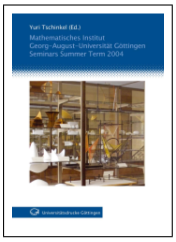

Appeared in: Y. Tschinkel (ed.), Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Summer Term 2004 (pp. 75-82). Universitätsdrucke Göttingen, 2004.

ZARISKI CHAMBERS AND STABLE BASE LOCI

THOMAS BAUER

Abstract. In joint work with A. Küronya and T. Szemberg we study certain asymptotic invariants of linear series: the stable base locus and the volume. In particular we are interested in the question how these invariants behave under small perturbations in the Néron-Severi space. We show that both invariants lead to a partition of the big cone into suitable subcones, and that – somewhat surprisingly – these two partitions coincide. This phenomenon is explained by the fact that both problems are closely related to the variation of the Zariski decomposition, which is an interesting problem quite on its own.

1. Introduction

We report here on our recent joint work [BKS] with A. Küronya and T. Szemberg on asymptotic invariants of linear series. Let us start by considering three questions.

Stable base loci. Let X be a smooth projective variety and L a line bundle on X. We denote by $\mathrm{SB}(L)$ the *stable base locus* of L, i.e., the intersection of the base loci of the linear series $|kL|$ for all positive integers k. More generally, we will consider the stable base loci of $\mathbb{Q}$-line bundles L by passing to an integral multiple of L; this is well-defined, since the stable base locus is invariant under taking multiples (i.e. tensor powers) of the line bundle.

Stable base loci were recently studied by Nakamaye ([N1], [N2]). He showed in particular that for a big and nef divisor L, an ample divisor A, and for small $\varepsilon > 0$, the stable base locus $\mathrm{SB}(L - \varepsilon A)$ is the union of all subvarieties $V \subset X$ such that $L^{\dim V} \cdot V = 0$. So in particular the stable base locus remains constant when a big and nef line bundle is perturbed in anti-ample directions. We ask quite generally:

Question 1. *How does* $\mathrm{SB}(L)$ *vary when* L *moves in the big cone of* X*?*

One needs to be a bit more precise here: the stable base locus is not invariant under numerical equivalence, and hence it is not a function on the big cone. Following [ELMNP] we therefore consider a modified version of the stable base locus, the *stabilized base locus*, defined as

$$\mathrm{B}_+(L) = \mathrm{SB}(L - A) \ ,$$

where A is a sufficiently small ample bundle. These stabilized base loci in fact turn out to be numerical invariants.

Volumes. Consider a line bundle L on a smooth projective variety X of dimension n. The Riemann–Roch problem is concerned with the study of the behaviour of $h^0(X, kL)$ as a function of k. While the exact determination of these dimensions is difficult in general, they grow typically (i.e. for big line bundles) like k^n. The *volume* of L, introduced by Cutkosky, is then defined as

$$\mathrm{vol}_X(L) =_{\mathrm{def}} \limsup_k \frac{h^0(X,kL)}{k^n/n!}.$$

This concept readily extends to $\mathbb{Q}$-divisors, and in fact it has recently been established (in [PAG]) that it defines a continuous function on the Néron-Severi space. It is thus natural to ask:

Question 2. *How does* $\mathrm{vol}(L)$ *vary when* L *moves in the big cone of* X*?*

Zariski decompositions. Let now X be a smooth projective surface. Recall ([Z], [KMM, Theorem 7.3.1]) that a (pseudo-)effective $\mathbb{R}$-divisor D on X admits a unique Zariski decomposition, i.e., there exists a unique effective $\mathbb{R}$-divisor $N_D = \sum_{i=1}^m a_i N_i$ such that

(i) $P_D = D - N_D$ ist nef,
(ii) N_D is either zero or its intersection matrix $(N_i \cdot N_j)$ is negative definite,
(iii) $P_D \cdot N_i = 0$ for $i = 1, \ldots, m$.

The Zariski decomposition is determined by the numerical equivalence class of D, so it makes sense to study it as a function on the Néron-Severi space. We ask:

Question 3. *How does the Zariski decomposition of* L *vary when* L *moves in the big cone of* X*?*

Each of the three problems leads to a decomposition of the big cone into subsets where the invariant in question behaves nicely. Our result [BKS] says that, somewhat surprisingly, on surfaces the underlying decompositions in fact agree:

Theorem. *Let* X *be a smooth projective surface over the complex numbers. Then there is a locally finite decomposition of the big cone of* X *into rational locally polyhedral subcones such that the following holds:*

(i) *In each subcone the support of the negative part of the Zariski decomposition of the divisors in the subcone is constant.*
(ii) *On each of the subcones the volume function is given by a single polynomial of degree two.*
(iii) *In the interior of each of the subcones the stable base loci are constant.*

As Zariski decompositions do not in general exist on higher-dimensional varieties, part (i) is specific to surfaces. On the other hand, it is natural to ask for higher-dimensional analogues of statements (ii) and (iii). A statement as clean as that of the theorem above, however, cannot be expected: already in dimension three there are examples where the volume is not locally polynomial (see [BKS], Section 3.3); and the subsets, where the stable base locus is constant, need not have rational boundaries (see [BKS], Example 2.11). Nonetheless it would be interesting to know whether the statements (ii) and (iii) might have higher-dimensional analogues at least for certain types of varieties.

2. The decomposition

In this section we will focus on explaining the decomposition of the big cone as stated in the theorem. Our purpose is to convey some feeling for the geometry underlying the three problems in question, and to sketch the main ideas. Details and complete proofs can be found in [BKS].

Consider an $\mathbb{R}$-divisor D on a smooth projective surface X with Zariski decomposition

$$D = P_D + N_D \ .$$

We consider the set of *null curves* and the set of *negative curves* of D, defined as

$$\mathrm{Null}(D) =_{\mathrm{def}} \{\, C \mid C \text{ irreducible curve with } D \cdot C = 0 \,\}$$

and

$$\mathrm{Neg}(D) =_{\mathrm{def}} \{\, C \mid C \text{ irreducible component of } N_D \,\}$$

respectively. On always has $\mathrm{Neg}(D) \subset \mathrm{Null}(P_D)$, but it may well happen that some of the null curves do not appear as components of N_D.

Now we can specify the subcones of the big cone mentioned in the theorem. To this end, consider for a big and nef $\mathbb{R}$-divisor P the set

$$\Sigma_P = \{\, D \in \mathrm{Big}(X) \mid \mathrm{Neg}(D) = \mathrm{Null}(P) \,\} \ .$$

It is immediate to check – using the properties of the Zariski decomposition – that Σ_P is a convex cone. (It will in general be neither open nor closed.) One shows (see [BKS], Lemma 1.6) that these cones yield a decomposition of the big cone, i.e.,

$$\mathrm{Big}(X) = \bigcup_{P \text{ big and nef}} \Sigma_P \ , \tag{1}$$

where $\Sigma_P = \Sigma_{P'}$ or $\Sigma_P \cap \Sigma_{P'} = \varnothing$ for any two big and nef $\mathbb{R}$-divisors P and P'. The main point in proving (1) is that, given a big divisor D, one is able to find a big and nef divisor P such that $\mathrm{Neg}(D) = \mathrm{Null}(P)$.

As far as part (i) of the theorem is concerned, two things remain to be shown.

Proposition 1. (a) *The cone Σ_P is locally polyhedral.*
(b) *The decomposition (1) is locally finite.*

(See [BKS], Proposition 1.10 and Proposition 1.15.) In order to prove assertion (a), we provide an explicit description of Σ_P as follows:

$$\overline{\Sigma}_P \cap \mathrm{Big}(X) = (\mathrm{Big}(X) \cap \mathrm{Face}(P)) + V^{\geqslant 0}(\mathrm{Null}(P)) \ . \tag{2}$$

Here

$$\mathrm{Face}(P) =_{\mathrm{def}} \mathrm{Null}(P)^{\perp} \cap \mathrm{Nef}(X)$$

is the smallest face of the nef cone that contains P, and $V^{\geqslant 0}(\mathrm{Null}(P))$ denotes the cone generated by the null curves of P. As the part of the nef cone that is contained in the big cone is locally polyhedral ([BKS], Corollary 1.4), statement (a) follows.

The description of the chambers in (2) is particularly instructive as it shows that each chamber Σ_P corresponds to a face of the nef cone. This fact is also illustrated in the example that we provide in Section 3.

Turning to assertion (b), suffice it to say that it is essentially a consequence of the

Main Lemma ([BKS], Lemma 1.14). *If D is a big $\mathbb{R}$-divisor and A an ample $\mathbb{R}$-divisor, then*

$$\mathrm{Neg}(D + \lambda A) \subset \mathrm{Neg}(D) \qquad \textit{for all } \lambda \geqslant 0 \ .$$

The nice fact about the main lemma is that it admits a pleasant elementary proof, which shows exactly how Zariski decompositions behave when a divisor moves in ample directions.

Sketch of Proof. Let $D = P + \sum_{i=1}^r a_i N_i$ be the Zariski decomposition of D. We show:

$(*)$ *There is a real number $\lambda_0 > 0$ and there are decreasing affine-linear functions f_i on $\mathbb{R}$ such that for $0 \leqslant \lambda \leqslant \lambda_0$ the Zariski decomposition of $D + \lambda A$ is given as*

$$D + \lambda A = \left(P + \lambda A + \sum_{i=1}^r (a_i - f_i(\lambda)) N_i \right) + \sum_{i=1}^r f_i(\lambda) N_i \ ,$$

and such that λ_0 is a zero of one of the functions f_i.

From this statement our lemma follows by induction on r. Turning to the proof of $(*)$, let us for real numbers $x_1, \ldots, x_r$ consider the divisor

$$P' =_{\text{def}} P + \lambda A + \sum_{i=1}^r (a_i - x_i) N_i \ .$$

The Zariski decomposition of $D + \lambda A$ is $P' + \sum_{i=1}^r x_i N_i$ if and only if the following conditions are satisfied:

$$0 \leqslant x_i \leqslant a_i \qquad \text{for } i = 1, \ldots, r, \tag{3}$$

$$P' \cdot N_i = 0 \qquad \text{for } i = 1, \ldots, r, \tag{4}$$

$$P' \text{ is nef.} \tag{5}$$

As $P + \lambda A$ is ample, condition (5) follows from (3) and (4). Condition (4) is equivalent to a system of linear equations in the indeterminates x_i, whose coefficient matrix is just the intersection matrix $S =_{\text{def}} (N_i \cdot N_j)$. As S is invertible, there is certainly no problem in solving for the x_i, but the whole point is whether the solutions x_i satisfy (3), i.e., whether $x_i \leqslant a_i$ for all i. Luckily, this is a consequence of the following statement:

$(**)$ *Let $S = (s_{ij})$ be a negative definite $r \times r$-matrix over the reals such that $s_{ij} \geqslant 0$ for $i \neq j$. Then all entries of the inverse matrix S^{-1} are $\leqslant 0$.*

Finally, the proof of $(**)$ is a nice exercise in linear algebra. (To be honest, the argument that we found is slightly tricky. In case of doubt see [BKS, Lemma 4.1].) $\square$

Note that the proof of the main lemma in fact shows exactly how the Zariski decomposition of $D + \lambda A$ varies as a function of λ: The coefficients of the negative part $N_{D+\lambda A}$ are decreasing affine-linear functions of λ, and as soon as one of these functions reaches zero, the component in question disappears from the negative part.

Let us conclude this section by briefly commenting on parts (ii) and (iii) of the theorem. For (iii) we show in [BKS] that for every rational divisor class D in the interior of a chamber the stable base locus $\mathrm{SB}(D)$ is given by the support of the negative part N_D of the Zariski decomposition, and that the stable base locus agrees with the stabilized base locus $\mathrm{B}_+(D)$. The essential point for (ii) is that the growth of $h^0(X, kD)$ is determined by the positive part in the Zariski decomposition of D.

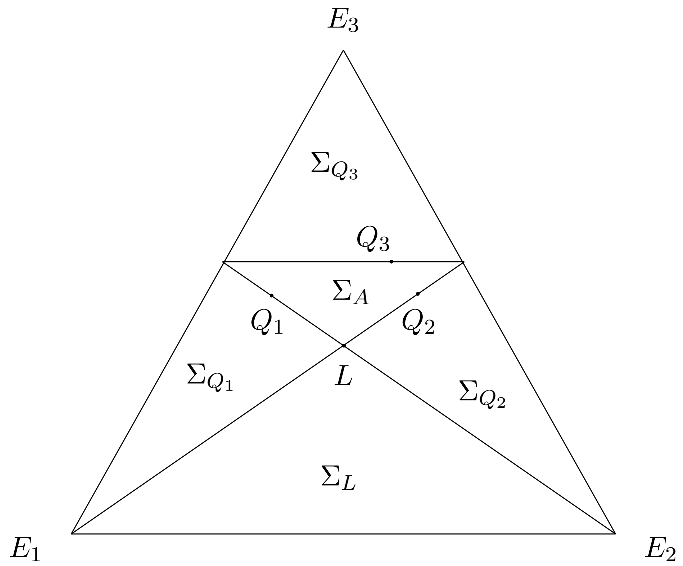

FIGURE 1. Two-point blow-up of the projective plane. The big cone consists of five chambers: Σ_A, Σ_{Q_1}, Σ_{Q_2}, Σ_{Q_3}, Σ_L.

3. Example: Two-point blow-up of the plane

Consider the blow-up $X \to \mathbb{P}^2$ of the projective plane in two points. On X there are exactly three irreducible curves with negative self-intersection: the exceptional divisors E_1 and E_2, and the proper transform E_3 of the line through the two blown-up points, whose class is $L - E_1 - E_2$. These three curves generate the closure of the big cone.

Figure 1 shows a cross-section of the (closure of the) big cone. The nef cone has five faces that contain big divisors, leading to five chambers:

- the chamber Σ_A of an ample class A (so that Σ_A is just the ample cone),
- chambers $\Sigma_{Q_1}, \Sigma_{Q_2}, \Sigma_{Q_3}$ associated to divisors Q_1, Q_2, Q_3 on the boundary of the nef cone as indicated in Figure 1,
- the chamber Σ_L.

Note that – in accordance with (2) – the dimension of a face and the number of the corresponding null curves add up to the dimension of the chamber, i.e., the Picard number of X:

divisor	dimension of face	null curves
A	3	0
Q_i	2	1
L	1	2

More information about the general situation on del Pezzo surfaces, as well as on K3 surfaces, can be found in Sections 3.1 and 3.2 of [BKS].

References

[BKS] Bauer, T., Küronya, A., Szemberg, T.: Zariski chambers, volumes, and stable base loci. J. reine angew. Math. (to appear)

[ELMNP] L.Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, M. Popa: Asymptotic invariants of base loci. Preprint, arXiv AG 0308116.

[KMM] Kawamata, Y. et al.: Introduction to the minimal model problem. Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[PAG] Lazarsfeld, R.: Positivity in Algebraic Geometry, book in preparation

[N1] Nakamaye, M.: Stable base loci of linear series Math. Ann. **318** (2000), 837–847

[N2] Nakamaye, M.: Base loci of linear series are numerically determined. Trans. Amer. Math. Soc. **355** (2003), 551–566

[Z] Zariski, O.: The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. Ann. Math. 76, 560-615 (1962)

Thomas Bauer, Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany.

E-mail address: `tbauer@mathematik.uni-marburg.de`