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Moduli as Algebraic Spaces

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Abstract.

We give a general criterion for the existence of a coarse moduli space as an algebraic space.

§1. Introduction

Given the class of inhomogeneously polarized, projective manifolds over $\mathbb{C}$, whose Hilbert polynomial is fixed, Matsusaka's big theorem ensures the boundedness of the corresponding moduli functor so that a coarse moduli space arises from an open subscheme of a certain Hilbert scheme. Mumford proved in [7, p. 217 ff.] that a coarse moduli space for non-uniruled manifolds exists and carries the structure of an algebraic space over $\mathbb{C}$ under the additional assumption that the automorphism groups of all objects are finite. Viehweg extended in [16, Theorem 9.16] the proof that the quotient of a scheme by an equivalence relation is an algebraic space, to those equivalence relations, whose equivalence classes are equidimensional. In this note, we show that in moduli theoretic situations the equidimensionality follows automatically.

§2. Fibered groupoids

In this section, we provide the formal framework. We denote by $p : \mathbf{F} \to \mathbf{A}$ a fibered groupoid in the sense of Grothendieck (categorie cofibrè en groupoides, [1, 9]). In our case $\mathbf{A}$ shall denote either the category $\mathbf{A}_c$ of complex spaces or the category $\mathbf{A}_s$ of schemes over $\mathbb{C}$ (separated and of finite type). By definition $p : \mathbf{F} \to \mathbf{A}$ is characterized by the following properties.

(i) For any morphism $g : R \to S$ in $\mathbf{A}$ and any $a \in Obj(\mathbf{A})$ over S there is a $g' : b \to a$ in $\mathbf{F}$ with $p(g') = g$. The object b is sometimes denoted by g^*a or $a \times_S R$.

(ii) Let $\alpha : a \to c$ and $\beta : b \to c$ be morphisms in $\mathbf{F}$ such that there exists a morphism $\varphi : p(a) \to p(b)$ in $\mathbf{A}$ with $p(\beta) \circ \varphi = p(\alpha)$.

Then there exists a unique morphism $\gamma : a \to b$ over φ such that $\beta \circ \gamma = \alpha$.

Property (ii) justifies the notation in (i).

Let S be in $Obj(A)$, then $\mathbf{F}(S)$ is by definition the category whose objects are objects in $\mathbf{F}$, which are mapped to S under p with morphisms over id_S. Passing to direct limits, one can assign to any fibered groupoid over the category of complex spaces a fibered groupoid over the category of complex space germs.

For any fibered groupoid there is an induced moduli functor $\mathfrak{M} : \mathbf{A} \to (Sets)$ where $\mathfrak{M}(S) = \overline{F(S)}$ is the set of isomorphism classes from $F(S)$, and where morphisms are defined in the obvious way. It is necessary in most cases to assign a sheafified moduli functor $\mathcal{M}$ to $\mathfrak{M}$, where the topology is the classical or étale topology depending on the choice of $\mathbf{A}$).

For the category of complex spaces $\mathbf{A}_c$ a coarse moduli space M is a morphism of functors

$$\Phi : \mathcal{M} \to M$$

with the following property

(i) for any complex space N and any morphism of functors $F : \mathcal{M} \to N$ there exists a unique morphism $f : M \to N$ such that $f \circ \Phi = F$.

(ii) the map $\mathcal{M}(Spec(\mathbb{C})) \to M(Spec(\mathbb{C}))$ is bijective.

(Here the complex space M is identified with $Hom(-, M)$).

If $\mathbf{A} = \mathbf{A}_s$, a coarse moduli space will be an *algebraic space* over $\mathbb{C}$: First to any scheme X the functor $Hom(-, X)$ from the category of affine schemes to the category of sets is assigned, inducing a sheaf of sets with respect to the étale topology. The latter is by definition a $\mathbb{C}$-space. An equivalence relation in the latter category is defined for all affine U and defines a quotient presheaf, and the corresponding sheaf is finally an algebraic space.

Let $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ be given. Then $\mathbf{F}_{a_0}$ denotes the induced fibered groupoid over the category of spaces with base point or space germs, whose objects are morphisms $a_0 \to a$ over $0 \to S$ implying a relationship to deformation theory: The usual deformation functor D_{a_0} from the category of complex space germs to the category of sets is equal to $D_{a_0}(S) = \overline{\mathbf{F}_{a_0}(S)}$, where the latter denotes the set of isomorphy classes of objects from $\mathbf{F}_{a_0}(S)$. Such deformation functors satisfy the axioms of Schlessinger [10] automatically, if the following condition holds (cf. [14, Lemma 2.6,2.7]):

(D) For all $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ there exists a semi-universal object in D_{a_0} over the category of complex spaces germs.

Let $\mathbf{A} = \mathbf{A}_c$, and let $a, b \in \mathbf{F}(S)$ for some complex space S. The functor $Isom_S(a,b)$: (Complex spaces $/S$) $\to$ (Sets) assigns to any complex space $R \to S$ the set of isomorphisms $a \times_S R \to b \times_S R$, and for morphisms in the category of complex spaces over S this functor is defined in an obvious way. We shall assume below that any such $Isom_S(a,b)$ is representable by a complex space over S. At the same time, we consider the functor $Isom_S(a,b)$ for a fibered groupoid over schemes. Because of the base change property, any such fibered groupoid defines a fibered groupoid over the category of $\mathbb{C}$-spaces. If $Isom_S(a,b)$ is representable by a $\mathbb{C}$-scheme (for any a, b), then the induced $\mathbb{C}$-space represents the induced functor for $\mathbb{C}$-spaces.

Now we come to the typical situation of a fibered groupoid $p : \mathbf{F} \to \mathbf{A}_s$, which is the restriction of a fibered groupoid $p_c : \mathbf{F}_c \to \mathbf{A}_c$ with $\mathbf{F}$ being a subcategory of $\mathbf{F}_c$. Under the assumption that for all Artinian schemes S_0 the category $\mathbf{F}(S_0)$ is a full subcategory of $\mathbf{F}_c(S_0)$ we call p_c a *complexification* of p. Let S be a $\mathbb{C}$-scheme and $a, b \in Obj(\mathbf{F}(S))$ and let $I = Isom_S(a,b) \to S$ represent the isomorphism functor. If follows easily that the induced morphism of corresponding complex spaces provides a representation of the isom-functor for $\mathbf{F}_c$. We state the following condition:

(Pr) For any a, b in $\mathbf{F}$ with $p(a) = p(b) = S$ the functor $Isom_S(a,b)$ is representable by a scheme $I = Isom_S(a,b) \to S$ proper over S.

Remark 1. *Let H be a scheme and $a \in \mathbf{F}(H)$, then $\psi : Isom_{H\times H}(a \times H, H \times a) \to H \times H$ defines an equivalence relation on H.*

We mention two technical conditions, which will be satisfied in our applications.

(R1) The morphism $p_2 = pr_2 \circ \psi : Isom_{H\times H}(a \times H, H \times a) \to H$ is smooth.

(R2) For $h \in H$ the morphism $\psi_h : p_2^{-1}(h) \to H \times \{h\}$ induced by ψ is smooth over its image.

Theorem 1. *Let $p : \mathbf{F} \to \mathbf{A}_s$ be a fibered groupoid with complexification $p_c : \mathbf{F}_c \to \mathbf{A}_c$, where semi-universal objects exist for the induced deformation functors. Suppose that the above condition (Pr) holds. Assume that for any $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ there exists an object a over a scheme H which induces a complete deformation of a_0 such that also (R1) and (R2) hold for a. Then there exists a coarse moduli space in the category of algebraic spaces over $\mathbb{C}$.*

We first show that p_c possesses a coarse moduli space M_c (in the category of complex spaces). Observe that any object a_0 from $\mathbf{F}$ over

$Spec(\mathbb{C})$ is the restriction of some b from $\mathbf{F}$ such that the restriction of b to some classical open set induces a semi-universal deformation. This fact, together with the representability of the deformation functor is sufficient for the proof from [12, 13], (and [14] for the nonreduced case). Let $c \in \mathbf{F}(S)$. We write $Aut_S(c) = Isom_S(c,c)$. Let S be connected, and denote by $Aut_S^0(c)$ the connected component, which contains the identity section.

Proposition 1. *The fibers of $Aut_S^0(c) \to S$ are complex Lie groups of constant dimension.*

Corollary 1. *Under the above conditions any semi-universal deformation of $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ is universal.*

The proposition and the corollary follow like in [13] and [14, Theorem 5.1] from the existence of a semi-universal deformation and the properness assumption.

Now the theorem is a consequence of [16, Thm. 9.1] (cf. [7, App. 5A.]).

§3. Applications to polarized varieties

Let (X, λ_X) be a projective manifold equipped with an inhomogeneous polarization λ_X i.e. an ample divisor up to numerical equivalence, we write λ_X as $c_{1,\mathbb{R}}(L)$ for some ample line bundle L. Let $h \in \mathbb{Q}[T]$ with $h(\mathbb{Z}) \subset \mathbb{Z}$, and consider those (X, λ_X) with $\chi(X, \mathcal{O}_X(L^k)) = h(k)$. Families of polarized projective manifolds with fixed h define a groupoid $p : \mathbf{F} \to \mathbf{A}_s$ with complexification. Let $\mathcal{M}_h : \mathbf{A}_s \to$ (Sets) be the induced sheafified moduli functor. Let $m > 0$ satisfy the statement of Matsusaka's theorem [5]: For all such X and L (with given dimension, and fixed Hilbert polynomial) the m-th powers L^m are very ample, and $H^j(X, \mathcal{O}_X(L^m)) = 0$ for all $j > 0$. In particular, the linear system of all sections of L^m provides an embedding of X into $\mathbb{P}_N$, where $N = h(m) - 1$.

Denote by $\mathcal{H} \subset \mathrm{Hilb}_{\mathbb{P}_N}^{h(m\cdot t)}$ the Zariski open subspace of all smooth $X \subset \mathbb{P}_N$ with Hilbert polynomial $h(m\cdot t)$ such that $\mathcal{O}_X(1)$ is divisible by m in $\mathrm{Pic}(X)$. (Assume that $\mathcal{H}$ is connected). The induced family $\mathcal{X} \hookrightarrow \mathbb{P}_N \times \mathcal{H} \to \mathcal{H}$ over $\mathcal{H}$ gives rise to the set-theoretic moduli space M_h, which always carries a natural topology induced by the classical topology on $\mathcal{H}$. Let $\psi : Isom_{\mathcal{H}\times\mathcal{H}}(\mathcal{X} \times \mathcal{H}, \mathcal{H} \times \mathcal{X}) \to \mathcal{H} \times \mathcal{H}$ be the canonical map. The necessary Hausdorff condition for M_h is the properness of $\mathrm{Im}(\psi) \to \mathcal{H}\times\mathcal{H}$ with respect to the classical topology. A slightly stronger condition is the properness of ψ.

Theorem 2. *Suppose that $\psi : Isom_{\mathcal{H}\times\mathcal{H}}(\mathcal{X} \times \mathcal{H}, \mathcal{H} \times \mathcal{X}) \to \mathcal{H} \times \mathcal{H}$ is proper. Then there exists a coarse moduli space, which is an algebraic space over $\mathbb{C}$.*

We verify the assumptions of Theorem 1: The properness of the isomorphism functor for any two given families of polarized projective manifolds can be proved easily using Hilbert schemes, and properties (R1) and (R2) follow like in [7] and [16]: (R1) follows from the Hilbert scheme construction, and (R2) is essentially Proposition 1 (cf. also [14]).

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