QUASI-PROJECTIVITY OF MODULI SPACES OF POLARIZED VARIETIES

GEORG SCHUMACHER AND HAJIME TSUJI

Abstract. By means of analytic methods the quasi-projectivity of the moduli space of algebraically polarized varieties with a not necessarily reduced complex structure is proven including the case of non-uniruled polarized varieties.

Contents

1. Introduction

In algebraic geometry, it is fundamental to study the moduli spaces of algebraic varieties. As for the existence of moduli spaces, it had been known that there exists an algebraic space as a coarse moduli space of non-uniruled polarized projective manifolds with a given Hilbert polynomial. Here an algebraic space denotes a space which is locally a finite quotient of an algebraic variety. Actually the notion of algebraic spaces was introduced to describe the moduli spaces ([AR1]). According to the theory of algebraic spaces by M. Artin ([AR1, AR2, KT]), the category of proper algebraic spaces of finite type defined over $\mathbb{C}$ is equivalent to the category of Moishezon spaces. Hence the moduli spaces of non-uniruled polarized manifolds have abundant meromorphic functions and were considered to be not far from being quasiprojective. Various attempts were made to prove the quasiprojectivity of the moduli spaces of non-uniruled, polarized algebraic varieties (cf. [K-M, KN, KO1, V]). E. Viehweg ([V]) developed a theory to construct positive line bundles on moduli spaces. He used results on the weak semipositivity of the direct images of relative multicanonical bundles. In particular he could prove the quasiprojectivity of the moduli spaces of canonically polarized manifolds ([V]). J. Kollár studied the Nakai-Moishezon criterion for ampleness on certain complete moduli spaces in [KO1], with applications to the projectivity of the moduli space of stable curves and certain moduli spaces of stable surfaces under boundedness conditions. However, his approach appears quite different from our present methods, which do not require the completeness of moduli spaces. His result was used to show the projectivity of the compactified moduli spaces of surfaces with ample canonical bundles by V. Alexeev ([AL]).

The main result in this paper is the quasi-projectivity of the moduli space of non-uniruled polarized manifolds. However, non-uniruledness is not used here. All we need is the existence of a moduli space.

In fact, given a polarized projective manifold, a universal family of embedded projective manifolds over a Zariski open subspace $\mathscr{H}$ of a Hilbert scheme is determined after fixing the Hilbert polynomial.

The identification of points of $\mathscr{H}$, whose fibers are isomorphic as polarized varieties, defines an analytic equivalence relation $\sim$ such that the set theoretic moduli space is $\mathscr{M} = \mathscr{H}/\sim$. The quotient is already a complex space, if the equivalence relation is proper. Moreover, in this situation, it follows that $\mathscr{M}$ is an algebraic space. If the above equivalence relation is induced by the action of a projective linear group G, properness of $\sim$ means properness of the action of G. In this moduli theoretic case $\mathscr{H}/\sim$ is already a geometric quotient.

Theorem 1. *Let $\mathscr{K}$ be a class of polarized, projective manifolds such that the moduli space $\mathscr{M}$ exists as a proper quotient of a Zariski open subspace of a Hilbert scheme. Then $\mathscr{M}$ is quasi-projective.*

The proof of the theorem consists of two steps. The first step is to construct a line bundle on the compactified moduli space with a singular hermitian metric of strictly positive curvature on the interior.

The method is based upon the curvature formula for Quillen metrics on determinant line bundles ([BGS]), the theory of Griffiths about period mappings ([GRI]), and moduli of framed manifolds.

The second step is to construct sufficiently many holomorphic sections of a power of the above line bundles in terms of L^2-estimates of the $\overline{\partial}$-operator. The key ingredient here is the theory of closed positive $(1,1)$-currents, which controls the multiplier ideal sheaf of a singular hermitian metric. This step can be viewed as an extension of the Kodaira embedding theorem to the quasi-projective case.

2. Singular hermitian metrics

Definition 1. *Let X be a complex manifold and L a holomorphic line bundle on X. Let h_0 be a hermitian metric on L of class C^∞ and $\varphi \in L^1_{loc}(X)$. Then $h = h_0 \cdot \mathbf{e}^{-\varphi}$ is called a singular hermitian metric on L.*

Following the notation of [DE4] we set

$$d^c = \frac{\sqrt{-1}}{2\pi}(\partial - \overline{\partial})$$

and call the real $(1,1)$-current

$$\Theta_h = dd^c(-\log h) = -\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log h \tag{1}$$

the "curvature current" of h. It differs from the Chern current by a factor of 2.

A real current Θ of type $(1,1)$ on a complex manifold of dimension n is called *positive*, if for all smooth $(1,0)$-forms $\alpha_2, \ldots, \alpha_n$

$$\Theta \wedge \sqrt{-1}\alpha_2 \wedge \overline{\alpha}_2 \wedge \ldots \wedge \sqrt{-1}\alpha_n \wedge \overline{\alpha}_n$$

is a positive measure. We write $\Theta \geq 0$.

A singular hermitian metric h with positive curvature current is called *positive*. This condition is equivalent to saying that the locally defined function $-\log h$ is plurisubharmonic.

Let $W \subset \mathbb{C}^n$ be a domain, and Θ a positive current of degree (q,q) on W. For a point $p \in W$ one defines

$$\nu(\Theta, p, r) = \frac{1}{r^{2(n-q)}} \int_{\|z-p\|<r} \Theta(z) \wedge (dd^c\|z\|^2)^{n-q}.$$

The *Lelong number* of Θ at p is defined as

$$\nu(\Theta, p) = \lim_{\substack{r \longrightarrow 0 \\ r>0}} \nu(\Theta, p, r).$$

If Θ is the curvature of $h = \mathbf{e}^{-u}$, u plurisubharmonic, one has

$$\nu(\Theta, p) = \sup\{\gamma \geq 0; u \leq \gamma \log(\|z-p\|^2) + O(1)\}.$$

The definition of a singular hermitian metric carries over to the situation of reduced complex spaces.

Definition 2. *Let Z be a reduced complex space and L a holomorphic line bundle. A* singular *hermitian metric h on L is a singular hermitian metric h on $L|Z_{reg}$ with the following property: There exists a desingularization $\pi : \widetilde{Z} \longrightarrow Z$ such that h can be extended from Z_{reg} to a singular hermitian metric $\widetilde{h}$ on π^*L over $\widetilde{Z}$.*

The definition is independent of the choice of a desingularization under a further assumption. Suppose that $\Theta_{\widetilde{h}} \geq -c \cdot \omega$ in the sense of currents, where $c > 0$, and ω is a positive definite, real $(1,1)$-form on $\widetilde{Z}$ of class C^∞. Let $\pi_1 : Z_1 \longrightarrow Z$ be a further desingularization. Then $\widetilde{Z} \times_Z Z_1 \longrightarrow Z$ is dominated by a desingularization Z' with projections $p : Z' \longrightarrow \widetilde{Z}$ and $p_1 : Z' \longrightarrow Z_1$. Now $p^* \log \widetilde{h}$ is of class L^1_{loc} on Z' with a similar lower estimate for the curvature. The push-forward $p_{1*}p^*\widetilde{h}$ is a singular hermitian metric on Z_1. In particular, the extension of h to a desingularization of Z is unique. □

In [G-R] for plurisubharmonic functions on a normal complex space the Riemann extension theorems were proved, which will be essential for our application. The relationship with the theory of distributions was treated in [DE].

For a reduced complex space a *plurisubharmonic* function u is by definition an upper semi-continuous function $u : X \longrightarrow [-\infty, \infty)$ whose restriction to any local, smoothly parameterized analytic curve is either identically $-\infty$ or subharmonic.

A function $u : X \longrightarrow [-\infty, \infty)$ from $L^1_{loc}(X)$, which is locally bounded from above is called *weakly plurisubharmonic*, if its restriction to the regular part of X is plurisubharmonic.

Differential forms with compact support on a reduced complex space are by definition locally extendable to an ambient subspace, which is an open subset U of some $\mathbb{C}^n$. Hence the dual spaces of differential C^∞-forms on such U define currents on analytic subsets of U. The positivity of a real $(1,1)$-current is defined in a similar way as above involving expressions of the form (1).

For functions locally bounded from above of class L^1_{loc} the weak plurisubharmonicity is equivalent to the positivity of the current $dd^c u$. It was shown that these functions are exactly those, whose pull-back to the normalization of X are plurisubharmonic. We note

Definition 3. *Let L be a holomorphic line bundle on a reduced complex space X. Then a singular hermitian metric h is called positive, if the functions, which define locally $-\log h$ are weakly plurisubharmonic.*

This definition is compatible with Definition 2: Let L be a holomorphic line bundle on a complex space Z equipped with a positive, singular hermitian metric h_r on $L|Z_{reg}$. If $\pi : \widetilde{Z} \longrightarrow Z$ is a desingularization, and $\widetilde{h}$ a positive, singular hermitian metric on π^*L, extending $h|Z_{reg}$, we see that $-\log h_r$ is locally bounded from above at the singularities of Z so that $\widetilde{h}$ induces a singular, positive metric on L over Z.

3. Deformation theory of framed manifolds – V-structures

Let X be a compact complex manifold and $D \subset X$ a smooth (irreducible) divisor. Then (X, D) is called a logarithmic pair or a *framed manifold.*

For *any* $m \in \mathbb{N}$ an associated V-structure $\widetilde{X}_m$ on X is defined in terms of local charts $\pi : W \longrightarrow U$, $U \subset X$, $W \subset \mathbb{C}^n$ such that π is just an isomorphism, if $U \cap D = \emptyset$ or a cyclic Galois covering of order m with branch locus $U \cap D$.

By definition, the differential forms and vector fields on X with respect to the V-structure, which are V-differentiable or V-holomorphic, are defined on $X\backslash D$ with the property that the local lifts under $\pi|W\backslash\pi^{-1}(D) : W\backslash\pi^{-1}(D) \longrightarrow U\backslash D$ can be extended in a holomorphic or differentiable way to W.

With m being fixed, we denote by $\mathscr{T}_X^V$ and $\mathscr{A}_X^{V,q}(\mathscr{T}_X^V)$ resp. the sheaves of V-holomorphic vector fields and V-differentiable q-forms with values in $\mathscr{T}_X^V$ resp.

Lemma 1. (i) *For any $m \in \mathbb{N}$ the Dolbeault complex*

$$0 \longrightarrow \mathscr{T}_X^V \longrightarrow \mathscr{A}_X^{V,\bullet}(\mathscr{T}_X^V)$$

is well-defined and exact.

(ii) *The sheaf $\mathscr{T}_X^V$ is canonically isomorphic to $\Omega_X^1(\log D)^\wedge$.*

By definition, a family $(\mathscr{X}_s, \mathscr{D}_s)_{s \in S}$ of framed manifolds, parameterized by a complex space S is given by a smooth, proper, holomorphic map $f : \mathscr{X} \longrightarrow S$ together with a divisor $\mathscr{D} \subset \mathscr{X}$, such that $f|\mathscr{D}$ is proper and smooth, such that $\mathscr{X}_s = f^{-1}(s)$ and $\mathscr{D}_s = \mathscr{D} \cap \mathscr{X}_s$. A local deformation of a framed manifold (X, D) over a complex space S with base point $s_0 \in S$ is a deformation of the embedding $i : D \hookrightarrow X$, i.e. induced by a family $\mathscr{D} \hookrightarrow \mathscr{X} \longrightarrow S$ together with an isomorphism $(X, D) \xrightarrow{\sim} (\mathscr{X}_{s_0}, \mathscr{D}_{s_0})$, where two such objects are identified, if these are isomorphic over a neighborhood of the base point. The existence of versal deformations (i.e. complete and semi-universal deformations) of these objects is known. We denote by $T^\bullet(X) \simeq H^\bullet(X, \mathscr{T}_X)$ and $T^\bullet(X, D)$ resp. the tangent cohomology of X and (X, D) resp.

Corollary 1. *The space of infinitesimal deformations of* (X, D) *equals* $T^1(X, D) = H^1(\Gamma(X, \mathscr{A}_X^{V,\bullet}(\mathscr{T}_X^V)))$. *It can also be computed in terms of Cech cohomology as* $\check{H}^1(\mathfrak{U}, \mathscr{T}_X^V)$ *of* V*-holomorphic vector fields, where* $\mathfrak{U}$ *is a* G*-invariant* $\mathscr{T}_X^V$*-acyclic covering.*

We have the following exact sequence:

$$
\begin{aligned}
(2) \qquad 0 \longrightarrow T^0(X, D) \longrightarrow T^0(X) \longrightarrow H^0(D, \mathscr{O}_D(D)) \longrightarrow \\
T^1(X, D) \longrightarrow T^1(X) \longrightarrow H^1(D, \mathscr{O}_D(D)).
\end{aligned}
$$

We denote by $T^1_1(X) \subset T^1(X)$ the image of $T^1(X, D)$. The composition of $H^1(X, \mathscr{T}_X)) \longrightarrow H^1(D, \mathscr{O}_D(D))$ with the natural map $H^1(D, \mathscr{O}_D(D)) \longrightarrow H^2(X, \mathscr{O}_X)$ equals the map induced by the cup-product with the Chern class of D. The latter is induced by the Atiyah sequence for the pair $(X, \mathscr{O}_X(D))$, and its kernel $T^1_0(X)$ consists of those infinitesimal deformations for which the isomorphism class of the line bundle $[D]$ extends. Assume that D is an ample divisor on X, and $\lambda_X = c_1(D)$ its (real) Chern class. Then the pair (X, λ_X) is a *polarized variety*, and $T^1_0(X)$ is the space of infinitesimal deformations of (X, λ_X). Studying moduli spaces of polarized varieties, we are free to replace the ample divisor D by a uniformly chosen multiple, in which case $T^1_0(X)$ and $T^1_1(X)$ can be identified.

The group of infinitesimal automorphisms $T^0(X, D)$ vanishes, if $K_X + [D]$ is positive. Like in the case of canonically polarized manifolds, in a family of such *framed* manifolds the relative automorphism functor (or more generally isomorphism functor) is represented by a space such that the natural map to the base is finite and proper. Moreover, general deformation theory implies that any versal deformation is universal.

4. Cyclic coverings

Let X be a compact complex manifold, and D, D' effective divisors such that $D \sim m \cdot D'$ for some $m \in \mathbb{N}$. Denote by E and E' resp. bundle spaces for the corresponding line bundles. Let

$$\begin{tikzcd} E' \arrow[rr, "\ell"] \arrow[dr, "\pi'"'] & & E \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \tag{3}$$

be the morphism over X, which sends a bundle coordinate α to α^m.

Let σ be a canonical section of π. Then we define $X_m = V(\ell - \sigma \circ \pi') \subset E'$. If D is a smooth divisor, the subspace $X_m \subset E$ is a manifold, and $\pi'|X_m : X_m \longrightarrow X$ is a cyclic Galois covering with branch locus $D \subset X$.

We assume now that D is very ample, providing an embedding $\Phi : X \longrightarrow \mathbb{P}_N$. We denote by P the dual projective space, and by $\Sigma \subset \mathbb{P}_N \times P \longrightarrow P$ the tautological hyperplane with divisor $\mathscr{D} = \Sigma \cap (X \times P) \subset X \times P \longrightarrow P$ and bundle space $\mathscr{E} \longrightarrow X \times P$. Let $\mathscr{D}_t = \Sigma_t \cap X$ for $t \in P$.

We have flat families over $X \times P$ and P resp.

$$\begin{tikzcd} \underline{X}_m \arrow[r] \arrow[dr, "\mu"] \arrow[ddr, "\pi"'] & \underline{E}' \arrow[r] \arrow[d] & \underline{E} \arrow[dl] \\ & X \times P \arrow[d] & \\ & P & \end{tikzcd} \tag{4}$$

Here the bundle $\underline{E}$ comes from the globally defined divisor $\mathscr{D}$. The bundle $\underline{E}'$ is first defined locally with respect to P. The obstructions against defining E' globally are in the first cohomology over P with coefficients in the locally constant sheaf $\mathbb{C}^*$, which vanishes.

Proposition 1. *The total space $\underline{X}_m$ is smooth. In particular, the dualizing sheaf $\omega_{\underline{X}_m/P}$ equals the relative canonical sheaf $K_{\underline{X}_m/P} := K_{\underline{X}_m} \otimes \pi^* K_P^{-1}$.*

Proof. As $\underline{X}_m \subset \underline{E}'$ is of codimension one, it is sufficient, to find a local function for any $x_0 \in \underline{X}_m$, which vanishes at x_0, and whose gradient at this point is non-zero. Let again σ be a canonical section of the line bundle $\underline{E}$ over $X \times P$. We denote by t_0 the image of x_0 in P, and take local coordinates t of P around t_0. Let α be a local bundle coordinate of $\underline{E}'$ around t_0, and z a local coordinate on X so that x_0 is given by (z_0, α_0, t_0). Now $t_0 \in P$ corresponds to a section $\sigma_{t_0}(z)$ of $\underline{E}|X \times \{t_0\}$. The space $\underline{X}_m$ is defined by $g(z,\alpha,t) := \sigma_t(x) - \alpha^m = 0$ around x_0. If $\alpha_0 \neq 0$, we have $(\partial g/\partial\alpha)(x_0) \neq 0$. If $\alpha_0 = 0$ holds, $\sigma_{t_0}(z_0) = 0$. Since D is very ample on X, we find a section of $\underline{E}|X \times \{t_0\}$, which does not vanish at x_0. This section gives some $t_1 \in P$, i.e. some σ_{t_1}. Let $\sigma_{t(\tau)} = \sigma_{t_0} + \tau\sigma_{t_1}$ be the line through t_0 and t_1. Then $(\partial g/\partial\tau)|_{\tau=0} \neq 0$. $\square$

The analogous statement is true for smooth families $f : \mathscr{X} \longrightarrow S$: Let $\mathscr{D}'$ be a family of very ample divisors, which provide an embedding $\mathscr{X} \hookrightarrow \mathbb{P}(V) \times S \hookrightarrow \mathbb{P}(S^m V)$, V a finite dimensional $\mathbb{C}$-vector space. Then the family $m \cdot \mathscr{D}'$ defines an embedding $\mathscr{X} \hookrightarrow \mathbb{P}(W)$ for some W. These embeddings are compatible with respect to the canonical rational map $\mathbb{P}(S^m V) \rightharpoonup \mathbb{P}(W)$. As above, we denote by P he dual space to $\mathbb{P}(W)$. Let $\mathscr{E}'$ be the total space of the line bundle induced by $\mathscr{D}'$, and pulled back to $\mathscr{X} \times P$. Let $\mathscr{D} \subset \mathscr{X} \times P$ be the divisor $\Sigma \cap (\mathscr{X} \times P)$, where $\Sigma \subset \mathbb{P}(W) \times P$ denotes the tautological hyperplane like in the beginning of this section. The bundle $\mathscr{E}$ possesses a canonical section given by $\mathscr{D}$, and we have a map $\mathscr{E}' \longrightarrow \mathscr{E}$, which is the m-th power fiberwise. Again, we obtain a subspace $\mathscr{X}_m \subset \mathscr{E}'$.

Remark 1. *There is a natural diagram*

$$\begin{tikzcd}
\mathscr{X}_m \arrow[r] \arrow[dr, "\mu"'] \arrow[ddr, "f_m"'] & \mathscr{E}' \arrow[r] \arrow[d] & \mathscr{E} \arrow[dl] \\
& \mathscr{X} \times P \arrow[d, "f \times \mathrm{id}"] & \\
& S \times P &
\end{tikzcd} \tag{5}$$

where the induced map $\mathscr{X}_m \longrightarrow S$ is smooth. In particular, the canonical and dualizing sheaves $K_{\mathscr{X}_m/S\times P} = K_{\mathscr{X}_m} \otimes f_m^* K^{-1}_{S\times P}$ and $\omega_{\mathscr{X}_m/S\times P}$ resp. are isomorphic, if S is smooth.

Let (X, D) be a framed manifold, and $D \sim mD'$ for some effective D' as above. Again, let $G = \mathbb{Z}_m$ denote the Galois group, X is isomorphic to the quotient X_m/G, and the group G acts on $H^1(X_m, \mathscr{T}_{X_m})$ with invariant subgroup $H^1(X_m, \mathscr{T}_{X_m}) \supset H^1(X_m, \mathscr{T}_{X_m})^G$. The average over the group defines a retraction. Next, we identify $H^1(X_m, \mathscr{T}_{X_m})^G$ with the V-tangent cohomology group $\overset{\vee}{H}{}^1(\mathfrak{U}, \mathscr{T}_X^V)$ in the sense of Section 3: The morphisms $C^\bullet(\mathfrak{U}, \mathscr{T}_{X_m})^G \hookrightarrow C^\bullet(\mathfrak{U}, \mathscr{T}_{X_m}) \xrightarrow{r} C^\bullet(\mathfrak{U}, \mathscr{T}_{X_m})^G$ descend to the cohomology and $C^\bullet(\mathfrak{U}, \mathscr{T}_{X_m})^G \simeq C^\bullet(\mathfrak{U}, \mathscr{T}_X^V)$. This argument avoids any smoothing of invariant differential forms.

Remark 2. *The infinitesimal deformations of a framed manifold (X, D) can be identified with $T^1(X, D) = H^1(\Gamma(X, \mathscr{A}_X^{V,\bullet}(\mathscr{T}_X^V))) = H^1(\mathfrak{U}, \mathscr{T}_X^V) = H^1(X_m, \mathscr{T}_{X_m})^G$.*

5. Canonically polarized framed manifolds

We call a framed manifold (X, D) *canonically polarized*, if

$$K_X + [D] > 0,$$

and *m-framed* under the condition

$$(*)_m \qquad K_X + \frac{m-1}{m}[D] > 0$$

for some $m \geq 2$.

In the sequel we always assume condition $(*)_m$ for some fixed m. We note that for the Galois covering $\mu : X_m \longrightarrow X$ with smooth X_m the relation

$$\mu^*(K_X + \frac{m-1}{m}[D]) = K_{X_m}$$

holds. In our applications the divisor D will always be ample so that $(*)_m$ is slightly stronger than the first condition. We will still use the term "canonically polarized framed manifold" in this case. This will also be justified later.

Proposition 2. *Let $D' \subset X$ be a very ample divisor as above, and $m > 2$. Let $D \subset X$ be a smooth divisor $D \sim m \cdot D'$ such that*

$$K_X + \frac{m-2}{m} D$$

is very ample. Then the canonical bundle K_{X_m} is very ample.

Proof. The sheaf $\mathscr{O}_X(K_X + \frac{m-1}{m}D) \subset \mu_*(\mathscr{O}_{X_m}(K_{X_m}))$ is a direct summand. Let $\mathbb{Z}_m \simeq G \hookrightarrow \mathrm{Aut}(X_m)$ be the group of deck transformations with a generator γ, and denote by ζ a primitive m-the root of unity. Let

$\oplus_{j=1}^{m} E_j$ be an eigenspace decomposition of the space of global sections of K_{X_m} with respect to the eigenvalues ζ^j of γ. It follows that the spaces E_j can be identified with the space of global sections of $K_X+(m-j)\cdot D'$, again with $j=1,\ldots,m$. The pull-backs of sections of such a space are sections of $K_{X_m}-(j-1)A$, where $A\subset X_m$, $A\simeq D'$, is the branching divisor of μ, so that the identification $\Gamma(X,\mathscr{O}_X(K_X+(m-j)\cdot D'))\simeq E_j$ is a multiplication with a canonical section of $[(j-1)A]$.

The space E_1 clearly separates points, whose image under μ are different.

Let $p,q\in X_m$ with $\mu(p)=\mu(q)=x$. Then there exist sections of $[K_X+(m-2)D']$ and $[K_X+(m-1)D']$ which do not vanish at x. A suitable linear combination of the induced elements of E_1 and E_2 separates p and q. The argument is also applicable to tangent vectors. □

Now we consider the situation given in diagram (5), where S need not be smooth. Let $\mathscr{A}\subset S\times P$ be the locus of singular divisors D. Over its complement the direct image of the relative canonical sheaf is certainly locally free.

We write $\mathscr{X}'_m := \mathscr{X}_m\backslash f_m^{-1}(\mathscr{A})$, $T:=P\times S$, $T':=T\backslash\mathscr{A}$, and f'_m for the restriction of the map f_m. In a similar way we restrict $\widetilde{f}:=f\times\mathrm{id}$ to T' and get $\widetilde{f}':(\mathscr{X}\times P)'\longrightarrow T'$.

Proposition 3. *The locally free sheaf $f'_{m*}K_{\mathscr{X}'_m/T'}$ possesses a natural, locally free extension.*

Proof. We use the decomposition $f_{m*}K_{\mathscr{X}'_m/T'}=\oplus_{j=0}^{m-1}\widetilde{f}'_*(K_{(\mathscr{X}\times P)'/T'}+j\cdot[\mathscr{D}'|(\mathscr{X}\times P)'])$ from the proof of Proposition 2. Now for the family $(\mathscr{X}\times P)'\longrightarrow T'$, with relatively (very) ample divisor $\mathscr{D}'$, the Kodaira-Nakano vanishing theorem and the Grothendieck-Grauert comparison theorem show that for $j>0$ the sheaves $\widetilde{f}_*(K_{(\mathscr{X}\times P)/T}+j\cdot[\mathscr{D}'])$ are locally free on T (Here the divisor $\mathscr{D}'$ corresponds to the line bundle $\mathscr{E}'$). Let $j=0$. Since $f'_{m*}(K_{\mathscr{X}'_m/T'})$ is locally free on T', also $\widetilde{f}_*(K_{\mathscr{X}\times P/T})$ is locally free, when restricted to T'. On the other hand, it does not involve the divisor $\mathscr{D}'$, and $\widetilde{f}_*(K_{\mathscr{X}\times P/T})$ is the pull-back of the direct image of $K_{\mathscr{X}/S}$, so it is constant along all fibers of $T\longrightarrow S$, and locally free in the interior, hence also $\widetilde{f}_*(K_{K_{\mathscr{X}\times P/T}})$ is locally free. □

Next, we want to recover the above extension of the relative canonical sheaf. We have the diagram (5). The fibers of f_m are branched along the $\mathscr{D}_s$ with singularities over the singularities of the branching divisors. By definition the map f_m is flat with Cohen-Macaulay fibers. According to results of Kleiman [KL] for such morphisms relative dualizing sheaves commute with base change. Again, we denote by the letter ω dualizing sheaves.

It follows from the universal property of dualizing sheaves that

$$\begin{aligned} f_{m*}(\omega_{\mathscr{X}_m/T}) &\simeq \mathit{Hom}_{\mathscr{O}_T}(R^n f_{m*}(\mathscr{O}_{\mathscr{X}_m}), \mathscr{O}_T) \\ &= \mathit{Hom}_{\mathscr{O}_T}((R^n f_*)(\mu_* \mathscr{O}_{\mathscr{X}_m}), \mathscr{O}_T) \\ &\simeq \mathit{Hom}_{\mathscr{O}_T}((R^n f_*)(\oplus_{j=0}^{m-1}(\mathscr{O}_{\mathscr{X}\times P}(-j\cdot \mathscr{D}')), \mathscr{O}_T) \\ &\simeq f_* \mathit{Hom}_{\mathscr{O}_{\mathscr{X}\times P}}(\mu_* \mathscr{O}_{\mathscr{X}_m}, \omega_{\mathscr{X}\times P/T}) \end{aligned}$$

Altogether, we have

Lemma 2.

$$f_{m*}(\omega_{\mathscr{X}_m/T}) \simeq \oplus_{j=0}^{m-1} f_*(\omega_{\mathscr{X}\times P/T}(j\cdot \mathscr{D}')).$$

In particular, the extended sheaf from Proposition 3 equals $f_{m*}(\omega_{\mathscr{X}_m/T})$ (which is compatible with further pull-backs). Later we will consider this sheaf from a Hodge theoretic viewpoint.

6. Singular Hermitian metrics for families of canonically polarized framed manifolds

We first recall some facts concerning the period map in the sense of Griffiths [GRI] for families $f : \mathscr{Y} \longrightarrow S$ of manifolds with very ample canonical bundle. We will apply the results to families of the form $f_m : \mathscr{X}_m \longrightarrow S$ with relative dimension n from Section 4. The direct image under f of the relative canonical sheaf $K_{\mathscr{Y}/S}$ is also called Hodge bundle $\mathscr{E}_0$. It is equipped with the flat metric from $\mathrm{R}^n f_* \mathbb{C}$. Explicitly, for any two holomorphic n-forms ϕ and ψ on a manifold $\mathscr{Y}_{s_0}$ we have

$$(\phi, \psi) := (\sqrt{-1})^{n^2} \int_{\mathscr{Y}_{s_0}} \phi \wedge \overline{\psi}.$$

Let $\partial/\partial s$ be a tangent vector at a point s_0 Then the contraction with the Kodaira-Spencer class $[A^{\alpha}_{s_0\bar\beta}\frac{\partial}{\partial z^\alpha}dz^{\bar\beta}] \in H^1(\mathscr{Y}_{s_0}, \mathscr{T}_{\mathscr{Y}_{s_0}})$ induces a linear map

$$\sigma_0(\frac{\partial}{\partial s}\,|_{s_0}) : H^0(\mathscr{Y}_{s_0}, \Omega^n_{\mathscr{Y}_{s_0}}) \longrightarrow H^1(\mathscr{Y}_{s_0}, \Omega^{n-1}_{\mathscr{Y}_{s_0}}). \tag{6}$$

The natural metric on the latter space is again induced by the integration of exterior products of differential forms, after we provide the fibers with a family of auxiliary Kähler structures (e.g. of Kähler-Einstein type). Following Griffiths [GRI, Theorem (5.2)] the curvature Θ_0 of this hermitian metric is given by the formula

$$(\Theta_0\phi,\psi) = (\sqrt{-1})^{n^2}\int_{\mathscr{Y}_{s_0}} H\left(\sigma_0(\frac{\partial}{\partial s})(\phi)\right)\wedge\overline{H\left(\sigma_0(\frac{\partial}{\partial s})(\psi)\right)}, \tag{7}$$

which is defined in terms of cohomology classes. (Here H denotes the harmonic projection). So Θ_0 is semi-positive as well as its trace $\mathrm{tr}(\Theta_0)$. If $\mathrm{tr}(\Theta_0)(\frac{\partial}{\partial s},\frac{\partial}{\partial s})|_{s_0} = 0$, also $\Theta_0(\frac{\partial}{\partial s},\frac{\partial}{\partial s})|_{s_0}$ vanishes. The auxiliary Kähler metric is only needed to show the positivity of the curvature, the metric on the relative canonical bundle is independent of the choice. The sheaf $R^1f_*\Omega^{n-1}_{\mathscr{Y}/S}$ is usually denoted by $\mathscr{E}_1$.

Denote by $\mathbf{D}$ the period domain of Hodge structures, and by $\Phi : S \longrightarrow \mathbf{D}$ the induced (multivalued) period map. Then $\mathrm{Hom}(\mathscr{E}^0 \otimes_{\mathscr{O}_S} \mathbb{C}(s), \mathscr{E}^1 \otimes_{\mathscr{O}_S} \mathbb{C}(s))$ is a subspace of the tangent space of $\mathbf{D}$ at the point $\Phi(s)$, and it carries the natural L^2-inner product (cf. (7)). We denote this metric by ds_0^*. If $S \simeq \Delta^{*k}\times\Delta^\ell$, one knows $\Phi^* ds_0^* \leq \text{const.}\, ds^2_{Poinc}$, where ds^2_{Poinc} denotes the Poicaré metric.

On the other hand, for $f : \mathscr{Y} \longrightarrow S$, by (7), the trace of the curvature of the flat metric restricted to a bundle $\mathscr{E}_0$ gives exactly ds_0^2. This argument shows:

Lemma 3. *Let $\mathscr{Y} \longrightarrow S$, $S = \Delta^{*k}\times\Delta^\ell$ be a holomorphic family of canonically polarized manifolds. Let h_S be the natural C^∞ hermitian metric on $\det f_*\mathscr{K}_{\mathscr{Y}/S}$. Then the curvature Θ_S is semi-positive (in the sense of C^∞-forms), and dominated by a constant multiple of the Kähler form ω_S induced by ds^2_{Poinc}.*

For effectively parameterized families $f_m : \mathscr{X}_m \longrightarrow T$ and large m the map $\sigma_0 : H^1(\mathscr{X}_{m,s_0}, \mathscr{T}_{\mathscr{X}_{m,s_0}}) \longrightarrow \mathrm{Hom}(H^0(\mathscr{X}_{s_0}, \Omega^n_{\mathscr{X}_{s_0}}), H^1(\mathscr{X}_{s_0}, \Omega^{n-1}_{\mathscr{X}_{s_0}}))$ is in fact injective. This was shown in a general setting by Ivinskis, and attributed to Griffiths in [IV] for the special case of cycling coverings.

One can find a uniformly valid power m of $[\mathscr{D}_s]$ so that [IV, Theorem 2.4] holds. It has to be chosen in a way that the assumption of Donagi's Lemma (cf. [IV]) holds, i.e. $H^1(\mathscr{X}_s \times \mathscr{X}_s, \mathscr{F} \otimes \mathscr{O}_{\mathscr{X}_s}(m \cdot \mathscr{D}_s) \boxtimes \mathscr{O}_{\mathscr{X}_s}(m \cdot \mathscr{D}_s))$ vanishes for all $s \in S$, where $\mathscr{F}$ denotes a certain given coherent sheaf on $\mathscr{X} \times_S \mathscr{X}$.

Now the base S is equipped with the line bundle $\lambda_{fr} = \det f_{m*}\mathscr{K}_{\mathscr{X}_m/T}$ (which equals the determinant line bundle in the sense of the derived category, because of the Kodaira vanishing theorem). Then the curvature of the induced hermitian metric h on λ is $\Theta_h = \mathrm{tr}(\Theta_0)$. Altogether:

Proposition 4. *The curvature Θ_h of (λ_{fr}, h) is semi-positive. It is strictly positive in all directions, where the family is effectively parameterized.*

Now we return to the notation of Section 5. The main theorem is stated for non-singular base spaces.

Theorem 2. *The determinant (invertible) sheaf* $\det f_{m*}K_{\mathscr{X}_m/T}$ *carries a natural positive hermitian metric, whose Lelong numbers vanish everywhere. Moreover, for all $p \in \mathbb{N}$, the exterior powers Θ_h^p of its curvature form Θ_h are well-defined (p,p)-currents, whose Lelong numbers vanish everywhere as well.*

We shall apply the theorem in two different situations: Over the interior of the moduli space we deal with families of manifolds of the type X_m, where in the limit we have singular Galois coverings $X_m \longrightarrow X$ (cf. Section 5). Here the key point is that the total space $\mathscr{X}_m$ is already smooth according to Proposition 1 so that we can identify the relative dualizing sheaf with the relative canonical sheaf. The other situation occurs at the boundary of the moduli space, where we are free to modify the boundary.

The theorem follows from the known results in the theory of mixed Hodge structures. We show here an upper estimate for a singular Hermitian metric. Together with the positivity of this metric the vanishing of the Lelong numbers follows.

Concerning singular base spaces of holomorphic families, we observe that the L^2-inner products (for tangent vectors of the base) are well-defined for singular bases spaces. For our applications we will need that the construction is functorial, i.e. compatible with base changes like restrictions to closed subspaces and desingularizations in the view of Definition 2.

For a family $f_m : \mathscr{X}_m \longrightarrow T$ (T is smooth), we denote by $\mathscr{A} \subset T$ the set of points with singular fibers. Let $\nu : \widetilde{T} \longrightarrow T$ be given by a sequence of blow-ups with regular centers so that the preimage $\mathscr{B}$ of $\mathscr{A}$ is a normal crossings divisor. Let $\widetilde{\mathscr{X}_m} \longrightarrow \mathscr{X}_m \times_T \widetilde{T}$ be a desingularization of the component of $\mathscr{X}_m \times_T \widetilde{T}$ that dominates $\widetilde{T}$, with the property that the preimage of $\mathscr{B}$ is a normal crossing divisor. Let

$$
\begin{tikzcd}
\widetilde{\mathscr{X}_m} \arrow[r, "\nu'"] \arrow[d, "\widetilde{f}_m"'] & \mathscr{X}_m \arrow[d, "f_m"] \\
\widetilde{T} \arrow[r, "\nu"'] & T
\end{tikzcd}
\tag{8}
$$

be the induced commutative diagram. We denote by a prime accent the restriction of fiber spaces to the resp. complements of normal crossing divisors.

An argument of Deligne shows that the local monodromy of $R^n f_{m*}\mathbb{C}$ on T' is unipotent around generic points of $\mathscr{A}$, i.e. in codimension one. And since it is locally abelian on $\widetilde{T}'$, this holds everywhere. For our purpose the unipotent reduction is sufficient. We need a local statement with respect to the base T. The argument is known: Around each component of the normal crossings divisor $\mathscr{B}$ the eigenvalues of the local monodromy transformation on $R^n \widetilde{f}'_{m*}\mathbb{C}$ are certain roots of unity [B]. After taking a finite morphism $\kappa : \check{T} \longrightarrow \widetilde{T}$, branched over $\widetilde{B}$ the local monodromy groups become unipotent. We consider

$$
\begin{tikzcd}
\check{\mathscr{X}}_m \arrow[r, "\kappa'"] \arrow[d, "\check{f}_m"'] & \widetilde{X}_m \arrow[d, "\widetilde{f}_m"] \\
\check{T} \arrow[r, "\kappa"'] & \widetilde{T}
\end{tikzcd}
\tag{9}
$$

The canonical extension of $R^n \check{f}'_{m*}\mathbb{C}_{\check{\mathscr{X}}'_m} \otimes \mathscr{O}_{\check{T}'}$ to $\check{T}$ ([DL]) is a coherent sheaf. By a theorem of W. Schmid [S], the subsheaf $\check{f}'_{m*}K_{\check{\mathscr{X}}'_m/\check{T}'}$ extends to a locally free sheaf on $\check{T}$. Kawamata's theorem [KA] states that this locally free extension is equal to $\check{f}_{m*}K_{\check{\mathscr{X}}_m}$. It is known that also $\widetilde{f}_{m*}K_{\widetilde{\mathscr{X}}_m}$ is locally free: Namely as κ' is a proper holomorphic map of equidimensional complex manifolds, $K_{\widetilde{\mathscr{X}}_m} \subset \kappa'_* K_{\check{\mathscr{X}}}$ is a direct summand, and hence $\widetilde{f}_{m*}K_{\widetilde{\mathscr{X}}_m} \subset \widetilde{f}_{m*}\kappa'_* K_{\check{\mathscr{X}}_m} = \kappa_* \check{f}_{m*}K_{\check{\mathscr{X}}_m}$ is a direct summand. Now the latter is locally free, as $\check{f}_{m*}K_{\check{\mathscr{X}}_m}$ is a locally free $\mathscr{O}_{\check{T}}$-module, and κ is a finite proper map of complex manifolds. We have $K_{\mathscr{X}_m} = \nu'_* K_{\widetilde{\mathscr{X}}_m}$ on the manifold $\mathscr{X}_m$ so that $f_{m*}K_{\mathscr{X}_m}$ is locally free.

Next, we use W. Schmid's description of sections of $\check{f}_{m*}K_{\check{\mathscr{X}}_m}$ around points of the normal crossing divisor. Let $\Delta^k \simeq U \subset \check{T}$ be an open subset such that the complement of the normal crossing divisor is $U' \simeq \Delta^{*\ell} \times \Delta^{k-\ell}$.

Let ϕ be a section of $K_{\check{\mathscr{X}}_m}$ over $\check{f}_m^{-1}(U)$. Over U' it can be expressed in terms of a basis $\{s_1, \ldots, s_M\}$ of multivalued (locally constant) sections of $R^n \check{f}_{m*}\mathbb{C}_{\check{\mathscr{X}}_m}$ over U'. So $\phi = \sum f_\nu \cdot s_\nu$ for certain multivalued holomorphic functions on U'. According to [S, (4.17)], the holomorphicity of ϕ in points of the normal crossing divisor is equivalent to the f_ν having at most logarithmic singularities. Next the L^2-norm is computed at points $t \in U'$. (We identify $K_{\check{\mathscr{X}}}$ with $K_{\check{\mathscr{X}}/\check{T}}$).

$$\|\phi(t)\|^2 = \int_{\check{\mathscr{X}}_{m,t}} \phi(t) \wedge \overline{\phi(t)} = \sum_{\nu,\mu} f_\nu(t)\overline{f_\mu(t)} \cdot \int_{\check{\mathscr{X}}_{m,t}} s_\nu(t) \wedge \overline{s_\mu(t)}$$

The latter integrals are independent of t, because the s_i are locally flat sections. So

$$\|\phi\|^2 \leq \sum_{j=1}^{\ell} c_j(-\log|t_j|)$$

for some constants $c_j > 0$.

The unipotent reduction preserves such estimates so that a similar estimate (with different constants) also holds for sections of $\widetilde{f}_{m*}K_{\widetilde{\mathscr{X}}_m}$.

This implies an estimate for sections of $f_{m*}K_{\mathscr{X}_m}$. We only note the following rough estimate: Let $W \subset T$ be an open subset. Then for any $\psi \in (f_{m*}K_{\mathscr{X}_m})(W)$ we have

$$\|\psi(x)\|^2 \leq \sum \alpha_j(-\log|\tau_j|)$$

for certain positive constants α_j and holomorphic functions τ_j, which vanish on $\mathscr{A}$. This proves the following lemma:

Lemma 4. *The holomorphic line bundle* $\det(f_{m*}K_{\mathscr{X}_m/T})$ *carries a singular hermitian metric h, which is of class C^∞ on $T\backslash\mathscr{A}$ such that in local holomorphic coordinates*

$$h \leq \sum_j \beta_j(-\log|\tau_j|) \tag{10}$$

for certain $\beta_j > 0$.

The above growth condition for the singular hermitian metric h, which is positive by Proposition 4 implies:

Corollary 2. *For any $x \in T = P \times S$, the Lelong numbers $\nu(h,x)$ vanish, in particular, the theorem holds for $p = 1$.*

The curvature form Θ satisfies a Poincaré growth condition on $\Delta^{*\ell} \times \Delta^{k-\ell}$ (cf. Lemma 3). In particular all powers Θ^p define closed (p,p)-currents. These estimates hold for the Hodge metrics over $\check{T}$, $\widetilde{T}$, and since $\widetilde{T} \longrightarrow T$ is a modification of complex manifolds, also the Θ_h^p on T are closed currents. We show the last statement of Theorem 2.

Let $x \in P \times S$ be a point and $z_1, \ldots, z_k$ local coordinates such that $x = 0$. Let (locally) $h = \mathbf{e}^{-u}$, with u plurisubharmonic, and define $\varphi = \log\|z\|^2$. For any positive (p,p)-current R and small $r > 0$ the quantity $\nu(R,x,r)$ is defined by

$$\nu(R,x,r) = \frac{1}{r^{2p}} \int_{\|z\|<r} R \wedge (dd^c\|z\|^2)^{k-p},$$

and in terms of Demailly's generalized Lelong numbers

$$\nu(R,x,r) = \nu(R,\varphi,\log r),$$

where

$$\nu(R,\varphi,t) = \int_{\varphi(z)<t} R \wedge (dd^c\varphi)^{k-p}.$$

In a straightforward way a generalized Jensen formula can be proved:

$$\begin{aligned}\int_r^{r_1} \nu((dd^cu)^p,\varphi,t)dt = &\int_{\varphi=r_1} u(dd^cu)^{p-1} \wedge d^c\varphi \wedge (dd^c\varphi)^{k-p}\\ &- \int_{\varphi=r} u(dd^cu)^{p-1} \wedge d^c\varphi \wedge (dd^c\varphi)^{k-p}\\ &- \int_{r<\varphi<r_1} u(dd^cu)^{p-1} \wedge (dd^c\varphi)^{k-p+1}\end{aligned}$$

It is known that for any fixed r_1

$$\nu((dd^c u)^p, x) = \lim_{r \longrightarrow -\infty} \left(-\int_r^{r_1} \nu((dd^c u)^p, \varphi, t) dt/r \right).$$

Now the proof follows immediately, because

(i) $u \geq -c \cdot \log(\sum_j \beta_j(-\log|\tau_j|))$ by Lemma 4
(ii) As a plurisubharmonic function u is (locally) bounded from above
(iii) $dd^c u$ satisfies a Poincaré growth condition on $\widetilde{T}$.

□

7. Convergence property of generalized Petersson-Weil metrics

Our study of moduli of polarized varieties is based on moduli of (canonically polarized) framed manifolds. We include the definition of generalized Petersson-Weil metrics, which can also be part of a conceptual approach. However, analytic difficulties had to be overcome, framed manifolds are "approximated" by m-framed manifolds, which are closely related to cyclic coverings. This fact is also expressed in a convergence theorem for generalized Petersson-Weil metrics for (m-)framed manifolds and canonically polarized varieties.

In the first place, generalized Petersson-Weil metrics are intrinsically defined Kähler metrics on the base spaces of universal deformations. Due to functoriality these will be seen to descend to moduli spaces.

In this section, we will assume that for all $\varepsilon \in \mathbb{Q}$ with $0 \leq \varepsilon \leq \varepsilon_0$ the divisor

$$K_X + (1-\varepsilon)D$$

is positive. This condition is satisfied for $\varepsilon_0 = 1/m_0$ in our basic situation, where (X, D) is m_0-framed and D positive. The methods of [TS1, K1, K2, T-Y] yield unique Kähler-Einstein metrics $\eta_{X,m}$ on the V-manifolds $\widetilde{X}_m$ (cf. Section 3) of Ricci-curvature -1. As in the smooth compact case we can see that the V-Kähler-Einstein metrics define generalized Petersson-Weil metric on the moduli space of framed manifolds as follows:

Let $\mathscr{D} \hookrightarrow \mathscr{X} \longrightarrow S$ define an effective holomorphic family of framed manifolds $(\mathscr{X}_s, \mathscr{D}_s)_{s\in S}$. Let $(X, D) = (\mathscr{X}_{s_0}, \mathscr{D}_{s_0})$. Let $m \geq m_0$, and let $\widetilde{X}_m$ be equipped with the Kähler-Einstein metric $\eta_{X,m}$. For any $v \in T_{s_0}S$ denote by

$$A_{m,v} = A^{\alpha}_{m,v,\bar\beta} \frac{\partial}{\partial z^\alpha} dz^{\bar\beta} \in \Gamma(X, \mathscr{A}^{V,1}_X(\mathscr{T}^V_X))$$

the representative of the Kodaira-Spencer class of v according to Remark 2 in $T^1(X, D)$, which is harmonic with respect to $\eta_{X,m}$.

Definition 4. *Let $v, w \in T_{s_0}S$, and $A_{m,v}$, $A_{m,w}$ corresponding harmonic Kodaira-Spencer forms. Then the Petersson-Weil inner product is*

$$< v, w >_{PW} = \int_X < A_{m,v}, A_{m,w} > \omega^n_{X,m}.$$

The Kähler property of the induced form $\omega_{PW,m}$ on S can be shown in the same way as for the case of smooth, canonically polarized varieties. Also a fiber integral formula holds for the Petersson-Weil form, and a line bundle equipped with a Quillen metric can be constructed, whose curvature form equals $\omega_{PW,m}$ up to a constant [BGS].

On the other hand the tangent cohomology $T^1(X, D)$ can be computed in terms of the complete Kähler-Einstein metric $\omega_{X'}$ on $X' = X\backslash D$ as the $H^1_{(2)}(X', \mathscr{T}_{X'})$ the L^2-cohomology group of the sheaf of holomorphic vector fields $\mathscr{T}_{X'}$ [SCH1]. The L^2-structure on the tangent cohomology defines a Petersson-Weil metric $\omega_{PW,fr}$ on $\mathscr{M}_{fr}$.

Let $\Omega_{\mathscr{X}/S}$ be the relative volume form, i.e. a hermitian metric on $\bigwedge^n \mathscr{T}_{\mathscr{X}/S}$, induced by all $\eta_{\mathscr{X}_s,m}$, and denote by $\eta_{\mathscr{X},m}$ the negative of its curvature form on the total space. Its restrictions to all fibers are the Kähler-Einstein forms on the fibers. Let $v = \partial/\partial s \in T_sS$ be a tangent vector, and $\partial/\partial s + a^\alpha(\partial/\partial z^\alpha)$ the horizontal lift with respect to $\eta_{\mathscr{X},m}$. Also in the case of V-structures, its exterior derivative $\overline{\partial}(a^\alpha) = (\partial a^\alpha/\partial z^{\bar\beta})(\partial/\partial z^\alpha)dz^{\bar\beta}$, restricted to the fiber X, equals the harmonic Kodaira-Spencer form $A_{m,v}$. For a more detailed discussion of the Petersson-Weil inner product and Petersson-Weil forms for singular base spaces cf. also [F-S].

Denote by $\eta_{\mathscr{X}_s}$ the usual Kähler-Einstein metrics, and by $\eta_{\mathscr{X}}$ the negative of its Ricci form on the total space.

Measuring convergence in $C^{k,\alpha}(X')$-spaces with respect to quasi-coordinates on $X' = X \backslash D$ the $\eta_{X,m}$ tend to the complete Kähler-Einstein metric $\omega_{X'}$ on X' [TS2]. In a holomorphic family of framed manifolds, this convergence yields a convergence of the relative volume forms $\Omega_{\mathscr{X}/S,m}$ to the relative volume form $\Omega_{\mathscr{X}'/S}$ of the smooth Kähler-Einstein metrics in the spaces $C^{k,\alpha}(\mathscr{X}')$, $\mathscr{X}' = \mathscr{X} \backslash \mathscr{D}$. Together with the above fact about the characterization of harmonic Kodaira-Spencer forms we see immediately that the harmonic Kodaira-Spencer forms $A_{m,v}$ converge to the harmonic L^2-integrable Kodaira-Spencer forms $A_{fr,v}$ on X' with respect to the complete Kähler-Einstein metrics on X'.

Let m be fixed $\mathscr{D} \hookrightarrow \mathscr{X} \longrightarrow S_{m,fr}$ be a local universal holomorphic family of m-framed manifolds and $\mathscr{X}_m \longrightarrow \mathscr{X} \longrightarrow S_{m,fr}$ the induced family of branched coverings with $\mathscr{X}_{m,s}$ canonically polarized such that $S_{m,fr}$ embeds into a base of a universal family of canonically polarized manifolds, giving rise to $\kappa : S_{m,fr} \hookrightarrow S_c$, where S_c carries the usual Petersson-Weil form $\omega_{PW,can}$.

Proposition 5. *For the generalized Petersson-Weil metrics on moduli spaces of framed manifolds*

$$\lim_{m \longrightarrow \infty} \omega_{PW,m} = \omega_{PW,fr}$$

holds in any C^k-topology. The forms $\omega_{PW,m}$ are induced by the Petersson-Weil form for moduli of canonically polarized varieties:

$$\omega_{PW,m} = \frac{1}{m} \kappa^*(\omega_{PW,can}).$$

We have to show the second claim: We have the V-structures on the fibers $\mathscr{X}_s$, and the usual Kähler-Einstein metrics induce Kähler-Einstein V-metrics on the quotients $\mathscr{X}_{m,s}/\mathbb{Z}_m$. Any harmonic Kodaira-Spencer V-form lifts to a harmonic Kodaira-Spencer form on $\mathscr{X}_{m,s}$. the factor $1/m$ is due to the integration over m sheets as opposed to the integration over the V-manifold. □

8. Moduli spaces of framed manifolds

In this section, we make some basic remarks. In the analytic case, a *polarization* of a framed manifold (X, D) is the assignment of a Kähler class $\lambda_X \in H^2(X,\mathbb{R})$. Polarizations, which are images of integer valued cohomology classes, coincide with *inhomogeneous* polarizations in the sense of Mumford (cf. [M-F-K]). (Here, we can also allow rational coefficients and consider $\mathbb{Q}$-divisors.)

The following definition is also sensible for inhomogeneously polarized framed projective varieties (X, D, λ_X) (over $\mathbb{C}$).

Definition 5. (i) *A compact Kähler manifold X is called uniruled over a smooth divisor D, if there exists a surjective meromorphic map $\varphi : \mathbb{P}_1 \times Y \longrightarrow X$ with the following properties: The map φ does not allow a meromorphic factorization over $pr_2 : P_1 \times Y \longrightarrow Y$. The restriction of pr_2 to the proper transform of D under φ is a modification.*

(ii) *A polarized framed manifold (X, D, λ) is called non-uniruled, if the Kähler manifold D is non-uniruled, and if X is not uniruled over D.*

In the analytic category, the (coarse) moduli space of non-uniruled polarized Kähler manifolds exists.

For non-uniruled, polarized, projective framed manifolds (X, D, λ_X), first the Hilbert polynomials $P(x)$ for λ_X on X and $Q(x)$ for $\lambda_X|D$ are of interest. (If the polarization λ_X is represented by D, we have $Q(x) = P(x) - P(x-1)$).

Let λ_X be represented by a basic polar divisor and corresponding ample line bundle $\mathscr{L}_X$. As usual, Matsusaka's big theorem ([MA, L-M]) is applied to $(X, \mathscr{L}_X)$: There exists an integer $c > 0$ only depending on $P(x)$, such that for all $m \geq c$ the sheaves $\mathscr{L}_X^{\otimes m}$ are very ample.

Theorem 3. *There exists an algebraic space $\mathscr{M}_{fr}$ in the sense of Artin, which is the coarse moduli space of isomorphism classes of non-uniruled, polarized, framed projective manifolds (X, D, λ_X) with fixed Hilbert polynomials $P(x)$and $Q(x)$.*

As non-uniruledness is an open and closed condition for polarized varieties, we can also impose the condition that both X, and D are non-uniruled. Then the assignment $(X, D, \lambda_X) \mapsto (X, \lambda_X)$ (with Hilbert polynomials fixed) defines a natural map $\mathscr{M}_{fr} \longrightarrow \mathscr{M}$ of algebraic spaces, where $\mathscr{M}$ denotes the moduli space of uniruled polarized manifolds. If the divisors D are very ample and represent the polarization λ_X (and X is non-uniruled), D may also be singular giving rise to a moduli space $\widehat{\mathscr{M}}$ equipped with a natural morphism $\widehat{\nu} : \widehat{\mathscr{M}} \longrightarrow \mathscr{M}$.

Proof. First, $c > 0$ as above is taken and $m \geq c$ fixed and for all polarized varieties X with Hilbert polynomial $P(x)$ a corresponding projective embedding $X \hookrightarrow \mathbb{P}_N$ induced by global sections of $\mathscr{L}_X^{\otimes m}$ considered. As subvarieties of $\mathbb{P}_N$ these X have $P(m \cdot x)$ as Hilbert polynomials. We denote by $Hilb^P_{\mathbb{P}_N}$ the Hilbert scheme of all subvarieties with $P(m \cdot x)$ in the sense of Grothendieck [GRO]. The locus $\mathscr{H} \subset Hilb^P_{\mathbb{P}_N}$ of all smooth subvarieties is quasi-projective. Let

$$\begin{tikzcd} \mathscr{X} \arrow[r, "i"] \arrow[dr, "f"'] & \mathscr{H} \times \mathbb{P}_N \arrow[d, "pr_1"] \\ & \mathscr{H} \end{tikzcd} \tag{11}$$

be the universal flat family. Here the fibers $\mathscr{X}_s = f^{-1}(s)$ for $s \in \mathscr{H}$ carry the polarization $\mathscr{O}_{\mathscr{X}_s}(1) = \mathscr{L}^{\otimes m}_{\mathscr{X}_s}$

Next we fix the Hilbert polynomial $Q(x)$ with respect to D and $\mathscr{L}|D$. Again, by ([GRO]), Théorème 3.1 we are looking at a functor represented by a projective, flat $\mathscr{H}$-scheme $\widetilde{\nu} : \widehat{\mathscr{H}} \longrightarrow \mathscr{H}$ equipped with a universal flat family $\mathscr{D} \longrightarrow \widehat{\mathscr{H}}$. The locus $\mathscr{H}_{fr}$ of smooth divisors $\widehat{\mathscr{H}} \supset \mathscr{H}_{fr} \xrightarrow{\pi} \mathscr{H}$ is a quasi-projective variety. Explicitly, let P be the dual of $\mathbb{P}_N$, then $\mathscr{H}_{fr} \subset \widehat{\mathscr{H}} = \mathscr{H} \times P$ is a Zariski open subspace. We have

$$\begin{tikzcd} \mathscr{D} \arrow[r, "j"] \arrow[dr] & \widehat{\mathscr{X}} \arrow[r] \arrow[d, "\widehat{f}"] & \mathscr{X} \arrow[d, "f"] \\ & \widehat{\mathscr{H}} \arrow[r, "\widetilde{\nu}"'] & \mathscr{H} \end{tikzcd} . \tag{12}$$

The graph $\widehat{\Gamma} \subset \widehat{\mathscr{H}} \times \widehat{\mathscr{H}}$ of the equivalence relation identifying embedded manifolds with singular framings is mapped properly to the graph $\Gamma \subset \mathscr{H} \times \mathscr{H}$, which defines the moduli space $\mathscr{M}$ of polarized projective manifolds. By assumption, the natural map $\Gamma \longrightarrow \mathscr{H}$ is proper, so $\widehat{\Gamma}$ also defines a proper equivalence relation. This ensures the existence of a natural complex structure on $\widehat{\mathscr{M}}$. (Observe that this statement can also be proved in the non-reduced category). Finally $\widehat{\mathscr{M}}$ carries the structure of an algebraic space (cf. [SCH2]). The construction is compatible with the restriction to $\mathscr{H}_{fr}$. If the above equivalence relations are given by the action of $G = PGL(N+1,\mathbb{C})$ on $\widehat{\mathscr{H}}$ and $\mathscr{H}$ resp. the moduli spaces $\widehat{\mathscr{M}}$, $\mathscr{M}_{fr}$ and $\mathscr{M}$ are eventually geometric quotients. In the analytic case the statement of the Matsusaka-Mumford theorem is also valid (cf. [SCH1]) for framed polarized manifolds.

Later we will consider compactifications of the algebraic spaces $\mathscr{M}$ and $\widehat{\mathscr{M}}$ by normal crossings divisors with a morphism $\overline{\widehat{\mathscr{M}}} \longrightarrow \overline{\mathscr{M}}$. We can assume that it is induced by a flat morphism $\overline{\widehat{\mathscr{H}}} \longrightarrow \overline{\mathscr{H}}$ of suitably compactified Hilbert schemes of the similar type. $\square$

The moduli space $\mathscr{M}$ is induced by a smooth family of the form (11) with hyperplane section $\mathscr{D}' \subset \mathscr{X}$, such that the very ample divisors $\mathscr{D}'_s$ represent a fixed multiple of the polarizations on $\mathscr{X}_s$. Let $n = \dim \mathscr{X}_s$ as before. According to Fujita's theorem [FU], the divisors $K_{\mathscr{X}_s} + m\mathscr{D}'_s$ are ample for $m \geq n+2$. We fix $m > n+3$ and represent $m[\mathscr{D}'_s]$ by all possible divisors $\mathscr{D}_s$. This gives rise to a diagram of the form (12). We pull back the divisor $\mathscr{D}'$ to $\widehat{X}$ and obtain a bundle space $\underline{E}' \longrightarrow \widehat{\mathscr{X}}$. Let $\underline{E} \longrightarrow \widehat{\mathscr{X}}$ be the bundle associated to $\mathscr{D}$. Like in Section 4 we construct a family of cyclic coverings $f_m : \mathscr{X}_m \longrightarrow \widehat{\mathscr{H}}$ and a diagram

$$\begin{tikzcd} \mathscr{X}_m \arrow[r, "\mu"] \arrow[dr, "f_m"'] & \widehat{\mathscr{X}} \arrow[d, "\widehat{f}"] \\ & \widehat{\mathscr{H}} \end{tikzcd} \tag{13}$$

where the branch locus of μ is $\mathscr{D} \subset \widehat{\mathscr{X}}$. The fibers $\mathscr{X}_{m,s}$ are smooth for $s \in \mathscr{H}_{fr}$.

The above construction gives rise to a morphism of algebraic spaces κ from $\mathscr{M}_{fr}$ to a component $\mathscr{M}_c$ of the moduli space of canonically polarized (smooth) varieties. Let (X, D) be a fixed framed manifold with branched covering $X_m \longrightarrow X$ as above, and let $\widetilde{R}$ and R resp. denote base spaces of universal deformations. Then by Remark 2 there exists a closed holomorphic embedding $\widetilde{\kappa} : \widetilde{R} \longrightarrow R$ which induces the map κ in a neighborhood of the corresponding moduli point, where it is a finite map of the form $\widetilde{R}/\mathrm{Aut}(X, D) \longrightarrow R/\mathrm{Aut}(X_m)$. We observe that the group of deck transformations $\mathbb{Z}_m \subset \mathrm{Aut}(X_m)$ acts on R leaving the subspace $\widetilde{R} \subset R$ pointwise fixed, since the group action can be lifted to all of its fibers.

9. Fiber integrals and determinant line bundles for morphisms

We will use the method of *generalized determinant line bundles.* Let $F : Z \longrightarrow S$ be a proper, holomorphic map of complex spaces and $\mathscr{L}$ a coherent $\mathscr{O}_Z$-module.

The direct image $R^\bullet F_*\mathscr{L}$ of $\mathscr{L}$ under the proper map F in the derived category can be locally represented by a sequence $\mathscr{F}^\bullet$ of finite, free $\mathscr{O}_S$-modules, which is bounded to the right. If the morphism is flat, the sequence can be chosen as bounded, and the tensor product of the determinant sheaves of the $\mathscr{F}^i$ with alternating exponents ± 1 is by definition the determinant line bundle $\lambda = \det(\mathscr{L})$, and the latter is globally well-defined.

Let $\mathscr{L} = \mathscr{O}_Z(L)$ be a holomorphic line bundle equipped with a hermitian metric of class C^∞. According to Bismut, Gillet and Soulé, [BGS], under the assumption of F being a smooth Kähler morphism of complex manifolds (or reduced complex spaces [F-S]), the Chern form of the Quillen metric h^Q on $\det(\mathscr{L})$ is equal to the component of degree two of a fiber integral:

$$c_1(\lambda, h^Q) = -\left[\int_{Z/S} \mathrm{td}(Z/S)\mathrm{ch}(\mathscr{L})\right]_{(2)}, \tag{14}$$

where td and ch resp. define the Todd and Chern character resp. (This holds also, when $\mathscr{L}$ is replaced by a hermitian vector bundle.)

By functoriality and universal properties, this equation extends to $\mathscr{L}$ replaced by an element of the Grothendieck group, i.e. a virtual holomorphic vector bundle. For any n the virtual bundle $(\mathscr{L}-\mathscr{L}^{-1})^{n+1}$ has rank zero, and the lowest term in $\mathrm{ch}((\mathscr{L}-\mathscr{L}^{-1})^{n+1})$ is $2^{n+1}c_1(\mathscr{L})$. If n denotes the fiber dimension, the only contribution of the Todd character in (14) is equal to 1. Hence the Chern form of $\det((\mathscr{L} - \mathscr{L}^{-1})^{n+1})$ equals

$$-2^{n+1}\int_{Z/S} c_1(\mathscr{L}, h)^{n+1}. \tag{15}$$

Now we return to the situation of moduli spaces like in Section 8. The Hilbert scheme $\mathscr{H}_{fr}$ carries the determinant line bundle λ_{fr} with singular hermitian metric h_{fr} according to Proposition 4 and Lemma 2. It is important that the line bundle λ_{fr} on $\mathscr{H}_{fr}$ was extended to the line bundle $\widehat{\lambda}$ on $\widehat{\mathscr{H}}$. Let $\pi : \mathscr{H}_r \longrightarrow \mathscr{H}$ be a desingularization with fiber product $\nu_r : \widehat{\mathscr{H}_r} \longrightarrow \mathscr{H}_r$ and pull-back $\widehat{\lambda}_r$ of $\widehat{\lambda}$. Since ν_r is a smooth map with fiber isomorphic to $\mathbb{P}_N$, we can apply the above methods and consider the determinant bundle $\det((\widehat{\lambda}_r - \widehat{\lambda}_r^{-1})^{N+1})$.

We now apply these methods to singular hermitian metrics on singular spaces (cf. Section 2), and $(1,1)$-currents.

So far we are given a smooth holomorphic map $\widehat{\nu} : \widehat{\mathscr{H}} \longrightarrow \mathscr{H}$ and a holomorphic line bundle on $\widehat{\lambda}$ on $\widehat{\mathscr{H}}$, whose restriction λ_{fr} to $\mathscr{H}_{fr}$ carries the C^∞ hermitian metric h_{fr} with curvature form Θ_{fr}.

We use the above arguments to extend the determinant line bundle $\det((\widehat{\lambda} - \widehat{\lambda}^{-1})^{N+1})$ as a coherent sheaf from $\mathscr{H}$ to $\overline{\mathscr{H}}$. We denote by $\widehat{\Theta}$ the curvature current of $\widehat{\lambda}$. Let $\ell = \dim \mathscr{H}$. In order to define a fiber integral

$$\int_{\widehat{\mathscr{H}}/\mathscr{H}} \widehat{\Theta}^{N+1},$$

for any $(\ell-1, \ell-1)$-form φ of class C^∞ with compact support, we set

$$\Theta^Q(\varphi) = \int_{\widehat{\mathscr{H}}/\mathscr{H}} \Theta_{fr}^{N+1} \wedge \widetilde{\nu}^* \varphi,$$

with $\Theta_{fr} = \widehat{\Theta}|\mathscr{H}_{fr}$.

At this point, we may blow up $\widehat{\mathscr{H}}$ with exceptional set in $\widehat{\mathscr{H}} \backslash \mathscr{H}_{fr}$ and realize $\mathscr{H}_{fr}$ as a complement of a divisor with only normal crossings singularities so that the assumptions of Lemma 3 are satisfied. The upper Poincaré growth estimate for Θ_{fr} implies that the above integral is finite, and it vanishes, if φ is d-exact. So Θ^Q is well-defined as a d-closed $(1,1)$-current. Also Lemma 3 implies that Θ^Q is positive (in the sense of currents).

Proposition 6. *At all points $\mathscr{H}$ the Lelong numbers of Θ^Q vanish.*

The above statement also holds after descending to the moduli space at points of the boundary, as we can always achieve the situation of Section 2 after blowing up the boundary.

Proof. The proof follows immediately from Theorem 2. □

Lemma 5. *The current $(1/2\pi)\Theta^Q$ on $\mathscr{H}$ represents the Chern-class of the bundles $\det((\widehat{\lambda} - \widehat{\lambda}^{-1})^{N+1})$ on $\mathscr{H}$.*

Proof. We use an auxiliary C^∞ hermitian metric h_a on $\widehat{\lambda}$ with curvature form Θ_a. Then the fiber integral $\int_{\widehat{\mathscr{H}}/\mathscr{H}} \Theta_a^{N+1}$ exists and represents up to a numerical constant the Chern class $c_1(\det((\widehat{\lambda}-\widehat{\lambda}^{-1})^{N+1}))$ on $\mathscr{H}$. On $\mathscr{H}_{fr}$ the difference $\Theta_{fr}-\Theta_a$ is (globally) of the form $\sqrt{-1}\partial\overline{\partial}u$. Now

$$\Theta_{fr}^{N+1} = \sqrt{-1}\partial\overline{\partial}u \wedge \Omega + \Theta_a^{N+1},$$

where $\Omega = \sum_{j=0}^{N} \Theta_{fr}^j \wedge \Theta_a^{N-j}$.

Basic properties of the L^2-Dolbeault-complex on $\Delta^{*k}\times\Delta^l$ (cf. [Z]) show that u, and $\overline{\partial}u$ can be chosen as locally L^2-integrable (with respect to metrics with Poincaré growth condition). So $\int_{\mathscr{H}_{fr}/\mathscr{H}} \sqrt{-1}\,\overline{\partial}u\wedge\Omega$ defines actually a current. We claim that in the sense of currents

$$\int_{\mathscr{H}_{fr}/\mathscr{H}} \sqrt{-1}\,\partial\overline{\partial}u\wedge\Omega = -d\int_{\mathscr{H}_{fr}/\mathscr{H}} \sqrt{-1}\,\overline{\partial}u\wedge\Omega \tag{16}$$

holds.

In fact, the right hand side applied to a C^∞-form with compact support equals

$$-d\int_{\mathscr{H}_{fr}} \overline{\partial}u\wedge\Omega\wedge\widetilde{\nu}^*d\varphi = \int_{\mathscr{H}_{fr}} \partial\overline{\partial}u\wedge\Omega\wedge\widetilde{\nu}^*\varphi = \left(\int_{\mathscr{H}_{fr}} \partial\overline{\partial}u\wedge\Omega\right)(\varphi).$$

$\square$

Corollary 3. *There exists a singular hermitian metric h^Q for $\det((\widehat{\lambda}-\widehat{\lambda}^{-1})^{N+1})$ on $\mathscr{H}$, whose curvature is positive in the sense of currents.*

Remark 3. *Furthermore, it follows from the construction that for any subspace of $\mathscr{H}$, in particular for any curve in $\mathscr{H}$, the restriction of h^Q and Θ^Q resp. exist as singular metric and d-closed current resp. If $C\subset\mathscr{H}$ is a local analytic curve through a point p, representing a direction, where $\mathscr{X}\longrightarrow\mathscr{H}$ is effective, the current is strictly positive in this direction.*

The latter fact follows immediately, because the form Θ_{fr} is strictly positive on the preimage of C in $\mathscr{H}_{fr}$.

After blowing up the boundary λ^Q possesses a line bundle extension $\overline{\lambda^Q}$ on $\overline{\mathscr{H}}$. The result of this paragraph concerning Hilbert schemes is so far:

Theorem 4. *The compactified Hilbert scheme $\overline{\mathscr{H}} \supset \mathscr{H}$ carries a line bundle $\overline{\lambda}^Q$ with a singular hermitian metric $\overline{h^Q}$ whose curvature $\overline{\Theta^Q}$ is positive. The Lelong numbers vanish everywhere, and Θ^Q is strictly positive in effective directions of the family $\mathscr{X} \longrightarrow \mathscr{H}$. Moreover, on $\mathscr{H}$ the construction is functorial with respect to base changes of families concerning the line bundle and its curvature.*

In a final step we descend to the moduli space $\overline{\mathscr{M}}$.

The automorphism groups of the polarized manifolds act on local universal deformation spaces in a finite way (with uniformly bounded orders). By functoriality, a certain power $(\lambda^Q)^\mu$ descends from $\mathscr{H}$ to some $\lambda_{\mathscr{M}}$ on $\mathscr{M}$ together with a singular, positive hermitian metric $h_{\mathscr{M}}$. On $\overline{\mathscr{M}}$ the line bundle $\overline{\lambda^Q}$ gives rise to a coherent sheaf. As $\mu \cdot \Theta^Q$ is invariant under the action of the projective linear group on $\mathscr{H}$, it descends to the curvature current $\Theta_{\mathscr{M}}$ on $\mathscr{M}$. We look at the natural map $u : \mathscr{H} \longrightarrow \mathscr{M}$ extended to $\overline{u} : \overline{\mathscr{H}} \longrightarrow \overline{\mathscr{M}}$. The current $\Theta_{\mathscr{M}}$ will now be extended to $\overline{\mathscr{M}}$: Let φ be a C^∞ differential form of degree $(\dim \mathscr{M} - 1, \dim \mathscr{M} - 1)$ with compact support. We take a closed subvariety $S \subset \overline{\mathscr{H}}$, so that the map $S \longrightarrow \overline{\mathscr{M}}$ is generically finite, and dominant. The following definition is independent of the choice of S:

$$\Theta_{\overline{\mathscr{M}}}(\varphi) = \int_{\mathscr{M}} \Theta_{\mathscr{M}} \wedge \varphi := \frac{1}{\alpha} \int_S \mu \cdot \Theta^Q \wedge \overline{u}^*(\varphi),$$

where α denotes the generic degree of the map $u|S : S \longrightarrow \overline{\mathscr{M}}$. With $\varphi = d\psi$ we see the closedness of the current. Again, we have a positive d-closed current $\Theta_{\overline{\mathscr{M}}}$. It realizes the Chern class of $\lambda_{\mathscr{M}}$ on $\mathscr{M}$, which is the restriction of a coherent sheaf on $\overline{\mathscr{M}}$. Again, after blowing up the boundary and taking a suitable power of the line bundle, it possesses a line bundle extension $\lambda_{\overline{\mathscr{M}}}$ with a corresponding singular hermitian metric $h_{\overline{\mathscr{M}}}$ constructed from the current $\Theta_{\overline{\mathscr{M}}}$.

Theorem 5. *The moduli space $\mathscr{M}$ possesses a compactification $\overline{\mathscr{M}}$ as an algebraic space and a holomorphic line bundle $\overline{\lambda}$ with a singular hermitian metric h of positive curvature form Θ_h such that*

(i) *for all $p \in \overline{\mathscr{M}}$ and any holomorphic curve $C \subset \overline{\mathscr{M}}$ through p with $C \cap \mathscr{M} \neq \emptyset$ the (positive, d-closed) current $\Theta_h|C$ is well-defined, and the Lelong number $\nu(\Theta_h|C, p)$ vanishes,*

(ii) *for any smooth locally closed subspace $Z \subset \mathscr{M}$ the current $\Theta_h|_Z$ is well-defined, and $\Theta_h|_Z \geq \eta_Z$ in the sense of currents, where η_Z denotes some C^∞ hermitian form on Z.*

10. L^2-methods

In this section, we gather some results based upon Hörmander's techniques (cf. also the result by Ohsawa and Takegoshi [O-T]).

Let (Y, ω_Y) be a complete Kähler manifold, and (L, h) be a hermitian line bundle on Y. We write

$$\omega_Y = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}},$$

and use the semi-colon notation for covariant derivatives with respect to the metric tensor. Moreover the components of the connection form of the line bundle are

$$\theta_\alpha = \frac{h_{;\alpha}}{h},$$

and we denote by $\Theta_{\alpha\overline{\beta}}$ the coefficients of the curvature tensor. We use ∇_α for covariant derivatives of L-valued tensors, and $\|..\|$, $\|..(p)\|$ resp. for norms and pointwise norms resp. Let $\varphi = \varphi_{\overline{\beta}}\, dz^{\overline{\beta}}$ be any L-valued $(0,1)$-form of class C^∞. Then

$$\overline{\partial}^* \varphi = -g^{\bar{\beta}\alpha} \nabla_\alpha \varphi_{\bar{\beta}} = -g^{\bar{\beta}\alpha}(\varphi_{\bar{\beta};\alpha} + \varphi_{\bar{\beta}} \theta_\alpha)$$

is the formal adjoint of the $\overline{\partial}$-operator.

The *rough Laplacian* is defined by

$$\Delta\!\!\!\!\!-\,\varphi = -g^{\bar{\delta}\gamma} \nabla_\gamma \nabla_{\bar{\delta}} \varphi_{\bar{\beta}} dz^{\bar{\beta}},$$

and the Bochner-Kodaira-Nakano-Weitzenboeck formula for this situation reads

$$\Box\varphi = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\varphi = \Delta\!\!\!\!\!-\,\varphi + g^{\bar{\delta}\alpha} \varphi_{\bar{\delta}} (R_{\alpha\bar{\beta}} + \Theta_{\alpha\bar{\beta}}) dz^{\bar{\beta}},$$

where $R_{\alpha\bar{\beta}}$ denotes the Ricci-tensor of ω_X. (The contribution $R_{\alpha\bar{\beta}}$ cancels out, if we replace L by $L + K_Y$). The formula implies

$$\|\overline{\partial}\varphi\|^2 + \|\overline{\partial}^*\varphi\|^2 \geq \int_Y \varphi_{\bar{\beta}} (R_{\gamma\bar{\delta}} + \Theta_{\gamma\bar{\delta}}) \overline{\varphi}_\alpha g^{\bar{\beta}\gamma} g^{\bar{\delta}\alpha}\, h\, g\, dV_\omega \tag{17}$$

for all C^∞-forms with compact support. According to Andreotti and Vesentini [A-V] the estimate (17) holds (use cut-off functions) for all square integrable forms φ, for which $\overline{\partial}\varphi$ and $\overline{\partial}^*\varphi$, taken in the distributional sense, are square integrable. Let H_1 and H_2 resp. be the Hilbert spaces of square integrable L-valued $(n,0)$- and $(n,1)$-forms resp. Then the exterior derivative $\overline{\partial}$ is a densely defined closed operator $T : H_1 \longrightarrow H_2$ whose adjoint T^* is given by $\overline{\partial}^*$ (cf. [A-V]).

Proposition 7. *Let (Y, ω_Y) be a Kähler manifold, which possesses also a complete Kähler metric, and let (L, h) be a holomorphic line bundle, with a singular hermitian metric. Suppose that*

$$\Theta_h \geq c(p) \cdot \omega_Y$$

for some continuous, everywhere positive function $c(p)$ on Y. Then for any L-valued $(n,1)$-current v with $\overline{\partial}v = 0$, and

$$\int_Y \frac{1}{c(p)} \|v(p)\|^2 dV_\omega < \infty$$

there exists an L-valued current u with $\overline{\partial}u = v$ and

$$\int_Y \|u(p)\|^2 dV_\omega \leq \int_Y \frac{1}{c(p)} \|v(p)\|^2 dV_\omega.$$

Proof. We assume first that h is of class C^∞ and that ω_Y is complete. We follow the argument of Hörmander and Demailly. The closed subspace $F \subset H_2$ of all $\overline{\partial}$-closed forms contains the range of T, and T^* vanishes on the orthogonal complement of F so that we can consider T as an operator from H_1 to F, and T^* as an operator from F to H_1. Now (17) implies for all $\psi \in F$, contained in the domain of T^* that

$$\|T^*(\psi)\|^2 \geq \int_Y c(p) \|\psi(p)\|^2 dV_\omega.$$

For any $\psi, v \in F$, with ψ in the domain of T^* and $\int (1/c(p)) \|v(p)\|^2 < \infty$, we have

$$|(\psi, v)|^2 \leq \int_Y \frac{1}{c(p)} \|v(p)\|^2 dV_\omega \cdot \int_Y c(p) \|\psi(p)\|^2 dV_\omega,$$

hence

$$|(\psi, v)| \leq \left(\int_Y \frac{1}{c(p)} \|v(p)\|^2 dV_\omega \right)^{1/2} \cdot \|T^*(\psi)\|.$$

For any such v there is a continuous linear functional on the range of T^* sending $T^*\psi$ to (ψ, v). The Hahn-Banach theorem implies the existence of some $u \in H_1$ such that $(T^*\psi, u) = (\psi, v)$ for all ψ in the domain of T^*, i.e. $v = Tu$. Moreover $\|u\|^2 \leq \int_Y (1/c(p)) \|v(p)\|^2 dV_\omega$.

The extension of this result by Demailly to arbitrary Kähler metrics in [DE1], and the generalization to singular hermitian metrics due to Nadel [NA] are also applicable to the above case involving a function $c(p)$. □

11. Multiplier ideal sheaves

Let (L, h) be a singular hermitian line bundle on a complex manifold M. The sheaf $\mathscr{L}^2(L, h)$ of square-integrable sections with respect to h is defined by

$$\mathscr{L}^2(L, h)(U) = \{\sigma \in \Gamma(U, \mathscr{O}_M(L)); h(\sigma, \sigma) \in L^1_{loc}(U)\},$$

for open subsets $U \subset M$. There exists an ideal sheaf $\mathscr{I}(h)$, called *multiplier ideal sheaf* such that

$$\mathscr{L}^2(L, h)(U) = (\mathscr{O}_M(L) \otimes \mathscr{I}(h))(U)$$

holds. If we write $h = \mathbf{e}^{-\varphi} \cdot h_0$, where h_0 is a hermitian metric of class C^∞, and $\varphi \in L^1_{loc}(M)$ is the weight function, we see that

$$\mathscr{I}(h) = \mathscr{L}^2(\mathscr{O}_M, \mathbf{e}^{-\varphi})$$

holds. We also use the notation $\mathscr{I}(\varphi)$ for this sheaf.

For any modification $\pi : \widetilde{M} \longrightarrow M$ of complex manifolds, and any plurisubharmonic function χ the following identity of multiplier ideal sheaves is known (cf. [DE4, Prop. 5.8]):

$$\pi_*(\mathscr{O}_{\widetilde{M}}(K_{\widetilde{M}}) \otimes \mathscr{I}(\chi \circ \pi)) = \mathscr{O}_M(K_M) \otimes \mathscr{I}(\chi). \tag{18}$$

Definition 6. *A plurisubharmonic function φ on a complex manifold is said to have analytic singularities, if locally*

$$\varphi = \alpha \log(\sum_1^k |f_i|^2) + \varphi_0,$$

where the f_i denote holomorphic functions, φ_0 is a C^∞-function, and $\alpha \in \mathbb{R}_+$.

If σ_i are global sections of a line bundle L,

$$h^\alpha = \frac{\mathbf{e}^{-\varphi_0}}{(\sum |\sigma_i|^2)^\alpha}$$

defines a singular hermitian metric of positive curvature. In the above sense it will be called a metric with analytic singularities or algebraic singularities resp. (In the latter case $\alpha \in \mathbb{Q}_+$ is also required).

In the above situation the holomorphic functions f_i define some ideal $\mathscr{J} \subset \mathscr{O}_M$. We blow up M along the ideal $\mathscr{J}$, to make it locally free and in a way such that the exceptional set of the blow-up becomes a divisor $D = \sum D_I$ with normal crossings. We call the resulting modification $\pi : \widetilde{M} \longrightarrow M$. Now

$$K_{\widetilde{M}} = \pi^* K_M + R, \tag{19}$$

where $R = \sum \rho_j D_j$, $\rho_i \in \mathbb{N}$ is the exceptional divisor of π on $\widetilde{M}$.

The pull-back of $(\sum |f_i|^2)^\alpha$ to $\widetilde{M}$ vanishes on D, is of the form $\prod |\tau_i|^{2\beta_i}(1 + \sum |\widetilde{f}_j|^2)$, and

$$\varphi \circ \pi = \sum \beta_i \log |\tau_i|^2 + \widetilde{\varphi}_0, \tag{20}$$

where $\{\tau_i\}$ are defining functions of $\{D_i\}$, $\beta_i \in \mathbb{R}_{\geq 0}$, and $\widetilde{\varphi}_0$ is some C^∞-function. In this case the multiplier ideal sheaf can be computed explicitly as

$$\mathscr{I}(\varphi \circ \pi) = \mathscr{O}_{\widetilde{M}}(-\sum \lfloor \beta_i \rfloor D_i), \tag{21}$$

where $\lfloor \beta_i \rfloor$ is the Gaussian bracket. Together with (18) this implies

$$\mathscr{I}(\varphi) = \pi_* \mathscr{O}_{\widetilde{M}} \left(\sum (\rho_i - \lfloor \beta_i \rfloor) D_i \right). \tag{22}$$

In particular, $\mathscr{I}(\varphi \circ \pi)$ is locally free.

Proposition 8. *Let $\Delta^n \subset \mathbb{C}^n$ be a polydisk, φ a plurisubharmonic function with analytic singularities on Δ^n, and ψ a plurisubharmonic function such that $\sqrt{-1}\partial\overline{\partial}\psi$ is absolutely continuous on any local holomorphic curve $C \subset \Delta^n$ with $\varphi|C \not\equiv -\infty$. Then, after replacing Δ^n by any smaller, relatively compact polydisk, there exists real numbers γ arbitrarily close to 1 such that $\mathscr{I}(\gamma \cdot \varphi) = \mathscr{I}(\gamma \cdot \varphi + \psi)$ holds.*

Proof. In the sequel, we always allow Δ^n to be replaced by a slightly smaller polydisk. We first apply the above modification to $M = \Delta^n$ with respect to φ. Then we perform a further sequence of blow-ups and get a modification $\pi : \widetilde{\Delta} \longrightarrow \Delta^n$ so that also $\mathscr{J} = \mathscr{I}((\varphi + \psi) \circ \pi)$ is locally free, and such that with $\widetilde{M} = \widetilde{\Delta}$ the exceptional divisor is of the above form $D = \sum D_i$ with normal crossings. We still have (20,21) for φ.

For any point $x \in \widetilde{\Delta} \backslash D$ the function $\varphi \circ \pi$ is of class C^∞, and $\psi \circ \pi$ is absolutely continuous, when restricted to curves through x. Hence, by additivity of Lelong numbers, $\nu((\varphi + \psi) \circ \pi, x)$ vanishes. By [BO, SK] we have $\mathscr{J}_x = \mathscr{O}_{\widetilde{\Delta},x}$. So $V(\mathscr{J}) \subset D$. Hence $\mathscr{J} = \mathscr{O}_{\widetilde{\Delta}}(-\sum \beta_i' D_i)$ for some nonnegative integers β_i'.

Next, we use (18) as above and get

$$\mathscr{I}(\alpha \cdot \varphi) = \pi_* \left(\mathscr{O}_{\widetilde{\Delta}}(\sum (\rho_i - \lfloor \alpha \beta_i \rfloor) D_i) \right) \tag{23}$$

for all $\alpha > 0$.

We chose α so that $\alpha\beta_i \notin \mathbb{Z}$ for all $\beta_i \neq 0$. Next, we compute Lelong numbers. Let $x \in \widetilde{\Delta}$ and $\widetilde{C} \subset \widetilde{\Delta}$ a local analytic curve through x. If $\pi(\widetilde{C})$ is a point, at which ψ is different from $-\infty$, the Lelong number of $\psi \circ \pi$ vanishes. If $\pi(\widetilde{C})$ is a curve C, the assumption that $\psi|C$ is absolutely continuous implies that $\nu(\psi \circ \pi|\widetilde{C}, x) = 0$. Again, by additivity of Lelong numbers, $\nu(\pi^*(\alpha\varphi + \psi)|\widetilde{C}, x) = \nu(\pi^*(\alpha\varphi)|\widetilde{C}, x)$.

So far $\nu(\pi^*(\alpha\varphi + \psi), x) = \nu(\pi^*(\alpha\varphi), x)$ holds on $\widetilde{\Delta}$ (cf. [SI1]). For any point $x \in D_i \backslash \sum_{j \neq i} D_j$ this Lelong number is equal to $\nu_i := \alpha\beta_i \notin \mathbb{Z}$.

The latter fact allows us to compute the multiplier ideal sheaf from the Lelong number: As $\mathscr{I}((\alpha \cdot \varphi + \psi) \circ \pi)$ is locally free and the space is smooth, it is sufficient to compute it for points on the regular part of the normal crossings divisor D. Let D_i be the zero set of a coordinate function τ_i. Then

$$0 \leq \nu(|\tau_i|^{2\lfloor \nu_i \rfloor} \mathbf{e}^{-(\alpha\varphi + \psi)\circ\pi}, x) < 1$$

at some $x \in D_i \backslash \bigcup_{j \neq i} D_i$. It follows from [BO, SK] that

$$\mathscr{I}(|\tau_i|^{2\lfloor \nu_i \rfloor} \mathbf{e}^{-(\alpha\varphi + \psi)\circ\pi})_x = \mathscr{O}_{\widetilde{\Delta},x}$$

i.e. $\tau_i^{\lfloor \nu_i \rfloor} \in \mathscr{I}((\alpha\varphi + \psi) \circ \pi)_x$. We need to see that no lower power τ_i^k is contained in this multiplier ideal sheaf.

From the Lelong number of h, we get the known lower estimate

$$h \geq \frac{C}{\|z - x\|^{2\nu_i}}.$$

We use this estimate on a local analytic curve C_x, which intersects D_i in x transversally. So $\int_{C_x} h|\tau_i|^{2k} dV_{C_x} = \infty$. The same argument is used for all points on D_i near x. By Fubini's theorem τ_i^k is not in the multiplier ideal sheaf. Now equation (18) implies the claim. □

Remark 4. *The above proposition is still valid for the wider class of those plurisubharmonic functions, which differ from a plurisubharmonic function with analytic singularities, by a function, which is bounded by* $c \cdot \log(-\log \delta(x)))$*, where* $c > 0$ *is a constant, and* δ *is the distance of* x *from the singular set.*

12. A criterion for quasi-projectivity

Let X be a not necessarily reduced algebraic space with compactification $\overline{X}$ in the sense of algebraic spaces, and let L be a holomorphic line bundle on $\overline{X}$ with a *positive* singular hermitian metric h on $L|red(\overline{X})$ in the sense of Section 2.

Condition (P). *We say that the positivity condition (P) holds, if*

(i) *for all* $p \in \overline{X}$ *and any holomorphic curve* $C \subset \overline{X}$ *through* p *with* $C \cap X \neq \emptyset$ *the (positive, d-closed) current* $\Theta_h|C$ *is well-defined, and the Lelong number* $\nu(\Theta_h|C, p)$ *vanishes,*

(ii) *for any smooth locally closed subspace* $Z \subset X$ *the current* $\Theta_h|_Z$ *is well-defined, and* $\Theta_h|_Z \geq \gamma_Z$ *in the sense of currents, where* γ_Z *denotes some positive definite* C^∞ *hermitian form on* Z*.*

Now we state the criterion.

Theorem 6. *Let* X *be an irreducible, not necessarily reduced algebraic space with a compactification* $\overline{X}$*. Let* L *be a holomorphic line bundle on* $\overline{X}$*. The map*

$$\Phi_{|mL|} : \overline{X} \rightharpoonup \mathbb{P}_{N(m)},$$

where $N(m) = \dim |mL|$*, defines an embedding of* X *for sufficiently large* m*, if it satisfies condition (P).*

The above condition (P) can be relaxed in the sense that Lelong numbers need only vanish on $X \subset \overline{X}$ and that the hermitian metric has only analytic singularities at the boundary.

We will first assume that X is reduced and irreducible, and prove the theorem by induction over $n = \dim X$. The case $n = 1$ is obvious: Let X be an algebraic curve. If $\overline{X}$ is smooth, the assumption implies that $\deg(L) > 0$. Let $\overline{X}$ be a singular curve and $\pi : \widetilde{X} \longrightarrow \overline{X}$ be the normalization. Then $\deg(\pi^* L) > 0$ from the assumption so that $L^{\otimes \ell}$ defines an embedding of $\overline{X}$ into a projective space.

13. Bigness of L and weak embedding property

Compact spaces. Let X be a reduced, irreducible, compact complex space of dimension n, and $\mathscr{L} = \mathscr{O}_X(L) \in Coh(X)$ an invertible sheaf.

Definition 7. *The sheaf $\mathscr{L}$ is called big, if*

$$\limsup_{m \longrightarrow \infty} \frac{1}{m^n} h^0(X, \mathscr{L}^{\otimes m}) > 0.$$

In the sequel we denote by $\nu : Y \longrightarrow X$ the normalization of the (not necessarily locally irreducible) space X, and by $\rho : Z \longrightarrow Y$ a modification such that Z is smooth. If X is a Moishezon space, we assume also that Z is projective. Let $\pi = \rho \circ \nu$.

Proposition 9. *The following are equivalent:*

(i) *$\mathscr{L}$ is big*
(ii) *$\nu^* \mathscr{L}$ is big*
(iii) *$\pi^* \mathscr{L}$ is big*

Proof. We show that (ii) implies (i): Consider the exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \nu_* \mathscr{O}_Y \longrightarrow \mathscr{C} \longrightarrow 0,$$

where $\operatorname{supp}(\mathscr{C}) \subsetneq X$ is nowhere dense, and

$$0 \longrightarrow \mathscr{L}^{\otimes m} \longrightarrow \nu_* \nu^* \mathscr{L}^{\otimes m} \longrightarrow \mathscr{C} \otimes \mathscr{L}^{\otimes m} \longrightarrow 0.$$

The claim follows, because $h^0(\operatorname{supp}(\mathscr{C}), \mathscr{L}^{\otimes m} \otimes \mathscr{C}) = O(m^{n-1})$, and $h^0(Y, \nu^* \mathscr{L}^{\otimes m}) \sim m^n$.

The other implications are obvious. □

For any $m > 0$ with $h^0(X, \mathscr{L}^{\otimes m}) > 0$ we denote by $\Phi_{\mathscr{L}^{\otimes m}} : X \longrightarrow \mathbb{P}_N$, $N = N(m)$ the meromorphic map induced by global sections.

Proposition 10. *Let X be a (reduced) compact Moishezon space then the following are equivalent:*

(i) *$\mathscr{L}$ is big*
(ii) *$\Phi_{\mathscr{L}^{\otimes m}} : X \longrightarrow \mathbb{P}_N$ embeds some Zariski open subset of X for some $m > 0$*
(iii) *$\dim \Phi_{\mathscr{L}^{\otimes m}}(X) = \dim X$ for some $m > 0$*

Proof. We need to show that (i) implies (ii), the remaining implications are clear.

We consider as above the normalization and desingularization maps with Z projective. By Proposition 9, $\pi^*\mathscr{L}$ is big on Z. Let $\mathscr{A}$ be a very ample invertible sheaf. By Kodaira's lemma (cf. [K-O, App.]), for some $m > 0$ the sheaf $\pi^*\mathscr{L}^{\otimes m} \otimes \mathscr{A}^{-1}$ possesses a non-zero section with zero divisor E so that the sections of $\pi^*\mathscr{L}^{\otimes m}$ yield an embedding of $Z\backslash E$ into some $\mathbb{P}_N$. As $H^0(Z, \pi^*\mathscr{L}^{\otimes m}) = H^0(Y, \nu^*\mathscr{L}^{\otimes m})$, the invertible sheaf $\nu^*\mathscr{L}^{\otimes m}$ gives rise to an embedding of some Zariski open subset of Y into $\mathbb{P}_N$. Consider

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \nu_*\mathscr{O}_Y \longrightarrow \mathscr{C} \longrightarrow 0.$$

Let $\mathscr{I} \subset \mathscr{O}_X$ be the annihilator of $\mathscr{C}$. The zero set $V(\mathscr{I}) \subset X$, consisting of all non-normal points of X, is nowhere dense. We have $\mathscr{I} \cdot \nu_*\mathscr{O}_Y \subset \mathscr{O}_X$. Let $\mathscr{J} = \pi^*\mathscr{I} \subset \mathscr{O}_Z$. As $\mathscr{J} \cdot \mathscr{A}^{\otimes \ell}$ is globally generated for some $\ell > 0$, the linear system $H^0(Z, \mathscr{J} \cdot \mathscr{A}^{\otimes(\ell+1)}) \subset H^0(Z, \mathscr{A}^{\otimes(\ell+1)})$ embeds $Z\backslash V(\mathscr{J})$ into some projective space. Next, the multiplication with a canonical section of $\mathscr{O}_Z((\ell + 1)\cdot E)$ defines a map $H^0(Z, \mathscr{J} \cdot \mathscr{A}^{\otimes(\ell+1)}) \longrightarrow H^0(Z, \mathscr{J} \cdot \pi^*\mathscr{L}^{\otimes(\ell+1)}) \subset H^0(Z, \pi^*\mathscr{L}^{\otimes(\ell+1)}) = H^0(X, \nu_*\nu^*\mathscr{L}^{\otimes(\ell+1)})$, whose composition with $H^0(X, \nu_*\nu^*\mathscr{L}^{\otimes(\ell+1)}) \longrightarrow H^0(X, \mathscr{C} \otimes \mathscr{L}^{\otimes(\ell+1)})$ is identically zero. So the image of $H^0(Z, \mathscr{J} \cdot \mathscr{A}^{\otimes(\ell+1)})$ in $H^0(Z, \mathscr{J} \cdot \pi^*\mathscr{L}^{\otimes(\ell+1)})$ is contained in the subspace $H^0(X, \mathscr{L}^{\otimes(\ell+1)})$. Hence global sections of $\mathscr{L}^{\otimes(\ell+1)}$ embed a Zariski open subset of X. □

Compactified spaces. We return to the situation of Theorem 6, and we assume that $\overline{X} \supset X$ is reduced and irreducible. To show that L is big, we use the L^2-methods from Section 10.

Let $U \subset X_{reg}$ be a Zariski open subset, which is quasi-projective. We can find a smooth, projective compactification $\overline{U}$ together with a modification $\gamma : \overline{U} \longrightarrow \overline{X}$ such that the divisor $D = \overline{U} \backslash U$ has only normal crossings singularities and such that the singular hermitian metric h extends from U to $\overline{U}$ as a singular hermitian metric on $\gamma^* L$ (cf. Section 2). As usual one can construct a complete Kähler form η_U on U with Poincaré growth near the boundary from a Kähler form on $\overline{U}$ and a canonical section of D.

Lemma 6. *Let $x \in U$ be a point. Then there exists some $m_0 > 0$ so that for any $m \geq m_0$ there is a section*

$$\sigma \in H^0_{(2)}(U, \mathscr{O}_U(K_U + mL))$$

with $\sigma(x) \neq 0$.

Proof. We use Kodaira's argument. Let $W = \{(z_1, \ldots, z_n)\} \subset U$ be a coordinate neighborhood and ρ a cut-off function with support in W, which is identically equal to one a relatively compact neighborhood of x contained in W and has values between 0 and 1. We set

$$\psi_x = \rho(z) \cdot n \cdot \log(\sum |z_i|^2).$$

There exists some $m_0 > 0$ and a continuous strictly positive function $\widetilde{c}(p)$ on U so that

$$\sqrt{-1}\partial\overline{\partial}\psi_x + m_0 \cdot \Theta_h \geq \widetilde{c}(p) \cdot \eta_U.$$

Let $m \geq m_0$. We chose a local section $\sigma_x \in H^0(W, K_X + mL)$ on W with $\sigma_x(x) \neq 0$ and set

$$f = \overline{\partial}(\rho\sigma_x).$$

The metric $\mathbf{e}^{-\psi_x} h$ satisfies the assumptions of Proposition 7. Moreover f vanishes identically in a neighborhood of x. As the Lelong numbers of h vanish at all points of U,

$$\int_U \frac{1}{\widetilde{c}(p)} \mathbf{e}^{-\psi_x} ||f||^2 dV_\eta < \infty.$$

Now Proposition 7 implies the existence of a $(n, 0)$-form u of class C^∞ (since f is of class C^∞) with values in mL such that

$$\overline{\partial} u = f,$$

and

$$(\sqrt{-1})^{n^2} \int_U \mathbf{e}^{-\psi_x} h^m u \wedge \overline{u} < \infty.$$

The finiteness of the above integral (plus the fact that h is bounded from below, and that u is holomorphic on some neighborhood of x) imply that $u(x) = 0$. Now we can see that

$$\sigma = \rho \cdot \sigma_x - u$$

is an element of $H^0_{(2)}(U, \mathscr{O}_U(K_U + mL))$, which does not vanish at x. $\square$

In a similar way, by taking two points and directions at a point we obtain the following lemma.

Lemma 7. *For any compact set $K \subset U$ there exists a number $m(K) > 0$ so that for all $m \geq m(K)$ the linear system $|H^0_{(2)}(U, \mathscr{O}_U(K_U + mL))|$ gives an embedding of K into a projective space.*

We have the following extension property:

Lemma 8. *There is a canonical embedding:*

$$H^0_{(2)}(U, \mathscr{O}_U(K_U + mL)) \hookrightarrow H^0(\overline{U}, \mathscr{O}_{\overline{U}}(K_{\overline{U}} + m\gamma^* L))$$

Proof. In our situation h possesses an extension $\widetilde{h}$ as a singular metric on $\gamma^* L$ with positive curvature. In particular, $\widetilde{h}$ is locally bounded from below by a positive constant. For any $\sigma \in H^0_{(2)}(U, \mathscr{O}_U(K_U + mL))$ we have

$$(\sqrt{-1})^{n^2} \int_{\overline{U}} \widetilde{h}^m \gamma^* \sigma \wedge \overline{\gamma^* \sigma} < \infty.$$

So $\gamma^* \sigma$ extends holomorphically to $\overline{U}$. $\square$

We are given a singular hermitian metric on L over the reduced, and irreducible complex space $\overline{X}$, which amounts to a singular hermitian metric h on $\overline{X}_{reg}$, which can be extended from U as a singular metric $\widetilde{h}$ on $\gamma^* L$ over $\overline{U}$. The latter defines a multiplier ideal sheaf $\mathscr{I}(\widetilde{h}^m) \subset \mathscr{O}_{\overline{U}}$ which is defined by the following property: For all open subsets $W \subset \overline{U}$ the space

$$(\mathscr{O}_{\overline{U}}(K_{\overline{U}} + m\gamma^* L) \otimes \mathscr{I}(\widetilde{h}^m))(W)$$

consists of all

$$\sigma \in \mathscr{O}_{\overline{U}}(K_{\overline{U}} + m\gamma^* L)(W \cap U)$$

such that

$$(\sqrt{-1})^{n^2} \int_V \widetilde{h}^m \sigma \wedge \overline{\sigma} < \infty$$

for all $V \subset\subset W$.

Definition 8. *A bundle* (L,h) *is called* big *in the sense of singular hermitian bundles, if*

$$\limsup_{m\longrightarrow\infty} m^{-n} h^0(\overline{U}, \mathscr{O}_{\overline{U}}(m\gamma^* L)\otimes \mathscr{I}(\widetilde{h}^m)) > 0$$

holds.

For any such bundle, the pull-back $\widehat{L}$ of L to the normalization $\widehat{X} \longrightarrow \overline{X}$ satisfies $\limsup_{m\longrightarrow\infty} m^{-n} h^0(\widehat{X}, \mathscr{O}_{\widehat{X}}(m\widehat{L})) > 0$. According to Proposition 9, any such bundle is big in the usual sense, and Proposition 10 guarantees that associated linear systems embed certain Zariski open subsets.

We claim:

Proposition 11. *The above line bundle* (L,h) *is big on* $\overline{X}$*.*

Proof. Let

$$0 \neq \sigma_0 \in H^0_{(2)}(U, \mathscr{O}_{\overline{X}}(K_{\overline{X}} + m_0 L)),$$

be a section, which we extend to $\overline{U}$. We denote by $D_0 \subset \overline{U}$ the zero divisor. Next, we consider the restriction morphism

$$\begin{aligned} r_m : H^0_{(2)}(U, \mathscr{O}_U(K_U + mL|U)) = H^0(\overline{U}, \mathscr{O}_{\overline{U}}(K_{\overline{U}} + m\gamma^*L) \otimes \mathscr{I}(\widetilde{h}^m)) \\ \longrightarrow H^0(D_0, \mathscr{O}_{D_0}(K_{\overline{U}} + m\gamma^* L)). \end{aligned}$$

Since

$$h^0(D_0, \mathscr{O}_{D_0}(K_{\overline{U}} + m\gamma^*L)) = O(m^{n-1}),$$

and

$$\limsup_{m\longrightarrow\infty} m^{-n} \dim H^0_{(2)}(U, \mathscr{O}_{\overline{X}}(K_{\overline{X}} + mL)) > 0,$$

we see that

$$\limsup_{m\longrightarrow\infty} m^{-n} \dim \ker r_m > 0.$$

Now

$$\ker r_m \subset H^0(\overline{U}, \mathscr{O}_{\overline{U}}(K_{\overline{U}} + m\gamma^*L) \otimes \mathscr{I}(\widetilde{h}^m)) \cap H^0(\overline{U}, \mathscr{O}_{\overline{U}}(m-m_0)\gamma^*L).$$

Since $\widetilde{h}$ is locally bounded from below by some (positive) constant (in the appropriate measure theoretic sense), $\mathscr{I}(\widetilde{h}^m) \subset \mathscr{I}(\widetilde{h}^{m-m_0})$ holds. So $\ker r_m \subset H^0(\overline{U}, \mathscr{O}_{\overline{U}}((m-m_0)\gamma^*L) \otimes \mathscr{I}(\widetilde{h}^{m-m_0}))$. □

We state the following general fact, which implies that the above line bundle L, pulled back to a desingularization is nef.

Proposition 12. *Let Y be a projective manifold and (L,h) a positive, singular hermitian line bundle, whose Lelong numbers vanish everywhere. Then L is nef.*

Proof. Let A be an ample line bundle on X. For any $y \in Y$ one considers a finite, locally free resolution

$$\mathscr{P}^\bullet \longrightarrow \mathfrak{m}_{Y,y}$$

of the maximal ideal at y. Then we chose a multiple $\ell(y)\cdot A$ so that all $\mathscr{P}^j \otimes K_Y^{-1}(\ell(y)\cdot A)$ are positive. The value for $\ell(y)$ can be taken uniformly in a neighborhood of y. So we choose ℓ_0 uniformly on Y with this property. As the Lelong numbers vanish, the multiplier ideal sheaves $\mathscr{I}(h^m)$ are equal to $\mathscr{O}_Y$ for all $m > 0$. So $H^q(Y, \mathscr{P}^r \otimes \mathscr{O}(\ell_0 A + mL)) = 0$ for all r and $q, m > 0$ by the Nadel vanishing theorem. Now $H^1(Y, \mathfrak{m}_{Y,y} \otimes \mathscr{O}_Y(\ell_0 A + mL)) = 0$ for $m > 0$, and the sheaves $\mathscr{O}_Y(\ell_0 A + mL)$ are globally generated, in particular nef. Hence $L + \frac{\ell_0}{m} A$ is nef for any $m > 0$. With $m \longrightarrow \infty$ the claim follows. $\square$

14. Embedding of non-reduced spaces

Proposition 13. *Let X be a compact complex space, which possesses a holomorphic line bundle L, whose restriction to the reduction X_{red} is ample. Then L is ample.*

Proof. Let $\mathscr{O}_{X_{red}} = \mathscr{O}_X/\mathscr{I}$, and $X_j = (X_{red}, \mathscr{O}_X/\mathscr{I}^{j+1})$ so that $X_{red} = X_0$ and $X = X_k$ for some k. Let $\mathscr{L} = \mathscr{O}_X(L)$, and $\mathscr{L}_0 = \mathscr{L}|X_0$. Now

$$(\mathrm{E_j}) \qquad 0 \longrightarrow \mathscr{F}_j \longrightarrow \mathscr{O}_{X_{j+1}} \longrightarrow \mathscr{O}_{X_j} \longrightarrow 0$$

is a small extension, where $\mathscr{F}_j = \mathscr{I}^{j+1}/\mathscr{I}^{j+2}$ is a coherent $\mathscr{O}_{X_0}$-module.

We can assume from the beginning that $\mathscr{L}_0$ is very ample on X_0, and that $H^1(X_0, \mathscr{F}_j \otimes \mathscr{L}^{\otimes \ell}) = 0$ for all $\ell > 0$, and $j = 0, \dots k-1$, furthermore that $\mathscr{F}_j \otimes \mathscr{L}^{\otimes \ell}$ is globally generated for all j, and all $\ell > 0$. Now for all $\ell > 0$ the map $H^0(X, \mathscr{L}^{\otimes \ell}) \longrightarrow H^0(X_0, \mathscr{L}_0^{\otimes \ell})$ is surjective. So we have a holomorphic map $\Phi_{|L|} : X \longrightarrow \mathbb{P}_N$, whose restriction Φ_0 to X_0 is an embedding, i.e $\mathscr{O}_{\mathbb{P}_N} \longrightarrow \mathscr{O}_X$ followed by $\mathscr{O}_X \longrightarrow \mathscr{O}_{X_0}$ is surjective. We show by induction over j the existence of compatible maps $\alpha_j : \mathscr{O}_{\mathbb{P}_{N(j)}} \longrightarrow \mathscr{O}_{X_j}$. (In each step the number N will have to be raised.)

We assume that $\mathscr{O}_{\mathbb{P}_N} \longrightarrow \mathscr{O}_{X_j}$ is surjective. The pull-back of the small extension (E_j) of sheaves of analytic $\mathbb{C}$-algebras with respect to this map induces the direct sum of spaces $X_{j+1} \oplus_{X_j} \mathbb{P}_N$. As the surjective morphism α_j can be lifted to the morphism α_{j+1}, the induced small extension is trivial: we have the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}_j \arrow[r] & \mathscr{O}_{X_{j+1}} \arrow[r] & \mathscr{O}_{X_j} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F}_j \arrow[r] \arrow[u, equal] & \mathscr{O}_{\mathbb{P}_N}[\mathscr{F}_j] \arrow[r] \arrow[u, "\mathrm{surj}"] & \mathscr{O}_{\mathbb{P}_N} \arrow[r] \arrow[u, "\mathrm{surj}", "\alpha_j"'] & 0
\end{tikzcd}$$

Denote by $i : X_{j+1} \hookrightarrow \mathbb{P}_N[\mathscr{F}_j]$ and $\tau : \mathbb{P}_N[\mathscr{F}_j] \longrightarrow \mathbb{P}_N$ resp. the embedding and projection resp. Then $i^*\tau^*\mathscr{O}_{\mathbb{P}_N}(1) = \mathscr{L}$. So in

$$0 \rightarrow \mathscr{F}_j \otimes \mathscr{L}_0 \rightarrow (\mathscr{O}_{\mathbb{P}_N}[\mathscr{F}_j]) \otimes \mathscr{O}_{\mathbb{P}_N}(1) \rightarrow \mathscr{O}_{\mathbb{P}_N}(1) \rightarrow 0$$

we can identify the middle term with $\mathscr{O}_{\mathbb{P}_N}(1)[\mathscr{F}_j \otimes \mathscr{L}_0]$. Let $\{\sigma_0, \ldots, \sigma_N\} \subset H^0(\mathbb{P}_N, \mathscr{O}_{\mathbb{P}_N}(1))$ be a basis, and let $\mathscr{F}_j \otimes \mathscr{L}_0$ be generated by global sections $\tau_q, \ldots, \tau_r$. Let $\varepsilon^2 = 0$. Then the $\sigma_0, \ldots, \sigma_N, \varepsilon\tau_1, \ldots, \varepsilon\tau_r$ give rise to an embedding $\mathbb{P}_N[\mathscr{F}_j] \hookrightarrow \mathbb{P}_{N+r}$. Altogether $\mathscr{O}_{\mathbb{P}_{N+r}} \longrightarrow \mathscr{O}_{X_{j+1}}$ is surjective. $\square$

We need the above statement in a more general situation.

Let $\overline{Z}$ be a non-reduced complex space equipped with a holomorphic line bundle L, $\mathscr{L} = \mathscr{O}_{\overline{Z}}(L)$. Let $\overline{X} = \mathrm{red}(\overline{Z})$, and let $X \subset \overline{X}$ be a Zariski open subset. We denote by Z the restriction of the non-reduced structure to X. We consider the meromorphic map $\Phi = \Phi_{|L|_{\overline{X}}|} : \overline{X} \rightharpoonup \mathbb{P}_N$.

Proposition 14. *Assume that $\Phi|_X : X \longrightarrow \mathbb{P}_N$ is an embedding, and let the pull-back of L to some desingularization of $\overline{X}$ be also nef. Then for some multiple ℓ_0 the meromorphic map $\Phi_{|\ell_0 L|} : \overline{Z} \longrightarrow \mathbb{P}_M$ defines an embedding of an open subspace of Z.*

Proof. First, we take a (projective) desingularization of $\overline{X}$, and pull back the meromorphic map Φ. Then we eliminate the indeterminacy set by a sequence of blow-ups with smooth centers. This procedure is locally done by embedding the space in a smooth ambient space, blowing up the ambient space along smooth centers several times, and by taking in each step the proper transform of $\overline{X}$. We take locally embeddings of $\overline{X}$, which extend to embeddings of $\overline{Z}$. Let $\rho : \widetilde{Z} \longrightarrow \overline{Z}$

be the proper transform of $\overline{Z}$, together with the restriction $\pi : \widetilde{X} \longrightarrow \overline{X}$, which allows a morphism $\Psi : \widetilde{X} \longrightarrow \mathbb{P}_N$. We consider the k-th infinitesimal neighborhoods $\widetilde{Z}_k$ od $\widetilde{X}$ in $\widetilde{Z}$. These give rise to small extensions

$$0 \longrightarrow \mathscr{F}_j \longrightarrow \mathscr{O}_{\widetilde{Z}_{j+1}} \longrightarrow \mathscr{O}_{\widetilde{Z}_j} \longrightarrow 0,$$

(where the $\mathscr{F}_j$ are coherent $\mathscr{O}_{\widetilde{X}}$-modules).

Let $n = \dim X$. Denote by $\widetilde{L}$ the pull-back of L to $\widetilde{Z}$. We claim that $h^1(\widetilde{X}, \mathscr{F}_j(\ell \cdot \widetilde{L}|_{\widetilde{X}})) = O(\ell^{n-1})$ for any fixed j: The bundle $\widetilde{L}|_{\widetilde{X}}$ is big. After replacing $\widetilde{L}$ by a multiple, we write $\widetilde{L}|_{\widetilde{X}} = A + E$, where A is ample and E is effective by Kodaira's lemma. As $\widetilde{X}$ is projective we can fix a Kähler form $\eta_{\widetilde{X}}$, and since L is nef and big, for all $r > 0$ we can find hermitian metrics h_r on $\widetilde{L}|\widetilde{X}$ such that the curvature of h_r is greater or equal to $-(1/r)\eta_{\widetilde{X}}$. This shows the existence of some m_0 such that

$$H^1(\widetilde{X}, \mathscr{F}_j \otimes \mathscr{O}_{\widetilde{X}}(mA + \ell\widetilde{L})) = 0$$

for all $m \geq m_0$ and $\ell > 0$.

Let mE denote the non-reduced space with support E, induced by the divisor mE. Then

$$0 \longrightarrow \mathscr{F}_j(mA + \ell\widetilde{L}) \longrightarrow \mathscr{F}_j((m+\ell)\widetilde{L}) \longrightarrow \mathscr{F}_j((m+\ell)\widetilde{L})|_{mE} \longrightarrow 0$$

is exact, and for $m \geq m_0$

$$0 \longrightarrow H^1(\widetilde{X}, \mathscr{F}_j((m+\ell)\widetilde{L})) \longrightarrow H^1(mE, \mathscr{F}_j((m+\ell)\widetilde{L})|_{mE})$$

as well.

We fix $m = m_0$ and look at $\ell \gg 0$. Then $h^1(\widetilde{X}, \mathscr{F}_j((m_0+\ell)\widetilde{L})) = O(\ell^{n-1})$. Now

$$\begin{aligned} H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}_{j+1}}((m_0+\ell)\widetilde{L})) \longrightarrow H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}_j}((m_0+\ell)\widetilde{L})) \\ \longrightarrow H^1(\widetilde{X}, \mathscr{F}_j((m_0+\ell)\widetilde{L})) \end{aligned}$$

is exact, and $h^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}_j}(m_0+\ell)\widetilde{L})$ grows like ℓ^n, because we can assume by induction that high powers of $\widetilde{L}$ embed a Zariski open subset of $\widetilde{X}_j$, so $h^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}_{j+1}}((m_0+\ell)\widetilde{L})) \sim \ell^n$. This means that the sections of $H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}_{j+1}}((m_0+\ell)\widetilde{L}))$ define a meromorphic map $\Xi : \widetilde{X}_{j+1} \rightharpoonup \mathbb{P}_M$, which embeds an open subset $W \cap \widetilde{X}$ of the reduction $\widetilde{X}$, where $W \subset \widetilde{X}_{j+1}$ is open (cf. Proposition 10). We assume that $\Xi(W \cap \widetilde{X})$ is closed in some open set $\mathbb{P}_M\backslash B$ (everywhere with respect to the Zariski topology). The sets B and W can also be chosen in a way that

$$0 \longrightarrow (\Xi_*\mathscr{F}_j)|\mathbb{P}_M\backslash B \longrightarrow (\Xi_*\mathscr{O}_{\widetilde{Z}_{j+1}})|\mathbb{P}_M\backslash B \longrightarrow (\Xi_*\mathscr{O}_{\widetilde{Z}_j})|\mathbb{P}_M\backslash B \longrightarrow 0$$

is still exact. As in the proof of Proposition 13 we consider the fibered sum of complex spaces $W \oplus_{W\cap\widetilde{X}_j} (\mathbb{P}_M\backslash B)$, which is isomorphic to the trivial extension $(\mathbb{P}_M\backslash B)[\mathscr{F}_j|W]$, which is clearly quasi-projective. The rest follows like in the proof of Proposition 10. $\square$

15. Proof of the quasi-projectivity criterion

We first need the following fact:

Lemma 9. *Let $\pi : Y \longrightarrow X$ be a proper holomorphic map of reduced, compact, not necessarily normal, complex spaces. Let $S \subset X$ be a closed subspace such that π is an isomorphism over $X\backslash S$. Let $\mathscr{I} = \mathscr{I}_S \subset \mathscr{O}_X$ be the vanishing ideal of S. Then for any coherent sheaf $\mathscr{F}$ on X there is a number $m > 0$ and morphism $\mu : \mathscr{I}_S^m \cdot (\pi_*\pi^*\mathscr{F}) \longrightarrow \mathscr{F}$, which is an isomorphism over $X\backslash S$.*

Proof. We consider the short exact sequence $0 \longrightarrow \mathscr{F} \longrightarrow \pi_*\pi^*\mathscr{F} \longrightarrow \mathscr{C} \longrightarrow 0$, where $\mathrm{supp}(\mathscr{C}) \subset S$. Now the zero set of the annihilator ideal $V(Ann_{\mathscr{O}_X}(\mathscr{C}))$ is contained in S so that $\mathscr{I}^m \cdot \mathscr{C} = 0$ for some $m > 0$. $\square$

We consider the situation of Theorem 6 and assume that $\overline{X}$ is reduced and irreducible. Let

$$S = \{x \in X; |mL| \text{ does not define an embedding around } x \text{ for all } m > 0\}.$$

From Proposition 11, we know that the line bundle L in $\overline{X}$ is big, and by Proposition 10 the linear system $|mL|$ provides an embedding of some Zariski open subspace. Using Noether induction, we see that there is a number $m_0 > 0$ such that $\Phi_{|m_0L|}$ embeds $X\backslash S$. In particular, it embeds X, if S is empty.

Lemma 10. *Let $\mathscr{F}$ be a coherent $\mathscr{O}_{\overline{X}}$-module. Then there exists some $\ell_0 > 0$ such that for all $\ell \geq \ell_0$ the sheaf $\mathscr{F} \otimes \mathscr{O}_{\overline{X}}(\ell m_0 L)$ is generated by global sections at all points $x \in X\backslash S$.*

Proof. Let $\Phi = \Phi_{|m_0L|}$, and denote the graph of Φ by Γ_Φ. We have a diagram

$$\begin{tikzcd} \Gamma_\Phi \arrow[d, "\pi"'] \arrow[dr, "\Psi"] & \\ \overline{X} \arrow[r, "\Phi"'] & \mathbb{P}_N . \end{tikzcd}$$

Let $\widetilde{S} = S \cup (\overline{X}\backslash X)$, $T = \Psi(\pi^{-1}\widetilde{S})$, and $\mathscr{I}_T \subset \mathscr{O}_{\mathbb{P}_N}$ the corresponding ideal.

By Lemma 9 we can choose $m_1 > 0$ so that $\mathscr{I}_{\widetilde{S}}^{m_1}\pi_*\pi^*\mathscr{F} \subset \mathscr{F}$ holds.

According to Serre's theorem, for any m_2 the sheaf $\mathscr{I}_T^{m_2} \cdot (\Psi_*\pi^*\mathscr{F} \otimes \mathscr{O}_{\mathbb{P}_N}(\ell))$ is generated by global sections for all $\ell \geq \ell_0(m_0) > 0$.

From the construction, we have a morphism of sheaves $\Psi^*\mathscr{O}_{\mathbb{P}_N}(1) \longrightarrow \pi^*\mathscr{O}_{\overline{X}}(m_0L)$, which is an isomorphism over $\Gamma_\Phi\backslash\pi^{-1}(\widetilde{S})$. We have the following morphisms.

$$\begin{aligned} \mathscr{I}_T^{m_2} \cdot ((\Psi_*\pi^*\mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}_N}(\ell)) &\longrightarrow \mathscr{I}_T^{m_2} \cdot \Psi_*\pi^*(\mathscr{F} \otimes \mathscr{O}_{\overline{X}}(m_0\ell L)) \\ &\longrightarrow \Psi_*(\mathscr{I}^{m_2}_{\pi^{-1}\widetilde{S}} \cdot \pi^*(\mathscr{F} \otimes \mathscr{O}_{\overline{X}}(m_0\ell L))) \end{aligned}$$

Now

$$\begin{aligned} &H^0(\mathbb{P}_N, \mathscr{I}^{m_2} \cdot (\Psi_*\pi^*\mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}_N}(\ell)) \longrightarrow H^0(\Gamma_\Phi, \mathscr{I}^{m_2}_{\pi^{-1}\widetilde{S}} \cdot \pi^*(\mathscr{F} \otimes \mathscr{O}_{\overline{X}}(m_0\ell L))) \\ &\qquad = H^0(\overline{X}, \pi_*(\mathscr{I}^{m_2}_{\pi^{-1}\widetilde{S}} \cdot \pi^*\mathscr{F}) \otimes \mathscr{O}_{\overline{X}}(m_0\ell L)) \longrightarrow \\ &\quad H^0(\overline{X}, \mathscr{I}^{m_1}_{\widetilde{S}} \cdot \pi_*\pi^*\mathscr{F} \otimes \mathscr{O}_{\overline{X}}(m_0\ell L)) \longrightarrow H^0(\overline{X}, \mathscr{F} \otimes \mathscr{O}_{\overline{X}}(m_0\ell L)). \end{aligned}$$

Here, we chose $m_2 \gg 0$ large enough so that $\pi_*(\mathscr{I}^{m_2}_{\pi^{-1}(\widetilde{S})}) \cdot \mathscr{O}_{\overline{X}} \subset \mathscr{I}_S^{m_1}$.

Over $X\backslash S$ the above morphisms of sheaves are isomorphisms so that we can produce enough global sections, which generate $\mathscr{F} \otimes \mathscr{O}_{\overline{X}}(m_0\ell L)$ over $X\backslash S$. □

In the above situation we also need the case, where $\mathscr{F}$ is an ideal in $\mathscr{I} \subset \mathscr{O}_{\overline{X}}$. If $\overline{X}$ is smooth or normal, we have automatically $\pi_*\pi^*\mathscr{I} = \mathscr{I}$.

Definition 9. *Let Y be a reduced complex space of pure dimension n. The L^2-dualizing sheaf $\omega_Y^{(2)}$ of Y is defined by*

$$\begin{aligned} \omega_Y^{(2)}(W) = \{\eta \in \Gamma(W_{reg}, \mathscr{O}(K_{Y_{reg}})); \\ (\sqrt{-1})^{n^2} \int_V \eta \wedge \bar{\eta} < \infty \textit{ for every } V \subset\subset W\}, \end{aligned}$$

where W runs through the open sets of Y.

If $\alpha : \widetilde{Y} \longrightarrow Y$ is a desingularization such that the singular locus of Y corresponds to a normal crossings divisor in $\widetilde{Y}$, we have $\omega^{(2)}_Y = \alpha_* \mathscr{O}_{\widetilde{Y}}(K_{\widetilde{Y}})$. In particular, $\omega^{(2)}_Y$ is coherent.

Now we set

$S_+ = \{x \in X; \omega^{(2)}_{\overline{X}} \otimes L^{\otimes m}$ *is not generated by global sections at x for all m*$\}$,

and

$S_- = \{x \in X; \omega^{(2)\vee}_{\overline{X}} \otimes L^{\otimes m}$ *is not generated by global sections at x for all m*$\}$.

From Lemma 10, we know $S_+ \cup S_- \subset S$.

Let R be the non-normal locus of X. We denote by $\mathscr{I}_{\widetilde{S}\cup R} \subset \mathscr{O}_{\overline{X}}$ the ideal of functions that vanish on $\widetilde{S} \cup R$. Lemma 10 implies that there exists $m_1 > 0$ such that $\mathscr{O}_{\overline{X}}(m_0 m_1 L) \otimes \mathscr{I}_{\widetilde{S}\cup R}$ is generated by global sections at all points $x \in X \backslash S$. Let $\{\sigma_0, \ldots, \sigma_{N(m_0 m_1)}\}$ be a $\mathbb{C}$-basis of $\Gamma(\overline{X}, \mathscr{O}_{\overline{X}}(m_0 m_1 L) \otimes \mathscr{I}_{\widetilde{S}\cup R})$. Then

$$h_0 = \frac{1}{\sum |\sigma_i|^2}$$

defines a singular hermitian metric on $m_0 m_1 L$ over the space $\overline{X}$, whose singularities are contained in $\widetilde{S} \cup R$. Let m_+ and m_- resp. be integers such that $\omega^{(2)}_{\overline{X}} \otimes \mathscr{O}_{\overline{X}}(m_+ m_0 L)$ and $\omega^{(2)\vee}_{\overline{X}} \otimes \mathscr{O}_{\overline{X}}(m_- m_0 L)$ are generated by global sections over $X \backslash S$. Let $\{\sigma_{+,k}\}$ be a basis for the space of sections of the former sheaf over $\mathbb{C}$. Over $\overline{X}_{reg}$, we define a hermitian metric h_+ on $\omega^{(2)}_{\overline{X}} \otimes \mathscr{O}_{\overline{X}}(m_+ m_0 L)$ by

$$h_+ = \frac{1}{\sum |\sigma_{+,k}|^2}. \tag{24}$$

In a similar way a metric h_- on $\omega^{(2)\vee}_{\overline{X}} \otimes \mathscr{O}_{\overline{X}}(m_- m_0 L)$ is constructed. We chose a desingularization $\pi : \widetilde{X} \longrightarrow \overline{X}$ in such a way that also $\pi^* \omega^{(2)}_{\overline{X}}$ is invertible (after dividing by the torsion part). Then the pull-backs of the sections $\sigma_{+,k}$ and $\sigma_{-,k}$ define singular hermitian metrics over $\widetilde{X}$ on the corresponding line bundles. We impose a further condition: Let $U \subset X_{reg} \backslash S$ be a Zariski open subset, which is quasi-projective. Let $\overline{U}$ be a projective compactification that dominates $\widetilde{X}$ with modifications $\rho : \overline{U} \longrightarrow \widetilde{X}$ and $\gamma : \overline{U} \longrightarrow \overline{X}$. We pull back h, h_+, and h_- back to $\overline{U}$, and we assume as above that $\overline{U} \backslash U$ is a divisor with normal crossings singularities. We denote by $\eta_{\overline{U}}$ a Kähler form on $\overline{U}$ and by η_U a complete Kähler form on U with Poincaré growth condition near the boundary as above. By Proposition 11 the line bundle $\gamma^* L$ is big on $\overline{U}$. Kodaira's lemma provides an effective $\mathbb{Q}$-divisor A such that the $\mathbb{Q}$-divisor $\gamma^* L - A$ is ample, giving rise to a strictly positive hermitian metric h' of class C^∞ on the $\mathbb{Q}$-line bundle $\gamma^* L - A$. Let $a \cdot A$ be a Cartier divisor with $a \in \mathbb{N}$, and σ_A a section of $\mathscr{O}_{\overline{U}}(a \cdot A)$. Then

$$\frac{h'}{|\sigma_A|^{2/a}}$$

defines a singular hermitian metric on $\gamma^* L$, whose curvature Θ satisfies

$$\Theta \geq \alpha \cdot \eta_{\overline{U}}$$

for some $\alpha > 0$ on $\overline{U}$. Let D_j be the components of the normal crossings divisor $D = \overline{U} \backslash U$. We equip the bundles $[D_j]$ with a C^∞ hermitian metric. We can find canonical sections τ_j and some $\beta > 0$ such that the curvature of the modified hermitian metric

$$h'' = \frac{h'}{|\sigma_A|^{2/a}} \cdot \prod \left(-\log \|\tau_j\|\right)^\beta$$

over U satisfies

$$\Theta_{h''} \geq \varepsilon \cdot \eta_{\overline{U}}.$$

The following considerations apply to the above line bundles on $\overline{U}$.

Let $p, r \in \mathbb{N}$, and let $1 > \delta > 0$. Then

$$\widehat{h} := h_{p,r,\delta} := \gamma^*(h_0^{p-\delta} h_+^r h_-^{r+1}) h''^{\,\delta m_0 m_1}$$

is a singular hermitian metric on

$$\gamma^*(L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee})$$

with $e = m_0(pm_1 + rm_+ + (r+1)m_-)$ and $\Theta_{\widehat{h}} \geq \varepsilon \delta m_0 m_1 \cdot \eta_{\overline{U}}$. Hereafter we shall consider $\widehat{h}$ as a singular hermitian metric on $L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee}$.

Because of the definition (24) of h_0, any point of $\widetilde{S} \cup R$ is a pole of h_0, we can choose $p > 0$ large enough so that $\mathscr{I}(h_0^{p-1})$ annihilates $\pi_* \mathscr{O}_{\overline{U}} / \mathscr{O}_{\overline{X}}$.

Although as a singular hermitian metric on a *line bundle* $\widehat{h}$ is only defined over $\overline{X}_{reg}$, a coherent multiplier ideal sheaf $\mathscr{I}(\widehat{h}) \subset \mathscr{O}_{\overline{X}}$ can be given a meaning as follows: For $W \subset \overline{X}$ open, we define

$$(\mathscr{I}(\widehat{h}) \otimes \mathscr{O}_{\overline{X}}(L^{\otimes e}))(W) =$$

$$\{\sigma \in \Gamma(W, \mathscr{O}_{\overline{X}}(L^{\otimes e})); \int_{V\cap(\overline{X}_{reg})} |\sigma|^2 \widehat{h} < \infty \ \ \textit{for all } V \subset\subset W\}.$$

Observe that $\widehat{h}|\overline{X}_{reg}$ is a (singular) hermitian metric on $(L^{\otimes e}|_{\overline{X}_{reg}}) \otimes K^{-1}_{\overline{X}_{reg}}$. The ideal is $\mathscr{I}(\widehat{h})$ is coherent, since

$$\mathscr{I}(\widehat{h}) \otimes \mathscr{O}_{\overline{X}}(L^{\otimes e}) = \mathscr{I}(\gamma^*\widehat{h}) \cdot \gamma_*(K_{\overline{U}} \otimes \gamma^*(L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee})) \tag{25}$$

holds by the usual definition of the usual multiplier ideal sheaf $\mathscr{I}(\gamma^*\widehat{h})$ for the singular hermitian metric $\gamma^*\widehat{h}$ on $\gamma^*(L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee})$.

Now we specify the value of $\delta > 0$.

Remark 5. *We can see from the definition of $\widehat{h}$ on $\overline{X}$ that for sufficiently large p the zero set $V(\mathscr{I}(\widehat{h})) \cap X = S$. Furthermore for large p the embedding dimension of the non-reduced space defined by $\mathscr{I}(\widehat{h})$ is equal to* $\dim \overline{X}$.

Proof. For large p the contribution of h_0^{p-1} to $\gamma^*(h_0^{p-\delta} h_+^r h_-^{r+1})$ dominates the rest, in the sense that the zero set of the multiplier ideal sheaf is contained in $\widetilde{S} \cup R$ and contains $\overline{S} \cup R$. Next $\delta > 0$ is chosen small enough: The term $1/|\sigma_A|^2$ is equipped with a small exponent so that the L^2-integrability condition for holomorphic sections is not affected, and $V(\mathscr{I}(\widehat{h})) \cap X = S \cup R$ still holds. For large p also the second statement is satisfied. $\square$

Proposition 15. *The canonical map*

$$H^0(\overline{X}, \mathscr{O}_{\overline{X}}(L^{\otimes e})) \longrightarrow H^0\left(\overline{X}, \mathscr{O}_{\overline{X}}(L^{\otimes e}) \otimes \left(\mathscr{O}_{\overline{X}}/\mathscr{I}(\widehat{h})\right)\right)$$

is surjective.

Proof. Let $\tau \in H^0(\overline{X}, \mathscr{O}_{\overline{X}}(L^{\otimes e}) \otimes (\mathscr{O}_{\overline{X}}/\mathscr{I}(\widehat{h})))$ be a section. For any neighborhood $\widetilde{W}$ of $\widehat{S} = V(\mathscr{I}(\widehat{h}))$ we can find a C^∞ section $\widetilde{\tau}$ of $L^{\otimes e}$, whose restriction to $(\widehat{S}, \mathscr{O}_{\overline{X}}/\mathscr{I}(\widehat{h}))$ equals τ with $\mathrm{supp}(\widetilde{\tau}) \subset \widetilde{W}$. We consider $\gamma^*\overline{\partial}\widetilde{\tau} = \overline{\partial}\gamma^*\widetilde{\tau}$ on $\overline{U}$.

Since the L^2-cohomology $H^1_{(2)}(U, \gamma^*(L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee}) \otimes \mathscr{O}_{\overline{U}}(K_{\overline{U}}))$ with respect to $\widehat{h}$ vanishes by Nadel's theorem (or Hörmander's theorem on L^2-estimates resp.), there exists a C^∞-section u of $\gamma^*(L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee}) \otimes$

$\mathscr{O}_{\overline{U}}(K_{\overline{U}})$ on U, which is square-integrable with respect to the singular hermitian metric and the complete Kähler metric η_U on U such that $\overline{\partial} u = \overline{\partial}\gamma^*\widetilde{\tau}$. So $0 = \overline{\partial}(\gamma^*\widetilde{\tau} - u)$, i.e. $v = \gamma^*\widetilde{\tau} - u \in H^0(U, \gamma^*(L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee}) \otimes \mathscr{O}_{\overline{U}}(K_{\overline{U}}))$.

We claim that v is square-integrable: By (25) $\|u\|^2\widehat{h}$ is integrable over $\overline{X}_{reg}$. Since $\widehat{h}$ is a singular hermitian metric of positive curvature, it is locally bounded from below by a positive constant. Moreover $\widetilde{\tau}$ is of class C^∞, and U carries the complete metric η_U (with Poincaré growth condition). So v extends holomorphically to $\overline{U}$. Then v gives rise to a holomorphic section of $L^{\otimes e}$ on $\overline{X}$, which coincides with τ, when restricted to the subspace $(V(\mathscr{I}(\widehat{h})), \mathscr{O}_{\overline{X}}/\mathscr{I}(\widehat{h}))$ (cf. equation (25)). □

Proof of the Theorem. For large k, by induction hypothesis, $kL|\overline{S}$ defines an embedding of S. In the last step, we need to raise the power e of L without affecting the multiplier ideal sheaf $\mathscr{I}(\widehat{h})$. We replace the singular hermitian metric $\widehat{h}$ on $\gamma^*(L^{\otimes e} \otimes \omega_{\overline{X}}^{(2)\vee})$ by $\widehat{h} \cdot h^t$ on $\gamma^*(L^{\otimes(e+t)} \otimes \omega_{\overline{X}}^{(2)\vee})$, where h is the singular hermitian metric on L from he first part with vanishing Lelong numbers.

Since the curvature of h is absolutely continuous, by Proposition 8, we may assume that $\mathscr{I}(\widehat{h}) = \mathscr{I}(\widehat{h} \cdot h^t)$ holds over X, if we perturb δ by a small amount (i.e. we perturb $\widehat{h}$). Although this metric is not of analytic singularities, but a singularity of type $\log(-\log(\delta(x)))$ is negligible (cf. Remark 4). We chose t large enough so that $L^{\otimes(e+t)}$ defines an embedding of a Zariski open subspace of $(\overline{S}, \mathscr{O}_{\overline{X}}/\mathscr{I}(\widehat{h}))$ by Proposition 14. Now $L^{\otimes(e+t)}$ embeds $X\backslash S$ as well as a non-empty open subset of S, and it also separates normal directions of this set in X. This contradicts the choice of S, and proves Theorem 6 for reduced, irreducible spaces $\overline{X}$.

For non-reduced spaces, again we use induction over the dimension. Let L be a line bundle on $\overline{X}$ with the above assumptions. We know that for some $m > 0$ the meromorphic map $\Phi_{|mL|red(\overline{X})|}$ embeds $red(X)$. By Proposition 14 we can choose $m > 0$ so that $\Phi_{|mL|}$ embeds a Zariski open subspace $X' \subset X$. Let $T = red(X)\backslash red(X')$. According to the above proof, a multiple of L, restricted to a high infinitesimal neighborhood T_{inf} of T gives rise to a linear system, which embeds a Zariski open subspace of T_{inf}. Finally, by Proposition 15 global sections over T_{inf} can be extended to all of $\overline{X}$ contradicting the choice of T. □

Finally Theorem 5 and Theorem 6 imply Theorem 1.

References

[AL] Alexeev, V.: Boundedness and K^2 for log surfaces. Int. J. Math. **5** 779–810 (1994).

[A-V] Andreotti, A., Vesentini, E.: Carleman estimates for the Laplace-Beltrami equation on complex manifolds. Inst. Hautes Etud. Sci. Publ. Math. **25**, 313–362 (1965).

[A-S] Anghern, U. and Siu, Y.-T.: Effective freeness and point separation for adjoint bundles, Invent. Math. **122** 291–308 (1995).

[AR2] Artin, M.: Algebraization of formal moduli: II, Existence of modification, Ann. of Math. **91** 88–135 (1970).

[AR1] Artin, M.: Algebraization of fomal moduli: I, in Global Analysis, Papers in honor of K. Kodaira (D.C. Spencer, S. Iyanaga Ed.), Univ. of Tokyo Press and Princeton Univ. Press (1979).

[B-S] Bănică, C., Stanaşila, O.: Algebraic Methods in the Global Theory of Complex Spaces. Bucureşti, London, 1976

[BGS] Bismut, J.M., Gillet, H., Soulé, C.: Analytic torsion and holomorphic determinant bundles I, II, III. Commun. Math. Phy. **115** 49–78 (1988); **115** 79–126 (1988); **115** 301–351 (1988).

[BO] Bombieri, E.: Algebraic values of meromorphic maps, Invent. math. **10** (1970) 267–287 and Addendum Invent. math. **11** 163–166 (1970).

[B] Borel, A.: Introduction aux groupes arithmétiques: Paris, Hermann 1969.

[DL] Deligne, P.: Théorie de Hodge II. Publ. Math. IHES **40** 5–57 (1972).

[DE] Demailly, J.-P: Mésures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines. Mém. Soc. Math. de France (N.S.) **19** 1–124 (1985).

[DE1] Demailly, J.-P: Singular Hermitian metrics on positive line bundles. Complex algebraic varieties, Proc. Conf., Bayreuth 1990, Lect. Notes Math. **1507** 87–104 (1992).

[DE2] Demailly, J.-P: Regularization of closed positive currents and intersection theory. J. Algebr. Geom. **1** 361–409 (1992).

[DE3] Demailly, J.-P: A numerical criterion for very ample line bundles, J. of Diff. Geom. **37** 323–374 (1993).

[DE4] Demailly, J.-P: L^2 vanishing theorems for positive line bundles and adjunction theory, CIME Lectures Cetraro, 1994

[F-S] Fujiki, A., Schumacher, G.: The moduli space of compact extremal Kähler manifolds and generalized Petersson-Weil metrics. Publ. RIMS, Kyoto Univ. **26** 101–183 (1990).

[FU] Fujita, T.: Classification theories of polarized varieties, Cambridge University Press, New York, 1990.

[FU1] Fujita, T.: On Kähler fiber spaces over curves. J. Math. Soc. Japan, **30** 779–794 (1978)

[G-R] Grauert, H., Remmert, R.: Plurisubharmonische Funktionen in komplexen Räumen. Math. Z. **65** 175–194 (1956).

[GRO] Grothendieck, A.: Techniques de construction et théorèmes en géométrie algébrique IV: Les schémas de Hilbert, Séminaire Bourbaki **221** (1960/61).

[GRI] Griffiths, Ph.: Periods of integrals on algebraic manifolds. III: Some global differential-geometric properties of the period mapping, Publ. Math., Inst. Hautes Etud. Sci. **38** 125–180 (1970).

[HÖ] Hörmander, L.: L^-2 estimates and existence theorems for the ∂-operator Acta Math. **113** 89–152 (1965).

[IV] Ivinskis, K.: Das infinitesimale Torelli Problem für einfache Ueberlagerungen von hohem Grad. Manuscr. Math. **76** (1992) 137–145.

[J-S] Ji, S., Shiffman, B.: Properties of compact complex manifolds carrying closed positive currents. J. Geom. Anal. **3** 37–61 (1993).

[KA] Kawamata, Y.: Characterization of Abelian Varieties. Compos. Math. **43** 253–276 (1981).

[KL] Kleiman, St.: Relative duality for quasi-coherent sheaves. Compositio math. **41** 39–60 (1980).

[K-M] Knudsen, F. and Mumford, D.: The projectivity of the moduli of stable curves I, Math. Scand. **39** 39–55 (1971).

[KN] Knudsen, F.: The projectivity of the moduli of stable curves, II,III, Math. Scand. **52** 161–277 (1983).

[KT] Knutson, D.: Algebraic spaces, Springer Lect. Note in Math. **203** (1971).

[K1] Kobayashi, R.: Einstein-Kähler metrics on open algebraic surfaces of general type, Tohoku Math. J. **37**, 43–77 (1985)

[K2] Kobayashi, R.: Kähler-Einstein metric on an open algebraic manifold, Osaka J. Math. **21** 399–418 (1984)

[K-O] Kobayashi, S.-Ochiai, T.: Mappings into compact complex manifolds with negative first Chern class, Jour. Math. Soc. Japan 23 (1971), 137–148.

[KO1] Kollár, J.: Projectivity of complete moduli, J. of Diff. Geom. 32 (1990), 235–268.

[KO2] Kollár, J.: Higher direct images of dualizing sheaves. Ann. of Math. **123** 11–42 (1986).

[L-M] Liebermann, D., Mumford, D.: Matsusaka's big theorem. Algebraic Geometry, Arcata 1974. Proc. Symp. in Pure Math **29** AMS 1975

[MA] Matsusaka, T.: Polarized varieties with a given Hilbert polynomial. Amer. J. Math. **94** 1027–1077 (1972).

[M-F-K] Mumford, D., Fogarty, J., Kirwan, F.: Geometric Invariant Theory, 3 rd. edtn. Springer-Verlag, Berlin 1994.

[NA] Nadel, A.M.: Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature. Ann. of Math. **132** 549–596 (1990).

[O-T] Ohsawa, T., Takegoshi, K.: L^2-extension of holomorphic functions, Math. Z. **195** 197–204 (1987).

[P] Popp, H.: Moduli theory and classification theory of algebraic varieties, Springer Lect. Notes in Math. **620** (1977).

[S] Schmid, W.: Variation of Hodge structure: the singularities of the period mapping. Invent. math. **22** 211–319 (1973).

[SC] Schneider, M.: Halbstetigkeitssätze für relativ analytische Räume. Invent. math. **16** 161–176 (1972).

[SCH1] Schumacher, G.: Moduli of framed manifolds, Invent. math. **134** 229–249 (1998).

[SCH2] Schumacher, G.: Moduli as algebraic spaces, Proc. Oka100 Conference, Kyoto 2001, to appear in Adv. Stud. Pure Math.

[SI1] Siu, Y.-T.: Analyticity of sets associated to Lelong numbers and the estension of closed positive currents. Invent. math. **27** 53–156 (1974).

[SI2] Siu, Y.-T.: An effective Matsusaka big theorem. Ann. Inst. Fourier **43** 1387–1405 (1993).

[SK] Skoda, H.: Sous-ensembles analytiques d'ordre fini ou infini dans $\mathbb{C}^n$. Bull. Soc. Math. Francd **100** 353–408 (1972).

[T-Y] Tian, G., Yau, S.-T.: Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. Mathematical aspects of string theory (San Diego, Calif., 1986), 574–628, Adv. Ser. Math. Phys. **1**, World Sci. Publishing, Singapore (1987).

[TS1] Tsuji, H.: An inequality of Chern numbers for open algebraic varieties. Math. Ann. **277** 483–487 (1987).

[TS2] Tsuji, H.: A characterization of ball quotients with smooth boundary. Duke Math. J. **57** 537–553 (1988).

[V] Viehweg, E.: Weak positivity and stability of certain Hilbert points I,II,III, Invent. Math. **96** 639–669 (1989). Invent. Math. **101** 191–223 (1990), Invent. Math.**101** 521–543 (1990).

[Z] Zucker, St.: Hodge theory with degenerating coefficients: L_2-cohomology in the Poincaré metric. Annals of Math. **109**, 415–476 (1979).

Fachbereich Mathematik der Philipps-Universität, Hans-Meerwein-Strasse, Lahnberge, D-35032 Marburg, Germany

E-mail address: `schumac@mathematik.uni-marburg.de`

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ohokayama, Meguro 152–8551, Japan

E-mail address: `tsuji@math.titech.ac.jp`