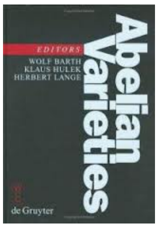


Appeared in:
Abelian Varieties (Egloffstein, 1993),
De Gruyter, Berlin New York, 1995, pp. 19-23.

# Abelian Threefolds in $(\mathbb{P}_2)^3$

Th. Bauer [1], T. Szemberg [2]

March 3, 1994

## 1. Introduction

The elliptic curves in a projective plane are the smooth cubics. In [3] Hulek proved that the only abelian surfaces in the product space $\mathbb{P}_2 \times \mathbb{P}_2$ are the obvious ones, i.e. the products of two plane cubics. Here we consider the analogous question for abelian threefolds in $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2$.

We prove:

**Theorem.** *Let $A$ be an abelian threefold over $\mathbb{C}$, embedded in $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2$. Then $A$ is a product $E_1 \times E_2 \times E_3$, where $E_1$, $E_2$ and $E_3$ are smooth plane cubics.*

We note that the existence of abelian threefolds in 6-dimensional products of projective spaces was recently studied by Birkenhake [1] in the case of two factors.

## 2. The Projections

Let $\varphi = (\varphi_1, \varphi_2, \varphi_3) : A \hookrightarrow (\mathbb{P}_2)^3$ be an embedding of an abelian threefold $A$ over $\mathbb{C}$ given by line bundles $L_1$, $L_2$, $L_3$. Further, let $\pi_i : (\mathbb{P}_2)^3 \longrightarrow \mathbb{P}_2^{(i)}$ denote the projection onto the $i$-th factor and $h_i := [\pi_i^* \mathcal{O}_{\mathbb{P}_2}(1)] \in H^2((\mathbb{P}_2)^3, \mathbb{Z})$. By the Künneth formula the class of $A$ in $H^6((\mathbb{P}_2)^3, \mathbb{Z})$ is of the form

$$[A] = a h_1 h_2 h_3 + \sum_{\substack{i,j=1,2,3 \\ i \neq j}} a_{ij} h_i^2 h_j \qquad (*)$$

with integers $a$, $a_{ij} \geq 0$.

[1] Supported by DFG grant Ba 423/3-3

[2] Partially supported by Daimler-Benz Stiftung project number 2.92.34 and KBN grant number 2 1077 91 01

**Lemma 2.1** *The coefficients of* $[A]$ *in* $(*)$ *satisfy the equation*

$$a(a-27) = \sum_{\sigma \in S_3} a_{\sigma(1),\sigma(3)}(9 - a_{\sigma(2),\sigma(3)})$$

*Proof.* The total Chern class of the normal bundle $\mathcal{N}_{A/(\mathbb{P}_2)^3}$ is

$$c(\mathcal{N}_{A/(\mathbb{P}_2)^3}) = \prod_{i=1}^{3}(1 + 3h_i + 3h_i^2) \cdot [A],$$

thus

$$c_3(\mathcal{N}_{A/(\mathbb{P}_2)^3}) = (27h_1h_2h_3 + 9\sum_{i \neq j} h_i^2 h_j) \cdot [A] = 27a + 9\sum_{i \neq j} a_{ij}.$$

On the other hand we have

$$A^2 = a^2 + \sum_{\sigma \in S_3} a_{\sigma(1),\sigma(3)} a_{\sigma(2),\sigma(3)}$$

Now our assertion follows from the self-intersection formula $A^2 = c_3(\mathcal{N}_{A/(\mathbb{P}_2)^3})$ ([2], p.103). □

In the sequel we will need the following

**Lemma 2.2** *Let* $A$ *be an abelian threefold,* $\psi : A \longrightarrow \mathbb{P}_2$ *a morphism and* $E \subset A$ *an elliptic curve such that all the restrictions* $\psi | t_a^* E$, $a \in A$, *are embeddings. Then* $\psi(t_a^* E) = \psi(E)$ *for all* $a \in A$.

*Proof.* Denote by $P := \mathbb{P}(H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(3)))$ the projective space of plane cubics and define a map

$$\begin{array}{rcl} \Phi : A & \longrightarrow & P \\ a & \longmapsto & \psi(t_a^* E) \end{array}$$

We choose ten points $e_1, \dots, e_{10} \in E$. Then

$$\begin{array}{rcl} Z & := & \{(a, C) \in A \times P \mid C \text{ contains } \psi(e_1 - a), \dots, \psi(e_{10} - a)\} \\ & = & \{(a, C) \in A \times P \mid C = \psi(t_a^* E)\} \end{array}$$

is a subvariety of $A \times P$. The projection $p : Z \longrightarrow A$ is bijective, hence an isomorphism by Zariski's Main Theorem. The map $\Phi$ is just the composition $\Phi = q \circ p^{-1}$, where $q : Z \longrightarrow P$ is the second projection. So $\Phi$ is a morphism and the image $\Phi(A)$ is a subvariety of $P$. If $\Phi(A)$ is of dimension $\geq 1$, then $\Phi(A)$ meets the hypersurface

$$\{ \text{ singular plane cubics } \} \subset P,$$

Since this contradicts the assumption that all images of $\Phi$ are smooth curves, we conclude that $\Phi(A)$ is a point. □

Further, we will frequently apply the following useful Lemma from [1]:

**Lemma 2.3** *Let $X$ be an abelian variety of dimension $g$ and $\varphi : X \longrightarrow \mathbb{P}_N$ a morphism with $\dim \varphi(X) = n < g$. Then $L := \varphi^* \mathcal{O}_{\mathbb{P}_N}(1)$ is semipositive of rank $n$ and $\varphi$ fits into a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & & & & \overset{\varphi}{\searrow} & \downarrow f & & \\ & & & & & & \mathbb{P}_N & & \end{array}$$

*where the upper row is an exact sequence of abelian varieties and $f$ is a morphism, which is finite onto its image.*

Now we are ready to prove:

**Proposition 2.4** *At least one of the projections $\varphi_1$, $\varphi_2$, $\varphi_3$ is not surjective.*

*Proof.* Suppose to the contrary that all of them are surjective. Because of the surjectivity of $\varphi_1$ Lemma 2.3 gives a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & S_1 & \longrightarrow & 0 \\ & & & & & \overset{\varphi_1}{\searrow} & \downarrow f_1 & & \\ & & & & & & \mathbb{P}_2^{(1)} & & \end{array}$$

where the upper row is an exact sequence of abelian varieties, $E_1$ being an elliptic curve and $S_1$ an abelian surface, and $f_1$ is a finite morphism of degree $d_1$, say.

By Riemann-Roch on $S_1$ and [4], Theorem 3.3.3, we have

$$3 \leq h^0(L_1) = \frac{1}{2} d_1,$$

hence $d_1 \geq 6$. Since $\varphi_1(E_1)$ is a point, we have

$$[E_1] = \alpha h_1^2 h_2^2 h_3 + \beta h_1^2 h_3^2 h_2$$

with $\alpha$, $\beta \geq 0$.

*Claim:* We have $\alpha \neq 1$ and $\beta \neq 1$.

*Proof:* By symmetry it is enough to consider $\alpha$. Applying the projection formula we get

$$\alpha = E_1 \cdot h_3 = (\varphi_3)_*(E_1) \cdot \mathcal{O}_{\mathbb{P}_2}(1) = \deg(\varphi_3 | E_1) \cdot \deg \varphi_3(E_1).$$

If we had $\alpha = 1$, then the morphism $\varphi_3 | E_1 : E_1 \longrightarrow \varphi_3(E_1)$ would be of degree 1 onto a line in $\mathbb{P}_2$, which of course is impossible.

Let us distinguish between two cases:

*Case I:* $\alpha = 0$ or $\beta = 0$.

Suppose $\alpha = 0$, i.e. $\varphi_3(E_1)$ is a point. Since both of $\varphi_1(E_1)$ and $\varphi_3(E_1)$ are then points, $\varphi_2$ must embed $E_1$ and all of its translates $t_a^* E_1$, $a \in A$, into $\mathbb{P}_2$. By Lemma

2.2 then $\varphi_2(t_a^* E_1) = \varphi_2(E_1)$ for all $a \in A$. Since every point of $A$ lies on a translate of $E_1$, we conclude that $\varphi_2$ is not surjective and the Proposition is proved in this case.

*Case II:* $\alpha \geq 2$ and $\beta \geq 2$.
Let $F_1$ be a general fibre of $\varphi_1$. Then we obtain

$$[F_1] = [A] \cdot h_1^2 = a_{23} h_1^2 h_2^2 h_3 + a_{32} h_1^2 h_3^2 h_2 .$$

Furthermore, we have $[F_1] = d_1 \cdot [E_1]$, hence

$$a_{23} = d_1 \cdot \alpha \geq 6 \cdot 2 = 12$$

and also $a_{32} \geq 12$. Arguing in the same way with the projections $\varphi_2$ and $\varphi_3$ we obtain

$$a_{ij} \geq 12 \qquad \text{for } i,j = 1,2,3, \ i \neq j .$$

Lemma 2.1 then yields

$$-183 \leq a(a-27) = \sum_{(i,j,k) \in S_3} a_{ij}(9 - a_{kj}) \leq -216,$$

a contradiction. We conclude that not all of the projections $\varphi_1$, $\varphi_2$ and $\varphi_3$ can be surjective. □

## 3. The Product Decomposition

Now we can prove the Theorem stated in the Introduction:

**Theorem 3.1** *Let $A$ be an abelian threefold over $\mathbb{C}$, embedded in $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2$. Then $A$ is a product $E_1 \times E_2 \times E_3$, where $E_1$, $E_2$ and $E_3$ are smooth plane cubics.*

*Proof.* By Proposition 2.4 we may assume that $\varphi_1$ is not surjective. By Lefschetz hyperplane theorem there are no abelian threefolds in $\mathbb{P}_2 \times \mathbb{P}_2$, since $\mathbb{P}_2 \times \mathbb{P}_2$ is simply connected. Thus the image $\varphi_1(A) \subset \mathbb{P}_2^{(1)}$ must be a curve. Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_1 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & 0 \\ & & & & & \searrow^{\varphi_1} & \downarrow {\scriptstyle f_1} & & \\ & & & & & & \mathbb{P}_2^{(1)} & & \end{array}$$

where $E_1$ is an elliptic curve, $S_1$ an abelian surface and $f_1$ a morphism, which is finite onto its image. Since the image $\varphi_1(S_1)$ is a point, $S_1$ is embedded into $\mathbb{P}_2^{(2)} \times \mathbb{P}_2^{(3)}$ by $(\varphi_2, \varphi_3)$. According to [3], 2.1, $S_1$ is then a product of elliptic curves $E_2 = \varphi_2(S_1)$ and $E_3 = \varphi_3(S_1)$. Identifying $S_1$ with its image under $(\varphi_2, \varphi_3)$ we may consider $E_2$, $E_3$ as elliptic curves on $A$.

Furthermore, we have $t_a^* S_1 = t_a^* E_2 \times t_a^* E_3$ for all $a \in A$. Since $\varphi_1(t_a^* S_1)$ is again a point $(\varphi_2, \varphi_3)$ embeds $t_a^* S_1$. In particular $\varphi_2$ embeds each translate $t_a^* E_2$. According to Lemma 2.2 we must have $\varphi_2(t_a^* E_2) = E_2$. Hence $\varphi_2$ is not surjective, i.e. $\varphi_2(A) = E_2$. Thus we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_2 & \longrightarrow & A & \longrightarrow & E_2' & \longrightarrow & 0 \\ & & & & & \searrow^{\varphi_2} & \downarrow f_2 & & \\ & & & & & & \mathbb{P}_2^{(2)} & & \end{array}$$

with an abelian surface $S_2$, an elliptic curve $E_2'$ and a finite morphism $f_2$. Since $\varphi_2(S_2)$ is a point, $S_2$ is embedded into $\mathbb{P}_2^{(1)} \times \mathbb{P}_2^{(3)}$ by $(\varphi_1, \varphi_3)$ and again $S_2 = \varphi_1(S_2) \times \varphi_3(S_2)$ according to [3]. In fact $S_2 = E_1 \times E_3$. The morphism $f_2$ is an isomorphism because $\varphi_2$ embeds $E_2$, hence $E_2' \cong E_2$. Since $E_2$ is contained in $A$ the exact sequence

$$0 \longrightarrow S_2 \longrightarrow A \longrightarrow E_2 \longrightarrow 0$$

splits. Then it follows

$$A \cong S_2 \times E_2 \cong E_1 \times E_2 \times E_3$$

and the Theorem is proved. □

We conclude with the following

**Question.** *Is every abelian variety of dimension $n$ in $(\mathbb{P}_2)^n$ a product of smooth plane cubics?*

*Acknowledgements.* We would like to thank Prof. W. Barth for helpful discussions.

## References

[1] Birkenhake, Ch: Abelian Threefolds in Products of Projective Spaces. To appear.
[2] Fulton, W.: Intersection Theory. Springer-Verlag, New York 1984
[3] Hulek, K.: Abelian Surfaces in Products of Projective Spaces. Algebraic Geometry L'Aquila, SLN 1417 (1988)
[4] Lange, H., Birkenhake, Ch.: Complex Abelian Varieties. Springer-Verlag, Grundlehren 302 (1992)

Th. Bauer, T. Szemberg
Mathematisches Institut der Universität
Bismarckstr. 1 1/2
D-91054 Erlangen
email: bauerth@mi.uni-erlangen.de
szember@mi.uni-erlangen.de

*permanent address of the second author:*
Instytut Matematyki
Uniwersytet Jagielloński
Reymonta 4
PL-30-059 Kraków