GENERALIZED PETERSSON-WEIL METRIC ON THE DOUADY SPACE OF EMBEDDED MANIFOLDS

INDRANIL BISWAS AND GEORG SCHUMACHER

1. Introduction

An adapted generalized Petersson-Weil metric is the tool of choice to investigate the Kähler geometry of various moduli spaces. Like in the classical case of moduli of compact (or punctured) Riemann surfaces, a natural L^2-inner product of distinguished representatives of Kodaira-Spencer classes defines a hermitian structure on the moduli space and eventually yields a Kähler structure. In various higher dimensional settings a generalized Petersson-Weil form has been constructed. Surprisingly, a generalized Petersson-Weil metric on the Douady space of submanifolds $X \hookrightarrow Z$ of a fixed Kähler manifold (Z, ω_Z) had not yet been constructed. In this note, we give a definition, which also applies to the relative Douady space in the sense of Pourcin [P]. The underlying hermitian inner product in the absolute case is an L^2-product of holomorphic sections of the normal bundle $\mathcal{N}_{X/Z}$. These are represented by differentiable vector fields orthogonal to the reference manifold X with respect to ω_Z. The generalized Petersson-Weil form satisfies a fiber-integral formula, which relates our construction to the results of Varouchas [V], and a construction for embedded Riemann surfaces relying on classical Teichmüller theory was undertaken recently by Maarouf [M]. The moduli space of divisors (divided by the automorphism group of the ambient space) occurred within the theory of moduli of framed manifolds [SCH]. As an application of the fiber integral formula for projective varieties the generalized Petersson-Weil form is understood as the curvature form of a determinant line bundle equipped with a Quillen metric.

2. *The relative Douady space*

Let C be a fixed complex space, and $An_{/C}$ the category of complex spaces over C whose objects are holomorphic maps $S \to C$ of complex spaces. We consider a fixed complex space $\mathcal{Z} \to C$ such that the fibers $(\mathcal{Z}_c, \mathcal{O}_{\mathcal{Z}_c})$, where $c \in C$, are complex manifolds. The existence of a relative Douady space of all compact complex subspaces follows from a theorem of G. Pourcin [P], which solves a global moduli problem. In the sequel we will compute its tangent space in an intrinsic way as space of infinitesimal deformations. We state the universal property.

For any object S in $An_{/C}$ we set $\mathcal{Z}_S := \mathcal{Z} \times_{\mathcal{C}} \mathcal{S}$ and denote by pr_j, $j = 1, 2$, the canonical projections.

Theorem 2.1. *(Pourcin) Let $\mathcal{Z}$ be a complex space over C. Then there exists a complex space $H \longrightarrow \mathcal{C}$ together with a closed subspace $\mathcal{X} \subset \mathcal{Z}_H$ satisfying the following properties:*

(i) *the map $pr_2 : \mathcal{Z}_H \longrightarrow H$ restricted to $\mathcal{X}$ is proper and flat;*
(ii) *for any space $S \longrightarrow \mathcal{C}$ and any subspace $\mathcal{Y} \subset \mathcal{Z}_S$ such that the canonical map $\mathcal{Y} \longrightarrow S$ is proper and flat, there exists a unique holomorphic map $S \longrightarrow H$ over $\mathcal{C}$ such that $\mathcal{X} \times_H \mathcal{S}$ equals $\mathcal{Y}$ as a subspace of $\mathcal{Z}_S$.*

Let $p \in H$ be a point with image 0 in $\mathcal{C}$. It corresponds to a compact subspace $X \subset Z$ of the fiber $Z := \mathcal{Z}_0$. We compute T_pH (equipped with a natural map $T_pH \longrightarrow T_0C$) and the Kodaira-Spencer map: The elements of T_pH correspond uniquely to holomorphic maps of the double point D to H (over C) carrying the underlying reduced point to p. General deformations are objects over punctured complex spaces (S, s_0) over $(C, 0)$ or germs of complex spaces. (Isomorphism classes need not be considered, since all objects come with an embedding onto a fixed space.) Hence, because of the universal property of H, the elements of T_pH are exactly infinitesimal relative deformations of $X \hookrightarrow Z$ over C, which are determined by the following data:

(a) a morphism $g : (D, 0) \longrightarrow (\mathcal{C}, 0)$
(b) a subspace $\mathcal{Y} \hookrightarrow g^*\mathcal{Z} = \mathcal{Z}_D$.

and a diagram

$$\begin{tikzcd}
 & Z \arrow[rr] \arrow[dd] & & \mathcal{Z}_D \arrow[dd, "F"] \\
X \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[ur] \arrow[dr, "f"'] & \\
 & 0 \arrow[rr] & & D
\end{tikzcd}$$

consisting of two embeddings and two Cartesian squares over $0 \hookrightarrow S$. The map F equals $pr_2 : Z \times_C D \longrightarrow D$. In the above situation, we consider the analytic extension of complex spaces together with the canonical epimorphisms:

$$\begin{tikzcd}
 & & \mathcal{O}_{\mathcal{Z}_D} \arrow[d, "\nu"'] \arrow[dr] & & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\alpha"'] & \mathcal{O}_{\mathcal{Y}} \arrow[r, "\rho"'] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

Denote by $\mathcal{J} \subset \mathcal{O}_{\mathcal{Z}_D}$ the ideal of X. Then $\nu(\mathcal{J}^2) = 0$, giving rise to a diagram

$$\begin{tikzcd}
O \arrow[r] & \mathcal{J}/\mathcal{J}^2 \arrow[r] \arrow[d, "\varphi"'] & \mathcal{O}_{\mathcal{Z}_D}/\mathcal{J}^2 \arrow[r] \arrow[d, "\overline{\nu}"] & \mathcal{O}_X \arrow[r] \arrow[d, "="] & O \\
O \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_{\mathcal{Y}} \arrow[r] & \mathcal{O}_X \arrow[r] & O
\end{tikzcd}$$

where also the first row is an analytic extension of $\mathcal{O}_X$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X)$. The map ν as well as $\overline{\nu}$ are $F^{-1}(\mathcal{O}_D)$-linear.

Conversely: Let $(D, 0) \to (C, 0)$ be a double point in our category of relative (punctured) complex spaces, and $\mathcal{J} \subset \mathcal{O}_{Z_D}$ the ideal of X. For any $\mathcal{O}_X-$linear map $\varphi : \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_X$ we define

$$\mathcal{O}_Y := \left(\mathcal{O}_{\mathcal{Z}_D}/\mathcal{J}^2\right) \Big/ \mathrm{Kernel}(\varphi)\,,$$

which is a fibred sum of a sheaf of $\mathbb{C}-$algebras and an $\mathcal{O}_{\mathcal{X}}$-module.

Proposition 2.2. *The tangent space T_pH is identified with the space of global homomorphisms $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X)$. Denoting the projection of $\mathcal{Z}$ onto C by q, the obvious restriction of global homomorphisms, namely*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X) \longrightarrow \mathrm{Hom}_{q^{-1}(\mathcal{O}_C)}(q^{-1}(\mathcal{I}_{q(p)}/\mathcal{I}^2_{q(p)}), q^{-1}(\mathcal{O}_C)) = T_{q(p)}C\,,$$

where $\mathcal{I}_{q(p)} \subset \mathcal{O}_C$ is the ideal sheaf for $q(p)$, coincides with the natural projection $T_pH \longrightarrow T_{q(p)}C$.

From now on we will consider the Douady space of embedded compact complex manifolds, where the family $\mathcal{Z} \to C$ is by assumption proper and smooth.

Proposition 2.3. *Let $t \in T_0C$. Then the fiber of t in T_pH consists of all holomorphic sections of the normal bundle of X in $\mathcal{Z}$ which project down to t under $\mathcal{Z} \to C$.*

3. The generalized Petersson-Weil metric

We fix an embedding of compact manifolds $i_0 : X \hookrightarrow Z$, and consider a deformation over a one dimensional smooth base $0 \in S \subset \mathbb{C}$. In other words, we have a commutative diagram

$$\begin{tikzcd}
& Z \arrow[rr] \arrow[dd] & & \tilde{\mathcal{Z}} \arrow[dd, "F"] \\
X \arrow[ur, "i_0"] \arrow[rr] \arrow[dr] & & \mathcal{X} \arrow[ur, "i"] \arrow[dr, "f"'] & \\
& 0 \arrow[rr] & & S
\end{tikzcd} \qquad (*)$$

We need to describe the infinitesimal deformation in adapted local coordinates, namely $\mathcal{Z} = \{(z, w, s)\}$ where the components of z are z^α; $\alpha = 1, \ldots, n$, and the components of

w are w^i; $i = 1, \ldots, m$. These can be chosen such that $\mathcal{X} = \{(z, 0, s)\}$ and $F(z, w, s) = s$. The tangent vector $v = \frac{\partial}{\partial s}|_{s=0} \in T_0 S$ gives rise to a holomorphic section $\rho = \rho(v)$ of the normal bundle $\mathcal{N}_{X|\tilde{\mathcal{Z}}}$, which is mapped to v under F_*. In the above local coordinates (z, w, s) it is again given by $\frac{\partial}{\partial s}$. (The holomorphic coefficients of $\frac{\partial}{\partial w^i}$ vanish identically.) Next a real, locally $\partial\overline{\partial}$-exact $(1,1)-$form $\tilde{\omega}$ on $\tilde{\mathcal{Z}}$ is fixed, which is positive semi-definite and is positive definite in the fiber direction of F. The generalized Petersson-Weil form will be constructed with respect to this form.
We assign to $\rho(\frac{\partial}{\partial s}|_{s=0})$ a differentiable complex vector field $\eta = \eta(\frac{\partial}{\partial s}|_{s=0})$ of type $(1,0)$ on X with values in $\mathcal{T}_{\mathcal{X}}$ as follows: For $x \in X$

(i) $\eta(x)$ projects to $\rho(x)$
(ii) $F_* \eta(x) = \frac{\partial}{\partial s}|_{s=0}$
(iii) $\eta(x)$ is perpendicular to X with respect to $\tilde{\omega}$.

The vector field η is uniquely determined. The form $\tilde{\omega}$ induces a natural sesquilinear form on all tangent spaces $T_x\tilde{\mathcal{Z}}$, and by integration over X a semi-norm $\|\eta\|_{PW}$. We denote the coefficients of $\tilde{\omega}$ by $g_{\alpha\overline{\beta}}$; $g_{\alpha\overline{j}}, g_{\alpha\overline{s}}$ etc. following the above convention on indices. Special attention is paid to $\tilde{\omega}|X = \sqrt{-1} g_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}$. The inverse of $g_{\alpha\overline{\beta}}$ is denoted by $g^{\overline{\beta}\alpha}$ and the corresponding volume form on X will be called $g \cdot dv$, where $g = \det(g_{\alpha\overline{\beta}})$ and dv is the Euclidean volume element. A short calculation gives:

Lemma 3.1. *With respect to local coordinates:*

(i) $\eta = \frac{\partial}{\partial s} - g_{s\overline{\beta}} g^{\overline{\beta}\alpha} \frac{\partial}{\partial Z^\alpha}$ *on* X
(ii) $\|\eta(z)\|^2_{PW} = g_{s\overline{s}} - g_{s\overline{\beta}} g_{\alpha\overline{s}} g^{\overline{\beta}\alpha} \geq 0$
(iii) $\|\frac{\partial}{\partial S}|_0\|^2_{PW} = \|\eta\|^2_{PW} = \int\limits_X (g_{s\overline{s}} - g_{s\overline{\beta}} g_{\alpha\overline{s}} g^{\overline{\beta}\alpha}) g dv \geq 0.$

The above considerations can be carried out at any point $s \in \mathcal{S}$, using for each base point local coordinates that are adapted to the situation. Thus, a positive semi-definite $(1,1)$-form $\omega^{PW}_{\mathcal{S}}$ on S is constructed:

$$\omega^{PW}_{\mathcal{S}} := \sqrt{-1} \|\eta(s)\|^2_{PW} ds \wedge \overline{ds},$$

where $\eta(s) = \eta(\frac{\partial}{\partial S}|_s)$ for $s \in \mathcal{S}$.

Lemma 3.2. *The Petersson-Weil form on $\mathcal{S}$ equals the following fiber integral*

$$\omega^{PW}_S = \frac{1}{(n+1)!} \int\limits_{\mathcal{X}/S} \omega^{n+1}_{\tilde{\mathcal{Z}}} | \mathcal{X},$$

where $n = \dim X$.

Returning to the general deformations of Section 2 we see that the above construction is functorial and yields a semi-positive form ω^{PW} on the moduli space.

Let $\mathcal{Z} \to \mathcal{C}$ be a holomorphic family of compact complex manifolds over a complex space C, which may also be non-reduced. The total space $\mathcal{Z}$ is equipped with a fixed positive definite real $(1,1)-$form $\omega_{\mathcal{Z}}$, which possesses locally a $\partial\overline{\partial}$-potential of class C^∞, a notion which also makes sense in the non-reduced case.
Let now $S \to C$ denote an arbitrary complex space over C equipped with an embedded family of complex spaces, i.e. we are given a diagram of type $(*)$ with $\tilde{\mathcal{Z}} = \mathcal{Z} \times_C S$. The above construction is compatible with base change and also valid for a one-dimensional base space replaced by a double point D.
Let

$$\begin{tikzcd}
 & Z \arrow[rr] \arrow[dd] & & \mathcal{Z}_D \arrow[r, "A"] \arrow[dd, "F"] & \mathcal{Z} \arrow[dd] \\
X \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[ur] \arrow[dr, "f"'] & & \\
 & 0 \arrow[rr] & & D \arrow[r, "\alpha"'] & C
\end{tikzcd}$$

be a deformation over the double point D, represented by η according to Lemma 3.1.

Lemma 3.3. *The Petersson-Weil form is positive definite on the moduli space; i.e.* $||\eta||^{PW} = 0$ *holds, if and only if the above infinitesimal deformation is trivial.*

Proof: Let $||\eta||^{PW} = 0$. We set $\tilde{\mathcal{Z}} = \mathcal{Z}_D$ and observe that a fiber integral

$$\int_{\mathcal{X}/D} (\omega_{\tilde{\mathcal{Z}}}^{n+1} | \mathcal{X})$$

yields a well defined form on D, which vanishes by assumption. Since $\omega_{\mathcal{Z}}^{(n+1)}$ is a positive $(n+1, n+1)-$form, this means that for all $z \in X$ the normal vector $\eta(z)$ of X in $\mathcal{Z}_D$ at Z is mapped to zero under A_*. According to Section 2 the deformation must be trivial.
The above results yield the theorem stated below. Let $\mathcal{Z} \to C$ be a family of compact complex manifolds and $\omega_{\mathcal{Z}}$ a Kähler form on $\mathcal{Z}$ (with a local $\partial\overline{\partial}$-potential of class C^∞). Let $H \to \mathcal{C}$ be the relative Douady space of embedded submanifolds of dimension n.

Theorem 3.4. *There exists a naturally defined generalized Petersson-Weil form* ω^{PW} *on* H. *Let*

$$\begin{tikzcd}
 & \mathcal{X} \arrow[rr] \arrow[dd] & & \mathcal{Z} \arrow[dd] \\
\mathcal{Z}_H \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{Y} \arrow[ur, "B"] \arrow[dr, "\beta"'] & \\
 & H \arrow[rr] & & C
\end{tikzcd}$$

denote the universal family. Then

$$\omega^{PW} = \frac{1}{(n+1)!} \int_{\mathcal{X}/H} (B^* \omega_{\mathcal{Z}}^{n+1} | \mathcal{X})$$

holds. In particular, ω^{PW} *is a Kähler form (with a local* $\partial\overline{\partial}$*-potential).*

In the sequel, we restrict ourselves to the absolute Douady space, where $C = 0$. Furthermore we assume that Z is a projective variety equipped with a positive hermitian line bundle $(\mathcal{L}, h)$ and $\omega_Z = c_1(\mathcal{L}, h)$. Let $(\mathcal{L}_{\mathcal{X}}, h_{\mathcal{X}})$ be the pull-back to $\mathcal{X}$, and denote by $((\mathcal{L}_{\mathcal{X}} - \mathcal{L}_{\mathcal{X}}^{-1})^{n+1}, \tilde{h})$ the induced virtual vector bundle of rank zero. Then

$$\omega^{PW} = \frac{1}{2^{n+1}(n+1)!} \int_{\mathcal{X}/\mathcal{S}} \mathrm{ch}((\mathcal{L}_{\mathcal{X}} - \mathcal{L}_{\mathcal{X}}^{-1})^{n+1}, \tilde{h})$$

(c.f. also [F-S, B-S]).
The right-hand side integral is invariant under the multiplication with the relative Todd form $\mathrm{td}(\mathcal{X}/S)$. The resp. Kähler metric on $\mathcal{X}$ is immaterial, since only the term in degree zero matters. Invoking a theorem of Bismut, Gillet and Soulé [B-G-S] we get

Theorem 3.5. *The generalized Petersson-Weil form is the curvature form of the determinant bundle $\lambda = \det((\mathcal{L}_{\mathcal{X}} - \mathcal{L}_{\mathcal{X}}^{-1})^{n+1})$ equipped with the Quillen metric h_Q.*

Acknowledgments : One of us (I.B.) acknowledges the generous hospitality of the Philipps-Universität, Marburg, where the work was carried out. The second named author would like to thank the Tata Institute of Fundamental Research, Bombay, where our collaboration was initiated, for its generous hospitality.

References

[B-G-S] Bismut, J.M., Gillet, H., Soulé, C.: Analytic torsion and holomorphic determinant bundles I, II, III. Commun. Math. Phy. **115** (1988), 49–78; **115** (1988), 79–126; **115** (1988), 301–351.

[B-S] Biswas, I., Schumacher, G.: Determinant bundle, Quillen metric, and Petersson-Weil form on moduli spaces, Geom. and Funct. Analysis (to appear).

[F-S] Fujiki, A., Schumacher, G.: The moduli space of compact extremal Kähler manifolds and generalized Petersson-Weil metrics Publ. RIMS, Kyoto Univ. **26**, 101–183 (1990).

[M] Maarouf, M.: Problémes d'hyperbolicité sur l'espace de Douady et ses variantes, Ph.D Thesis, Institut Fourier (1996).

[P] Pourcin, G.: Théorème de Douady au-dessus de S. Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat., III. Ser. **23**, 451–459 (1969).

[SCH] Schumacher, G.: Asymptotics of Kähler-Einstein Metrics on Quasi-Projective Manifolds and an Extension Theorem on Holomorphic Maps, Math. Ann. **311**, 631–645 (1988).

[V] Varouchas, J.: Stabilité de la classe des variétés Kaehleriennes par certains morphismes propres. Invent. Math. 77, 117–127 (1984).

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

E-mail address: indranil@math.tifr.res.in

Fachbereich Mathematik der Philipps-Universität, Hans-Meerwein-Strasse, Lahnberge, D-35032 Marburg, GERMANY

E-mail address: schumac@mathematik.uni-marburg.de