Appeared in:
Math. Ann. 309, 475-481 (1997)

Seshadri constants of quartic surfaces

Thomas Bauer

September 27, 1996

0. Introduction

Let L be a nef line bundle on a smooth projective variety X. The *Seshadri constant* of L at a point $x \in X$ is defined to be the real number

$$\varepsilon(L, x) =_{\text{def}} \sup\{\varepsilon \in \mathbb{R} \mid f^*L - \varepsilon E \text{ is nef}\} ,$$

where $f : \widetilde{X} \longrightarrow X$ is the blow-up of X at x and $E \subset \widetilde{X}$ the exceptional divisor. The *global Seshadri constant* of L is the infimum

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x) .$$

By Seshadri's criterion, L is ample if and only if $\varepsilon(L) > 0$.

Recent interest in Seshadri constants derives on the one hand from their application to adjoint linear systems. In fact, a lower bound on the Seshadri constant of L gives a bound on the number of jets that the adjoint line bundle $\mathcal{O}_X(K_X + L)$ separates (see [2] and [3]). On the other hand, Seshadri constants are very interesting invariants of polarized varieties in their own right. It is this second aspect that we investigate in the present paper.

Specifically, consider a smooth surface $X \subset \mathbb{P}^3$ and think of the projective embedding as fixed. We then simply write

$$\varepsilon(X, x) =_{\text{def}} \varepsilon(\mathcal{O}_X(1), x) \quad \text{and} \quad \varepsilon(X) =_{\text{def}} \varepsilon(\mathcal{O}_X(1))$$

and refer to these numbers as the *Seshadri constants of* X. One has a priori the following estimates:

$$1 \leq \varepsilon(X) \leq \sqrt{\deg(X)} ,$$

where the second inequality follows from Kleiman's theorem (see [3, Remark 1.8]). It is clear that for surfaces of degree ≤ 3 we have $\varepsilon(X) = 1$, since any such surface contains a line. Furthermore, recent work of A. Steffens [6] implies that $\varepsilon(X, x) \geq \left\lfloor \sqrt{\deg(X)} \right\rfloor$ for the very general point $x \in X$, if the Picard number of X equals 1.

1991 *Mathematics Subject Classification:* Primary 14C20; Secondary 14J28.

In general, however, the numbers $\varepsilon(X)$ and their potential geometric interpretation seem to be unknown up to now.

In this note we consider the first non-trivial case $\deg(X) = 4$. Our result shows that, somewhat surprisingly, there are only three possible values of $\varepsilon(X)$, where the sub-maximal ones account for special geometric situations. We prove:

Theorem. *Let $X \subset \mathbb{P}^3$ be a smooth quartic surface. Then the following statements on the Seshadri constant $\varepsilon(X)$ hold:*

(a) $\varepsilon(X) = 1$ *if and only if the surface X contains a line,*
(b) $\varepsilon(X) = \frac{4}{3}$ *if and only if there is a point $x \in X$ such that the Hesse form $\mathcal{H}_X$ vanishes at x and X does not contain any lines,*
(c) $\varepsilon(X) = 2$, *otherwise.*

The cases (a) and (b) occur on sets of codimension one in the space of quartic surfaces.

Here the *Hesse form* $\mathcal{H}_X$ of a smooth surface $X \subset \mathbb{P}^3$ is a quadratic form on the tangent bundle TX (see Sect. 1). The theorem implies in particular that for a generic quartic surface one has $\varepsilon(X) = 2$.

Notation and Conventions. We work throughout over the field $\mathbb{C}$ of complex numbers.

Numerical equivalence of divisors or line bundles will be denoted by $\equiv$.

1. Hesse forms of projective surfaces

We start with a discussion of Hesse forms. Consider a smooth surface $X \subset \mathbb{P}^3$ of degree $d \geq 2$, and let $f \in H^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)\right)$ be a homogeneous equation of f. The second order derivatives of f give rise to a map of vector bundles

$$d^2 f : \mathcal{O}_X(1)^{\oplus 4} \otimes \mathcal{O}_X(1)^{\oplus 4} \longrightarrow \mathcal{O}_X(d) \ .$$

Consider its restriction

$$\sigma : \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{O}_X(d)$$

to the kernel $\mathcal{F}$ of the map $df : \mathcal{O}_X(1)^{\oplus 4} \longrightarrow \mathcal{O}_X(d)$ defined by the first order derivatives. We have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{F} & \longrightarrow & TX & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(1)^{\oplus 4} & \longrightarrow & T\mathbb{P}^3|_X & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle df} & & \downarrow & & \\
 & & & & \mathcal{O}_X(d) & = & N_{X/\mathbb{P}^3} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The symmetric form σ will descend to the tangent bundle TX, if the image of $\mathcal{O}_X$ in $\mathcal{F}$ is contained in the radical subbundle

$$\mathrm{Rad}(\sigma) = \bigcup_{x \in X} \left\{ v \in \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \;\middle|\; \sigma(v,w) = 0 \text{ for all } w \in \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \right\} .$$

But this is a consequence of the Euler formula. So we get in effect a quadratic form

$$\mathcal{H}_X : \mathrm{Sym}^2(TX) \longrightarrow \mathcal{O}_X(d)$$

on the tangent bundle of X with values in $\mathcal{O}_X(d)$ which we will call the *Hesse form* of X.

The geometrical significance of $\mathcal{H}_X$ is summarized in the following proposition. For this denote for $k \geq 1$ by

$$D_k^s(\mathcal{H}_X) = \{x \in X \mid \mathcal{H}_X \text{ is of rank} \leq k \text{ at } x\}$$

the k-th (symmetric) degeneracy locus of $\mathcal{H}_X$, equipped with its natural scheme structure.

Proposition 1.1 *Let $X \subset \mathbb{P}^3$ be a smooth projective surface of degree ≥ 2, and let $\mathcal{H}_X$ be its Hesse form. Then we have:*

(a) The isotropic lines of $\mathcal{H}_X$ on T_xX correspond to the principal tangents of X, i.e. the lines ℓ such that we have

$$i(x, X \cdot \ell) \geq 3$$

for the local intersection multiplicity at x.

(b) *The degeneracy locus* $D_1^s(\mathcal{H}_X)$ *is a divisor of degree* $4d(d-2)$*, where* $d = \deg(X)$*. The open subset*

$$D_1^s(\mathcal{H}_X) - D_0^s(\mathcal{H}_X)$$

consists of the points $x \in X$ *such that there is only one principal tangent at* x*.*

(c) *The locus* $D_0^s(\mathcal{H}_X)$ *is finite; it consists of the points at which there are infinitely many principal tangents.*

Proof. The statements on the principal tangents follow easily from the definition of $\mathcal{H}_X$. Turning to the assertion about the dimension and the degree of $D_1^s(\mathcal{H}_X)$, let us first assume that $D_1^s(\mathcal{H}_X) = X$. Then the rank of the differential

$$T_x\gamma_X : T_xX \longrightarrow T_x(\mathbb{P}^3)^*$$

of the Gauß map $\gamma_X : X \longrightarrow (\mathbb{P}^3)^*$ is at most 1 for all points $x \in X$, and hence $\dim \gamma_X(X) \leq 1$. But this is impossible, because the smoothness of X implies that γ_X is finite. Since the codimension of $D_1^s(\mathcal{H}_X)$ is in any event at most 1, we see that $D_1^s(\mathcal{H}_X)$ is in fact a divisor. Its class in $H^2(X,\mathbb{Z})$ is then

$$\begin{aligned}
[D_1^s(\mathcal{H}_X)] &= 2c_1\left((TX)^\vee \otimes \sqrt{\mathcal{O}_X(X)}\right) \\
&= 2c_1\left((TX)^\vee\right) + \operatorname{rank}(\mathcal{H}_X)\cdot[\mathcal{O}_X(X)] \\
&= [\mathcal{O}_X(4\deg(X) - 8)]
\end{aligned}$$

(see [4, Theorem 10] and [5, Theorem 2]). It remains to show that $D_0^s(\mathcal{H}_X)$ is finite. In the alternative case the Hesse form $\mathcal{H}_X$, and hence the differential $T\gamma_X$, would vanish along a curve in X. But of course this again contradicts the finiteness of γ_X. □

In the next section we will need the following statement on hyperplane sections of smooth surfaces:

Lemma 1.2 *A smooth surface* $X \subset \mathbb{P}^3$ *of degree* ≥ 2 *admits no tropes, i.e. any hyperplane section of* X *is reduced.*

This follows (as in the proof of the preceding proposition) from the finiteness of the Gauß map of a smooth surface.

2. Seshadri constants

Let X be a smooth projective variety, L a nef line bundle on X and $x \in X$ a point. Recall that the Seshadri constant $\varepsilon(X,x)$ can alternatively be defined as

$$\varepsilon(X,x) = \inf_{C \ni x} \left\{ \frac{LC}{\operatorname{mult}_x(C)} \right\} ,$$

where the infimum is taken over all irreducible curves C on X passing through x.

We state now a lemma which in the surface case allows to determine local Seshadri constants by producing curves with high multiplicity at a given point. In the statement of the lemma the abbreviation

$$\varepsilon_{C,x} =_{\text{def}} \frac{LC}{\text{mult}_x(C)}$$

will be used.

Lemma 2.1 *Let X be a smooth projective surface, $x \in X$ a point and L an ample line bundle on X. Suppose that there is an irreducible curve C on X such that $C \equiv kL$ for some $k \geq 1$ and*

$$\varepsilon_{C,x} \leq \sqrt{L^2} \ .$$

Then $\varepsilon(L,x) = \varepsilon_{C,x}$.

More generally, let $D = \sum_{i=1}^r d_i D_i$ be an effective divisor such that $D \equiv kL$ for some $k \geq 1$ and assume that

$$\varepsilon_{D_i,x} \leq \sqrt{\frac{rd_i \cdot LD_i}{k}} \qquad \text{for } 1 \leq i \leq r$$

Then $\varepsilon(L,x) = \min\limits_{1 \leq i \leq r} \varepsilon_{D_i,x}$.

Proof. Of course the first assertion follows from the second one. In order to prove the second assertion, assume to the contrary that $\varepsilon(L,x) < \min\{\varepsilon_{D_i,x} \mid 1 \leq i \leq r\}$. So there is an irreducible curve $C' \subset X$ with $\varepsilon_{C',x} < \varepsilon_{D_i,x}$ for $1 \leq i \leq r$. These inequalities in particular force C' and D to intersect properly, so we get

$$LC' = \frac{1}{k} DC' = \frac{1}{k} \sum_{i=1}^r d_i D_i C' \geq \frac{1}{k} \sum_{i=1}^r d_i \, \text{mult}_x(D_i) \cdot \text{mult}_x(C') \ . \qquad (*)$$

Using the assumption $\varepsilon_{C',x} < \varepsilon_{D_i,x}$ and the fact that by definition

$$\text{mult}_x(D_i) \cdot \text{mult}_x(C') = \frac{LD_i \cdot LC'}{\varepsilon_{D_i,x} \varepsilon_{C',x}}$$

we obtain from $(*)$ that

$$LC' > LC' \frac{1}{k} \sum_{i=1}^r d_i \frac{LD_i}{\varepsilon_{D_i,x}^2} \ .$$

So we arrive at a contradiction with the assumption on $\varepsilon_{D_i,x}$ in the statement of the lemma, and this completes the proof. □

Using Proposition 1.1 and Lemma 2.1 we now prove the following statement which implies the theorem stated in the introduction:

Theorem 2.2 *Let $X \subset \mathbb{P}^3$ be a smooth quartic surface and $x \in X$ a point. Then:*

(a) $\varepsilon(X) = 1$ *if and only if* X *contains a line.*
(b) If X *does not contain any lines, then*

$$\varepsilon(X,x) = \begin{cases} \frac{4}{3} & , \text{ if } x \in D_0^s(\mathcal{H}_X) \\ 2 & , \text{ otherwise.} \end{cases}$$

The subsets $\{\varepsilon(X) = 1\}$ *and* $\{\varepsilon(X) = \frac{4}{3}\}$ *of the space* $\mathcal{S} \subset \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)))$ *of smooth quartic surfaces are of codimension* 1.

Proof. Suppose first that X contains a line ℓ. Since in any event $\varepsilon(X) \geq 1$, we then clearly have $\varepsilon(X,x) = \varepsilon_{\ell,x} = 1$ for $x \in \ell$ and therefore $\varepsilon(X) = 1$. Assume henceforth that X does not contain any lines and for fixed $x \in X$ consider the divisor $D =_{\text{def}} X \cap T_x X \in |\mathcal{O}_X(1)|$. Certainly D is reduced (see Lemma 1.2).

Let us first consider the case that D is irreducible. If the Hesse form $\mathcal{H}_X$ vanishes on $T_x X$, so that any tangent to X at x is a principal tangent, then $\text{mult}_x(D) \geq 3$. On the other hand, since D is an irreducible plane quartic curve, we have in any event $\text{mult}_x(D) \leq 3$, thus $\varepsilon_{D,x} = \frac{4}{3}$. Because of

$$\varepsilon_{D,x} \leq 2 = \sqrt{\mathcal{O}_X(1)^2}$$

Lemma 2.1 gives $\varepsilon(X,x) = \frac{4}{3}$. If, however, $\mathcal{H}_X$ is of rank ≥ 1 at x, then $\text{mult}_x(D) = 2$ and we obtain $\varepsilon(X,x) = 2$.

Now suppose that D is reducible. Then D must consist of two smooth conics D_1 and D_2 meeting at x. So $\mathcal{H}_X$ cannot vanish at x and because of

$$\varepsilon_{D_i,x} = \sqrt{2 \cdot \mathcal{O}_X(1) D_i}$$

Lemma 2.1 implies $\varepsilon(X,x) = 2$.

It remains to show the assertion about the codimensions. In the space $\mathcal{S}$ consider the subsets

$$\mathcal{L} = \{X \mid X \text{ contains a line}\} \subset \mathcal{S}$$

and

$$\mathcal{H} = \{X \mid \text{rank}\,(\mathcal{H}_X(x)) = 0 \text{ for some point } x \in X\} \subset \mathcal{S}\ ,$$

so that $\{\varepsilon(X) = 1\} = \mathcal{L}$ and $\{\varepsilon(X) = \frac{4}{3}\} = \mathcal{H} - \mathcal{L}$. Clearly $\mathcal{L}$ is of codimension 1. As for $\mathcal{H}$, we consider the variety

$$V =_{\text{def}} \left\{(X,x,\pi) \;\middle|\; x \in X,\ T_x X = \pi,\ \text{rank}\,(\mathcal{H}_X(x)) = 0\right\} \subset \mathcal{S} \times \mathbb{P}^3 \times (\mathbb{P}^3)^*$$

and the projections

$$\begin{array}{ccccc} & & V & & \\ & \swarrow^{pr_1} & \downarrow^{pr_2} & \searrow^{pr_3} & \\ \mathcal{S} \supset \mathcal{H} & & \mathbb{P}^3 & & (\mathbb{P}^3)^* \end{array}$$

The dimension of the general fibre of the map $pr_2 \times pr_3 : V \longrightarrow (pr_2 \times pr_3)(V)$ is easily seen to be $\dim(\mathcal{S}) - 6$. Further, by Proposition 1.1(c) the first projection $pr_1 : V \longrightarrow \mathcal{H}$ is of finite degree. So we obtain that

$$\begin{aligned} \dim(\mathcal{H}) &= \dim(V) \\ &= \dim(\mathcal{S}) - 6 + \dim((pr_2 \times pr_3)(V)) \\ &= \dim(\mathcal{S}) - 1 , \end{aligned}$$

since the image of $pr_2 \times pr_3$ is the 5-dimensional incidence variety in $\mathbb{P}^3 \times (\mathbb{P}^3)^*$. So $\mathcal{H}$ is of codimension 1 as well, and this completes the proof of the theorem. □

Acknowledgements. This research was supported by DFG contract Ba 423/7-1. I would like to thank W. Barth for helpful discussions. Furthermore, I am grateful to the referee for his comments on the first version of this paper.

References

[1] Demailly, J.-P.: Singular Hermitian metrics on positive line bundles. Complex algebraic varieties (Bayreuth, 1990), Lect. Notes Math. 1507, Springer-Verlag, 1992, pp. 87-104.
[2] Ein, L., Lazarsfeld, R.: Seshadri constants on smooth surfaces. Journées de Géometrie Algébrique d'Orsay, Astérisque 218, 1993, pp. 177-186.
[3] Ein, L., Küchle, O., Lazarsfeld, R.: Local positivity of ample line bundles. J. Differential Geom. 42, 193-219 (1995)
[4] Harris, J., Tu, Loring W.: On symmetric and skew-symmetric determinantal varieties. Topology 23, 71-84 (1984)
[5] Józefiak, T., Lascoux, A., Pragacz, P.: Classes of determinantal varieties associated with symmetric and skew-symmetric matrices. Math. USSR Izvestija 18, 575-586 (1982)
[6] Steffens, A.: Remarks on Seshadri constants. Preprint

Thomas Bauer, Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstraße $1\frac{1}{2}$, D-91054 Erlangen, Germany
(e-mail: bauerth@mi.uni-erlangen.de)