betti number theorem implies a lower estimate for the diameter in the following sense: Under the restriction to curves, whose curvature is bounded from below by a number $-\kappa^2$, the diameter is bounded from below by $C\log(d)/\kappa$, where C denotes a positive constant. We use rather explicit methods to show the following result:

Theorem 1. *The diameter of a plane algebraic curve $C \subset {}_2(\ )$ of degree d, equipped with the Fubini-Study metric is bounded by*

$$(2d^2 - 2d + 1)(4d^2 + 1) \cdot \pi \ .$$

The theorem has immediate consequences for the diameter of complete intersections in ${}_n$. Our estimate seems to be also of interest concerning results of Y. Yomdin [6] and M. Briskin–Y. Yomdin [3] in the area of polynomial control problems.

Key words and phrases. Diameter, Plane Curve, Fubini-Study Metric.

1. Preparations

Our estimates will be based upon projections onto projective lines. Let $(x_0 : x_1 : x_2)$ denote homogeneous coordinates on $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{C})$. For $j = 0, 1, 2$ and $\{j, k, \ell\} = \{0, 1, 2\}$ we consider the lines L_j defined as zero sets $V(x_j)$ and the points P_j defined as zero sets $V(x_k, x_\ell)$. Furthermore, we have the canonical projections $\pi_j : \mathbb{P}_2\backslash P_j \to L_j$, defined by setting the j-th coordinate equal to zero. The projective plane $\mathbb{P}_2$ and the lines L_j resp. are equipped with the Fubini-Study forms $\omega_{\mathbb{P}_2}$ and $\omega_{\mathbb{P}_1}$ resp.

Lemma 1. *Let $\gamma : [0, 1] \longrightarrow \mathbb{P}_2\backslash\{P_0, P_1, P_2\}$ be a curve of class C^∞. Then the length is estimated by*

$$L(\gamma) \leq \sum_{j=0}^{2} L(\pi_j \circ \gamma),$$

or equivalently

$$\omega_{\mathbb{P}_2} \leq \sum_{j=0}^{2} \pi_j^*(\omega_{\mathbb{P}_1}).$$

on $\mathbb{P}_2\backslash \bigcup L_j$.

Proof. With respect to inhomogeneous coordinates $(1 : x_1 : x_2)$ we have

$$ds^2_{\mathbb{P}_2} = \frac{|dx_1|^2 + |dx_2|^2 + |x_1 dx_2 - x_2 dx_1|^2}{(1 + |x_1|^2 + |x_2|^2)^2}$$

$$\leq \frac{|dx_1|^2}{(1 + |x|_1^2)^2} + \frac{|dx_2|^2}{(1 + |x_2|^2)^2} + \frac{|d(\frac{x_2}{x_1})|^2}{(1 + |\frac{x_2}{x_1}|^2)^2}.$$

□

Obviously, it is sufficient to show an upper bound in terms of the degree only for a generic class of embedded curves: Let $\mathcal{C}_d$ be the set of all smooth plane curves C of degree d such that

(i) $P_j \notin C$ for $j = 0, 1, 2$
(ii) $\pi_2|C : C \longrightarrow L_2$ is a simple branched covering.

We estimate the length of a particular class of real algebraic curves. Let $C \in \mathcal{C}_d$, and let $L \subset L_2$ be a closed geodesic. We denote its preimage under $\pi_2|C : C \longrightarrow L_2$ by C .

Lemma 2. *The curves $\pi_0(C\ ) \subset L_0$ and $\pi_1(C\ ) \subset L_1$ are real algebraic of degree at most d^2.*

Proof. Let C be the zero set $V(F)$, where $F = F(x_0, x_1, x_2)$ is an irreducible homogeneous polynomial of degree d.

We first show the claim for the line $L = \overline{\{(t : 1 : 0); t \in \ \}} \subset L_2$ and $\pi_0(C\ )$ say.

Since C is irreducible of degree greater than one, it intersects any fiber of the map π_0 in a discrete set of points. Therefore, we restrict ourselves to the affine set $U = \ _2\backslash L_1 = \pi_0^{-1}(L_0\backslash\{P_2\})$. Then $\pi_0(C\ )$ is the closure of $\pi_0(U \cap C\ )$. Now $(0 : 1 : x_2) \in \pi_0(U \cap C\ )$, if and only if there exists $t \in \ $ such that $F(t, 1, x_2) = 0$.

Classical elimination theory yields the following. Denote by $R(x_2, \overline{x_2})$ the resultant of $\mathrm{Re}(F(t, 1, x_2))$ and $\mathrm{Im}(F(t, 1, x_2))$ with respect to t. We use the fact that for any two polynomials $g(y, z), h(y, z)$ of degree m and n resp. the resultant $R_{g,h}(z)$ (where y is eliminated) is a polynomial of degree at most $m \cdot n$. Hence $\pi_0(C\ )$ is real algebraic of degree at most d^2.

Let $L \subset L_2$ be an arbitrary geodesic. Then L is the closure of the set of all $(a+b\cdot t : c+d\cdot t : 0) \in L_2$; $t \in \ $, where the a, b, c, d determine an element of $SU(2)$. As above the resultant of $\mathrm{Re}(F(a+b\cdot t, c+d\cdot t, x_2))$ and $\mathrm{Im}(F(a+b\cdot t, c+d\cdot t, x_2))$ is a polynomial of degree at most d^2 in x_2 and $\overline{x_2}$. ☐

Lemma 3. *Let $C \subset \ _1(\ )$ be a real algebraic curve of degree δ. Then the length of C with respect to the Fubini-Study metric of $_1(\ )$ is at most $2\pi\delta$.*

Proof. We assume that $C \subset \mathbb{C} \subset \mathbb{P}_1$ is connected and chose a piecewise smooth parametrization $\gamma : [0,1] \longrightarrow C$; $\gamma(t) = u(t) + iv(t)$. Then

$$L(\gamma) = \int_0^1 \frac{(|u'|^2+|v'|^2)^{1/2}}{1+u^2+v^2}dt \le \int_0^1 \frac{|u'|}{1+u^2}dt + \int_0^1 \frac{|v'|}{1+v^2}dt$$

Since the projections $z \mapsto u$ and $z \mapsto v$, restricted to C have at most a number of δ sheets, the above integral is at most

$$2\delta \int_{-\infty}^{+\infty} \frac{du}{1+u^2} = 2\delta\pi.$$

□

2. Proof of the theorem

In the sequel, we describe a generic type of branching. Let $C \subset \mathbb{P}_2 \backslash \{P_0, P_1, P_2\}$ and denote by $\pi : C \longrightarrow \mathbb{P}_1$ the restriction of $\pi_2 : \mathbb{P}_2 \backslash P_2 \longrightarrow L_2$ to C. Again π must have only simple generic branch points B_j, and we impose in addition to (i) and (ii) that

(iii) the images $Q_j = \pi(B_j)$ of the branch points B_j are distinct, where $j = 1, \ldots, b$, with $b = d^2 - d$, and no three of these are contained in a closed geodesic.

Next we choose a point R in $\mathbb{P}_1 \backslash \bigcup_{j<k} L_{jk}$, where L_{jk} is the closed geodesic through Q_j and Q_k. Let S_j be the segment of the real projective line from R to $Q_j, j = 1, \ldots, b$, and $S = \cup S_j$. The complement $\mathbb{P}_1 \backslash S$ is simply connected and $\pi^{-1}(\mathbb{P}_1 \backslash S)$ decomposes into d isomorphic copies E_ν; $\nu = 1, \ldots, d$, where we set $E_\nu = \mathbb{P}_1^{(\nu)} \backslash \cup S_j^{(\nu)}$, with copies $\mathbb{P}_1^{(\nu)}$ and $S_j^{(\nu)}$ resp. of $\mathbb{P}_1$ and S_j resp. Let $R^{(\nu)} \in \mathbb{P}_1^{(\nu)}$ correspond to R.

Any branch point B_j is contained in the closure $\overline{E_\nu}$ of E_ν in C for exactly two values of ν. For all j with $B_j \notin \overline{E_\nu}$ we fill $S_j^{(\nu)} \backslash R^{(\nu)}$ into E_ν and obtain $\tilde{E}_\nu$, which is a copy of $\mathbb{P}_1$ with a certain number of segments removed, emanating from one point. We count boundary points of $\tilde{E}_\nu$ twice, except for the endpoints of line segments. The domain $\tilde{E}_\nu$ with boundary added is denoted by $\hat{E}_\nu$. Now C is obtained from $\cup \hat{E}_\nu$

by means of the usual gluing process. There is a natural projection $\rho_\nu : \hat{E}_\nu \longrightarrow {}_1$. We chose arbitrary sheets $\hat{E}_1$ and $\hat{E}_2$ say and points $R_j \in \hat{E}_j$; $j = 1, 2$ with $\rho_j(R_j) = R$; $j = 1, 2$. We want to connect the images of R_1 and R_2 in C by the images of line segments in the boundaries of $\hat{E}$, where sheets are switched at branch points. We give the construction. Let $S^{(\nu)}_{j,1}, S^{(\nu)}_{j,2} \subset \partial \hat{E}_\nu$ correspond to $S^{(\nu)}_j \subset {}^{(\nu)}_1$. We follow one of these segments from R_1 in $\hat{E}_1$ to the adjacent branch point. Either we switch sheets at the branch point, or go back on the opposite edge of the same $\hat{E}_\nu$. We follow the next edge on the present sheet to the next branch point and switch again sheets, or not, keeping the orientation, i.e. in a way such that the set $\hat{E}_\nu$ is always on the same side of the edge. After circulating a certain number of times we arrive at R_2. Now we need to visit any branch point at most once: otherwise we get a closed loop, which we can eliminate from our path. Therefore the total number of segments does not exceed twice the number of branch points $2b$.

To conclude the proof, it is sufficient to show the claim for generic $C \subset {}_2$ with $\pi : C \longrightarrow {}_1$ as above. Let two points of C be given. By a continuity argument one of these can play the role of R_2, whereas the other point is contained in the image of some other sheet $\hat{E}_\nu$ say $\hat{E}_2$ and can be connected with some R_2 located over R_1 on the corresponding boundary component. This amounts to a total of at most $2b + 1$ segments. According to Lemma 2 and Lemma 3 the length of each segment is at most $(4d^2 + 1)\pi$, which shows that the diameter is bounded from above by $(2b+1)(4d^2+1)\cdot\pi = (2d^2 - 2d + 1)(4d^2 + 1)\pi$.

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