Dr. sc. Helga Baum
Born in 1954 in Berlin. Studied Mathematics at the Humboldt University Berlin. Since 1980 at the Humboldt University. Received Dr. rer. nat. in 1981, Dr. sc. in 1989.

Prof. Dr. sc. Thomas Friedrich
Born in 1949 in Leipzig. Studied at the University of Wrocław from 1968 to 1973. Since 1973 at the Humboldt University Berlin. Received Dr. rer. nat. in 1974, Dr. sc. nat. in 1979. Currently full Professor at the Humboldt University since 1987.

Dr. Ralf Grunewald
Born in 1961 in Leipzig. Studied Mathematics in Torun and Berlin. Since 1986 at the Humboldt University Berlin. Received Dr. rer. nat. in 1986.

Dr. Ines Kath
Born in 1964 in Dresden. Studied Mathematics at the Humboldt University Berlin from 1982 to 1987. Received Dr. rer. nat. in 1990. Working since 1986 at the Humboldt University.

CIP-Titelaufnahme der Deutschen Bibliothek

Twistors and Killing spinors on Riemannian manifolds /
Helga Baum ... – Stuttgart ; Leipzig : Teubner, 1991
(Teubner-Texte zur Mathematik ; Bd. 124)
ISBN 3-8154-2014-8
NE: Baum, Helga ; GT

TEUBNER-TEXTE zur Mathematik · Band 124
ISSN 0138-502X

Printed in Germany
Gesamtherstellung: Druckerei „G. W. Leibniz" GmbH, Gräfenhainichen

TEUBNER-TEXTE zur Mathematik · Band 124

Helga Baum · Thomas Friedrich
Ralf Grunewald · Ines Kath

Twistors and Killing Spinors on Riemannian Manifolds

B. G. Teubner Verlagsgesellschaft
Stuttgart · Leipzig 1991

In this book we investigate, after an introductory section to Clifford algebras, spinors on manifolds etc., in particular solutions of the twistor equation as well as Killing spinors. New results on the construction and classification of Riemannian manifolds with real and imaginary Killing spinors, respectively, are the main subject of this book. Moreover, we consider the relations between solutions of the general twistor equation and Killing spinors.

In diesem Buch werden nach einem einleitenden Abschnitt über Clifford-Algebren, Spinoren auf Mannigfaltigkeiten etc. , insbesondere Lösungen der Twistor-Gleichung sowie Killing-Spinoren studiert. Den Hauptinhalt des Buches bilden neue Resultate zur Konstruktion und Klassifikation Riemannscher Mannigfaltigkeiten mit reellen bzw. imaginären Killing-Spinoren.
Desweiteren werden die Beziehungen zwischen Lösungen der allgemeinen Twistor-Gleichung und Killing-Spinoren untersucht.

Dans ce livre on etudie, après a un paragraphe introduisant dédié à des algèbres de Clifford, des spineurs sur des variétés etc., des solutions de l'équation twisteur ainsi que des spineurs de Killing.
Le contenue essentiel du livre est formé par des résultats nouveaux concernant la construction et la classification des variétés riemanniennes admettant des spineurs de Killing réels ou imaginaires. De plus on analyse les relations entre des solutions de l'équation twisteur générale et des spineurs de Killing.

В этом томе после вступительной части, посвященной алгебрам Клиффорда, спинорам на многообразиах ит. п., рассматриваются решения твисторного уравнения как и спиноры Киллинга.
Главное содержание тома состоит в новых результатах о построении и классификации римановых многообразий с вещественными или мнимыми спинорами Киллинга.
Кроме того, исследоваются связи между решениями общего твисторного уравнения и спинорами Киллинга.

Preface

This book is devoted to the so-called Killing and twistor spinors, special kinds of spinors on Riemannian manifolds appearing in Mathematical Physics as well as in a purely mathematical context.

In the first chapter we give an introduction to Clifford algebras, spin-representation and the spinor calculus on Riemannian manifolds. Furthermore, we investigate the two natural first order differential operators on spinors, the Dirac and the Twistor operator. The main subject of the present book is the construction and the classification of Riemannian manifolds with real and imaginary Killing spinors. The results described here were obtained during the last 5 years and are presented in a systematical and complete manner in this book for the first time.

Berlin, May 1990

Helga Baum
Thomas Friedrich
Ralf Grunewald
Ines Kath

Contents

Introduction

This book is devoted to the so-called Killing and twistor spinors, special kinds of spinors on Riemannian manifolds appearing in Mathematical Physics as well as in a purely mathematical context.

A Killing spinor is a spinor field φ on a Riemannian spin manifold satisfying the linear differential equation

$$\nabla_X \varphi = \beta X \cdot \varphi$$

for a complex number β and all vector fields X. Killing spinors were first introduced in General Relativity (see [22], [26]) as a technical tool to construct integrals of the free geodesic motion. More recently, they occurred in 10- and 11- dimensional supergravity theories (see [2,6,7,8,12,25]). When studying classical solutions without fermionic fields with a "residual supersymmetry" it was observed that this residual supersymmetry could give rise to a Killing spinor (see [5], [13], [14]). On the other hand, Killing spinors also played an important role in the construction of exact solutions by providing useful "Ansätze" for the matter field (see [3,4,17,27]).

In Geometry, Killing spinors appeared in 1980 in connection with eigenvalue problems of the Dirac operator D. If (M^n,g) is a compact Riemannian manifold with positive scalar curvature $R > 0$ and if R_0 denotes the minimum of R, Th. Friedrich proved (see [32]) the inequality for the first eigenvalue λ_1 of the Dirac operator D

$$\lambda_1^2 \geq \frac{1}{4} \frac{n R_0}{n-1} .$$

Moreover, if $\lambda = \pm \frac{1}{2} \sqrt{\frac{n R_0}{(n-1)}}$ is an eigenvalue of D with the eigenspinor φ, then φ satisfies the stronger equation

$$\nabla_X \varphi = \mp \frac{1}{2} \sqrt{\frac{R_0}{n(n-1)}} X \cdot \varphi$$

i.e. the eigenspinors to the smallest possible eigenvalue are

Killing spinors (see[32]). The existence of Killing spinor imposes algebraic conditions on the Weyl tensor of the space and on the covariant derivative of the curvature tensor; in particular, M^n has to be an Einstein space (see [32],[34]). Furthermore, in 1980 Th. Friedrich constructed an Einstein metric on the 5-dimensional Stiefel manifold $V_{4,2}$ admitting Killing spinors. Compact 7-dimensional Einstein spaces with Killing spinors were constructed by M. Duff, B. Nilsson and C. Pope (see [26],[27]) as well as by P. van Nieuwenhuizen and N. Warner (see [89]) in 1983. Using the twistor construction, Th. Friedrich and R. Grunewald obtained (see [40]) the first even-dimensional examples in 1985; they constructed Einstein metrics on $P^3(\mathbb{C})$ and on the flag manifold $F(1,2)$ with Killing spinors. In 1986 O. Hijazi proved (see [57]) the inequality

$$\lambda_1^2 \geq \frac{1}{4}\,\frac{n}{n-1}\,\mu_1,$$

where μ_1 is the first eigenvalue of the Yamabe operator $L = 4\,\frac{n-1}{n-2}\,\Delta + R$ (see also [82] for a general approach). However, if this lower bound is an eigenvalue of the Dirac operator, then the scalar curvature R is constant and the eigenspinor is a Killing spinor. A Kähler manifold does not admit Killing spinors (see [57]). Consequently, in case of a compact Kähler manifold there exists a better estimation for the first eigenvalue of the Dirac operator; this case has been investigated in a series of papers by K.D. Kirchberg (see [68],[69],[70]). Moreover, O. Hijazi proved that if a compact 8-dimensional manifold admits a Killing spinor, then it is isometric to the sphere S^8 (see [58]).

During the last years we investigated the relation between Killing spinors and other geometric structures on the underlying manifold. It has turned out that, in case of a compact odd-dimensional manifold, there is a link between Killing spinors and special contact structures; this observation yields a general construction principle of compact odd-dimensional Riemannian manifolds with Killing spinors as well as classification results in dimension $n = 5,7$ (see [41], [42],[43],[44]).

Furthermore, on compact even-dimensional manifolds - at least in dimension $n = 6$ - there exists a relation between Killing spinors and certain non-integrable almost complex structures (see [55]). The complete non-compact Riemannian manifolds with Killing spinors were classified by H. Baum (see [5],[6],[7]) in 1988.

Killing spinors are special solutions of the conformally invariant field equation

$$Y\cdot\nabla_X\varphi + X\cdot\nabla_Y\varphi = \frac{2}{n}\, g(X,Y)D\varphi,$$

the so-called twistor equation (see [89],[92]).
In mathematics, the twistor equation appeared as an integrability condition for the complex structure on the twistor space of a 4-dimensional Riemannian manifold (see [2]). A. Lichnerowicz (see [89]) started a systematical geometrical investigation of the solutions of the twistor equation in 1987. In particular, using the solution of the Yamabe problem he proved that on a compact manifold the space of all twistor spinors coincides - up to a conformal change of the metric-with the space of all Killing spinors (see [84]).
Th. Friedrich (see [38]) studied the zeros and "first integrals" of twistor spinors and their relation to Killing spinors in case of an arbitrary Riemannian manifold.
In the first chapter of the book we give a short introduction to the spinor calculus on Riemannian manifolds and the Dirac equation. We define the notion of Killing and twistor spinors, prove some elementary geometrical facts of manifolds admitting these kinds of spinors and investigate the relation between Killing spinors, twistor spinors and solutions of the Dirac equation. In Chapter 2 we investigate the properties of twistor spinors in detail. We study special twistor spinors satisfying the equation

$$\nabla_X\varphi + \frac{f}{n}\, X\cdot\varphi = 0$$

for a complex function f and the so-called equation (E), introduced by A. Lichnerowicz.
In Chapter 3 we give an interpretation of twistor spinors as a holomorphic linear section on a certain line bundle over the twistor space of a Riemannian 4-manifold.
The existence of a non-trivial Killing spinor on a Riemannian spin manifold (M^n,g) implies in particular that (M^n,g) is an Einstein space with constant scalar curvature $R = 4n(n-1)B^2$. Hence, the number B is real or purely imaginary and there are different types of Killing spinors:

real Killing spinors	$(B \in \mathbb{R}\setminus\{0\})$
imaginary Killing spinors	$(B \in i\,\mathbb{R}\setminus\{0\})$
parallel spinors	$(B = 0)$.

Assuming the completeness of (M,g), real Killing spinors occur only on compact manifolds and imaginary Killing spinors only on non-compact manifolds. In the Chapters 4 and 5 compact manifolds with real Killing spinors are studied.

If the dimension of M is odd, then Killing spinors are related to Einstein-Sasaki-structures (Chap. 4), in even dimension Killing spinors are related to non-integrable almost complex structures (Chap. 5). In Chapter 6 we present an overview on results about parallel spinors and Chapter 7 is devoted to the description of the structure of non-compact complete manifolds with imaginary Killing spinors.

Chapter 1: An Introduction to Killing and Twistor Spinors

In this Chapter we give a short introduction to the spinor calculus on Riemannian manifolds and the Dirac and twistor equation. In the first three parts we fix the notations and sum up basic facts concerning spinors and the Dirac operator on Riemannian manifolds. For proofs of the stated properties we refer to [4].
In the last two parts we introduce the notion of twistor and Killings spinors and prove some elementary geometric properties of manifolds admitting these kinds of spinors.

1.1. The Spin-Group and the Spinor Representation

Let us denote by $(e_1,\dots,e_n)$ the canonical basis of the Euclidean space $(\mathbb{R}^n, <\,,>)$ and by $Cliff(\mathbb{R}^n)$ the Clifford algebra of $\mathbb{R}^n$ with the bilinear form $-<,>$. $Cliff(\mathbb{R}^n)$ is an algebra over $\mathbb{R}$ that is multiplicatively generated by the vectors $e_1,\dots,e_n$ with the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\,\delta_{ij} \qquad i,j = 1,\dots,n.$$

In case $n = 2m$, the complexification $Cliff^c(\mathbb{R}^n)$ of the Clifford algebra is isomorphic to the algebra $M(2^m;\mathbb{C})$ of all complex matrices of rank 2^m. In case $n = 2m+1$, $Cliff^c(\mathbb{R}^n)$ is isomorphic to $M(2^m;\mathbb{C}) \oplus M(2^m;\mathbb{C})$. In this book we use the following identification of $Cliff^c(\mathbb{R}^n)$ with the matrix algebras mentioned above:
Denote

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\alpha(j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even.} \end{cases}$$

Then an isomorphism

$$\Phi_{2m}: Cliff^c(\mathbb{R}^{2m}) \longrightarrow M(2^m;\mathbb{C})$$

is given by the Kronecker product

$$\Phi_{2m}(e_j) := E \otimes \dots \otimes E \otimes g_{\alpha(j)} \otimes \underbrace{T \otimes \dots \otimes T}_{[(j-1)/2]\text{-times}} \qquad (1.1)$$

$$j = 1,\dots,2m.$$

An isomorphism

$$\Phi_{2m+1}: Cliff^c(\mathbb{R}^{2m+1}) \longrightarrow M(2^m;\mathbb{C}) \oplus M(2^m;\mathbb{C})$$

is given by

$$\begin{aligned}\Phi_{2m+1}(e_j) &:= (\Phi_{2m}(e_j), \Phi_{2m}(e_j)) \quad j=1,\ldots,2m \\ \Phi_{2m+1}(e_{2m+1}) &:= (i\, T\otimes\ldots\otimes T, -i\, T\otimes\ldots\otimes T)\end{aligned} \tag{1.2}$$

The group $Spin(n)$ is a double covering of the special orthogonal group $SO(n)$, which is universal iff $n\geq 3$. $Spin(n)$ can be realized as a subgroup in $Cliff(\mathbb{R}^n)$

$$Spin(n) := \{x_1\cdot\ldots\cdot x_{2k} \mid x_j\in\mathbb{R}^n, \|x_j\| = 1, k\in\mathbb{N}\}.$$

Its Lie algebra is the vector space

$$\underline{spin}\,(n) = span_{\mathbb{R}}(e_i\cdot e_j \mid 1\leq i < j\leq n) \subset Cliff(\mathbb{R}^n)$$

with the commutator $[v,w] = v\cdot w - w\cdot v$.

The map

$$\begin{aligned}\lambda: Spin(n) &\longrightarrow SO(n) \\ u &\longmapsto \lambda(u), \quad \lambda(u)x := uxu^{-1} \quad \text{for } x\in\mathbb{R}^n\end{aligned}$$

is a double covering of $SO(n)$ with the differential

$$\begin{aligned}\lambda_*: \underline{spin}(n) &\longrightarrow \underline{so}(n) \\ e_i\cdot e_j &\longmapsto 2E_{ij},\end{aligned}$$

where

$$E_{ij} = \begin{array}{c} \begin{pmatrix} \ldots & -1 & \ldots \\ \ldots +1 & & \ldots \end{pmatrix} \begin{array}{l} i \\ j \end{array} \\ \quad i \quad j \end{array} \qquad i<j$$

is the basis of the Lie algebra $\underline{so}(n)$ of $SO(n)$. If we restrict the map Φ_{2m} to $Spin(2m)$, we obtain a 2^m-dimensional representation of $Spin(2m)$, the so-called <u>spinor representation</u>

$$\Phi_{2m}|_{Spin(2m)} : Spin(2m) \longrightarrow GL(\mathbb{C}^{2^m})$$

We denote this representation as well as its representation space by Δ_{2m}: The modul Δ_{2m} splits into two irreducible unitary representations $\Delta_{2m} = \Delta^+_{2m} \oplus \Delta^-_{2m}$, given by the eigensubspaces of the endomorphism $\Phi_{2m}(e_1\cdot\ldots\cdot e_{2m})$ to the eigenvalues $\pm i^m$.

Let us denote by $u(\varepsilon)\in\mathbb{C}^2$ the vector

$$u(\varepsilon) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\varepsilon i\end{pmatrix}, \quad \varepsilon = \pm 1$$

and let

$$u(\mathcal{E}_1,\dots,\mathcal{E}_m) = u(\mathcal{E}_1)\otimes\dots\otimes u(\mathcal{E}_m)\ , \qquad \mathcal{E}_j = \pm 1.$$

$(u(\mathcal{E}_1,\dots,\mathcal{E}_m) \mid \prod_{j=1}^{m} \mathcal{E}_j = \pm 1)$ is an ON-basis of $\Delta^{\pm}_{2m}$ with respect to the standard scalar product of $\mathbb{C}^{2^m}$. In case $n = 2m+1$, we obtain two 2^m-dimensional irreducible unitary representations Δ_{2m+1} and $\hat{\Delta}_{2m+1}$ of Spin(2m+1) if we restrict Φ_{2m+1} to Spin(2m+1) and project onto the first and second component of $M(2^m;\mathbb{C}) \oplus M(2^m;\mathbb{C})$, respectively.

The isomorphism Φ_n also gives rise to a multiplication of vectors and spinors, the so-called <u>Clifford multiplication</u>

$$\mu : \mathbb{R}^n \otimes \overset{(\wedge)}{\Delta}_n \longrightarrow \overset{(\wedge)}{\Delta}_n$$

$$x \otimes u \longmapsto \mu(x\otimes u) = x\cdot u = \begin{cases} \Phi_n(x)u & n \text{ even} \\ \mathrm{proj}_j\cdot\Phi_n(x)u & n \text{ odd}\ ; \end{cases}$$

μ is invariant under the Spin(n)-action, where Spin(n) acts on $\mathbb{R}^n$ by the covering λ .

Using (1.1) and (1.2) the following properties are easy to verify:

1) For an element $u \in \Delta_{2m+1}$ let $u = u^+ \oplus u^-$ denote the decomposition of u with respect to the (non-invariant) subspaces $\mathrm{span}\left(u(\mathcal{E}_1,\dots,\mathcal{E}_m) \mid \prod_{j=1}^{m} \mathcal{E}_j = \pm 1\right)$.

   The map

   $$u = u^+ \oplus u^- \in \Delta_{2m+1} \longmapsto \hat{u} := u^+ - u^- \in \hat{\Delta}_{2m+1}$$

   is an isomorphism of the Spin(2m+1)-representations. The Clifford multiplication satisfies $\widehat{x\cdot u} = -x\cdot\hat{u}$.

2) The map

   $$\begin{aligned} \Delta_{2m+1}\big|_{\mathrm{Spin}(2m)} &\longrightarrow \Delta_{2m} \\ u &\longmapsto u \end{aligned}$$

   is an isomorphism of the Spin(2m) representations. By this identification the vector e_{2m+1} acts on $\Delta_{2m} = \Delta^+_{2m} \oplus \Delta^-_{2m}$ by

   $$e_{2m+1}\cdot(u^+ \oplus u^-) = (-1)^m i\ (u^+ - u^-).$$

3) The map

   $$\begin{aligned} \Delta_{2m+2}\big|_{\mathrm{Spin}(2m+1)} &\longrightarrow \Delta_{2m+1} \oplus \hat{\Delta}_{2m+1} \\ \begin{pmatrix}1\\0\end{pmatrix} \otimes u + \begin{pmatrix}0\\1\end{pmatrix} \otimes \hat{v} &\longmapsto u \oplus \hat{v} \end{aligned}$$

   is an isomorphism of the Spin(2m+1)-representations.

Due to this identification the vector e_{2m+2} acts on $\Delta_{2m+1} \oplus \hat{\Delta}_{2m+1}$ by

$$e_{2m+2} \cdot (u \oplus \hat{v}) = (-1)^m i (v \oplus \hat{u}).$$

1.2. The Spinor Calculus on Riemannian Spin Manifolds

Let (M^n, g) be an n-dimensional oriented Riemannian manifold and let $P = (P, p, M; SO(n))$ be the bundle of all $SO(n)$-frames of (M^n, g). A spinor structure of (M^n, g) is a pair (Q, f) of a $Spin(n)$-principal bundle $Q = (Q, q, M; Spin(n))$ and a continuous surjective map $f: Q \to P$ such that the diagramme

$$\begin{array}{ccccc} Q \times Spin(n) & \xrightarrow{\cdot} & Q & \searrow^{q} & \\ \downarrow f \times \lambda & & \downarrow f & & M \\ P \times SO(n) & \xrightarrow{\cdot} & P & \nearrow_{p} & \end{array}$$

commutes. An oriented Riemannian manifold (M, g) admits a spinor structure iff the second Stiefel-Whitney class $w_2(M)$ vanishes.. In case $w_2(M) = 0$, the isomorphy classes of spinor structures are classified by the first cohomology group $H^1(M; \mathbb{Z}_2)$. An oriented Riemannian manifold with a spinor structure (Q, f) is called a spin manifold. The complex vector bundle

$$S := Q \times_{Spin(n)} \Delta_n$$

associated with the $Spin(n)$-principal bundle Q by means of the spinor representation Δ_n is called a spinor bundle of (M^n, g). By $\langle \cdot, \cdot \rangle$ we denote the complex scalar product on S defined by the canonical hermitian product on Δ_n and by $(\cdot, \cdot) := \mathrm{Re} \langle \cdot, \cdot \rangle$ the corresponding real scalar product on S. A smooth section $\varphi \in \Gamma(S)$ of S is called a spinor field on (M, g).
In case of even dimension $n = 2m$, the spinor bundle S splits into two subbundles $S = S^+ \oplus S^-$

$$S^{\pm} := Q \times_{Spin(2m)} \Delta^{\pm}_{2m},$$

which we call the positive and negative part of S.
In odd dimension n we denote by $\hat{S}$ the bundle $\hat{S} := Q \times_{Spin(n)} \hat{\Delta}_n$, which is isomorphic to the spinor bundle S by

$$\begin{array}{c} S \xrightarrow{\hat{}} \hat{S} \\ [q, u] \longmapsto [q, \hat{u}]. \end{array}$$

By $\mu: TM \otimes S \longrightarrow S$
$X \otimes \varphi \longmapsto X\cdot\varphi$

we denote the Clifford multiplication on the bundle level, which is the bundle morphism defined by the Clifford multiplications on the fibres. In case of even n, μ exchanges the positive and negative part of S. In case of odd dimension we have $\widehat{X\cdot\varphi} = - X\cdot\hat{\varphi}$.
The Clifford multiplication can be extended to k-forms. Each k-form $\omega \in \Omega^k(M)$, $1\le k\le n$, acts as a bundle morphism on the spinor bundle S, which is defined by the local formula

$$\omega\cdot\varphi = \sum_{1\le i_1<\ldots<i_k\le n} \omega(s_{i_1},\ldots,s_{i_k})s_{i_1}\cdot\ldots\cdot s_{i_k}\cdot\varphi ,$$

where $(s_1,\ldots,s_n)$ is a local ON-basis of (M,g). In this book we often identify the tangent bundle TM with the cotangent bundle T^*M by means of the metric g

$$\begin{array}{ccc} TM & \xrightarrow{\ \sim\ } & T^*M \\ X & \longmapsto & g(X,.). \end{array}$$

In particular, the Clifford multiplication by a vector is the same as that by the dual covector . Now, we list some properties of the Clifford multiplication which are easy to verify by the definition.

1) If $\varphi\in\Gamma(S)$ is a spinor field without zeros and $X \in \mathfrak{X}(M)$ a vector field on M, then an equation of the form $X\cdot\varphi \equiv 0$ provides $X \equiv 0$.

2) For the Clifford multiplication,

$$X\cdot Y + Y\cdot X = -2g(X,Y)\, id_S \tag{1.3}$$

$$\left.\begin{aligned} X\cdot\omega &= (X\wedge\omega) - (X\lrcorner\,\omega) \\ \omega\cdot X &= (-1)^k \{X\wedge\omega + X\lrcorner\,\omega\} \end{aligned}\right. \tag{1.4}$$

are satisfied, where X,Y are vector fields and ω is a k-form on M.

3) With respect to the scalar product in S we have

$$\langle X\cdot\varphi,\psi\rangle = -\langle\varphi, X\cdot\psi\rangle \ , \ X\in\mathfrak{X}(M) \tag{1.5}$$

$$\langle\omega\cdot\varphi,\psi\rangle = (-1)^{k(k+1)/2}\langle\varphi,\omega\cdot\psi\rangle \ , \omega\in\Omega^k(M) \tag{1.6}$$

$$(X\cdot\varphi \ , \ Y\cdot\varphi \ ) = g(X,Y)|\varphi|^2 \tag{1.7}$$

The Levi-Civita connection ∇^M on (M,g) defines a covariant derivative $\nabla^S : \Gamma(S) \longrightarrow \Gamma(TM \otimes S)$ in S, the so-called <u>spinor derivative</u>. Locally, ∇^S is given by

$$\nabla^S_X\varphi = X(\varphi) + \frac{1}{2}\sum_{1\le k<l\le n} \omega_{kl}(X)s_k\cdot s_l\cdot\varphi \ , \tag{1.8}$$

where $\omega_{kl} = g(\nabla^M s_k, s_l)$ are the connection forms of ∇^M with respect to a local ON-basis $(s_1,\ldots,s_n)$. For even n the spinor derivative respects the positive and negative part of the spinor bundle S. In odd dimension we have $\widehat{\nabla^S_X\varphi} = \nabla^S_X\hat{\varphi}$. The spinor derivative satisfies the following rules:

$$X\langle\varphi,\psi\rangle = \langle\nabla^S_X\varphi,\psi\rangle + \langle\varphi,\nabla^S_X\psi\rangle \tag{1.9}$$

$$\nabla^S_X(Y\cdot\varphi) = \nabla^M_X Y\cdot\varphi + Y\cdot\nabla^S_X\varphi \tag{1.10}$$

$$\nabla^S_X(\omega\cdot\varphi) = \nabla^M_X\omega\cdot\varphi + \omega\cdot\nabla^S_X\varphi \ , \tag{1.11}$$

where X,Y are vector fields, ω a k-form, and φ,ψ spinor fields on M.

Let us denote the curvature tensor in (S,∇^S) by $\mathcal{R}^S : \Lambda^2 M \longrightarrow End(S)$

$$\mathcal{R}^S(X,Y) = \nabla^S_X\nabla^S_Y - \nabla^S_Y\nabla^S_X - \nabla^S_{[X,Y]} \ .$$

The endomorphism $\mathcal{R}^S(X,Y)$ on S can be expressed by the curvature of (M,g). Let us consider the curvature tensor of (M,g) to be a bundle morphism $\mathcal{R} : \Lambda^2 M \longrightarrow \Lambda^2 M$ on the bundle of 2-forms $\Lambda^2 M$

$$\mathcal{R}(\sigma^i\wedge\sigma^j) := \sum_{k<l} R_{ijkl}\,\sigma^k\wedge\sigma^l ,$$

where $R_{ijkl} = g(\mathcal{R}^M(s_i,s_j)s_k,s_l)$ are the components of the curvature tensor $\mathcal{R}^M$ of (M,g) and $(\sigma^1,\ldots,\sigma^n)$ is the dual basis to the ON-basis $(s_1,\ldots,s_n)$. Furthermore, we consider the Ricci-tensor of (M,g) to be a bundle map $Ric: TM \longrightarrow TM$ of the tangent bundle

$$Ric(X) := \sum_{k=1}^n Ric(X,s_k)s_k \ .$$

Then, using the local formula (1.8) for the spinor derivative ∇^S we obtain

$$\mathcal{R}^S(X,Y)\varphi = \frac{1}{2}\mathcal{R}(X\wedge Y)\cdot\varphi \tag{1.12}$$

After applying the first Bianci-identity for the curvature tensor $\mathcal{R}^M$ of (M,g), this relation yields

$$\sum_{k=1}^n s_k\cdot\mathcal{R}^S(X,s_k)\varphi = -\frac{1}{2}Ric(X)\cdot\varphi \ . \tag{1.13}$$

Now we recall the behaviour of the spinor calculus by conformal change of the metric g (comp. [4], 3.2.4.). Let (M,g) be a spin manifold and let $\tilde{g} := \sigma\cdot g$, $\sigma\in C^\infty(M)$, be a conformally equivalent metric. Then there is an identification $\sim$ of the spinor bundle S of (M,g) and $\tilde{S}$ of $(M,\tilde{g})$ such that

$$\tilde{X}\cdot\tilde{\varphi} = \widetilde{X\cdot\varphi} \tag{1.14}$$

$$\nabla^{\tilde{S}}_{\tilde{X}}\tilde{\varphi} = \sigma^{-1/2}\widetilde{\nabla_X\varphi} + \frac{1}{2}\widetilde{X\cdot\mathrm{grad}(\sigma^{-1/2})\cdot\varphi} + \frac{1}{2}X(\sigma^{-1/2})\tilde{\varphi}\,, \tag{1.15}$$

where $\tilde{X} = \sigma^{-1/2}X$ for $X \in TM$.

Finally, we collect some formulas for the spinor calculus on submanifolds of codimension one and on warped products with intervals.

Let (M^n,g) be a spin manifold with spinor structure (Q,f) and spinor bundle S and let $F^{n-1}\subset M$ be an oriented submanifold of codimension one with induced metric. We denote by ξ the normal vector field on F^{n-1} given by the orientation of F and M. The reduction of $Q|_F$ with respect to ξ induces a spinor structure (Q_F,f_F) on $(F^{n-1},g|_F)$. Let $S_F := Q_F \times_{Spin(n-1)} \Delta_{n-1}$ denote the spinor bundle of $(F,g|_F)$. Applying the algebraic properties of the spinor representation restricted to a Spin group of codimension one (Chapter 1.1) and formula (1.8) for the spinor derivative, one obtains

1.) If $n = 2m+1$, the restriction of the spinor bundle S to F^{n-1} is isomorphic to the bundle S_F, where ξ acts on S_F by

$$\xi\cdot(\varphi^+\oplus\varphi^-) = (-1)^m i(\varphi^+ - \varphi^-) \tag{1.16}$$

and the spinor derivative of $\varphi\in\Gamma(S)$ is given by

$$\nabla^S_X\varphi = \nabla^{S_F}_X(\varphi|_F) - \frac{1}{2}\nabla^M_X\xi\cdot\xi\cdot\varphi \tag{1.17}$$

for all $X\in T_xF$.

2.) If $n = 2m+2$, the restriction of the spinor bundle S to F^{n-1} is isomorphic to the bundle $S_F \oplus \hat{S}_F$, where ξ acts on $S_F \oplus \hat{S}_F$ by

$$\xi\cdot(\varphi_1\oplus\hat{\varphi}_2) = (-1)^m i(\varphi_2\oplus\hat{\varphi}_1) \tag{1.18}$$

and the spinor derivative of $\varphi\in\Gamma(S)$ is given by

$$\nabla^S_X\varphi = \nabla^{S_F}_X\varphi_1\oplus\nabla^{S_F}_X\hat{\varphi}_2 - \frac{1}{2}\nabla^M_X\xi\cdot\xi\cdot\varphi \tag{1.19}$$

for all $X\in T_xF$ (with the denotation $\varphi|_F = \varphi_1\oplus\hat{\varphi}_2$).

Now, let (F^{n-1},h) be a spin manifold with the spinor structure (Q_F,f_F), $I = (a,b)\subseteq\mathbb{R}$ be an open interval and $\sigma\in C^\infty(I,(0,\infty))$ be a smooth positive function. We consider the warped product

$$(M^n,g) := F^{n-1}\,{}_\sigma\!\times I := (F\times I,\ \sigma(t)h\oplus dt^2).$$

(M^n,g) admits a spinor structure (Q,f) which reduces itself with respect to $\xi(x,t) = \frac{\partial}{\partial t}$ to a Spin(n-1)-structure $(\hat{Q},\hat{f})$ realizing, over each fibre $F\times\{t\}$, the spinor structure of $(F,\sigma(t)\cdot h)$ that is conformally equivalent to the spinor structure (Q_F,f_F) of (F,h). Let $\pi: F \times I \longrightarrow F$ be the projection. For a section $\varphi\in\Gamma(\pi^*S_F)$ we denote by $\varphi_t\in \Gamma(S_F)$ the spinor field $\varphi_t(x) := \varphi(x,t)$. For a vector field X on F let $\tilde{X}$ be the vector field $\tilde{X}(x,t) := \sigma(t)^{-\frac{1}{2}} X(x)$ on M. On the warped product $M = F\,{}_\sigma\!\times I$ the Levi-Civita connection satisfies

$$g(\nabla^M_{\tilde{X}}\, \tilde{s}_\alpha,\ \tilde{s}_\beta) = \sigma^{-\frac{1}{2}}\ h(\nabla^F_X\, s_\alpha, s_\beta)$$

$$g(\nabla^M_{\tilde{X}}\, \tilde{s}_\alpha, \xi) = -\frac{1}{2}\sigma^{-1}\sigma'\ \ h(s_\alpha, X)$$

$$g(\nabla^M_{\xi}\, \tilde{s}_\alpha, \xi) = g(\nabla^M_{\xi}\, \tilde{s}_\alpha, \tilde{s}_\beta) = 0,$$

where $(s_1,\dots,s_{n-1})$ denotes a local ON-basis on (F,h) and $X\in T_xF$. Applying the formulas (1.14) and (1.15) for the spinor calculus on conformally equivalent manifolds we obtain the following relations between the spinor bundles of (F^{n-1},h) and $(M^n,g) := F\,{}_\sigma\!\times I$:

1.) If $n = 2m+1$, the spinor bundle S of (M,g) can be identified with the bundle π^*S_F by

$$\pi^*S_F \xrightarrow{\ \sim\ } S = \hat{Q}\times_{Spin(2m)} \Delta_{2m+1}$$

$$\varphi=[q,u(x,t)] \longrightarrow \tilde{\varphi}:=[\tilde{q},u(x,t)] \quad ,$$

where $\tilde{q}$ denotes the element of $\hat{Q}_{(x,t)}$ corresponding to $q\in Q_{F,x}$ with respect to the conformal equivalence of Q_F and $\hat{Q}|_{F\times\{t\}}$. By this identification the Clifford multiplication satisfies

$$\begin{aligned} \tilde{X}(x,t)\cdot \tilde{\varphi}(x,t) &= \widetilde{X(x)\cdot\varphi_t(x)}, \qquad X(x)\in T_xF \\ \xi\cdot \widetilde{(\varphi^+\oplus\varphi^-)} &= (-1)^m i\ \widetilde{(\varphi^+ - \varphi^-)} \end{aligned} \tag{1.20}$$

and the spinor derivative is given by

$$\begin{aligned} \nabla^S_X\tilde{\varphi} &= \sigma^{-1/2}\ \widetilde{\nabla^{S_F}_X \varphi_t} - \frac{1}{4}\sigma^{-1}\sigma'\ \ \tilde{X}\cdot\xi\cdot\tilde{\varphi} \qquad , X\in T_xF \\ \nabla^S_\xi\tilde{\varphi} &= \widetilde{\frac{\partial}{\partial t}(\varphi)}. \end{aligned} \tag{1.21}$$

2.) If $n = 2m+2$, the spinor bundle S of (M,g) can be identified with the bundle $\pi^*S_F \oplus \pi^*\hat{S}_F$ by

$$\pi^* S_F \oplus \pi^* \hat{S}_F \xrightarrow{\;\sim\;} S = \hat{Q} \times_{Spin(n-1)} \Delta_n$$

$$\varphi := \varphi_1 \oplus \hat{\varphi}_2 \longmapsto \tilde{\varphi} := \tilde{\varphi}_1 \oplus \tilde{\hat{\varphi}}_2$$

$$= [q, u(x,t)] \oplus [q, \hat{v}(x,t)] \qquad = [\tilde{q}, \binom{1}{0} \otimes u(x,t) + \binom{0}{1} \otimes \hat{v}(x,t)] .$$

By this identification the Clifford multiplication satisfies

$$\left.\begin{aligned} \tilde{X}(x,t) \cdot \tilde{\varphi}(x,t) &= \widetilde{X(x) \cdot \varphi_{1t}(x)} \oplus \widetilde{X(x) \cdot \varphi_{2t}(x)}, \quad X \in T_x F \\ \xi \cdot \tilde{\varphi} &= (-1)^m i \, (\tilde{\varphi}_2 \oplus \tilde{\hat{\varphi}}_1) \end{aligned}\right\} \qquad (1.22)$$

and the spinor derivative is given by

$$\begin{aligned} \nabla^S_{\tilde{X}} \tilde{\varphi} &= \sigma^{-\frac{1}{2}} \widetilde{\{\nabla^{S_F}_X \varphi_1 \oplus \nabla^{\hat{S}_F}_X \hat{\varphi}_2\}} - \frac{1}{4} \sigma^{-1} \sigma' X \cdot \xi \cdot \tilde{\varphi} \qquad , X \in T_x F \\ \nabla^S_{\xi} \tilde{\varphi} &= \widetilde{\frac{\partial}{\partial t}(\varphi)} \; . \end{aligned} \qquad (1.23)$$

In the following chapters we will often omit the symbols S and M in the covariant derivatives ∇^S and ∇^M for simplicity and denote all covariant derivatives by ∇ .

1.3. The Dirac Operator of a Riemannian Spin Manifold

Let (M^n, g) be a Riemannian spin manifold with the spinor bundle S. The Dirac operator of (M^n, g) is the first order differential operator defined by

$$D: \Gamma(S) \xrightarrow{\nabla^S} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S) .$$

Locally D can be expressed by

$$D = \sum_{k=1}^{n} s_k \cdot \nabla^S_{s_k} \, , \qquad (1.24)$$

where $(s_1, \dots, s_n)$ is a local ON-basis of (M^n, g).
Let f be a function, X a vector field and w a k-form on M.
Using (1.24), (1.3), (1.4), (1.10) and (1.11) the following commuting rules are easy to verify

$$D(f\varphi) = f\, D\varphi + \operatorname{grad} f \cdot \varphi \qquad (1.25)$$

$$D(X \cdot \varphi) = -X \cdot D\varphi - 2\nabla_X \varphi + \sum_{j=1}^{n} s_j \cdot \nabla_{s_j} X \cdot \varphi \qquad (1.26)$$

$$D(w \cdot \varphi) = (-1)^k w \cdot D\varphi + (d + \delta) w \cdot \varphi - 2 \sum_{j=1}^{n} (s_j \lrcorner w) \cdot \nabla_{s_j} \varphi \qquad (1.27)$$

where d is the exterior differential and δ its adjoint.
The Dirac operator is elliptic and formally selfadjoint on the space $\Gamma_c(S)$ of smooth sections with compact support with respect to the scalar product

$$\langle \varphi, \psi \rangle := \int_M \langle \varphi(x), \psi(x) \rangle \, dM.$$

If (M^n,g) is complete, D is essentially selfadjoint in the space of L^2-sections $L_2(S)$ defined by the completion of $(\Gamma_c(S), \langle , \rangle)$. If M is compact, the spectrum of D (as an elliptic and formally selfadjoint operator) contains only discrete real eigenvalues of finite multiplicity (see [91], Chap. 11). In case of even dimension n, D exchanges the positive and negative part of S. Hence, for an eigenspinor $\varphi = \varphi^+ \oplus \varphi^- \in \Gamma(S^+ \oplus S^-)$ to an eigenvalue $\lambda \in \mathbb{R}$ we have

$$D\varphi^+ = \lambda \varphi^- \qquad \text{and} \qquad D\varphi^- = \lambda \varphi^+$$

and, therefore,

$$D(\varphi^+ - \varphi^-) = (-\lambda)(\varphi^+ - \varphi^-).$$

Hence, the spectrum of D on a compact manifold in case of even dimension is symmetric to zero.

In 1963 A. Lichnerowicz ([81]) proved the following Weitzenböck formula for the square of the Dirac operator

$$D^2 = \frac{1}{4} R + \Delta^S,$$

where R denotes the scalar curvature of (M,g) and Δ^S the Bochner-Laplace operator of ∇^S:

$$\Delta^S := \nabla^{S*} \circ \nabla^S = - \sum_{k=1}^{n} (\nabla^S_{s_k} \nabla^S_{s_k} + \operatorname{div}(s_k) \nabla^S_{s_k}).$$

Let $f \in C^\infty(M)$ be a smooth real-valued function and $\nabla^f : \Gamma(S) \longrightarrow \Gamma(TM \otimes S)$ the metric covariant derivative on S defined by

$$\nabla^f_X \varphi := \nabla^S_X \varphi + f\, X \cdot \varphi \ .$$

Then we have the following generalization of Lichnerowicz' formula:

<u>Theorem 1</u> ([32]):

$$(D-f)^2 = \Delta^f + \frac{1}{4} R + (1-n) f^2, \tag{1.28}$$

where Δ^f denotes the Bochner-Laplace operator of the covariant derivative ∇^f.

<u>Proof:</u> From (1.25) it follows that

$$(D-f)^2 = D^2 - 2fD - \operatorname{grad} f \cdot + f^2.$$

Using the Lichnerowicz formula for D^2 we obtain

$$(D-f)^2 = \Delta^S + \frac{1}{4} R - 2fD - \operatorname{grad} f \cdot + f^2.$$

Applying the rules (1.9) and (1.10) it follows for the Bochner-Laplace operators Δ^S and Δ^f

$$\Delta^f = \Delta^S - 2fD - \operatorname{grad} f \cdot + nf^2.$$

Hence

$$(D-f)^2 = \Delta^f + \frac{1}{4} R + (1-n)f^2.$$

Theorem 1 implies the following Corollary for the spectrum of the Dirac operator on compact manifolds.

<u>Corollary 1</u> ([32]): Let (M^n,g) be a compact Riemannian spin manifold of positive scalar curvature $R > 0$. Then

1) The first positive and negative eigenvalue λ_+ and λ_- of D satisfy the estimates

$$|\lambda_{\pm}| \geq \frac{1}{2}\sqrt{\frac{R_o n}{n-1}} ,$$

where R_o is the minimum of the scalar curvature R.

2) If the lowest bound $+\frac{1}{2}\sqrt{\frac{R_o n}{n-1}}$ or $-\frac{1}{2}\sqrt{\frac{R_o n}{n-1}}$ is an eigenvalue of D and φ a corresponding eigenspinor, then φ satisfies the differential equation

$$\nabla_X \varphi + \frac{1}{2}\sqrt{\frac{R_o}{n(n-1)}}\; X \cdot \varphi = 0$$

resp.

$$\nabla_X \varphi - \frac{1}{2}\sqrt{\frac{R_o}{n(n-1)}}\; X \cdot \varphi = 0$$

for all vector fields X on M.

<u>Proof:</u> Assume that λ is an eigenvalue of D with $\lambda^2 \leq \frac{1}{4} R_o \frac{n}{n-1}$ and $\varphi \not\equiv 0$ is an eigenspinor to λ. By integration of (1.28) with the function $f = \frac{\lambda}{n}$ we obtain

$$\int_M \langle (D-\tfrac{\lambda}{n})^2 \varphi, \varphi\rangle dM = \int_M \left\{ (\tfrac{1}{4}R + \tfrac{1-n}{n^2}\lambda^2)|\varphi|^2 + |\nabla^{\frac{\lambda}{n}}\varphi|^2 \right\} dM$$

and because of $D\varphi = \lambda\varphi$

$$0 = \int_M \left\{ (\tfrac{1}{4} R + \tfrac{1-n}{n}\lambda^2)|\varphi|^2 + |\nabla^{\frac{\lambda}{n}}\varphi|^2 \right\} dM. \qquad (*)$$

In case $\lambda^2 < \frac{1}{4} R_o \frac{n}{n-1}$, equation (*) requires $\varphi \equiv 0$, which is a contradiction. In case $\lambda^2 = \frac{1}{4} R_o \frac{n}{n-1}$, (*) particularly yields $\nabla^{\frac{\lambda}{n}}\varphi \equiv 0$. This proves the second statement.

Let (M^n,g), $n \geq 3$, be a compact connected spin manifold. In [57] O. Hijazi proved a lower estimate for the eigenvalues λ of the

Dirac operator of (M^n,g) using a conformal invariant bound:

$$\lambda^2 \geq \frac{n}{4(n-1)} \mu_1 \geq \frac{n}{4(n-1)} R_o,$$

where μ_1 is the first eigenvalue of the conformal scalar Laplacian

$$L = 4 \frac{n-1}{n-2} \Delta + R.$$

If λ_1 is an eigenvalue of D satisfying $\lambda_1^2 = \frac{n}{4(n-1)} \mu_1 > 0$, then (M,g) has constant scalar curvature $R \equiv \mu_1$, i.e. the two bounds are equal.

A compact spin manifold with positive scalar curvature has no harmonic spinors (by a harmonic spinor we mean a non-trivial element of $\mathcal{H} := \{\varphi \in \Gamma(S) \mid D\varphi \equiv 0\}$).

In general, the dimension of $\mathcal{H}$ depends on the metric as well as on the chosen spinor structure (see [61]).

If $\tilde{g} = \sigma g$ is a conformally equivalent metric, the Dirac operator $\tilde{D}$ of $(M,\tilde{g})$ satisfies

$$\tilde{D}(\tilde{\varphi}) = \sigma^{-\frac{n+1}{4}} \widetilde{D(\sigma^{\frac{n-1}{4}} \varphi)} \tag{1.29}$$

(cf. [61] or [4] Chap. 3.2.4).

Thus, the dimension of the space of harmonic spinors is a conformal invariant.

1.4. The Twistor Operator of a Riemannian Spin Manifold

Let (M^n,g) be a Riemannian spin manifold with the spinor bundle S and let μ: $TM \otimes S \longrightarrow S$ be the Clifford multiplication. Then $\ker \mu$ is a subbundle of $TM \otimes S$ and there exists a projection p: $TM \otimes S \longrightarrow \ker \mu$ onto $\ker \mu$ given by the formula

$$p(X \otimes \varphi) = X \otimes \varphi + \frac{1}{4} \sum_{k=1}^{n} s_k \otimes s_k \cdot X \cdot \varphi \quad ,$$

where $(s_1,\ldots,s_n)$ is a local ON-basis.

Definition 1: The twistor operator $\mathcal{D}$ of (M,g) is the composition of the spinor derivative ∇^S and the projection p

$$\mathcal{D} := p \circ \nabla^S : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\mathrm{Ker}\ \mu).$$

Locally we have

$$\begin{aligned} \mathcal{D}\varphi = p\nabla^S\varphi = p\left(\sum_{k=1}^{n} s_k \otimes \nabla_{s_k}\varphi\right) = \\ = \sum_{k=1}^{n} s_k \otimes \left\{\nabla_{s_k}\varphi + \frac{1}{n} s_k \cdot D\varphi\right\} \end{aligned} \tag{1.30}$$

Definition 2: A spinor field $\varphi \in \Gamma(S)$ is called a twistor spinor if φ lies in the kernel of the twistor operator $\mathcal{D}$.

Theorem 2: Let $\varphi \in \Gamma(S)$. The following conditions are equivalent:

1) φ is a twistor spinor
2) φ satisfies the so-called twistor equation

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \tag{1.31}$$

   for all vector fields X.
3) For all vector fields X, Y it holds

$$X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi = \frac{2}{n} g(X,Y) D\varphi . \tag{1.32}$$

4) The spinor field

$$X \cdot \nabla_X \varphi$$

   does not depend on the unit vector field X.

Proof: Equation (1.30) shows that the condition $\mathcal{D}\varphi \equiv 0$ is equivalent to the twistor equation

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \qquad \text{for all } X \in \mathfrak{X}(M).$$

Multiplying this by a vector field we have

$$Y \cdot \nabla_X \varphi + \frac{1}{n} Y \cdot X \cdot D\varphi = 0 \qquad \text{and}$$

$$X \cdot \nabla_Y \varphi + \frac{1}{n} X \cdot Y \cdot D\varphi = 0.$$

Using condition (1.3), the sum of these equations provides (1.32). Conversely, let (1.32) be valid. We multiply (1.32) for $Y = s_j$ by s_j and sum up over $j = 1,\dots,n$. Then

$$\begin{aligned} \frac{2}{n} X \cdot D\varphi &= \sum_{j=1}^{n} s_j \cdot X \cdot \nabla_{s_j} \varphi - n \nabla_X \varphi \\ &= - X \cdot D\varphi - 2 \nabla_X \varphi - n \nabla_X \varphi \\ &= - X \cdot D\varphi - (n+2) \nabla_X \varphi . \end{aligned}$$

This shows (1.31). Finally, from (1.32) it follows that

$$X \cdot \nabla_X \varphi = \frac{1}{n} D\varphi$$

for all unit vector fields X. Conversely, if $X \cdot \nabla_X \varphi$ do not depend on the unit vector field X, we obtain, by setting

$\psi := X \cdot \nabla_X \varphi$,

$$D\varphi = n\psi \qquad \text{and} \qquad \nabla_X \varphi = - X \cdot X \cdot \nabla_X \varphi = - X \cdot \psi .$$

This provides (1.31).

Let $\varphi \in \Gamma(S)$ be an arbitrary spinor field on (M,g). By $u_\varphi \in C^\infty(M)$ we denote the function

$$u_\varphi(x) := \langle \varphi(x), \varphi(x) \rangle ,$$

which we call the <u>length function of φ</u>. By $T_\varphi \in \mathfrak{X}(M)$ we understand the vector field

$$T_\varphi := \sum_{j=1}^{n} (\varphi, s_j \cdot D\varphi) s_j ,$$

where $(s_1,\dots,s_n)$ is a local ON-basis.

The following formulas are convenient for the calculus with twistor spinors.

<u>Theorem 3 ([83]):</u> Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then the following conditions are satisfied:

$$D^2\varphi = \frac{1}{4} R \frac{n}{n-1} \varphi \tag{1.33}$$

$$\nabla_X(D\varphi) = \frac{n}{2(n-2)} \left(R \cdot \frac{1}{2(n-1)} X - \mathrm{Ric}(X)\right) \cdot \varphi \tag{1.34}$$

$$\Delta u_\varphi = \frac{R}{2(n-1)} u_\varphi - \frac{2}{n} \langle D\varphi, D\varphi \rangle \tag{1.35}$$

$$\mathrm{grad}\, u_\varphi = -\frac{2}{n} T_\varphi \tag{1.36}$$

$$\mathrm{div}(T_\varphi) = -\langle D\varphi, D\varphi \rangle + \frac{1}{4} \frac{n}{n-1} R\, u_\varphi . \tag{1.37}$$

Here R denotes the scalar curvature and $\mathrm{Ric}: TM \to TM$ the Ricci curvature of (M^n,g).

<u>Proof:</u> Let $x \in M$. In the following calculations we use an ON-basis $(s_1,\dots,s_n)$ arising from one in T_xM by parallel displacement along geodesics. Then

$$\mathrm{div}(s_j)(x) = 0, \quad [s_i,s_j](x) = 0 \quad \text{and} \quad (\nabla s_j)(x) = 0.$$

If we differentiate the twistor equation (1.31) and use rule (1.10), we obtain in $x \in M$

$$0 = \sum_{j=1}^{n} \nabla_{s_j} \nabla_{s_j} \varphi + \frac{1}{n} \nabla_{s_j}(s_j \cdot D\varphi)$$

$$= -\Delta^S \varphi + \frac{1}{n} D^2\varphi .$$

Applying the Weitzenböck formula $D^2 = \Delta^S + \frac{1}{4} R$ it follows

$$D^2\varphi = \frac{1}{4} R \frac{n}{n-1} \varphi .$$

Furthermore, let X be a local vector field arising from a vector in T_xM by parallel displacement along geodesics. Then, the twistor equation implies in $x \in M$

$$\nabla_{s_j}\nabla_X\varphi + \frac{1}{n}X\cdot\nabla_{s_j}(D\varphi) = 0 \quad \text{and}$$
$$\nabla_X\nabla_{s_j}\varphi + \frac{1}{n}s_j\cdot\nabla_X(D\varphi) = 0 \qquad j=1,\dots,n.$$

Hence,

$$\mathcal{R}^S(X,s_j)\varphi = -\frac{1}{n}s_j\cdot\nabla_X(D\varphi) + \frac{1}{n}X\cdot\nabla_{s_j}(D\varphi).$$

Using condition (1.13) we obtain

$$\begin{aligned}\mathrm{Ric}(X)\cdot\varphi &= -2\sum_{j=1}^n s_j\cdot\mathcal{R}^S(X,s_j)\varphi = \\ &= -2\nabla_X(D\varphi) - \frac{2}{n}\sum_{j=1}^n s_j\cdot X\cdot\nabla_{s_j}(D\varphi) \\ &= -2\nabla_X(D\varphi) + \frac{2}{n}X\cdot D^2\varphi + \frac{4}{n}\nabla_X(D\varphi).\end{aligned}$$

Applying $D^2\varphi = \frac{1}{4}R\frac{n}{n-1}\varphi$ it follows

$$\nabla_X(D\varphi) = \frac{n}{2(n-2)}\left\{\frac{R}{2(n-1)}X - \mathrm{Ric}(X)\right\}\cdot\varphi \quad .$$

For the Laplacian of the length-function $u_\varphi = \langle\varphi,\varphi\rangle$ formula (1.9) provides

$$\begin{aligned}\Delta u_\varphi &= -\sum_{j=1}^n s_j s_j(u_\varphi) = -\sum_{j=1}^n s_j\{\langle\nabla_{s_j}\varphi,\varphi\rangle + \langle\varphi,\nabla_{s_j}\varphi\rangle\} \\ &= \{\langle\Delta^S\varphi,\varphi\rangle + \langle\varphi,\Delta^S\varphi\rangle\} - 2\sum_{j=1}^n\langle\nabla_{s_j}\varphi,\nabla_{s_j}\varphi\rangle \quad .\end{aligned}$$

Using the twistor equation (1.31), the Lichnerowicz-formula $D^2 = \Delta^S + \frac{1}{4}R$ and (1.33) we obtain

$$\begin{aligned}\Delta u_\varphi &= \frac{R}{2(n-1)}\cdot u_\varphi - \frac{2}{n^2}\sum_{j=1}^n\langle s_j\cdot D\varphi, s_j\cdot D\varphi\rangle \\ &= \frac{R}{2(n-1)}u_\varphi - \frac{2}{n}\langle D\varphi, D\varphi\rangle \ .\end{aligned}$$

Furthermore, the twistor equation yields

$$\begin{aligned}\mathrm{grad}\, u_\varphi &= \sum_{j=1}^n s_j(u_\varphi)s_j = \sum_{j=1}^n\{\langle\nabla_{s_j}\varphi,\varphi\rangle + \langle\varphi,\nabla_{s_j}\varphi\rangle\}s_j \\ &= 2\sum_{j=1}^n(\varphi,\nabla_{s_j}\varphi)s_j \\ &= -\frac{2}{n}\sum_{j=1}^n(\varphi, s_j\cdot D\varphi)s_j \\ &= -\frac{2}{n}T_\varphi\ .\end{aligned}$$

The last two equations give

$$\begin{aligned}\mathrm{div}(T_\varphi) &= -\frac{n}{2}\mathrm{div}(\mathrm{grad}\, u_\varphi) = \frac{n}{2}\Delta u_\varphi \\ &= \frac{1}{4}R\frac{n}{n-1}u_\varphi - \langle D\varphi, D\varphi\rangle \ .\end{aligned}$$

Now we can prove a further condition for φ being a twistor spinor. Let $K: TM \longrightarrow TM$ denote the bundle map

$$K(X) := \frac{1}{n-2}\left\{ \frac{R}{2(n-1)} X - Ric(X) \right\}.$$

We consider the bundle $E := S \oplus S$ and the covariant derivative ∇^E in E defined by

$$\nabla^E_X := \begin{pmatrix} \nabla^S_X & \frac{1}{n} X \cdot \\ -\frac{n}{2} K(X) & \nabla^S_X \end{pmatrix}$$

<u>Theorem 4</u> ([38]): For any twistor spinor $\varphi \in \Gamma(S)$ it holds that $\nabla^E \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} \equiv 0$. Conversely, if $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \Gamma(E)$ is ∇^E-parallel, then φ is a twistor spinor and $\psi = D\varphi$.

<u>Proof:</u> Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then

$$\nabla^E_X \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} = \begin{pmatrix} \nabla_X \varphi + \frac{1}{n} X \cdot D\varphi \\ \nabla_X D\varphi - \frac{n}{2} K(X) \cdot \varphi \end{pmatrix}$$

The twistor equation and formula (1.34) provide $\nabla^E_X \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} = 0$.
Now, let $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \Gamma(E)$ be a ∇^E-parallel section : $\nabla^E \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv 0$. Then, in particular, by definition of ∇^E we have, $\nabla_X \varphi + \frac{1}{n} X \cdot \varphi = 0$ for all vector fields X.

Multiplying this by X and using (1.3) and (1.24) this shows that $D\varphi = \psi$ and that φ is a solution of the twistor equation (1.31).

By Theorem 4 the twistor spinors correspond to the ∇^E-parallel sections of the bundle E. Hence, a twistor spinor φ is defined by its values $\varphi(m_o)$, $D\varphi(m_o)$ at some points $m_o \in M$. In particular, we obtain

<u>Corollary 2:</u> The dimension of the space of twistor spinors on a connected, n-dimensional Riemannian manifold is less than or equal to $2^{[\frac{n}{2}]+1}$.

<u>Corollary 3:</u> Let φ be a twistor spinor on a connected Riemannian manifold such that φ and $D\varphi$ vanish at some point $m_o \in M$. Then the twistor φ is trivial, i.e. $\varphi \equiv 0$.

Let us denote the components of the tensor K introduced above by $K_{ij} = \frac{1}{n-2}\left\{\frac{R}{2(n-1)} g_{ij} - R_{ij}\right\}$. The Weyl-tensor W is given by the formula

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - g_{\beta\delta}K_{\alpha\gamma} - g_{\alpha\gamma}K_{\beta\delta} + $$
$$+ g_{\beta\gamma}K_{\alpha\delta} + g_{\alpha\delta}K_{\beta\gamma} .$$

We understand the curvature tensor $\mathcal{R}$ as well as the Weyl-tensor W as endomorphisms of $\Lambda^2 M^n$ by the rule

$$\mathcal{R}(\sigma^i \wedge \sigma^j) = \sum_{k<l} R_{ijkl}\sigma^k \wedge \sigma^l$$

$$W(\sigma^i \wedge \sigma^j) = \sum_{k<l} W_{ijkl}\sigma^k \wedge \sigma^l .$$

Using this notation a straightforward calculation provides for the curvature tensor $\mathcal{R}^E$ of the covariant derivative ∇^E in the bundle E:

$$\mathcal{R}^E(X,Y)\binom{\varphi}{\psi} = \begin{pmatrix} \frac{1}{4} W(X \wedge Y)\varphi \\ \frac{1}{4} W(X \wedge Y)\psi + \frac{n}{2}((\nabla_Y K)(X) - (\nabla_X K)(Y))\varphi \end{pmatrix} .$$

<u>Theorem 5:</u> Let $\varphi \in \operatorname{Ker} \mathcal{D}$ be a twistor spinor on a Riemannian manifold M^n, then, for any 2-form $\eta = Y \wedge Z$ and any vector X, we have

$$W(\eta)\cdot\varphi = 0 \tag{1.38}$$
$$W(\eta)\cdot D\varphi + 2n\{(\nabla_Z K)(Y) - (\nabla_Y K)(Z)\}\cdot\varphi = 0 \tag{1.39}$$
$$(\nabla_X W)(\eta)\cdot\varphi = -2X\cdot\{(\nabla_Z K)(Y) - (\nabla_Y K)(Z)\}\cdot\varphi + \frac{2}{n}(X \lrcorner W(\eta))\cdot D\varphi \tag{1.40}$$

<u>Proof:</u> The equations (1.38) and (1.30) follow directly from the formula for the curvature tensor $\mathcal{R}^E$ and Theorem 4. We differentiate the equation

$$W(\eta)\cdot\varphi = 0$$

with respect to X and obtain - using (1.38) and (1.39) -

$$\begin{aligned}(\nabla_X W)(\eta)\cdot\varphi &= \nabla_X(W(\eta))\cdot\varphi - W(\nabla_X\eta)\cdot\varphi = \\ &= \nabla_X(W(\eta))\cdot\varphi - 0 \\ &= \nabla_X(W(\eta)\cdot\varphi) - W(\eta)\nabla_X\varphi \\ &= -W(\eta)\nabla_X\varphi \\ &= \frac{1}{n} W(\eta)\cdot X\cdot D\varphi .\end{aligned}$$

From formula (1.4) we obtain the following commuting rule for a

2-form w and a vector X

$$w \cdot X = (X \cdot w) + 2(X \lrcorner w),$$

Hence

$$(\nabla_X W)(\eta) \cdot \varphi = \frac{1}{n} X \cdot W(\eta) \cdot D\varphi + \frac{2}{n}(X \lrcorner W(\eta)) \cdot D\varphi$$
$$= -2X \cdot \{(\nabla_Z K)(Y) - (\nabla_Y K)(Z)\} \cdot \varphi + \frac{2}{n}(X \lrcorner W(\eta)) \cdot D\varphi .$$

We derive now a well-known relation between the covariant derivatives of the Weyl-tensor W and the tensor K. Consider the Bianchi identity

$$\nabla_i R_{jk\alpha\beta} + \nabla_j R_{ki\alpha\beta} + \nabla_k R_{ij\alpha\beta} = 0.$$

Contracting this equation with respect to $i = \beta$ we obtain

$$\nabla_\beta R_{jk\alpha\beta} - \nabla_j R_{k\alpha} + \nabla_k R_{j\alpha} = 0.$$

Contracting again with respect to $j = \alpha$ it follows

$$\nabla_k(R) = 2\nabla_\beta R_{k\beta}.$$

Now we calculate

$$\nabla_\alpha K_{\alpha\beta} = \frac{1}{n-2}\left(\frac{\nabla_\alpha(R)}{2(n-1)}\delta_{\alpha\beta} - \nabla_\alpha R_{\alpha\beta}\right) =$$
$$= \frac{1}{n-2}\left(\frac{\nabla_\alpha(R)}{2(n-1)}\delta_{\alpha\beta} - \frac{1}{2}\nabla_\beta(R)\right) =$$
$$= -\frac{\nabla_\beta(R)}{2(n-1)} .$$

Using the latter formula as well as the definition of the Weyl-tensor W a direct computation yields

$$\nabla_\alpha W_{\beta\gamma\delta\alpha} = (3-n)(\nabla_\beta K_{\gamma\delta} - \nabla_\gamma K_{\beta\delta}) .$$

A Riemannian manifold (M^n, g) is called <u>conformally symmetric</u> if

$$\nabla W = 0$$

(compare [107]). The above formula particularly proves that

$$(\nabla_X K)(Y) - (\nabla_Y K)(X) = 0$$

holds in any conformally symmetric space.

<u>Theorem 6:</u>

1) Let (M^n, g) be a conformally symmetric Riemannian manifold with a non-trivial twistor spinor φ and suppose that $D\varphi$ vanishes on a discrete set only. Then M is a conformally flat space, i.e. $W \equiv 0$.
2) A connected three-dimensional Riemannian manifold with a non-trivial twistor is conformally flat.

Proof: Suppose $\nabla W = 0$. By Theorem 5, formula (1.40), and the previous calculation we have

$$(X \lrcorner W(\eta^2))\cdot D\varphi = 0$$

for any vector X and any 2-form η^2. Since $D\varphi$ vanishes only on a discrete set, this yields that the 1-form $X \lrcorner W(\eta^2)$ equals zero for any vector X, i.e. $W(\eta^2) = 0$ for any 2-form η^2.
In case M is a 3-dimensional manifold, we have $W \equiv 0$ and, consequently, the integrability condition (1.39) of Theorem 5 implies

$$(\nabla_X K)(Y) - (\nabla_Y K)(X) = 0$$

for all vector fields X,Y on the set $\{m \in M^3 \mid \varphi(m) \neq 0\}$. This set is dense in M^3 (see Chapter 2 of this book) and therefore this equation holds in all points of M^3. On the other hand, the last equation in case of a 3-dimensional Riemannian manifold is equivalent to the conformal flatness of the space ([94]).

Example 1: Let us consider the Euclidean space $M^n = \mathbb{R}^n$ and a twistor spinor $\varphi : \mathbb{R}^n \longrightarrow \Delta_n$ on it. According to Theorem 3, formula (1.34), we have $\nabla(D\varphi) = 0$, i.e. $D\varphi =: \varphi_1$ is constant. Now we integrate the twistor equation

$$0 = \nabla_X \varphi + \frac{1}{n} X\cdot D\varphi = \nabla_X \varphi + \frac{1}{n} X\cdot \varphi_1$$

along the line $\{sx \mid 0 \le s \le 1\}$ and obtain

$$\varphi(x) - \varphi(0) = -\frac{1}{n} x\cdot \varphi_1 .$$

Consequently, the set of all twistor spinors on $\mathbb{R}^n$ is given by

$$\varphi(x) = \varphi_o - \frac{1}{n} x\cdot \varphi_1$$

with $\varphi_o, \varphi_1 \in \Delta_n$. In particular, we have $\dim \operatorname{Ker} \mathcal{D} = 2^{[\frac{n}{2}]+1}$.
Moreover, any twistor spinor on $\mathbb{R}^n$ vanishes at most at one point, since $x\cdot\varphi_1 = 0$ implies $x = 0$ or $\varphi_1 = 0$.

The twistor equation is conformally invariant in the following sense: Let $\tilde{g} = \sigma g$ be a conformally equivalent metric to g and let $\tilde{\mathcal{D}}$ be the twistor operator of $(M,\tilde{g})$. Then we have

Theorem 7: For each spinor field $\varphi \in \Gamma(S)$,

$$\tilde{\mathcal{D}}\tilde{\varphi} = \sigma^{-1/4}\, \widetilde{\mathcal{D}(\sigma^{-1/4}\varphi)} .$$

In particular, $\varphi \in \Gamma(S)$ is a twistor spinor on (M,g) iff $\sigma^{1/4}\tilde{\varphi} \in \Gamma(\tilde{S})$ is a twistor spinor on $(M,\tilde{g})$.

Proof: Using (1.14), (1.15) and (1.29) we obtain by a straightforward calculation

$$\tilde{\nabla}^{\tilde{S}}_{\tilde{X}}\tilde{\varphi} + \frac{1}{n}\tilde{X}\cdot\tilde{D}\tilde{\varphi} = \sigma^{-1/4}\left\{\nabla^{S}_{X}(\sigma^{-1/4}\varphi) + \frac{1}{n}X\cdot D(\sigma^{-1/4}\varphi)\right\}.$$

With (1.30) the statement follows.

1.5. Killing Spinors on Riemannian Spin Manifolds

A special example of twistor spinors are the so-called Killing spinors.

Definition 3: A spinor field $\varphi \in \Gamma(S)$ is called a Killing spinor to the Killing number $B \in \mathbb{C}$ if the differential equation

$$\nabla_X\varphi = BX\cdot\varphi \tag{1.41}$$

is satisfied for all vector fields X on M.

By $\mathcal{K}(M^n,g)_B$ we denote the space of all Killing spinors of (M^n,g) to the Killing number B. Obviously, each Killing spinor is a twistor spinor satisfying formally the eigenvalue equation $D\varphi=-nB\varphi$ for the Dirac operator. Since Killing spinors are parallel with respect to the covariant derivative $\nabla_X - BX\cdot$ on the spinor bundle, a non-trivial Killing spinor φ on a connected manifold has no zeros. In particular, its length function is positive.
If φ is a Killing spinor to a real Killing number B, the real vector field

$$X_\varphi := \sum_{j=1}^{n} i\langle\varphi, s_j\cdot\varphi\rangle s_j$$

is a Killing vector field (if it is not identically zero):
From

$$\begin{aligned}\nabla_Y(X_\varphi) &= i\sum_{j=1}^{n}\{\langle\nabla_Y\varphi, s_j\cdot\varphi\rangle + \langle\varphi, s_j\cdot\nabla_Y\varphi\rangle\}\, s_j\\ &= iB\left\{\sum_{j=1}^{n}\langle\varphi,(s_j\cdot Y - Y\cdot s_j)\cdot\varphi\rangle\, s_j\right\}\end{aligned}$$

it follows for the Lie derivative that

$$\begin{aligned}\mathcal{L}_{X_\varphi} g(Y,Z) &= g(\nabla_Y X_\varphi, Z) + g(Y, \nabla_Z X_\varphi)\\ &= iB\{\langle\varphi,(Z\cdot Y - Y\cdot Z)\cdot\varphi\rangle + \langle\varphi,(Y\cdot Z - Z\cdot Y)\cdot\varphi\rangle\}\\ &= 0.\end{aligned}$$

This is the origin of the name Killing spinor.
There is a fundamental geometric condition for (M^n,g) admitting Killing spinors:

Theorem 8 [32]: Let (M^n,g) be a connected spin manifold with a non-trivial Killing spinor $\varphi \in \mathcal{K}(M,g)_B$. Then (M^n,g) is an Einstein space with the scalar curvature $R = 4n(n-1)B^2$.

Proof: Let $\varphi \in \mathcal{K}(M,g)_B$ be a non-trivial Killing spinor. Then the Killing equation (1.41) yields for the curvature tensor in S

$$\begin{aligned}
\mathcal{R}^S(X,Y)\varphi &= \nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi \\
&= \nabla_X(BY\cdot\varphi) - \nabla_Y(BX\cdot\varphi) - B[X,Y]\cdot\varphi \\
&= B(\nabla^M_X Y - \nabla^M_Y X - [X,Y])\cdot\varphi + B(Y\cdot\nabla_X\varphi - X\cdot\nabla_Y\varphi) \\
&= B^2(Y\cdot X - X\cdot Y)\cdot\varphi \\
&= 2B^2(Y\cdot X + g(X,Y))\cdot\varphi .
\end{aligned}$$

Using (1.13) we obtain for the Ricci tensor

$$\begin{aligned}
Ric(X)\cdot\varphi &= -2\sum_{k=1}^{n} s_k\cdot\mathcal{R}^S(X,s_k)\varphi \\
&= -4B^2\sum_{k=1}^{n} s_k\cdot(s_k\cdot X + g(X,s_k))\varphi \\
&= 4B^2(n-1)X\cdot\varphi .
\end{aligned}$$

Since φ has no zeros,

$$Ric(X) = 4B^2(n-1)X.$$

Therefore, (M^n,g) is an Einstein space of constant scalar curvature $R = 4B^2n(n-1)$.

Let $\varphi \in \mathcal{K}(M,g)_B$ be a non-trivial Killing spinor. In particular, Theorem 8 shows that the Killing number B is either real or imaginary. In case $B \in \mathbb{R}\setminus\{0\}$, we call φ a real Killing spinor; in case $B \in i\mathbb{R}\setminus\{0\}$, we call φ an imaginary Killing spinor; in case $B = 0$, φ is of course a ∇^S-parallel spinor field.

There is an important difference between the real and the imaginary case.

Theorem 9: Let (M^n,g) be a complete connected spin manifold with a non-trivial Killing spinor φ. If φ is real, (M^n,g) is a compact Einstein space of positive scalar curvature. If φ is imaginary, (M^n,g) is a non-compact Einstein space of negative scalar curvature.

Proof: Let φ be real. Then, on account of Theorem 8, (M^n,g) is a complete Einstein space of positive scalar curvature and the Theorem of Myers ([74], Chap. VIII.5) provides the compactness of M.

Let $\varphi \in \mathcal{K}(M,g)_{\mu i}$ be imaginary and assume that M is compact. Then φ satisfies the eigenvalue equation $D^2\varphi = -n^2\mu^2\varphi$ and

$$0 \le \int_M \langle D\varphi, D\varphi\rangle \, dM = \int_M \langle D^2\varphi, \varphi\rangle \; dM = -n^2\mu^2 \int_M \langle \varphi, \varphi\rangle dM$$

implies $\varphi \equiv 0$.

Theorem 9 and Corollary 1 show that the compact manifolds admitting non-trivial Killing spinors are just those compact manifolds of positive scalar curvature R that have the smallest possible first eigenvalue λ_+ or λ_- of the Dirac operator D: $\varphi \in \Gamma(S)$ is an eigenspinor of D to the eigenvalue $\pm \frac{1}{2}\sqrt{\frac{R_0 n}{n-1}}$ iff φ is a non-trivial Killing spinor to the Killing number $\mp \frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}$.

Moreover, in the compact case <u>all</u> twistor spinors can be obtained by Killing spinors using a conformal deformation of the metric.

<u>Theorem 10</u> ([83]): Let (M^n,g) be a compact Riemannian manifold. Then there exists a conformal equivalent Riemannian metric $\tilde{g} = \sigma \cdot g$ of constant scalar curvature R such that

$$\sigma^{1/4}\,\widetilde{\operatorname{Ker}\mathcal{D}} = \mathcal{K}(M,\tilde{g})_{\frac{1}{2}\sqrt{\frac{\tilde{R}}{n(n-1)}}} \oplus \mathcal{K}(M,\tilde{g})_{-\frac{1}{2}\sqrt{\frac{\tilde{R}}{n(n-1)}}} .$$

<u>Proof:</u> From the solution of the Yamabe-problem one knows that there exists a conformal change $\tilde{g} = \sigma \cdot g$ of the metric g such that the Riemannian manifold $(M,\tilde{g})$ has constant scalar curvature $\tilde{R}$. By Theorem 7 we have for the twistor spinors on (M,g) and $(M,\tilde{g})$

$$\operatorname{Ker}\tilde{\mathcal{D}} = \sigma^{1/4}\,\widetilde{\operatorname{Ker}\mathcal{D}}.$$

According to Theorem 3, formulae (1.33), each twistor spinor $\tilde{\varphi}$ of $(M,\tilde{g})$ is an eigenspinor of $\tilde{D}^2$ to the eigenvalue $\frac{1}{4}\tilde{R}\frac{n}{n-1}$. Since

$$\operatorname{Ker}(\tilde{D}^2 - \lambda^2) = \operatorname{Ker}(\tilde{D} + \lambda) \oplus \operatorname{Ker}(\tilde{D} - \lambda)$$

holds on a compact manifold for all $\lambda \in \mathbb{R}$, the statement follows from the above mentioned connection between Killing spinors and eigenvalues of the Dirac operator on compact manifolds.

<u>Theorem 11</u> ([83],[57]): Let (M^n,g) be a connected spin manifold admitting a non-trivial Killing spinor φ to the Killing number $B \neq 0$. Then, there are no non-trivial parallel k-forms, $k \neq 0,n$, on M^n. In particular, such a manifold is non-Kählerian. If w is a harmonic k-form and M is compact, then $w \cdot \varphi \equiv 0$.

Proof: Let $\varphi \neq 0$ be a Killing spinor to the Killing number $\beta \neq 0$ and let w be a parallel k-form, $k \neq 0,n$.
In particular, w is harmonic and the application of formula (1.27) provides

$$\begin{aligned} D(w\cdot\varphi) &= (-1)^k w\cdot D\varphi + (d+\delta)w\cdot\varphi - 2\sum_{j=1}^{n}(s_j \lrcorner w)\cdot\nabla_{s_j}\varphi \\ &= (-1)^{k+1} n\beta w\cdot\varphi - 2\beta\sum_{j=1}^{n}(s_j \lrcorner w)s_j\cdot\varphi \\ &= (-1)^{k+1}\beta(n-2k)w\cdot\varphi \, . \end{aligned} \tag{i}$$

This shows that $D^2(w\cdot\varphi) = \beta^2(n-2k)^2\, w\cdot\varphi$ for each harmonic k-form w. If M is compact, the smallest eigenvalue of D^2 is $n^2\beta^2$ (Corollary 1, Theorem 8). Hence, in this case $w\cdot\varphi \equiv 0$ follows for each harmonic form w.

Using $D^2 = \frac{1}{4}R + \Delta^S$ and $R = 4n(n-1)\beta^2$, (i) yields

$$\Delta^S(w\cdot\varphi) = \beta^2\{(n-2k)^2 - n(n-1)\}\, w\cdot\varphi . \tag{ii}$$

On the other hand, by (1.11) we have for each parallel form

$$\nabla^S(w\cdot\varphi) = \nabla w\cdot\varphi + w\cdot\nabla^S\varphi = w\cdot\nabla^S\varphi$$

and, hence,

$$\begin{aligned} \Delta^S(w\cdot\varphi) &= w\cdot\Delta^S\varphi = w\cdot(D^2\varphi - \tfrac{1}{4}R\varphi) \\ &= \beta^2(n^2 - n(n-1))w\cdot\varphi \end{aligned} \tag{iii}$$

Since $k \neq 0,n$, (ii) and (iii) provide $w\cdot\varphi \equiv 0$.
Differentiating $w\cdot\varphi = 0$ we obtain

$$\nabla_X(w\cdot\varphi) = w\cdot\nabla_X\varphi = \beta w\cdot X\cdot\varphi = 0.$$

Using (1.4) it follows

$$\begin{aligned} 0 = w\cdot X\cdot\varphi &= (-1)^k\{X\cdot w\cdot\varphi + 2(X\lrcorner w)\cdot\varphi\} \\ &= 2(-1)^k\{(X\lrcorner w)\cdot\varphi\} \quad . \end{aligned}$$

Hence, $(X\lrcorner w)\cdot\varphi = 0$ for all vector fields X on M. Using $\nabla_Y(X\lrcorner w) = \nabla_Y X\lrcorner w + X\lrcorner\nabla_Y w$ and differentiating again, we obtain

$$\begin{aligned} 0 = \nabla_{X_2}((X_1\lrcorner w)\cdot\varphi) &= \nabla_{X_2}(X_1\lrcorner w)\cdot\varphi + (X_1\lrcorner w)\cdot\nabla_{X_2}\varphi \\ &= (\nabla_{X_2}X_1\lrcorner w)\cdot\varphi + \beta(X_1\lrcorner w)\cdot X_2\cdot\varphi \\ &= 2(-1)^{k-1}\beta\,\{X_2\lrcorner(X_1\lrcorner w)\cdot\varphi\} \quad . \end{aligned}$$

Consequently, $(X_2\lrcorner(X_1\lrcorner w))\cdot\varphi = 0$ for all vector fields X_1, X_2 on M. By further differentiation in the same way one obtains

$$w(X_1,\ldots,X_k)\cdot\varphi = 0$$

for all vector fields $X_1,\ldots,X_k$ on M.

This implies $w \equiv 0$. A Kähler form is parallel, hence (M,g) cannot be Kählerian.

Remark: In [68],[69] and [70], K.-D. Kirchberg studied the eigenvalues of the Dirac operator on compact Kähler manifolds (M^n,g) of positive scalar curvature R. He proved the following lower bound for the eigenvalues:

$$|\lambda| \geq \frac{1}{2}\sqrt{\frac{n+2}{n} R_o}, \tag{*}$$

where R_o is the minimum of R. If $\lambda^2 = \frac{1}{4}\frac{n+2}{n} R_o$ is an eigenvalue of D^2, the complex dimension m $(n=2m)$ of M is odd and (M,g) is Kähler-Einstein. In case of even complex dimension m, the eigenvalues are bounded by

$$|\lambda| \geq \frac{1}{2}\sqrt{\frac{n}{n-2} R_o}. \tag{**}$$

The estimations (*) and (**) are sharp in the sense that there are Kähler manifolds with equality in (*) and (**). In case $m = 1$ and $m = 3$, the only Kähler manifolds with equality in (*) are the sphere $S^2 = \mathbb{C}P^1$, the complex projective space $\mathbb{C}P^3$ and the flag manifold $F(\mathbb{C}^3)$. For the 4-dimensional Grassmannian manifold $G_{2,4} = S^2 \times S^2$ the equality in (**) is valid.

Finally we prove further geometric conditions for a manifold (M,g) admitting Killing spinors.

Theorem 12: Let $\varphi \in \mathcal{K}(M^n,g)_\beta$ be a Killing spinor on (M^n,g). Then, for the curvature tensor $\mathcal{R}$ and the Weyl tensor W the following conditions are satisfied:

$$W(\eta)\cdot\varphi = 0 \tag{1.42}$$

$$(\nabla_X W)(\eta)\cdot\varphi = -2\beta\{X \lrcorner W(\eta)\}\cdot\varphi \tag{1.43}$$

$$\{\mathcal{R}(\eta) + 4\beta^2\eta\}\cdot\varphi = 0 \tag{1.44}$$

$$(\nabla_X \mathcal{R})(\eta)\cdot\varphi = -2\beta\{X \lrcorner (\mathcal{R}(\eta)+4\beta^2\eta)\}\cdot\varphi \tag{1.45}$$

for all 2-forms η and all vectors X.

Proof: Each Killing spinor is a twistor spinor. Hence equation (1.42) follows from formula (1.38) of Theorem 6. Since the Killing spinor φ satisfies $D\varphi = -n\beta\cdot\varphi$, the formulas (1.39) and (1.30) from Theorem 5 show that

$$((\nabla_Z K)(Y) - (\nabla_Y K)(Z))\cdot\varphi = 0$$

for all vectors Y and Z. Thus, (1.43) follows from (1.40). According to Theorem 8 (M^n,g) is an Einstein space of scalar curvature $R = 4n(n-1)B^2$. Therefore, the components of the Weyl tensor with respect to an ON-basis are

$$W_{ijkl} = R_{ijkl} + 4B^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Hence, the Weyl-tensor and the curvature tensor considered to be bundle maps on $\Lambda^2 M$ are connected by

$$W(\eta) = \mathcal{R}(\eta) + 4B^2\eta .$$

Furthermore, we obtain

$$\begin{aligned}(\nabla_X W)(\eta) &= \nabla_X(W(\eta)) - W(\nabla_X\eta)\\ &= \nabla_X(\mathcal{R}(\eta)) + 4B^2\nabla_X\eta - \mathcal{R}(\nabla_X\eta) - 4B^2\nabla_X\eta\\ &= (\nabla_X\mathcal{R})(\eta).\end{aligned}$$

Thus, the equations (1.42) and (1.43) provide (1.44) and (1.45).

<u>Theorem 13:</u> Let (M^n,g) be a connected spin manifold with a non-trivial Killing spinor to the Killing number $B \neq 0$. Then (M,g) is locally irreducible. If (M,g) is locally symmetric or the dimension of M is not greater than 4, then (M,g) is a space of constant sectional curvature $4B^2$.

<u>Proof:</u> Let $\varphi \in \mathcal{K}(M^n,g)_B$ be a non-trivial Killing spinor. Let us assume that (M,g) is locally reducible in $x \in M$. Then there exists an open neighbourhood U of x, which is isometric to a Riemannian product $U_1 \times U_2$. For the 1-forms σ_1 and σ_2 on U_1 and U_2, respectively, we have $\mathcal{R}(\sigma_1 \wedge \sigma_2) = 0$ and $(\nabla_X\mathcal{R})(\sigma_1\wedge\sigma_2) = 0$ for all vectors X tangent to U_1 or U_2. Formula (1.45) implies $-8B^3(X \lrcorner (\sigma_1 \wedge \sigma_2))\cdot\varphi \equiv 0$ on U. Hence, $Z\cdot\varphi \equiv 0$ for all vector fields Z on U, which provides $\varphi \equiv 0$ on U - a contradiction.

Now we assume (M,g) to be locally symmetric, i.e. $\nabla\mathcal{R} \equiv 0$. Then (1.45) implies

$$\{X \lrcorner (\mathcal{R}(\eta) + 4B^2\eta)\}\cdot\varphi = 0$$

for all vectors X and all 2-forms η. Since φ has no zeros, we even obtain

$$\mathcal{R}(\eta) + 4B^2\eta = 0$$

for all 2-forms η. Hence, (M^n,g) is a space of constant sectional curvature $4B^2$.

Now, suppose for the dimension n of M: $n \leq 4$. An Einstein space of dimension $n \leq 3$ has constant sectional curvature. It remains to prove the statement in case n=4. In dimension 4 the Weyl tensor splits into a positive and a negative part

$$W = \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix}$$

with respect to the decomposition of $\Lambda^2 M$ in the subspace of self-dual and anti-selfdual 2-forms

$$\Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M.$$

For a 4-dimensional Einstein space there exists an ON-basis $(s_1, \ldots, s_4)$ in each tangent space $T_x M$ such that the Weyl tensor $W(x): \Lambda^2_x M \to \Lambda^2_x M$ is, with respect to the basis

$(s_1 \wedge s_2, s_1 \wedge s_3, s_1 \wedge s_4, s_3 \wedge s_4, s_4 \wedge s_2, s_2 \wedge s_3)$, of the form

$$W(x) = \frac{R(x)}{12} + \begin{pmatrix} A & B \\ B & A \end{pmatrix} ,$$

where

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} , \quad B = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} , \quad \mu_j, \lambda_j \in \mathbb{R},$$

$$\mu_1 + \mu_2 + \mu_3 = 0 \quad \text{and} \quad \lambda_1 + \lambda_2 + \lambda_3 = -\frac{R(x)}{12}$$

(cf. [97]). Furthermore,

$$W_+(x) = 0 \quad \text{iff} \quad \lambda_k + \mu_k = -\frac{1}{12} R(x), \qquad k=1,2,3$$

$$W_-(x) = 0 \quad \text{iff} \quad \lambda_k - \mu_k = -\frac{1}{12} R(x), \qquad k=1,2,3$$

(cf. [33], § 1). An even form respects the positive and negative part of the spinor bundle. Hence, from (1.42) it follows for the Killing spinor $\varphi = \varphi^+ \oplus \varphi^-$ that

$$s_i \cdot s_j \cdot W_x(s_i \wedge s_j) \cdot \varphi^{\pm}(x) = 0 \qquad (i,j) = (1,2),(1,3),(1,4).$$

Applying formula (1.1) for the Clifford-multiplication, in case $\varphi^+(x) \neq 0$ these equations give the algebraic condition

$$\det\left(\left(-\frac{R(x)}{12} - (\lambda_k + \mu_k)\right)E\right) = 0 \qquad k=1,2,3$$

and, in case $\varphi^-(x) \neq 0$, the condition

$$\det\left(\left(-\frac{R(x)}{12} - (\lambda_k - \mu_k)\right)E\right) = 0 \qquad k=1,2,3.$$

This implies $W_{\mp}(x) = 0$ if $\varphi^{\pm}(x) \neq 0$. Now, from the Killing equation one concludes

$$\nabla_X \varphi^+ = \beta \cdot X \cdot \varphi^- \qquad \text{and} \quad \nabla_X \varphi^- = \beta\, X \cdot \varphi^+ .$$

Since φ is non-trivial and $B \neq 0$, this shows that φ^+ as well as φ^- are non-trivial spinor fields on M.
Both spinors, φ^+ and φ^-, are twistor spinors on (M,g), hence their zero-set in M is discrete (see Chapter 2.1. of this book). Hence both, the positive part W_+ and the negative part W_- of the Weyl-tensor are identically zero. Because of

$$0 = W(\eta) = \mathcal{R}(\eta) + 4B^2\eta ,$$

(M^4,g) is a space of constant sectional curvature $4B^2$.

<u>Corollary 4:</u> Let (M^4,g) be a 4-dimensional compact, connected spin manifold with a non-trivial real Killing spinor to the Killing number $B \neq 0$. Then (M^4,g) is isometric to the standard sphere of radius $\frac{1}{2|B|}$.

<u>Proof:</u> According to Theorem 13, (M^4,g) is a space of constant sectional curvature $4B^2$. Hence, (M^4,g) is isometric to the projective space $\mathbb{R}P^4$ or to the sphere S^4 (with suitable normation of the metric) (see [106], Th. 2.5.1). However, $\mathbb{R}P^4$ is non-orientable and therefore no spin manifold.

The proof Theorem 13 also entails that $W_+ \equiv 0$ is valid on a 4-dimensional manifold with a non-trivial parallel spinor $\psi^\pm \in \Gamma(S^\pm)$.

<u>Example 2:</u> Let us consider the standard sphere (S^n,g_0). We identify $S^n \setminus \{\text{north pole}\}$ via stereographic projection with the Euclidean space $\mathbb{R}^n$. Then we have

$$g_0 = \frac{4}{(1+\|x\|^2)^2}\, g_{\mathbb{R}^n}$$

for the metric.
According to Theorem 7 and Example 1 the twistor spinors on $S^n \setminus \{\text{north pole}\}$ are all spinors $\tilde{\varphi}_{u,v}$ given by

$$\varphi_{u,v}(x) = \frac{u + x\cdot v}{\sqrt{1+\|x\|^2}} ,$$

where $u,v \in \Delta_n$ are constants.
By formula (1.15) we have for the spinor derivative of the twistor spinor $\tilde{\varphi} = \tilde{\varphi}_{u,v}$

$$\nabla^{\tilde{S}}_{\tilde{e}_j}\tilde{\varphi} = \frac{1+\|x\|^2}{2}\,\widetilde{\partial_j(\varphi)} + \frac{1}{2}\,\widetilde{e_j\cdot x\cdot\varphi} + \frac{1}{2}\,x_j\tilde{\varphi}$$
$$= \frac{1}{2}\,\frac{1}{\sqrt{1+\|x\|^2}}\,\widetilde{e_j\cdot(x\cdot u+v)}.$$

Now, suppose that $\tilde{\varphi} = \tilde{\varphi}_{u,v}$ is a Killing spinor on $S^n \setminus \{\text{north pole}\}$. Then we have

$$\nabla_{\tilde{e}_j}\tilde{\varphi} = \pm \frac{1}{2}\, \tilde{e}_j \cdot \tilde{\varphi} \ ,$$

which is equivalent to the condition

$$x \cdot u + v = \pm\, (u + x \cdot v) \qquad \text{for all} \quad x \in \mathbb{R}^n .$$

Setting $x = 0$, this implies $v = \pm\, u$.

Hence, each Killing spinor of $S^n \setminus \{\text{north pole}\}$ is of the form $\tilde{\varphi}_{u,\pm u}$, where $u \in \Delta_n$ is a constant. The functions $\varphi_{u,\pm u}$ extend to infinity.

Consequently, the set of all Killing spinors on S^n to the Killing number $\pm \frac{1}{2}$ are the spinors $\tilde{\varphi}_u$, which, on $S^n \setminus \{\text{north pole}\}$, are given by

$$\varphi_u(x) = \frac{(1 \pm x) \cdot u}{\sqrt{1 + \|x\|^2}} \ , \quad u \in \Delta_n \ \text{constant.}$$

<u>Example 3:</u> Let H^n be the hyperbolic space realized as an open unit ball in $\mathbb{R}^n$ with the metric

$$g = \frac{4}{(1 - \|x\|^2)^2}\, g_{\mathbb{R}^n} .$$

As in example 2 we obtain:

The twistor spinors on H^n are all spinors $\tilde{\varphi}_{u,v}$ given by

$$\varphi_{u,v}(x) = \frac{u + x \cdot v}{\sqrt{1 - \|x\|^2}}$$

where $u, v \in \Delta_n$ are constants.

The Killing spinors on H^n are

$$\mathcal{K}(H^n)_{\pm \frac{1}{2} i} = \left\{ \tilde{\varphi}_u \,\middle|\, \varphi_u(x) = \frac{1}{\sqrt{1 - \|x\|^2}}\, (u \pm i\, x \cdot u) \ , \quad u \in \Delta^n \right\}.$$

Chapter 2: The Properties of Twistor Spinors

2.1. The Zeros of a Twistor Spinor

We turn now to investigate the zeros of a solution φ of the twistor equation $\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$. First of all we shall prove that the zeros are isolated points.

Theorem 1 ([38]): Let (M^n, g) be a connected Riemannian manifold and $\varphi \neq 0$ a twistor spinor. Then $N_\varphi = \{m \in M^n : \varphi(m) = 0\}$ is a discrete subset of M^n.

Proof: Suppose $\varphi(m) = 0$. Using formula (1.34) we have $\nabla(D\varphi)(m) = 0$.
Concerning

$$\begin{aligned}
(YX\, u_\varphi)(m) &= 2(Y(\nabla_X \varphi, \varphi))(m) = \\
&= -\tfrac{2}{n}(Y(X \cdot D\varphi, \varphi))(m) = \frac{2}{n^2}(X \cdot D\varphi, Y \cdot D\varphi)(m) \\
&= \frac{2}{n^2} g(X,Y) |D\varphi(m)|^2
\end{aligned}$$

we see that the Hessian of the function $u_\varphi = |\varphi|^2$ at the point $m \in M^n$ is given by

$$\mathrm{Hess}_m\, u_\varphi(X,Y) = \frac{2}{n^2} g(X,Y) |D\varphi(m)|^2 .$$

In case $D\varphi(m) \neq 0$, m is a non-degenerate critical point of u_φ and consequently an isolated zero point of φ. In case $D\varphi(m) = 0$, we obtain $\varphi \equiv 0$ by Corollary 3 of Chapter 1.

We consider now a geodesic $\gamma(t)$ in M^n and a twistor spinor φ. Denote by $u(t)$, $v(t)$ the functions $u_\varphi(\gamma(t))$, $|D\varphi|^2(\gamma(t))$. Moreover, we introduce the functions

$$f_1(t) = g(K(\dot\gamma(t)), \dot\gamma(t))$$
$$f_2(t) = \frac{n^2}{2} |K(\dot\gamma(t))|^2 .$$

From the twistor equation (1.31) as well as formula (1.34) we obtain

$$\left.\begin{array}{l}
u''(t) = f_1(t)u(t) + 2n^{-2}v(t) \\
\text{and in the case that } \nabla_{\dot\gamma} K(\dot\gamma) \text{ is parallel to } \dot\gamma \\
v''(t) = f_2(t)u(t) + \frac{1}{2} n^2 f_1'(t)u'(t) + f_1(t)v(t)
\end{array}\right\} \quad (2.1)$$

Theorem 2 ([38]): Let $\varphi \neq 0$ be a twistor spinor and denote by $\gamma : [0,T] \longrightarrow M^n$ a geodesic joining of two zeros of φ. Then

a) $\mathrm{Ric}(\dot\gamma)$ is parallel to $\dot\gamma$

b) $\mathrm{grad}\, u_\varphi$ is parallel to $\dot\gamma$

c) $\frac{dv}{dt} = \frac{n^2}{2} g(K(\dot\gamma),\dot\gamma)\frac{du}{dt}$

d) $u\cdot v = \frac{n^2}{4}(\frac{du}{dt})^2$

<u>Proof:</u> Using the notation introduced above we have

$$u(0) = \frac{du}{dt}(0) = \frac{dv}{dt}(0) = 0, \qquad v(0) > 0$$
$$u(T) = \frac{du}{dt}(T) = \frac{dv}{dt}(T) = 0, \qquad v(T) > 0.$$

Since $u(t)$ and $v(t)$ satisfy the equations (2.1), we obtain

$$\frac{d}{dt}\left(\frac{dv}{dt} - \frac{n^2}{2} f_1 \frac{du}{dt}\right) = \left(f_2 - \frac{n^2}{2} f_1^2\right)u.$$

If $f_2 - \frac{n^2}{2} f_1^2 \not\equiv 0$ on the interval $[0,T]$, we have

$$0 = \frac{dv}{dt}(T) - \frac{n^2}{2} f_1(T)\frac{du}{dt}(T) =$$
$$= \int_0^T \left(f_2 - \frac{n^2}{2} f_1^2\right)u > 0$$

because $f_2 - \frac{n^2}{2} f_1^2 = \frac{n^2}{2}\left(|K(\dot\gamma)|^2 - g(K(\dot\gamma),\dot\gamma)^2\right) \geqq 0$,
a contradiction. In case $f_2 - \frac{n^2}{2} f_1^2 \equiv 0$, $\mathrm{Ric}(\dot\gamma)$ is parallel to $\dot\gamma$ and $\frac{dv}{dt} = \frac{n^2}{2} g(K(\dot\gamma),\dot\gamma)\frac{du}{dt}$.
Moreover, we calculate

$$\frac{d}{dt}\left(u\cdot v - \frac{n^2}{4}\left(\frac{du}{dt}\right)^2\right) = \frac{du}{dt} v + u\frac{dv}{dt} - \frac{n^2}{2}\frac{du}{dt}\frac{d^2u}{dt^2} =$$
$$= \frac{du}{dt} v + \frac{n^2}{2} f_1 u \frac{du}{dt} - \frac{n^2}{2}\frac{du}{dt}\left(f_1 u + \frac{2}{u^2} v\right) = 0,$$

i.e. $uv = \frac{n^2}{4}(\frac{du}{dt})^2$. Since φ is a twistor spinor vanishing at some point, we have $u_\varphi \cdot D\varphi = \frac{n}{2}\,\mathrm{grad}(u_\varphi)\cdot\varphi$ (see formula (2.3)). This implies $u\cdot v = \frac{n^2}{4}|\mathrm{grad}\; u_\varphi|^2$ and, consequently, $|\mathrm{grad}\, u_\varphi|^2 = (\frac{du}{dt})^2$, i.e. the gradient of u_φ is parallel to $\dot\gamma$.

<u>Corollary 1:</u> Let (M^n,g) be a complete connected Riemannian manifold and suppose that the (1,1)-tensor $K := \frac{1}{n-2}\left(\frac{R}{2(n-1)} - \mathrm{Ric}\right)$ is non-negative. Then any twistor spinor $\varphi \not\equiv 0$ vanishes at most at one point.

<u>Proof:</u> Suppose $u_\varphi(m_1) = 0 = u_\varphi(m_2)$, $m_1 \neq m_2$, and consider a geodesic $\gamma : [0,T] \to M^n$ from m_1 to m_2. Then

$$\frac{d^2u}{dt^2} = f_1 \cdot u + \frac{2}{u^2} v \geq 0$$

since K is non-negative. With respect to
$u(0) = u(T) = 0$ and $\frac{du}{dt}(0) = \frac{du}{dt}(T) = 0$

we conclude $u(t) = 0$ on $[0,T]$, i.e. φ vanishes on the curve $\gamma(t)$, a contradiction to Theorem 1.

Remark: The condition $K \geq 0$ is satisfied in particular if (M^n,g) is an Einstein space with scalar curvature $R \leq 0$. On the Euclidean space R^n and on the hyperbolic space H^n there exist twistor spinors vanishing at some point. Solving the twistor equation on certain warped products $M^n {}_{f^2}\!\times \mathbb{R}^1$ we shall construct examples of Riemannian manifolds admitting twistor spinors with an arbitrary number of zeros.

2.2. The Solutions of the Twistor Equation on Warped Products $M^{2m} {}_{f^2}\!\times \mathbb{R}^1$.

Let (M^{2m},g) be an even-dimensional Einstein space with scalar curvature $R \neq 0$. The decomposition of the spinor bundle $S = S^+ \oplus S^-$ yields a decomposition of the kernel of the twistor operator

$$\ker(\mathcal{D}) = \ker^+(\mathcal{D}) \oplus \ker^-(\mathcal{D})$$
$$\ker^\pm(\mathcal{D}) = \{\varphi \in \Gamma(S^\pm) : \mathcal{D}\varphi = 0\}.$$

Since M^{2m} is an Einstein space, it follows from the formulas (1.33) and (1.34) that the Dirac operator D maps $\ker^\pm(\mathcal{D})$ into $\ker^\mp(\mathcal{D})$. In particular, if $\varphi_1^+,\dots,\varphi_k^+$ is a basis of $\ker^+(\mathcal{D})$, then $D\varphi_1^+,\dots,D\varphi_k^+$ is a basis of $\ker^-(\mathcal{D})$.
We fix a function $f: \mathbb{R}^1 \longrightarrow (0,\infty)$ and consider the warped product $M^{2m} {}_{f^2}\!\times \mathbb{R}^1$ with the Riemannian metric $f^2(t)\, g \oplus dt^2$. Spinor fields on $M^{2m} {}_{f^2}\!\times \mathbb{R}^1$ are t-parametric families $\varphi(x,t)$ of spinor fields on M^2. We solve the twistor equation on $M^{2m} {}_{f^2}\!\times \mathbb{R}^1$ and obtain:

Theorem 3 ([87]): Let (M^{2m},g) be an Einstein space with scalar curvature $R \neq 0$ and denote by $\varphi_1^+,\dots,\varphi_k^+$ a basis of $\ker^+(\mathcal{D})$. The twistor spinors $\varphi(x,t)$ on the warped product $(M^{2m} \times \mathbb{R}^1,\ f^2(t)\, g \oplus dt^2)$ are given by

$$\varphi(x,t) = \sqrt{f}\sum_{j=1}^{k}\{a_j h_1(t)+b_j h_2(t)\}\varphi_j^+(x) +$$
$$+(\sqrt{f})^3 i\ (-1)^m \frac{4(2m-1)}{R}\sum_{j=1}^{k}\{a_j \dot h_1(t)+b_j \dot h_2(t)\} D\varphi_j^+(x)$$

where a_j, b_j are complex numbers and the functions $h_1(t), h_2(t)$ are equal

$$h_1(t) = \sin\left(\frac{1}{2}\sqrt{\frac{-R}{2m(2m-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right)$$

$$h_2(t) = \cos\left(\frac{1}{2}\sqrt{\frac{-R}{2m(2m-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right)$$

in case $R < 0$, or

$$h_1(t) = \sinh\left(\frac{1}{2}\sqrt{\frac{-R}{2m(2m-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right)$$

$$h_2(t) = \cosh\left(\frac{1}{2}\sqrt{\frac{R}{2m(2m-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right)$$

in case $R > 0$.

Proof: We recall that $\varphi(x,t)$ is a twistor spinor with respect to the metric $f^2(t)\, g \oplus dt^2$ if and only if $\hat\varphi := \frac{1}{\sqrt{f}}\varphi$ is a twistor spinor with respect to the metric $\hat g = g \oplus (\frac{dt}{f})^2$ (see Theorem 7 of Chapter 1). The vector field $s_{2m+1} := f\frac{\partial}{\partial t}$ is a unit vector field on $(M^{2m}\times \mathbb{R}^1, \hat g)$. Using the identification of the spin bundle of $M^{2m}\times\mathbb{R}^1$ with the spin bundle $S = S^+ \oplus S^-$ of M^{2m} the Clifford multiplication by $s_{2m+1} = f\frac{\partial}{\partial t}$ is given by

$$f\frac{\partial}{\partial t} = i(-1)^m \quad \text{on} \quad S^+$$
$$f\frac{\partial}{\partial t} = -i(-1)^m \quad \text{on} \quad S^-$$

(see formula (1.16)). We apply now the last condition of Theorem 2, Chapter 1, characterizing twistor spinors. It follows easily that if $\hat\varphi$ is a twistor spinor on $(M^{2m}\times\mathbb{R}^1,\ g\oplus(\frac{dt}{f})^2)$, then any restriction $\hat\varphi\big|_{M^{2m}\times\{t\}}$ is a twistor spinor on (M^{2m},g), too. Hence, $\hat\varphi$ has the following form

$$\hat\varphi(x,t) = \sum_{j=1}^{k} c_j^+(t)\varphi_j^+(x) + \sum_{j=1}^{k} c_j^-(t) D\varphi_j^+(x).$$

Consequently, we have only one condition for $\hat\varphi$ resulting from the twistor equation, namely $s_{2m+1}\nabla_{s_{2m+1}}\hat\varphi$ has to coincide with $s_i\nabla_{s_i}\hat\varphi$ for $i\le 2m$. Since M^{2m} is an Einstein space, we obtain from (1.32) and (1.34)

$$s_i \nabla_{s_i} \varphi_j^+ = \frac{1}{2m} D\varphi_j^+$$

$$s_i \nabla_{s_i} (D\varphi_j^+) = \frac{1}{2m} D^2\varphi_j^+ = \frac{R}{4(2m-1)} \varphi_j^+ \ .$$

Thus the condition $s_{2m+1} \nabla_{s_{2m+1}} \hat{\varphi} = s_i \nabla_{s_i} \hat{\varphi}$ yields the differential equations

$$\dot{C}_j^+ = -\frac{i(-1)^m R}{4(2m-1)} \frac{1}{f} C_j^-$$

$$\dot{C}_j^- = \frac{i(-1)^m}{2m} \frac{1}{f} C_j^+ \quad .$$

In particular, we obtain

$$\ddot{C}_j^+ = \frac{R}{8m(2m-1)} \frac{1}{f^2} C_j^+ - \frac{\dot{f}}{f} \dot{C}_j^+$$

and the fundamental solutions of this differential equation are the functions $h_1(t)$, $h_2(t)$ given above.

This provides

$$\hat{\varphi}(x,t) = \sum_{j=1}^{k} a_j h_1(t) + b_j h_2(t)\} \varphi_j^+(x) +$$
$$+ f \frac{4(2m-1)i(-1)^m}{R} \sum_{j=1}^{k} \{a_j \dot{h}_1(t) + b_j \dot{h}_2(t)\} D\varphi_j^+(x)$$

as well as the general solution $\varphi = \sqrt{f}\,\hat{\varphi}$ of the twistor equation on the warped product $(M^{2m} \times \mathbb{R}^1, f^2(t)g \oplus dt^2)$.

<u>Remark:</u> Consider an Einstein space (M^{2m},g) with negative scalar curvature and a basis $\varphi_1^+,\dots,\varphi_k^+$ in $\ker^+(\mathcal{D})$. Suppose that there exists a point $m_o \in M^{2m}$ such that the spinors

$$\varphi_1^+(m_o),\dots,\varphi_k^+(m_o)$$

as well as $D\varphi_1^+(m_o),\dots,D\varphi_k^+(m_o)$

are linearly dependent. Hence, there exist non-trivial linear combinations

$$\sum_{j=1}^{k} b_j \varphi_j^+(m_o) = 0 \qquad \sum_{j=1}^{k} a_j D\varphi_j^+(m_o) = 0.$$

We fix an integer $l \in N$ and a function $f: \mathbb{R} \longrightarrow (0,\infty)$ such that

$$2l \sqrt{\frac{2m(2m-1)}{-R}}\,\pi \le \int_0^{\infty} \frac{d\tau}{f(\tau)} < 2(l+1) \sqrt{\frac{2m(2m-1)}{-R}}\,\pi \quad .$$

Then $$\varphi(x,t) = \sqrt{f} \sum_{j=1}^{k} \{a_j h_1(t) + b_j h_2(t)\} \varphi_j^+(x) +$$
$$+ (\sqrt{f}\,)^3 i(-1)^m \frac{4(2m-1)}{R} \sum_{j=1}^{k} \{a_j \dot{h}_1(t) + b_j \dot{h}_2(t)\} D\varphi_j^+(x)$$

is a twistor spinor on the warped product $(M^{2m} \times \mathbb{R}^1, f^2(t)g \oplus dt^2)$ vanishing at 1 points. For example, the hyperbolic space H^{2m} admits $2^{m+1} = 2\cdot\dim(S)$ independent twistor spinors (see example 1.3). Consequently, the warped products $H^{2m} \times_{f^2} \mathbb{R}^1$ are spaces with twistor spinors.

2.3. The First Integrals C_φ and Q_φ on $\ker(\mathcal{D})$.

The kernel $\ker(\mathcal{D})$ of the twistor operator $\mathcal{D}$ on a connected Riemannian manifold (M^n,g) is a vector space and its dimension is bounded by $2^{[\frac{n}{2}]+1}$. On this vector space, there exist a quadratic form C and a form Q of order four defined by

$$C_\varphi = (D\varphi,\varphi) = \mathrm{Re}\langle D\varphi,\varphi\rangle$$
$$Q_\varphi = |\varphi|^2|D\varphi|^2 - (D\varphi,\varphi)^2 - \sum_{\alpha=1}^{n} (D\varphi, s_\alpha\cdot\varphi)^2,$$

where $s_1,\dots,s_n$ is an orthonormal frame on M^n. We prove that for any solution φ of the twistor equation the functions C_φ and Q_φ are constant on the manifold M^n and, consequently, we obtain well-defined first integrals $C,Q\colon \ker(\mathcal{D}) \rightarrow \mathbb{R}^1$ on $\ker(\mathcal{D})$.

<u>Theorem 4</u> ([38],[83]): Let φ be a twistor spinor on a connected Riemannian manifold M^n. Then C_φ and Q_φ are constant.

<u>Proof:</u> We differentiate C_φ with respect to the vector field X

$$\nabla_X(C_\varphi) = (\nabla_X(D\varphi),\varphi) + (D\varphi,\nabla_X\varphi).$$

Using the twistor equation $\nabla_X\varphi = -\frac{1}{n} X\cdot D\varphi$ as well as formula (1.34) we obtain $\nabla_X(C_\varphi) = 0$ since $(X\cdot\psi,\psi) = \mathrm{Re}\langle X\cdot\psi,\psi\rangle = 0$ for any spinor $\psi \in S$ and vector X. In the same way we prove $\nabla_X(Q_\varphi) = 0$. We have

$$\begin{aligned}\nabla_X(Q_\varphi) &= 2(\nabla_X\varphi,\varphi)|D\varphi|^2 + 2|\varphi|^2(\nabla_X(D\varphi),D\varphi)\\ &\quad - 2\sum_{\alpha=1}^{n} (D\varphi, s_\alpha\cdot\varphi)(\nabla_X(D\varphi), s_\alpha\varphi)\\ &\quad + \frac{2}{n}\sum_{\alpha=1}^{n} (D\varphi, s_\alpha\cdot\varphi)(D\varphi, s_\alpha\cdot X\cdot D\varphi).\end{aligned}$$

Using formula (1.7) we obtain

$$\sum_{\alpha=1}^{n} (s_\alpha\cdot\varphi, D\varphi)(s_\alpha\cdot X\cdot D\varphi, D\varphi) = -(X\cdot\varphi, D\varphi)|D\varphi|^2$$

$$\sum_{\alpha=1}^{n}(s_\alpha\cdot\varphi\ ,D\varphi)(s_\alpha\varphi\ ,Z\cdot\varphi)=(Z\cdot\varphi\ ,D\varphi)|\varphi|^2$$

and the twistor equation and formula (1.34) yield

$$\nabla_X(Q_\varphi) = 0.$$

Remark: Denote by $V\varphi$ the real subspace of S given by

$$V_\varphi = \{X\cdot\varphi : X \in TM\}.$$

Then we have

$$Q_\varphi = |\varphi|^2\ \mathrm{dist}^2(D\varphi\ ,\mathrm{Lin}_R(\varphi\ ,V\varphi)).$$

The vector field $T\varphi$ defined by

$$T\varphi = \sum_{\alpha=1}^{n}(\varphi\ ,s_\alpha\cdot D\varphi)s_\alpha$$

satisfies equation (1.36)

$$T\varphi = -\frac{n}{2}\,\mathrm{grad}(u_\varphi)$$

and an elementary calculation provides the formula

$$|C_\varphi\cdot\varphi - u_\varphi\cdot D\varphi - T_\varphi\cdot\varphi|^2 = u_\varphi Q_\varphi. \tag{2.2}$$

In particular, if φ is a twistor spinor such that $C_\varphi = 0 = Q_\varphi$, then

$$u_\varphi\cdot D\varphi = \frac{n}{2}\,\mathrm{grad}(u_\varphi)\cdot\varphi \tag{2.3}$$

holds. This occurs for example in case the twistor spinor φ has zeros.

Theorem 5 ([87]): Let (M^n,g) be an Einstein space with scalar curvature $R \neq 0$. Then any twistor spinor $\varphi\in\ker(\mathcal{D})$ is the sum of two real (in case $R>0$) or imaginary (in case $R<0$) Killing spinors.

Proof: Assume $\mathrm{Ric} = \frac{R}{n}g$. Then we have

$$K = \frac{1}{n-2}\left(\frac{R}{2(n-1)} - \mathrm{Ric}\right) = -\frac{R}{2(n-1)n}\,g$$

and, consequently, the twistor equation and formula (1.34) provide

$$\nabla_X\varphi = -\frac{1}{n}X\cdot D\varphi$$

$$\nabla_X(D\varphi) = -\frac{R}{4(n-1)}X\cdot\varphi\ .$$

We consider the spinor fields $\psi_\pm = \mp\frac{1}{2}\sqrt{\frac{Rn}{n-1}}\,\varphi + D\varphi$. Since $R\neq 0$, we have

$$\varphi = \sqrt{\frac{n-1}{Rn}}\,(\psi_- - \psi_+)\ .$$

On the other hand, $\psi_\pm$ are Killing spinors with Killing numbers $\lambda_\pm = \pm\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$:

$$\begin{aligned}
\nabla_X(\psi_\pm) &= \mp\frac{1}{2}\sqrt{\frac{Rn}{n-1}}\ \nabla_X\varphi + \nabla_X(D\varphi) = \\
&= \pm\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\ X\cdot D\varphi - \frac{R}{4(n-1)}\,X\cdot\varphi = \\
&= \pm\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\ X\cdot\Big(\mp\frac{1}{2}\sqrt{\frac{Rn}{n-1}}\,\varphi + D\varphi\Big) \\
&= \pm\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\ X\cdot\psi_\pm .
\end{aligned}$$

Theorem 6 ([38],[87]): Let (M^n,g) be a Riemannian spin manifold with a twistor spinor φ such that $|\varphi| \equiv 1$. Then (M^n,g) is an Einstein space with scalar curvature

$$R = \frac{4(n-1)}{n}\,(C_\varphi^2 + Q_\varphi).$$

Proof: Suppose $|\varphi| \equiv 1$, i.e. $u_\varphi \equiv 1$. Because of $T_\varphi = -\frac{n}{2}\,\mathrm{grad}(u_\varphi)$ we obtain $T_\varphi = 0$ and, consequently,

$$(X\cdot\varphi\,, D\varphi) \equiv 0$$

for any vector field X. Now we have

$$\begin{aligned}
&\frac{n}{2}\,g(K(X),Y) = (\nabla_X(D\varphi), Y\cdot\varphi) = \\
&= \nabla_X(D\varphi\,, Y\cdot\varphi) - (D\varphi\,, \nabla_X(Y\cdot\varphi)) = \\
&= 0 - (D\varphi, (\nabla_X Y)\cdot\varphi) - (D\varphi\,, Y\cdot\nabla_X\varphi) = \\
&= 0 - 0 + \frac{1}{n}(D\varphi\,, Y\cdot X\cdot D\varphi) = -\frac{1}{n}\,g(X,Y)\,|D\varphi|^2 .
\end{aligned}$$

This implies $g(K(X),Y) = -\frac{2}{n^2}\,g(X,Y)\,|D\varphi|^2$, and with respect to $K = \frac{1}{n-2}\big(\frac{R}{2(n-1)} - \mathrm{Ric}\big)$ we conclude that (M^n,g) is an Einstein space. Moreover,

$$\begin{aligned}
&C_\varphi^2 + Q_\varphi = |\varphi|^2\,|D\varphi|^2 - \sum_{\alpha=1}^{n}(D\varphi\,, s_\alpha\varphi)^2 = \\
&= 1\cdot|D\varphi|^2 - 0 = -\frac{n^2}{2}\,\frac{g(K(X),X)}{g(X,X)} = \\
&= -\frac{n^2}{2}\,\frac{1}{n-2}\Big(\frac{R}{2(n-1)} - \frac{R}{n}\Big) = \frac{n\,R}{4(n-1)} .
\end{aligned}$$

Consider a twistor spinor $\varphi \not\equiv 0$ on a connected Riemannian manifold (M^n,g). The set N_φ of zeros of φ is a discrete set. Outside N_φ we introduce the conformally equivalent metric

$$\tilde{g} = \frac{1}{|\varphi|^4} g.$$

Then $\psi := \frac{1}{|\varphi|} \tilde{\varphi}$ is a solution of the twistor equation on the Riemannian manifold $(M^n \setminus N_\varphi, \tilde{g})$ with length one, $|\psi| \equiv 1$ (see Theorem 7, Chapter 1).
Therefore we obtain

Corollary 2 ([38],[87]): Let (M^n,g) be a Riemannian spin manifold with a non-trivial twistor spinor φ. Then $(M^n \setminus N_\varphi, \frac{1}{|\varphi|^4} g)$ is an Einstein space with non-negative scalar curvature

$$\tilde{R} = \frac{4(n-1)}{n} (C_\varphi^2 + Q_\varphi).$$

In case $C_\varphi^2 + Q_\varphi > 0$, $\frac{1}{|\varphi|}\tilde{\varphi}$ is the sum of two real Killing spinors on $(M^n \setminus N_\varphi, \frac{1}{|\varphi|^4} g)$. If $C_\varphi^2 + Q_\varphi = 0$, then $\frac{1}{|\varphi|}\tilde{\varphi}$ is a parallel spinor.

Remark: We say that a twistor spinor φ is conformally equivalent to a Killing spinor if there exists a conformal change of the metric $\tilde{g} = \sigma g$ such that $\sigma^{1/4}\tilde{\varphi}$ is a Killing spinor with respect to the metric $\tilde{g}$. We introduce the function $f = \frac{1}{2}\sigma^{-1/2}$. Then the Killing equation

$$\nabla_X(\sigma^{1/4}\tilde{\varphi}) + \frac{a}{n} X \cdot (\sigma^{1/4}\tilde{\varphi}) = 0$$

becomes equivalent to

$$a\varphi - 2fD\varphi + n \operatorname{grad}(f)\cdot\varphi = 0.$$

The integrability conditions of the latter equation have been investigated in the paper [38]. For example it turns out that a twistor spinor φ is conformally equivalent to a real Killing spinor if and only if $C_\varphi \neq 0$ and $Q_\varphi = 0$.

2.4. A Characterization of Spaces of Constant Curvature

Theorem 7 ([7],[87]): Let (M^n,g) be a connected, complete Einstein space with spinor structure and non-positive scalar curvature $R \leq 0$. Suppose that $\varphi \not\equiv 0$ is a non-parallel twistor spinor such that the length function $u_\varphi = |\varphi|^2$ attains a minimum. Then (M^n,g) is isometric to the hyperbolic space H^n (in case $R < 0$) or to the Euclidean space $\mathbb{R}^n$ $(R=0)$.
We shall divide the proof of this theorem into several steps.

Let $\varphi \neq 0$ be a twistor spinor on an Einstein space M^n. A point $m \in M^n$ is a critical point of the length function u_φ if and only if $(D\varphi, X \cdot \varphi) = 0$ for all vectors $X \in T_m M^n$. Moreover, the Hessian of u_φ at this critical point is given by

$$\mathrm{Hess}_m u_\varphi(X,Y) = \left\{ \frac{2}{n^2} |D\varphi|^2 - \frac{R}{2n(n-1)} |\varphi|^2 \right\} g(X,Y).$$

In case $R < 0$, $\mathrm{Hess}_m u_\varphi$ is positive-definite. Suppose that the scalar curvature vanishes, $R = 0$. With respect to formula (1.34) it turns out that $D\varphi$ is a parallel spinor field and thus $|D\varphi|^2$ is constant. Since φ is a non-parallel twistor spinor, we conclude $|D\varphi|^2 = \mathrm{const} > 0$ and we obtain again that $\mathrm{Hess}_m u_\varphi$ is positive-definite. Consequently, in case $R \leq 0$ any critical point of the length function u_φ is a non-degenerate minimum. Suppose now that u_φ has two different critical points $m_1, m_2 \in M^n$ and consider a geodesic $\gamma(t)$ $(0 \leq t \leq T)$ from m_1 to m_2.

For the functions $u(t) = u_\varphi(\gamma(t))$ and $v(t) = |D\varphi|^2(\gamma(t))$ we obtain the following differential equations

$$\left.\begin{aligned} u'' &= \frac{2}{n^2} v - \frac{R}{2n(n-1)} u \\ v'' &= -\frac{Rn}{4(n-1)} u'' \end{aligned}\right\} \qquad (2.4)$$

from formula (2.1) as well as the conditions $u'(0) = u'(T) = 0$ and $v'(0) = v'(T) = 0$. In case $R = 0$, it follows that

$$v(t) \equiv C_1, \quad u(t) = \frac{C_1}{n^2} t^2 + C_2 t + C_3$$

for some constants C_1, C_2, C_3. With respect to $u'(0) = u'(T) = 0$ we conclude $C_1 = 0$, i.e. $D\varphi$ vanishes on $\gamma(t)$. Since M^n is an Einstein space, $D\varphi$ is a twistor spinor. According to Theorem 1 we have $D\varphi \equiv 0$ and, consequently, φ is a parallel spinor, which contradicts the assumption. In case $R<0$, we obtain in particular the equation

$$u'' = -\frac{R}{n(n-1)} u + C_1^*$$

and the conditions $u'(0) = u'(T) = 0$ imply $u \equiv 0$, also a contradiction. In order to summarize, we proved that if M^n is a connected, complete Einstein manifold with scalar curvature $R \leq 0$, then the length function u_φ of a non-parallel twistor spinor has at most one critical point. Moreover, if this critical point actually appears, then it is a non-degenerate minimum. Suppose now that this is the case and denote the unique critical point by $m_0 \in M^n$. Let $d(m, m_0)$ be the distance from an arbitrary point

$m \in M^n$ to m_o and denote by $\gamma(t)$ $(0 \leq t \leq T = d(m, m_o))$ a shorted geodesic from m_o to m.
We integrate the equations (2.4) along $\gamma(t)$ and obtain the following relations between the length function u_φ and the distance function $d(m, m_o)$ $(R < 0)$:

$$u_\varphi(m) = \left\{u_\varphi(m_o) - \frac{4(n-1)}{n \cdot R} v_\varphi(m_o)\right\} \sinh^2\left(\frac{1}{2}\sqrt{\frac{-R}{n(n-1)}} \cdot\right.$$
$$\left.\cdot\, d(m, m_o)\right) + u_\varphi(m_o)$$
$$|D\varphi(m)|^2 = v_\varphi(m) = \left\{v_\varphi(m_o) - \frac{n \cdot R}{4(n-1)} u_\varphi(m_o)\right\} \cosh^2\left(\frac{1}{2}\sqrt{\frac{-R}{n(n-1)}} \cdot\right.$$
$$\left.\cdot\, d(m, m_o)\right) + \frac{n \cdot R}{4(n-1)} u_\varphi(m_o).$$

If the scalar curvature R vanishes, we get

$$u_\varphi(m) = \frac{v_\varphi(m_o)}{n^2} \, d(m, m_o)^2 + u_\varphi(m_o)$$
$$|D\varphi(m)|^2 = v_\varphi(m) \equiv v_\varphi(m_o) > 0.$$

Since we already know that u_φ has only one critical point, we conclude that the distance function $d(m, m_o)$ is smooth on $M^n \setminus \{m_o\}$ and has no critical points in this set. In particular, the exponential map

$$\exp_{m_o} : T_{m_o} M^n \longrightarrow M^n$$

is a diffeomorphism and the geodesic spheres $S^{n-1}(m_o, r)$ around m_o coincide with the level surfaces of the function u_φ. They are smooth submanifolds of M^n. We denote by ξ the normal vector field to the geodesic spheres,

$$\xi = \frac{\operatorname{grad}(u_\varphi)}{\|\operatorname{grad}(u_\varphi)\|} .$$

We differentiate equation (1.36)

$$\operatorname{grad}(u_\varphi) = -\frac{2}{n} T_\varphi = -\frac{2}{n} \sum_{j=1}^{n} (\varphi, s_j \cdot D\varphi) s_j$$

with respect to the vector field X and apply the twistor equation as well as formula (1.34). Then we obtain

$$\nabla_X(\operatorname{grad}(u_\varphi)) = \left\{\frac{2}{n^2} v_\varphi - \frac{R}{2n(n-1)} u_\varphi\right\} X$$

for any vector $X \in TM^n$. The last formula yields

$$\nabla_X \xi = \frac{\left\{\frac{2}{n^2} v_\varphi - \frac{R}{2n(n-1)} u_\varphi\right\}}{\|\operatorname{grad}(u_\varphi)\|} \cdot \{X - g(X, \xi)\xi\}.$$

In particular, we have

$$\nabla_\xi \xi = 0. \tag{2.5}$$

Suppose for a moment that the scalar curvature is negative, $R < 0$
A simple calculation provides the formula

$$\frac{2}{n^2} v_\varphi(m) - \frac{R}{2n(n-1)} u_\varphi(m) = \left\{\frac{2}{n^2} v_\varphi(m_o) - \frac{R}{2n(n-1)} u_\varphi(m_o)\right\} \cdot \cosh\left(\sqrt{\frac{-R}{n(n-1)}}\, d(m,m_o)\right).$$

Moreover, starting with $|\mathrm{grad}\, u_\varphi|^2 = \frac{4}{n^2}|T\varphi|^2 = \frac{4}{n^2}\sum_{j=1}^{n}(D\varphi, s_j\cdot\varphi)^2 = \frac{4}{n^2}(u_\varphi\cdot v_\varphi - C_\varphi^2 - Q_\varphi)$ and recall that $C_\varphi^2 + Q_\varphi$ is constant, we obtain $C_\varphi^2 + Q_\varphi = u_\varphi(m_o)\cdot v_\varphi(m_o)$ and therefore

$$|\mathrm{grad}\, u_\varphi|^2 = \frac{4}{n^2}(u_\varphi v_\varphi - u_\varphi(m_o)v_\varphi(m_o)).$$

Now we calculate $\nabla_X \xi$ and come to the result

$$\nabla_X \xi(m) = \sqrt{\frac{-R}{n(n-1)}}\coth\left(\sqrt{\frac{-R}{n(n-1)}}\, d(m,m_o)\right)X \tag{2.6}$$

for all vectors $X \in T_m M^n$ orthogonal to $\xi(m)$.
A similar discussion in case $R = 0$ proves the formula

$$\nabla_X \xi(m) = \frac{1}{d(m,m_o)} X \tag{2.7}$$

for all vectors $X \in T_m M^n$ orthogonal to $\xi(m)$.

Let $\gamma_t(m)$ denote the integral curves of the vector field ξ normalized by the condition $\gamma_o(m) = m$. Consider the diffeomorphism

$$\psi : S^{n-1}(m_o,1) \times (0,\infty) \longrightarrow M^n \setminus \{m_o\}$$

given by $\psi(m,t) = \gamma_{t-1}(m)$. The formulas (2.5), (2.6) and (2.7) allow us to calculate the pull back $\psi^*(g)$ of the metric g:

$$\psi^*(g) = \frac{\sinh^2\left(\sqrt{\frac{-R}{n(n-1)}}\, t\right)}{\sinh^2\left(\sqrt{\frac{-R}{n(n-1)}}\right)} g_o \oplus dt^2 \quad \text{if} \quad R < 0$$

$$\psi^*(g) = t^2 g_o \oplus dt^2 \quad \text{if} \quad R = 0,$$

where $g_o = g\big|_{S^{n-1}(m_o,1)}$ is the restriction of the metric g to the geodesic sphere $S^{n-1}(m_o,1)$. We introduce the polar coordinates on $R^n = T_{m_o}M^n$

$$\Phi : S^{n-1} \times (0,\infty) \longrightarrow T_{m_o}M^n = \mathbb{R}^n$$

$\Phi(v,t) = t\cdot v$. Because of

$$\exp_{m_o} \Phi(v,t) = \psi(\exp_{m_o}(v),t)$$

we see that the metric $\hat{g} := \exp_{m_o}^* g$ on $\mathbb{R}^n = T_{m_o}M^n$ in polar coordinates has the form

$$\Phi^*(\hat{g}) = \sinh^2\left(\sqrt{\tfrac{-R}{n(n-1)}}\; t\right) h \oplus dt^2 \qquad \text{if} \quad R < 0 \tag{2.8}$$

$$\Phi^*(\hat{g}) = t^2\, h \oplus dt^2 \qquad \text{if} \quad R = 0, \tag{2.9}$$

where h is a metric on S^{n-1} defined by (in case $R<0$)

$$h = \frac{1}{\sinh^2\left(\sqrt{\tfrac{-R}{n(n-1)}}\right)} \quad \exp_{m_o}^*\left(g\big|_{S^{n-1}(m_o,1)}\right).$$

If we transform $\Phi^*(g)$ back into Euclidean coordinates $\varphi : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1} \times (0,\infty)$ $\varphi(x) = \left(\frac{x}{\|x\|}, \|x\|\right)$, we obtain for the coordinates of $\hat{g}$ with respect to the canonical basis of $\mathbb{R}^n$:

$$\hat{g}_{ij}(x) = \frac{\sinh^2\left(\sqrt{\tfrac{-R}{n(n-1)}}\,\|x\|\right)}{\|x\|^2} \; h_{\frac{x}{\|x\|}}\left(e_i - \frac{x_i \cdot x}{\|x\|^2},\; e_j - \frac{x_j \cdot x}{\|x\|^2}\right) + \\ + \frac{x_i x_j}{\|x\|^2}.$$

For $w \in S^{n-1}$ we denote by $a_i(w)$ the tangent vector $a_i(w) := e_i - \langle w, e_i\rangle\, w \in T_w S^{n-1}$. $\hat{g}$ is continuous on $\mathbb{R}^n$. Therefore, for all $w \in S^{n-1}$ we obtain

$$\begin{aligned}\hat{g}_{ij}(0) &= \lim_{t\to 0} \hat{g}_{ij}(t\,w) = \\ &= \left(\frac{-R}{n(n-1)}\right) h_w(a_i(w), a_j(w)) - \\ &\quad - \langle a_i(w), a_j(w)\rangle_{R^n} + \delta_{ij}\end{aligned}$$

Using $w = e_i$, $a_i(e_i) = 0$ implies that

$$\hat{g}_{ij}(0) = \delta_{ij}.$$

The vectors $a_1(w), \ldots, a_n(w)$ generate $T_w S^{n-1}$. Hence, the metric h is a multiple of the standard metric $g\big|_{S^{n-1}}$ of the sphere S^{n-1}.

Finally, $\Phi^*(\hat{g})$ is the metric of the hyperbolic space $(R<0)$ or the Euclidean space $(R=0)$ in polar coordinates, i.e. (M^n, g) is isometric to the hyperbolic space or to the Euclidean space. This proves Theorem 7.

2.5. The Equation $\nabla_X\varphi + \frac{f}{n} X\cdot\varphi = 0$

If $f: M^n \longrightarrow \mathbb{C}$ is a complex-valued function on a connected Riemannian spin manifold (M^n,g), we consider sections φ of the spinor bundle satisfying, for any vector $X \in TM^n$, the differential equation

$$\nabla_X\varphi + \frac{f}{n} X\cdot\varphi = 0. \tag{2.10}$$

This equation implies $D\varphi = \sum_{j=1}^{n} s_j\cdot\nabla_{s_j}\varphi = f\cdot\varphi$ and, hence $\nabla_X\varphi + \frac{1}{n} X\, D\varphi = 0$, i.e. any solution of equation (2.10) is a twistor spinor. Since equation (2.10) - restricted to a curve in the manifold - is an ordinary differential equation of first order, any non-trivial solution of equation (2.10) has no zeros. The first integrals of such a special twistor spinor are given by

$$C_\varphi = \mathrm{Re}(f)u_\varphi \quad\text{and}\quad Q_\varphi = (\mathrm{Im}(f))^2\{u_\varphi^2 - \sum_{\alpha=1}^{n}(i\varphi, s_\alpha\varphi)^2\}.$$

Theorem 8 ([83]): On a connected Riemannian spin manifold (M^n,g) of dimension $n \geq 3$ let $\varphi \not\equiv 0$ be a spinor such that $\nabla_X\varphi + \frac{f}{n} X\cdot\varphi = 0$, where f is a complex-valued function with the real part $\mathrm{Re}(f) \not\equiv 0$. Then f is constant and φ is a real Killing spinor.

Proof: φ is a twistor spinor and we can apply formula (1.34)

$$\nabla_X(D\varphi) = \frac{n}{2} K(X)\cdot\varphi$$

$$\nabla_X(f)\varphi - \frac{f^2}{n} X\cdot\varphi = \frac{n}{2} K(X)\cdot\varphi \tag{2.11}$$

Denote the real and the imaginary part of f by a and b, $f = a+ib$. Using the real part $(\ ,\ ) = \mathrm{Re}\langle,\rangle$ of the inner product for spinors we multiply by φ:

$$\nabla_X(a)u_\varphi - \frac{2ab}{n}(iX\cdot\varphi, \varphi) = 0 \tag{2.12}$$

Multiplying equation (2.11) by $X = s_\alpha$ we have

$$\sum_{\alpha=1}^{n} \nabla_{s_\alpha}(f)s_\alpha\varphi + \left(f^2 - \frac{n\cdot R}{4(n-1)}\right)\varphi = 0 \tag{2.13}$$

$$\sum_{\alpha=1}^{n} \nabla_{s_\alpha}(f)s_\beta s_\alpha\varphi + \left(f^2 - \frac{n\cdot R}{4(n-1)}\right)s_\beta\varphi = 0$$

$$\nabla_{s_\beta}(f)\varphi + \left(\frac{n\cdot R}{4(n-1)} - f^2\right)s_\beta\varphi + \frac{1}{2}\sum_{\alpha=1}^{n}\nabla_{s_\alpha}(f)$$

$$\cdot(s_\alpha s_\beta - s_\beta s_\alpha)\cdot\varphi = 0. \tag{2.14}$$

We multiply again by the spinor φ

$$\nabla_{s_\beta}(a)u_\varphi - 2ab(i\, s_\beta\cdot\varphi,\varphi) + \tag{2.15}$$
$$+\frac{1}{2}\sum_{\alpha=1}^{n}\nabla_{s_\alpha}(b)(i(s_\alpha s_\beta - s_\beta s_\alpha)\cdot\varphi,\varphi) = 0.$$

From the equations (2.12) and (2.14) we obtain

$$(1-n)\nabla_{s_\beta}(a)u_\varphi + \frac{1}{2}\sum_{\alpha=1}^{n}\nabla_{s_\alpha}(b)(i(s_\alpha s_\beta - s_\beta s_\alpha)\varphi,\varphi) = 0. \tag{2.16}$$

The latter equation we multiply by $\nabla_{s_\beta}(b)$ and take the sum over β:

$$(1-n)g(\mathrm{grad}(a),\mathrm{grad}(b))u_\varphi = 0.$$

Consequently we obtain $g(\mathrm{grad}(a),\mathrm{grad}(b)) \equiv 0.$ (2.17)

Since $C_\varphi = a\, u_\varphi$ is constant, and non-zero by the assumption $\mathrm{Re}(f) \not\equiv 0$ we conclude

$$g(\mathrm{grad}(u_\varphi),\mathrm{grad}(b)) \equiv 0 \Rightarrow 2\sum_{\alpha=1}^{n}\nabla_{s_\alpha}(b)(\nabla_{s_\alpha}\varphi,\varphi) \equiv 0 \Longrightarrow$$
$$-\frac{2}{n}\sum_{\alpha=1}^{n}\nabla_{s_\alpha}(b)((a+ib)s_\alpha\cdot\varphi,\varphi) \equiv 0 \Longrightarrow b(i\,\mathrm{grad}(b)\cdot\varphi,\varphi) = 0 \tag{2.18}$$

The inner product $\langle \mathrm{grad}(b)\cdot\varphi,\varphi\rangle$ is an imaginary number and therefore (2.18) is equivalent to

$$b\langle \mathrm{grad}(b)\cdot\varphi,\varphi\rangle \equiv 0. \tag{2.19}$$

Equation (2.13) can be written in the form

$$\mathrm{grad}(f)\varphi + \left(f^2 - \frac{n\cdot R}{4(n-1)}\right)\varphi \equiv 0.$$

We multiply by $b\cdot\varphi$ and obtain with respect to (2.19)

$$b\langle \mathrm{grad}(a)\cdot\varphi,\varphi\rangle + b\left(f^2 - \frac{n\cdot R}{4(n-1)}\right)u_\varphi \equiv 0.$$

From the real part of the latter equation we conclude

$$b\left(a^2 - b^2 - \frac{n\cdot R}{4(n-1)}\right) \equiv 0 \tag{2.20}$$

Denote by $U \subset M^n$ the set of all points $m \in M^n$ such that $a^2(m) - b^2(m) - \frac{n}{4(n-1)}R(m) \not\equiv 0$. U is an open subset of M^n and (2.20) yields that b vanishes on U. From equation (2.12) we see that a is constant on U, i.e. $\varphi|_U$ is a Killing spinor with Killing number $\beta = -\frac{f}{n}$. Theorem 8 of Chapter 1 provides

$R = 4n(n-1)\beta^2 = \frac{4(n-1)}{n} f^2 = \frac{4(n-1)}{n}(a^2-b^2)$ on the set U, which is a contradiction to the definition of this set. Thus, we have

$$a^2 - b^2 - \frac{n\cdot R}{4(n-1)} \equiv 0 \tag{2.21}$$

on the whole manifold M^n. We multiply equation (2.16) by $\nabla_{s_\beta}(a)$ and take the sum over β. Furthermore, we apply (2.17), i.e.

$\mathrm{grad}(a)\cdot\mathrm{grad}(b) = -\mathrm{grad}(b)\cdot\mathrm{grad}(a)$. Then we obtain

$$(1-n)|\mathrm{grad}(a)|^2 u_\varphi + (i\cdot\mathrm{grad}(b)\cdot\mathrm{grad}(a)\cdot\varphi , \varphi) \equiv 0. \tag{2.22}$$

We consider again the equation

$$\mathrm{grad}(f)\cdot\varphi + \left(f^2 - \frac{n\cdot R}{4(n-1)}\right)\varphi \equiv 0$$

and multiply it by $i\cdot\mathrm{grad}(b)$:

$$i\,\mathrm{grad}(b)\cdot\mathrm{grad}(a)\cdot\varphi + |\mathrm{grad}(b)|^2\varphi + \left(f^2 - \frac{n\cdot R}{4(n-1)}\right)i\cdot\mathrm{grad}(b)\cdot\varphi = 0.$$

We take the real part of the inner product of the last equation by φ:

$$(i\cdot\mathrm{grad}(b)\,\mathrm{grad}(a)\cdot\varphi , \varphi) + |\mathrm{grad}(b)|^2 u_\varphi +$$

$$+ \left(a^2-b^2 - \frac{n\cdot R}{4(n-1)}\right)(i\cdot\mathrm{grad}(b)\cdot\varphi , \varphi) = 0.$$

Equation (2.21) yields now

$$(i\cdot\mathrm{grad}(b)\,\mathrm{grad}(a)\cdot\varphi , \varphi) = -|\mathrm{grad}(b)|^2 u_\varphi$$

and from (2.22) we conclude

$$(n-1)|\mathrm{grad}(a)|^2 + |\mathrm{grad}(b)|^2 \equiv 0.$$

Now we have $\mathrm{grad}(a) \equiv \mathrm{grad}(b) \equiv 0$, i.e. f is constant. This means that φ is a Killing spinor and the assumption $\mathrm{Re}(f) = a \neq 0$ implies now $b = 0$. This proves Theorem 8.

Next we consider the case that the spinor field φ is a solution of the equation $\nabla_X\varphi + \frac{ib}{n} X\cdot\varphi = 0$ with some real-valued function $b: M^n \longrightarrow \mathbb{R}^1$. For any twistor spinor φ we introduce the real subspace

$$V_\varphi = \{X\cdot\varphi : X \in TM^n\} \subset S$$

as well as the function

$$H_\varphi = \mathrm{dist}^2(i\varphi , V_\varphi)$$

defined on the set $\{m \in M^n : \varphi(m) \neq 0\}$.

<u>Theorem 9</u> ([38]): If $\nabla_X\varphi + \frac{ib}{n} X\cdot\varphi = 0$ with a real-valued function $b: M^n \longrightarrow \mathbb{R}^1$, then

a) $u_\varphi H_\varphi$ is constant.

b) $Q_\varphi = b^2 u_\varphi H_\varphi$.

<u>Proof:</u> Suppose $\nabla_X\varphi + \frac{ib}{n} X\cdot\varphi = 0$. Then $D\varphi = ib\varphi$ and we obtain $Q_\varphi = b^2 u_\varphi H_\varphi$ by definition of Q_φ. Since

$$u_\varphi H_\varphi = u_\varphi^2 - \sum_{\alpha=1}^{n} (i\varphi, s_\alpha \varphi)^2$$

we calculate

$$\nabla_X(u_\varphi H_\varphi) = 4\, u_\varphi(\nabla_X \varphi, \varphi) -$$
$$- 2 \sum_{\alpha=1}^{n} (i\varphi, s_\alpha \varphi)(i \nabla_X \varphi, s_\alpha \varphi) -$$
$$- 2 \sum_{\alpha=1}^{n} (i\varphi, s_\alpha \cdot \varphi)(i\varphi, s_\alpha \cdot \nabla_X \varphi) =$$
$$= -\frac{4b}{n} u_\varphi (i\cdot X\cdot \varphi, \varphi) - \frac{2b}{n} \sum_{\alpha=1}^{n} (i\varphi, s_\alpha \varphi)(X\cdot\varphi, s_\alpha \varphi) +$$
$$+ \frac{2b}{n} \sum_{\alpha=1}^{n} (i\varphi, s_\alpha \cdot \varphi)(i\varphi, s_\alpha \cdot X\cdot i\cdot \varphi) =$$
$$= \frac{4b}{n} u_\varphi (i\varphi, X\cdot\varphi) - \frac{2b}{n}(i\varphi, X\cdot\varphi) u_\varphi - \frac{2b}{n} (i\varphi, X\cdot\varphi) u_\varphi = 0,$$

i.e. $u_\varphi H_\varphi$ is constant.

Corollary 3: If φ is a solution of the equation $\nabla_X \varphi + \frac{ib}{n} X\cdot\varphi = 0$ and $Q_\varphi \neq 0$, then b is constant and φ is an imaginary Killing spinor.

Corollary 4: If φ is a solution of the equation $\nabla_X \varphi + \frac{ib}{n} X\cdot\varphi = 0$ and $Q_\varphi = 0$, then $\frac{1}{\sqrt{u_\varphi}} \tilde{\varphi}$ is a parallel spinor with respect to the metric $\tilde{g} = \frac{1}{u_\varphi^2} g$.

Since $C_\varphi = (D\varphi, \varphi) = (ib\varphi, \varphi) = 0$, Corollary 4 is a special case of Corollary 2. In Chapter 7 we shall classify Riemannian manifolds with a non-trivial solution of the equation $\nabla_X \varphi + \frac{ib}{n} X\cdot\varphi = 0$ and $Q_\varphi = 0$. Moreover, we shall prove that a complete Riemannian manifold admitting an imaginary Killing spinor φ such that $Q_\varphi > 0$ is isometric to the hyperbolic space H^n.

2.6. The Equation E

A. Lichnerowicz (see [84]) introduced the so-called equation (E) for a spinor field:

$$\nabla_X(D\varphi) + \frac{R}{4(n-1)} X\cdot\varphi = 0. \tag{2.23}$$

We denote by $\ker(E)$ the space of all spinor fields $\varphi \in \Gamma(S)$ solving this equation. The existence of a non-trivial solution of the equation (E) implies that the scalar curvature of the manifold is constant.

Theorem 10 ([84]): Suppose that a connected Riemannian spin manifold (M^n, g) of dimension $n \geq 3$ admits a non-trivial solution of the equation (E). Then the scalar curvature R is constant.

Proof: Let $\varphi \not\equiv 0$ be a solution of the equation

$$\nabla_X(D\varphi) + \frac{R}{4(n-1)} X\cdot\varphi = 0.$$

Then

$$D^2\varphi = \sum_{\alpha=1}^{n} s_\alpha \nabla_{s_\alpha}(D\varphi) = \frac{n\cdot R}{4(n-1)}\varphi$$

$$D^3\varphi = \frac{n\cdot R}{4(n-1)} D\varphi + \frac{n}{4(n-1)} \operatorname{grad}(R)\cdot\varphi$$

and, consequently

$$\left\{D^2 - \frac{n\cdot R}{4(n-1)}\right\} D\varphi = \frac{n}{4(n-1)} \operatorname{grad}(R)\cdot\varphi \tag{2.24}$$

On the other hand, differentiating the equation (E) we obtain

$$\nabla_{s_\alpha} \nabla_{s_\alpha}(D\varphi) + \frac{R}{4(n-1)} s_\alpha \nabla_{s_\alpha}(\varphi) +$$

$$+ \frac{1}{4(n-1)} \nabla_{s_\alpha}(R) s_\alpha\cdot\varphi = 0$$

and, furthermore,

$$-\Delta(D\varphi) + \frac{R}{4(n-1)} D\varphi + \frac{1}{4(n-1)} \operatorname{grad}(R)\cdot\varphi = 0.$$

We apply now the formula $D^2 = \Delta + \frac{1}{4}R$ and obtain

$$\left\{D^2 - \frac{n\cdot R}{4(n-1)}\right\} D\varphi = \frac{1}{4(n-1)} \operatorname{grad}(R)\cdot\varphi \quad . \tag{2.25}$$

The equations (2.24) and (2.25) yield

$$\operatorname{grad}(R)\cdot\varphi = 0 \tag{2.26}$$

If $\varphi(m) \neq 0$, then $\operatorname{grad} R(m) = 0$ by (2.26). Suppose now $\varphi(m) = 0$. Since φ is a solution of the elliptic differential equation

$$D^2\varphi = \frac{n\cdot R}{4(n-1)}\varphi,$$

there exists a sequence of points m_i converging to m such that $\varphi(m_i) \neq 0$ (see [20]). Then we have $\operatorname{grad} R(m_i) = 0$ and with respect to the continuity of $\operatorname{grad}(R)$ we obtain again $\operatorname{grad} R(m) = 0$. Consequently, the gradient of the scalar curvature vanishes identically.

Any Killing spinor is a solution of the equation (E). Indeed, if $\nabla_X\varphi = \beta X\cdot\varphi$, then $D\varphi = -n\beta\varphi$ and $R = 4n(n-1)\beta^2$. This implies

$$\nabla_X(D\varphi) + \frac{R}{4(n-1)} X\cdot\varphi = -n\beta\nabla_X\varphi + n\beta^2 X\cdot\varphi =$$

$$= n\beta\left\{-\nabla_X\varphi + \beta X\cdot\varphi\right\} = 0.$$

In case of a compact manifold the kernel of the twistor operator, ker(E), and the Killing spinors coincide:

Corollary 5: Let (M^n, g) be a compact connected Riemannian spin manifold such that $\ker(E) \neq \{0\}$. Then

$$\ker(E) = \ker(\mathcal{D}) = \text{Killing-spinors}.$$

Proof: Since ker(E) is non-trivial, the scalar curvature R is constant and we already know (Theorem 10 of Chapter 1) that the kernel of the twistor operator is the space of all Killing spinors $\mathcal{K}$. Moreover, we have

$$\ker(\mathcal{D}) = \mathcal{K} \subset \ker(E).$$

Suppose now that $\varphi \in \ker(E)$. Then $D^2\varphi = \frac{n \cdot R}{4(n-1)}\varphi$, and φ is the sum of two Killing spinors (see Corollary 1 of Chapter 1). This proves $\ker(E) \subset \mathcal{K}$.

Theorem 11 ([46]):

$$\ker(E) = \ker\left(D^2 - \frac{n \cdot R}{4(n-1)}\right) \cap D^{-1}(\ker(\mathcal{D})).$$

Proof: If $\varphi \in \ker(E)$ we have

$$\nabla_X(D\varphi) + \frac{R}{4(n-1)} X \cdot \varphi = 0$$

and

$$D^2\varphi = \sum_{\alpha=1}^{n} s_\alpha \nabla_{s_\alpha}(D\varphi) = \frac{n \cdot R}{4(n-1)}\varphi.$$

In particular, ker(E) is contained in the kernel of the operator $D^2 - \frac{n \cdot R}{4(n-1)}$. Moreover,

$$\nabla_X(D\varphi) + \frac{1}{n} XD(D\varphi) = \nabla_X(D\varphi) + \frac{R}{4(n-1)} X \cdot \varphi = 0,$$

i.e. $D\varphi$ belongs to the kernel of the twistor operator. Conversely, suppose that $D^2\varphi = \frac{n \cdot R}{4(n-1)}\varphi$ and $D\varphi \in \ker(\mathcal{D})$.
Then we get

$$\nabla_X(D\varphi) + \frac{1}{n} XD(D\varphi) = 0$$

$$\nabla_X(D\varphi) + \frac{R}{4(n-1)} X \cdot \varphi = 0$$

and φ is a solution of the equation (E).

Theorem 12 ([46]): Let (M^n, g) be a connected Riemannian spin manifold with constant scalar curvature $R \neq 0$. The map

$$\ker(E) \ni \varphi \longrightarrow D\varphi \in \ker(\mathcal{D})$$

is an isomorphism between $\ker(E)$ and $\ker(\mathcal{D})$.

Proof: Suppose $\varphi \in \ker(E)$ and $D\varphi = 0$. Then we have $0 = D^2\varphi = \frac{n \cdot R}{4(n-1)}\varphi$ and, consequently $\varphi = 0$. This proves that the given map $\ker(E) \longrightarrow \ker(\mathcal{D})$ is injective. Suppose now $\varphi \in \ker(\mathcal{D})$ and consider $\varphi^* = \frac{4(n-1)}{n\,R} D\varphi$. Then we have

$D\varphi^* = \frac{4(n-1)}{n\,R} D^2\varphi = \varphi$ and

$$\nabla_X(D\varphi^*) + \frac{R}{4(n-1)} X \cdot \varphi^* = \nabla_X(\varphi) + \frac{1}{n} X \cdot D\varphi = 0.$$

Thus φ^* belongs to the kernel of (E) and is an inverse image of φ.

Corollary 6: If (M^n, g) is a connected Riemannian spin manifold with constant scalar curvature $R \neq 0$, then

$$\dim \ker(E) = \dim \ker(\mathcal{D}).$$

Theorem 13 ([46]):

a) If (M^n, g) is an Einstein space with scalar curvature $R \neq 0$, then

$$\ker(E) = \ker(\mathcal{D}).$$

b) Let (M^n, g) be a connected Riemannian spin manifold such that $\ker(E) \cap \ker(\mathcal{D}) \neq \{0\}$. Then M^n is an Einstein space.

Proof: Suppose first that M^n is an Einstein space, $\mathrm{Ric}(X) = \frac{R}{n} X$. If $\varphi \in \ker(E)$, we obtain from Theorem 12 and formula (1.34)

$$\nabla_X(D^2\varphi) = \frac{n}{2} K(X) \cdot D\varphi$$

$$\frac{1}{2}\frac{n \cdot R}{n-1} \nabla_X\varphi = \frac{n}{2} K(X) \cdot D\varphi = -\frac{R}{4(n-1)} X \cdot \varphi$$

and φ is a twistor spinor. Conversely, if $\varphi \in \ker(\mathcal{D})$ we use again formula (1.34)

$$\nabla_X(D\varphi) = \frac{n}{2} K(X) \cdot \varphi \; ,$$

which reduces in an Einstein space to

$$\nabla_X(D\varphi) + \frac{R}{4(n-1)} X \cdot \varphi = 0.$$

This means that, in an Einstein space, every twistor spinor is a solution of the equation (E). We consider now an arbitrary Riemannian manifold as well as a non-trivial solution $\varphi \in \ker(E) \cap \ker(\mathcal{D})$. Using the formulas (1.34) and (2.23) we obtain the condition

$$-\frac{R}{4(n-1)} X \cdot \varphi = \frac{n}{2} K(X) \cdot \varphi$$

and, finally,

$$\mathrm{Ric}(X)\cdot\varphi = \frac{R}{n} X\cdot\varphi \ .$$

Since φ is a twistor spinor, the zeros of φ are isolated points and we conclude

$$\mathrm{Ric}(X) = \frac{R}{n} X,$$

i.e. M^n is an Einstein space.

Chapter 3: A survey of Twistor Theory

3.1. Two-dimensional Conformal Geometry

Let M^2 be a 2-dimensional manifold with a fixed orientation. Two Riemannian metrics g_1, g_2 on M^2 are conformally equivalent if there exists a function $\lambda : M^2 \longrightarrow \mathbb{R}^1$ such that

$$g_1 = e^{2\lambda} g_2.$$

The set of all conformal structures $\mathrm{Conf}(M^2)$ is the set of all equivalence classes of Riemannian metrics. On the other hand, we consider the set $\mathrm{Complex}(M^2)$ of all complex structures $J: TM^2 \longrightarrow TM^2$ with the properties

a) $J^2 = -\mathrm{Id}$

b) for any vector $0 \neq X \in TM^2$ the pair $\{X, JX\}$ defines the given orientation.

Since the Nijenhuis tensor

$$[JX,JY]-[X,Y]-J[X,JY]-J[JX,Y]$$

vanishes identically in dimension two, any operator $J \in \mathrm{Complex}(M^2)$ defines a complex structure on M^2 (see [105]). If $[g] \in \mathrm{Conf}(M^2)$ is a class of conformally equivalent metrics, we consider the operator J^g being the rotation in positive direction around the angle $\frac{\pi}{2}$. The link between two-dimensional conformal geometry and one-dimensional complex analysis is now given by the following

Theorem 1: The map

$$\Phi : \mathrm{Conf}(M^2) \longrightarrow \mathrm{Complex}(M^2),\ \Phi[g] := J^g$$

is bijective.

Proof: Suppose that $\Phi[g_1] = \Phi[g_2]$. If $X \in T_m M^2$ is a non-trivial tangent vector, then $\{X, J^{g_1} X\} = \{X, J^{g_2} X\}$ is a basis in $T_m M^2$. An arbitrary vector $Y \in T_m M^2$ decomposes into

$$Y = AX + BJ^{g_1}X = AX + BJ^{g_2}X$$

and consequently we obtain

$$g_1(Y,Y) = (A^2+B^2)g_1(X,X)$$

$$g_2(Y,Y) = (A^2+B^2)g_2(X,X).$$

This means that the function $\dfrac{g_1(X,X)}{g_2(X,X)}$ does not depend on the vector $X \in T_m M^2$, but only on the point $m \in M^2$. Hence, we have a positive function $\lambda^*: M^2 \to R^1_+$ such that, for any vector $X \in T_m M^2$,

$$g_2(X,X) = \lambda^*(m) g_1(X,X)$$

holds, i.e. the Riemannian metrics g_1 and g_2 are conformally equivalent.

Finally, given a complex structure $J: TM^2 \to TM^2$ we fix an arbitrary Riemannian metric h on M^2 and consider the metric

$$g(X,Y) = h(X,Y) + h(JX,JY).$$

Then we have $g(JX,JY) = g(X,Y)$ and, in particular, $g(X,JX) = 0$. This implies $J = J^g$, i.e. the map Φ : Conf $M^2 \longrightarrow$ Complex (M^2) is also surjective.

Denote by $* : T^*M^2 \longrightarrow T^*M^2$ the Hodge operator on 1-forms with respect to the metric g. Using the identification of the tangent bundle with the cotangent bundle given by the Riemannian metric g we obtain the commutative diagramme

$$\begin{tikzcd} T^*M^2 \arrow[r, "*"] \arrow[d, dash, "\wr\, g"] & T^*M^2 \arrow[d, dash, "\wr g"] \\ TM^2 \arrow[r, "J^g"] & TM^2 \end{tikzcd} .$$

Indeed, if X_1, X_2 is an orthonormal basis of the fixed orientation in $T_m M^2$ and σ^1, σ^2 is the dual basis, we have

$$*\sigma^1 = \sigma^2 \qquad J^g X_1 = X_2$$

$$*\sigma^2 = -\sigma^1 \qquad J^g X_2 = -X_1.$$

Fix a covector $\omega \in T^*_m M^2$ as well as a vector $X \in T_m M^2$. The commutative diagramme immediately yields the relation

$$(*\omega)(X) = -\omega(J^g X)$$

between the complex structure J^g and the $*$-operator.
The complexification $T^*M^2 \otimes \mathbb{C}$ of the cotangent bundle splits into

$$T^*M^2 \otimes \mathbb{C} = \Lambda^{1,0} \otimes \Lambda^{0,1}$$

with

$$\Lambda^{1,0} = \{\omega \in T^*M^2 \otimes \mathbb{C} : *\omega = -i\omega\}$$

$$\Lambda^{0,1} = \{\omega \in T^*M^2 \otimes \mathbb{C} : *\omega = +i\omega\} .$$

Proposition 1: A smooth function $f: M^2 \longrightarrow \mathbb{C}$ is holomorphic if and only if its differential df is a section in the bundle $\Lambda^{1,0}$.

Proof: Since the differential df splits, according to the decomposition $T^*M^2 \otimes \mathbb{C} = \Lambda^{1,0} \otimes \Lambda^{0,1}$, into

$$df = \frac{df + i{*}df}{2} + \frac{df - i{*}df}{2} ,$$

we see that $df \in \Gamma(\Lambda^{1,0})$ is equivalent to $df = i{*}df$. This equation means

$$(i{*}df)(X) = -\, i\,df(J^g X) = df(X),$$

i.e. $df(J^g X) = i\,df(X)$ for any vector $X \in TM^2$.

Denote by

$$pr_{\Lambda^{1,0}}: T^*M^2 \otimes \mathbb{C} \longrightarrow \Lambda^{1,0}$$

$$pr_{\Lambda^{0,1}}: T^*M^2 \otimes \mathbb{C} \longrightarrow \Lambda^{0,1}$$

the projections of the complexified cotangent bundle onto $\Lambda^{1,0}$ and $\Lambda^{0,1}$, respectively.
We introduce the opeators

$$\partial_0 : C^\infty(M^2) \longrightarrow \Gamma(\Lambda^{1,0}),\ \partial_0 = pr_{\Lambda^{1,0}} \circ d$$

$$\bar{\partial}_0 : C^\infty(M^2) \longrightarrow \Gamma(\Lambda^{0,1}),\ \bar{\partial}_0 = pr_{\Lambda^{0,1}} \circ d$$

$$\partial_1 : \Gamma(T^*M^2 \otimes \mathbb{C}) \longrightarrow \Gamma(\Lambda^2 M^2 \otimes \mathbb{C}),\ \partial_1 = d \circ pr_{\Lambda^{0,1}}$$

$$\bar{\partial}_1 : \Gamma(T^*M^2 \otimes \mathbb{C}) \longrightarrow \Gamma(\Lambda^2 M^2 \otimes \mathbb{C}),\ \bar{\partial}_1 = d \circ pr_{\Lambda^{1,0}}.$$

Then we have $\bar{\partial}_1 \partial_0 f = d \circ pr_{\Lambda^{1,0}} \circ df = d\left(\frac{df + i{*}df}{2}\right) = \frac{1}{2}\, i\,d{*}df.$
On the other hand, the Laplace operator Δ on functions is defined by $\Delta f = -\, {*}d{*}df$. Now we obtain

$$2\,i\,\bar{\partial}_1 \partial_0 f = -d{*}df = \Delta(f)\cdot dM^2,$$

where dM^2 is the volume form of (M^2,g).

A 1-form ω is a holomorphic form if locally ω is the differential of a holomorphic function,

$$\omega = df = \partial_0 f.$$

Proposition 2: A 1-form ω is a holomorphic form if and only if $d\omega = 0$ and $*\omega = -i\omega$.

Proof: Suppose that ω is a holomorphic form. Then we have (locally) $\omega = \partial_0 f = df$. Hence, we obtain $d\omega = ddf = 0$ and $*\omega = *\partial_0 f = -i\,\partial_0 f = -i\omega$. Conversely, if $d\omega = 0$ and $*\omega = -i\omega$, then there exists locally a smooth function $f: M^2 \longrightarrow \mathbb{C}$ such that $df = \omega$ (Poincaré Lemma). Moreover, since $*\omega = -i\omega$, we conclude by Proposition 1 that f is a holomorphic function.

Corollary 1: A holomorphic 1-form on M^2 is a harmonic 1-form.

Corollary 2: A 1-form ω is a holomorphic form if and only if

$$\omega = \alpha + i * \alpha,$$

where α is a harmonic form.

Proof: Suppose that α is a harmonic form and consider $\omega = \alpha + i{*}\alpha$. Then

$$d\omega = d\alpha + i\, d{*}\alpha = 0$$
$$*\omega = *\alpha - i\alpha = -i\omega,$$

i.e. ω is a holomorphic form. Conversely, if ω is a holomorphic form, then we have

$$\omega = \frac{\omega}{2} + \frac{\omega}{2} = \frac{\omega}{2} + i * \frac{\omega}{2}$$

and $\alpha := \frac{\omega}{2}$ is a harmonic form.

To summarize, in real dimension $n=2$ there exists a one-to-one correspondence between Conformal Geometry and Complex Analysis. Moreover, solutions of certain real partial differential equations (harmonic forms) correspond to holomorphic objects on the underlying complex manifold (holomorphic forms). The algebraic background is the isomorphism of the groups $SO(2) \approx U(1)$. This isomorphism means that an Euclidean structure (conformal structure) in dimension two determines a unique complex structure, namely the rotation around $\frac{\pi}{2}$. The main idea of Penrose's twistor theory is the generalization of this point of view to the dimension $n = 4$. In this case the situation is more complicated since the group $U(2) \subset SO(4)$ does not coincide with $SO(4)$. The homogeneous space $SO(4)/U(2)$ is a two-dimensional sphere. Consequently, given an Euclidean vector space $(E^4, \langle , \rangle , \sigma)$ with a fixed orientation there is a $S^2 = \mathbb{C}P^1$-parameter family $J^-(E^4)$ of complex structures compatible with the Euclidean structure and the orientation. Starting with an oriented 4-dimensional Riemannian manifold M^4 we consider in any tangent space T_mM^4 the family $J^-(T_mM^4)$ as well as the 6-dimensional manifold

$$Z = \bigcup_{m \in M^4} J^-(T_mM^4).$$

Z is called the twistor space of M^4. In Section 3.3. we will describe the (almost-) complex structure of the twistor space and some of the links between tbe four-dimensional conformal geometry of M^4 and the complex analysis on the twistor space Z.

3.2. The Curvature Tensor of a 4-dimensional Manifold

We describe now the decomposition of the curvature tensor of a 4-dimensional Riemannian manifold. A general reference is for example [33]. Let (M^4,g) be an oriented Riemannian manifold of dimension four. The Hodge operator $* : \Lambda^2 \longrightarrow \Lambda^2$ on 2-forms is an involution, $** = 1$. Consequently, we obtain a decomposition of the bundle Λ^2 into

$$\Lambda^2 = \Lambda^2_+ \otimes \Lambda^2_- ,$$

where $\Lambda^2_\pm$ is the (± 1)-eigen-subspace of $*$. We understand the curvature tensor $\mathcal{R}$ as well as the Weyl tensor W as bundle morphisms

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2, \qquad W : \Lambda^2 \longrightarrow \Lambda^2.$$

Since the contraction of the Weyl tensor is zero, W maps $\Lambda^2_\pm$ into $\Lambda^2_\pm$. Consequently, the Weyl tensor splits into

$$W = \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix} , \quad W_\pm : \Lambda^2_\pm \to \Lambda^2_\pm .$$

Moreover, the curvature tensor $\mathcal{R}$ decomposes into

$$\mathcal{R} = \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix} + \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} - \frac{R}{12} ,$$

where R is the scalar curvature and $B: \Lambda^2_- \longrightarrow \Lambda^2_+$ is a bundle morphism. It is well known (see [33]) that $B \equiv 0$ if and only if (M^4, g) is an Einstein space. A Riemannian manifold M^4 is said to be self-dual if $W_- = 0$. The 4-dimensional sphere S^4, the complex projective space $\mathbb{C}P^2$ and the Riemannian product $S^2 \times H^2$ of the two-dimensional sphere by the hyperbolic plane are examples of self-dual Riemannian manifolds. In case of a compact manifold M^4, its signature $\sigma(M^4)$ is given by

$$\sigma(M^4) = \frac{1}{12\pi^2} \int_{M^4} (|W_+|^2 - |W_-|^2)$$

and the Gauß-Bonnet formula can be written in the form

$$\chi(M^4) = \frac{1}{8\pi^2} \int_{M^4} (|W_+|^2 + |W_-|^2) +$$

$$+ \frac{1}{48\pi^2} \int_{M^4} (R^2 - 3|Ric|^2).$$

In particular, if (M^4, g) is a compact Einstein manifold it holds that

$$\chi(M^4) - \frac{R^2 \, vol(M^4)}{192\pi^2} \geq \frac{3}{2} |\sigma(M^4)| .$$

3.3. The Twistor Space of a 4-dimensional Manifold

Denote by $(E^4, \langle\,,\rangle, \sigma)$ the 4-dimensional Euclidean vector space with inner product $\langle\,,\rangle$ and given orientation σ. Consider the set $J^-(E^4)$ of all endomorphisms $J: E^4 \longrightarrow E^4$ satisfying the following conditions:

(i) $J^2 = -id$

(ii) $\langle JX, JY\rangle = \langle X, Y\rangle$ for all vectors $X, Y \in E^4$

(iii) $\det(J) = 1$, i.e. J preserves the orientation σ

(iv) Setting $\Omega^J(X,Y) := \langle JX, Y\rangle$, then the given orientation σ equals $-\Omega^J \wedge \Omega^J = -(\Omega^J)^2$.

By definition, $J^-(E^4)$ describes a connected set of complex structures on E^4, compatible with the inner product as well as the given orientation.

For $J \in J^-(E^4)$ and a matrix $A \in SO(4)$, the composition AJA^{-1} is again in $J^-(E^4)$; the mapping $J \longmapsto AJA^{-1}$ for $A \in SO(4)$ defines a transitive $SO(4)$-action of $J^-(E^4)$ whose isotropy subgroup at a point $J_0 \in J^-(E^4)$ is equal to $U(2) \subset SO(4)$. In this way,

$J^-(E^4) = {}^{SO(4)}/U(2)$ is a symmetric space and isomorphic to the complex projective line $\mathbb{C}P^1$ (cp. [33]).
Now, let $(M^4,g,\mathcal{O})$ be a four-dimensional oriented Riemannian manifold, and denote by $P = (P,p,M^4;SO(4))$ the principal bundle of all orthonormal frames of M^4. Then the associated bundle
$Z = P \times_{SO(4)} {}^{SO(4)}/U(2) \cong {}^P/U(2)$ is called the <u>Twistor space</u> of $(M^4,g,\mathcal{O})$. Using the notion introduced above, the fibre Z_x of the Twistor space Z at a point $x \in M$ can be written as $Z_x = J^-(T_xM^4)$; therefore, the Twistor space parametrizes the almost complex structures on M^4, which are compatible with the metric and the orientation.
The Levi-Civita connection of (M^4,g) decomposes the tangent bundle TZ into horizontal and vertical subbundles, $TZ = T^hZ + T^vZ$.
Denoting the twistor-projection by $\pi : Z \longrightarrow M^4$, we get an almost complex structure I on Z preserving this decomposition and coinciding with the canonical complex structure on the fibres ${}^{SO(4)}/U(2) = \mathbb{C}P^1$. At the point $J \in Z$ the action of I on the horizontal part T^h_JZ of the tangent space at J is given by $I = \pi_*^{-1} \circ J \circ \pi_*: T^h_JZ \longrightarrow T^h_JZ$.

<u>Theorem 2</u> (see [2]): (Z,I) is a complex manifold if and only if (M^4,g) is a self-dual Riemannian manifold.

Consider now two conformally equivalent metrics $g_2 = e^{2\lambda}g_1$ on M^4. The corresponding twistor spaces $Z(g_1)$, $Z(g_2)$ coincide since the conditions (i)-(iv) defining the twistor space are conformally invariant. Moreover, an elementary calculation yields that the almost complex structures $I(g_1)$, $I(g_2)$ coincide, too (see [95]). Hence, the (almost-) complex manifold (Z,I) depends only on the conformal structure of the underlying space M^4. Moreover, using the complex manifold (Z,I) as well as the family of projective lines given by the fibres of the projection $\pi: Z \longrightarrow M^4$ one can reconstruct the conformal structure of M^4 from the holomorphic structure of (Z,I) (see [2],[95]).

The twistor space Z can also be described by the projective spin bundle $P(S^-)$. Consider the negative spin representation $Spin(4) \longrightarrow GL(\Delta_4^-)$. The group $SO(4)$ acts on the complex projective space $P(\Delta_4^-)$ and, consequently, the bundle

$$P(S^-) := P \times_{SO(4)} P(\Delta_4^-)$$

is well-defined over any 4-dimensional Riemannian manifold. If

$\{\psi^-\}$ is a projective spinor at the point $m \in M^4$, we define the operator $J_m\{\psi^-\}: T_mM^4 \longrightarrow T_mM^4$ by the formula

$$J_m\{\psi^-\}(X)\cdot\psi^- = i\cdot X\cdot\psi^-.$$

Then $J_m^{\{\psi^-\}}$ is a point in the twistor space Z over $m \in M^4$ and we obtain another interpretation of the twistor space (see [33]):

$$Z = P(S^-).$$

The decomposition of the tangent bundle of Z into horizontal and vertical parts also yields a one-parameter-family of Riemannian metrics on Z.
Therefore, on the Lie algebra $\check{\mathcal{M}}(2)$ we consider the positive-definite, $Ad(U(2))$-invariant inner product

$$B(X,Y) := -\frac{1}{2}\,Re(Tr(X\circ Y)), \quad X,Y \in \check{\mathcal{M}}(2)$$

and denote by ds^2 the standard Riemannian metric on $\mathbb{C}P^1 = U(2)/[U(1)\times U(1)]$ induced by B.

For a fixed positive real number t, a Riemannian metric g_t on Z is defined by taking the pull-back of the metric g to the horizontal part, and adding the t-fold of the fibre metric ds^2 in the vertical part of the tangent space of Z at an arbitrary point $J \in Z$, i.e.

$$g_t = \pi^* g + t\cdot ds^2, \qquad t > 0.$$

The study of the Kähler condition for g_t yields the following result, proved by Th. Friedrich/H. Kurke and N. Hitchin independently.

<u>Theorem 3</u> (see [45] or [63]): Let (M^4,g) be a self-dual Einstein space with positive scalar curvature R. Then the corresponding twistor space (Z,I,g_t) is a Kähler manifold if and only if $t = \frac{48}{R}$ holds. In this situation, g_t is also an Einstein metric with the same scalar curvature as (M^4,g). □

Examples of this situation are provided by the 4-dimensional sphere S^4 with twistor space $Z = \mathbb{C}P^3$, and the complex projective space $\mathbb{C}P^2$ with the complex flag manifold $F(1,2)$ as its twistor space. However, under the additional assumption of compactness (or, with respect to Myer's theorem, also completeness) of the four-dimensional manifold, they already exhaust the list of all possible examples, as the following proposition shows.

<u>Theorem 4</u> (see [45] or [63]): A compact four-dimensional self-dual Einstein space with positive scalar curvature is isometric either

to the sphere S^4 or to the complex projective plane $\mathbb{C}P^2$, both endowed with their standard metric. □

By construction of the metric g_t, the projection $\pi : (Z, g_t) \longrightarrow (M^4, g)$ is a Riemannian submersion. In this situation there are standard formulas to establish a relation between the curvature tensors of Z and M^4, and assertions about Einstein metrics on Z can be made.

<u>Theorem 5</u> (see [39]):

(1) Let (M^4, g) be a self-dual Einstein space with positive scalar curvature $R > 0$. If $t = {}^{48}/R$ or $t = {}^{24}/R$, then the metric g_t is an Einstein metric on the twistor space Z.

(2) Let (M^4, g) be a 4-dimensional Riemannian manifold. If its twistor space (Z, g_t) is an Einstein space for some $t > 0$, then (M^4, g) is a self-dual Einstein space with positive scalar curvature $R > 0$, and either $t = {}^{48}/R$ or $t = {}^{24}/R$ holds.

The above theorem is also valid for non-compact manifolds. However, we restrict our further considerations to the compact case: applying the twistor construction to S^4 and $\mathbb{C}P^2$, by Theorem 5 a further Einstein metric on $\mathbb{C}P^3$ and $F(1,2)$ will be obtained in addition to the standard Kähler-Einstein one. This second Einstein metric turns out to be non-Kähler in both cases, but it is still homogeneous under the action of $SO(5)$ and $U(3)$, respectively. We briefly describe these metrics, since they will be needed later in § 4 of Chapter 5.

a) <u>The case $M^4 = S^4$</u>

We decompose the Lie algebra $\underline{so}(5)$ into

$\underline{so}(5) = \underline{so}(4) \oplus \underline{\mathfrak{m}} = [\underline{\check{\mathfrak{u}}}(2) \oplus \underline{\mathfrak{u}}] \oplus \underline{\mathfrak{m}}$,

with $\underline{\mathfrak{m}} = \mathrm{Lin}\{E_{15}, E_{25}, E_{35}, E_{45}\}$,

$\underline{\mathfrak{u}} = \mathrm{Lin}\{E_{13} + E_{24}, E_{14} - E_{23}\}$ and

$\underline{\check{\mathfrak{u}}}(2) = \mathrm{Lin}\{E_{12}, E_{34}, E_{13} - E_{24}, E_{14} + E_{23}\}$,

where the matrices $\{E_{ij}\}_{i<j}$ are the standard basis elements of $\underline{so}(n)$ introduced in Chapter 1.

Using the inner product on $\underline{so}(5)$ given by

$$B_1(X,Y) := -\tfrac{1}{2}\,\mathrm{Tr}(X \circ Y), \qquad X, Y \in \underline{so}(5),$$

the metric g on $S^4 = {}^{SO(5)}/SO(4)$ induced by $B_1 \restriction \underline{\mathfrak{m}} \times \underline{\mathfrak{m}}$ is the standard Einstein metric on S^4 and has scalar curvature $R = 12$. Denoting by $\alpha : SO(4) \longrightarrow SO(\underline{m})$ the isotropy representation of S^4, the frame bundle $P \to S^4$ is given by $P = SO(5) \times_\alpha SO(\underline{m}) \cong SO(5)$, and the twistor space Z equals

$$Z = SO(5) \times_{SO(4)} SO(4)/U(2) \cong SO(5)/U(2).$$

The projection $Z \to S^4$ then corresponds to the imbedding $U(2) \hookrightarrow SO(4)$, and the family $\{g_t\}_{t>0}$ of Riemannian metrics on Z is expressed by $B_1|_{\underline{m}\times\underline{m}} + t\cdot ds^2$, $t > 0$.

Since the Riemannian metric on $\mathbb{C}P^1 = SO(4)/U(2)$ induced by $\lambda \cdot B_1|_{\underline{n}\times\underline{n}}$ is isometric to $t\cdot ds^2$ iff $\lambda = \frac{1}{2}t$, the Riemannian metric g_λ on Z induced by the bilinear form $B_1|_{\underline{m}\times\underline{m}} + \lambda B_1|_{\underline{n}\times\underline{n}}$ is an Einstein metric for $\lambda_1 = 2$ and $\lambda_2 = 1$ (see Theorem 5). The second parameter consequently yields the normal homogeneous metric on $SO(5)/U(2)$. To describe the Einstein metric corresponding to the parameter $\lambda_1 = 2$, we use the isomorphism

$\Phi : SO(5)/U(2) \longrightarrow \mathbb{C}P^3$ given in ([33], p.86):

If we regard $\mathbb{C}P^3 = U(4)/[U(3)\times U(1)]$ with $\check{\underline{u}}(4) = [\check{\underline{u}}(3) \oplus \check{\underline{u}}(1)] \oplus \underline{k}$, where

$$\underline{k} = \left\{ \begin{pmatrix} 0 & -\bar{A}^t \\ A & 0 \end{pmatrix} \; ; \; A \in \mathbb{C}^3 \right\} \cong \mathbb{C}^3,$$

the differential $d\Phi : \underline{n} + \underline{m} \longrightarrow \underline{k}$ is given by

$$\begin{array}{ll} E_{15} \longrightarrow (0,\frac{1}{2},0)^t & E_{25} \longrightarrow (0,\frac{i}{2},0)^t \\ E_{35} \longrightarrow (0,0,\frac{1}{2})^t & E_{45} \longrightarrow (0,0,\frac{i}{2})^t \\ Y_5 \longrightarrow (-1,0,0)^t & Y_6 \longrightarrow (i,0,0)^t . \end{array}$$

Here $Y_5 = E_{13}+E_{24}$, $Y_6 = E_{14}-E_{23}$ denotes the basis of $\underline{n}$. Since the vectors $E_{15}, E_{25}, E_{35}, E_{45}$ and $\frac{1}{\sqrt{2\lambda}}Y_5$, $\frac{1}{\sqrt{2\lambda}}Y_6$ are orthonormal with respect to $B_1|_{\underline{m}\times\underline{m}} + \lambda B_1|_{\underline{n}\times\underline{n}}$, for $\lambda = 2$ this bilinear form corresponds under $d\Phi$ to

$$\langle A,B \rangle := 2(\bar{A}^t B + \bar{B}^t A) \; ; \qquad A,B \in \underline{k} ,$$

and this scalar product on $\underline{k}$ describes the usual Kähler-Einstein metric on $\mathbb{C}P^3$.

b) <u>The case $M^4 = \mathbb{C}P^2$</u>

We represent the complex projective plane $\mathbb{C}P^2 = U(3)/[U(1)\times U(2)]$ as a homogeneous space and decompose the Lie algebra $\underline{u}(3)$ into

$\check{\mathfrak{u}}(3) = [\check{\mathfrak{u}}(1) \oplus \check{\mathfrak{u}}(2)] \oplus \mathfrak{m}$, with

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & 0 \\ -\bar{b} & 0 & 0 \end{pmatrix} \quad , \ a,b \in \mathbb{C} \right\} \cong \mathbb{C}^2 .$$

As an Ad-invariant, positive-definite inner product in $\check{\mathfrak{u}}(3)$ we take

$$B_2(X,Y) = -\tfrac{1}{2}\,\mathrm{Re}(\mathrm{Tr}(X \circ Y)) \quad ; \qquad X,Y \in \check{\mathfrak{u}}(3).$$

Then $B_2|_{\mathfrak{m}\times\mathfrak{m}}$ yields the homogeneous standard metric on $\mathbb{C}P^2$; with this metric, $\mathbb{C}P^2$ is a self-dual Einstein space of scalar curvature $R = 24$ (see [33]). Denoting by $\alpha : [U(1) \times U(2)] \longrightarrow SO(\mathfrak{m})$ the isotropy representation of $\mathbb{C}P^2$, the frame bundle of $\mathbb{C}P^2$ is given by $P = U(3) \times_\alpha SO(\mathfrak{m})$, and since $J^-(\mathfrak{m}) = {}^{SO(4)}/_{U(2)}$ is isomorphic to $\mathbb{C}P^1 = {}^{U(2)}/_{[U(1) \times U(1)]}$, the twistor space Z of $\mathbb{C}P^2$ is obtained by

$$Z = U(3) \times_\alpha J^-(\mathfrak{m}) \cong {}^{U(3)}/_{[U(1) \times U(1) \times U(1)]}.$$

Here the projection $\pi : Z \longrightarrow \mathbb{C}P^2$ is given by the imbedding $[U(1) \times U(1) \times U(1)] \hookrightarrow [U(1) \times U(2)]$. Geometrically, Z is the manifold $F(1,2)$ of $(1,2)$-flags in $\mathbb{C}^3$.
Using the additional notation

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\bar{c} & 0 \end{pmatrix} \quad ; \ c \in \mathbb{C} \right\} \cong \mathbb{C},$$

we obtain $\check{\mathfrak{u}}(3) = [\check{\mathfrak{u}}(1) \oplus \check{\mathfrak{u}}(1) \oplus \check{\mathfrak{u}}(1)] \oplus \mathfrak{m} \oplus \mathfrak{n}$; now remark that the metric ds^2 on $\mathbb{C}P^1$ coincides with the metric induced by $B_2|_{\mathfrak{n}\times\mathfrak{n}}$ and the corresponding imbedding $U(2) \hookrightarrow U(3)$. Thus, the family $\{g_t\}_{t>0}$ of Riemannian metrics on Z is determined by $B_2|_{\mathfrak{m}\times\mathfrak{m}} + t \cdot B_2|_{\mathfrak{n}\times\mathfrak{n}}$, and according to Theorem 5, Einstein metrics are obtained for $t_1 = 2$ and $t_2 = 1$. The second parameter corresponds to the normal homogeneous metric on $F(1,2)$; on the other hand, $t_1 = 2$ yields a Kähler-Einstein metric on $F(1,2)$ with the corresponding complex structure $I : \mathfrak{m} \oplus \mathfrak{n} \longrightarrow \mathfrak{m} \oplus \mathfrak{n}$ given at the end of example (a) in § 4, Chapter 5.

3.4. A Holomorphic Interpretation of the Twistor Equation

We consider a four-dimensional Riemannian spin manifold M^4 as well as a non-trivial solution $\psi_0^- \in \Gamma(S^-)$ of the twistor equation

$$\mathcal{D}\psi_0^- = 0.$$

The integrability condition

$$W(\eta^2)\psi_0^- = 0$$

(see Chapter 1, Theorem 12) and the fact that the zero points of ψ_0^- are isolated yield

$$W_- = 0,$$

i.e. M^4 is a self-dual Riemannian manifold. The twistor space $Z = P(S^-)$ is a complex manifold. Moreover, the manifolds $S^- \setminus 0$ and $(S^-)^* \setminus 0$ are complex manifolds , too. We recall that a spinor $0 \neq \psi^- \in S_m^-$ at the point $m \in M^4$ defines a complex structure on T_mM^4 by the formula

$$J^{\psi^-}(X)\cdot\psi^- = i\, X\psi^-.$$

Furthermore, by the rule

$$(X\cdot\xi^-)(\psi^-) := \xi^-(X\cdot\psi^-)$$

we introduce a Clifford multiplication of a vector $X \in T_mM^4$ by a dual spinor $\xi^- \in (S_m^-)^*$. If $0 \neq \xi^- \in (S_m^-)^*$ is a dual spinor, it defines a complex structure $J^{\xi^-} : T_mM^4 \longrightarrow T_mM^4$:

$$J^{\xi^-}(X)\cdot\xi^- = i\, X\cdot\xi^-.$$

Suppose that ξ^- is given by the Hermitian product on S_m^- and by a spinor $0 \neq \varphi^- \in S_m^-$:

$$\xi^-(\psi^-) = \langle\psi^-, \varphi^-\rangle.$$

Then we obtain for any spinor $\psi^- \in S_m^-$:

$$\begin{aligned}
\langle\psi^-, J^{\varphi^-}(X)\cdot\varphi^-\rangle &= \langle\psi^-, i\, X\varphi^-\rangle = \\
= i\langle X\psi^-, \varphi^-\rangle &= i\,\xi^-(X\psi^-) = \\
= i(X\cdot\xi^-)(\psi^-) &= (J^{\xi^-}(X)\cdot\xi^-)(\psi^-) = \\
= \xi^-(J^{\xi^-}(X)\cdot\psi^-) &= \langle J^{\xi^-}(X)\cdot\psi^-, \varphi^-\rangle = \\
= -\langle\psi^-, J^{\xi^-}(X)\varphi^-\rangle &
\end{aligned}$$

and, consequently,

$$J^{\xi^-} = -J^{\varphi^-}.$$

The tangent spaces $T_{\psi^-}(S^- \setminus 0)$, $T_{\xi^-}((S^-)^* \setminus 0)$ split into a vertical and a horizontal part. Since the vertical parts are canonically isomorphic to the complex vector spaces which are the fibres of the corresponding bundle they admit complex structures. On the horizontal subspaces we define the complex structures by

pulling back the operators J^{ψ^-}, J^{ξ^-} from T_mM^4.
Finally, $S^-\setminus 0$ and $(S^-)^*\setminus 0$ have universal almost complex structures, and they are integrable if and only if M^4 is a self-dual Riemannian manifold (see [2],[33]). We obtain the diagramme

$$\begin{tikzcd}
(S^-)^*\setminus 0 \arrow[r, "p"] \arrow[dr] & Z = P(S^-) \arrow[d] & S^-\setminus 0 \arrow[l, "q"'] \arrow[dl] \\
& M^4 &
\end{tikzcd}$$

where $(S^-)^*\setminus 0$, Z and $S^-\setminus 0$ are holomorphic manifolds. The projection $p: (S^-)^*\setminus 0 \longrightarrow Z$ is given by $p(\xi^-) = \mathrm{Ker}(\xi^-)$. This map is the projection of a holomorphic $\mathbb{C}^*$-prinicpal fibre bundle over the twistor space Z. We introduce the associated bundle

$$H = [(S^-)^*\setminus 0] \times_{\mathbb{C}^*} \mathbb{C}\,.$$

H is a holomorphic 1-dimensional vector bundle over Z.

Consider now an arbitrary section $\psi^- \in \Gamma(M^4;S^-)$. We define a function $\hat{\psi}^- : (S^-)^*\setminus 0 \longrightarrow \mathbb{C}$ by

$$\hat{\psi}^-(\xi^-) := \xi^-(\psi^-).$$

For any number $A \in \mathbb{C}^*$ we have

$$\hat{\psi}^-(A\cdot\xi^-) = A\cdot\psi^-(\xi^-).$$

The latter equation means that $\hat{\psi}^-$ is a section in the associated bundle H over the twistor space. Conversely, if $f \in \Gamma(Z;H)$ is a section, it is given by a function $f: (S^-)^*\setminus 0 \longrightarrow \mathbb{C}$ with the property

$$f(A\cdot\xi^-) = A\, f(\xi^-).$$

We say that f is a linear section of the bundle H if f satisfies

$$f(\xi_1^- + \xi_2^-) = f(\xi_1^-) + f(\xi_2^-).$$

A linear section is an element of $(S^-)^{**} \simeq S^-$, and consequently the space of all sections $\Gamma(M^4;S^-)$ is isomorphic to the space $\Gamma_{lin}(Z;H)$ of all linear sections:

$$\Gamma(M^4;S^-) = \Gamma_{lin}(Z;H).$$

The space $\mathcal{H}^o(Z;H)$ of all holomorphic sections is contained in $\Gamma_{lin}(Z;H)$:

$$\mathcal{H}^o(Z;H) \subset \Gamma_{lin}(Z;H).$$

Indeed, if f is a holomorphic section, then the restriction to any fibre $f: (S^-_m)^* \setminus 0 \longrightarrow \mathbb{C}$ is a holomorphic function with the property

$$f(A \cdot \xi^-) = A\, f(\xi^-).$$

The power series expansion of this function contains only linear terms, i.e. f is linear on any fibre. Next we prove that a spinor field $\psi^- \in \Gamma(S^-)$ is a solution of the twistor equation if and only if the section $\hat{\psi}^- \in \Gamma(Z;H)$ is holomorphic. We need the following algebraic lemma:

Lemma 1: Let $A: \mathbb{R}^4 \longrightarrow \Delta_4^-$ be a map satisfying the following conditions:

(i) A is $\mathbb{R}$-linear

(ii) For any algebraic spinor $\varphi^- \in \Delta_4^-$ and any vector $X \in \mathbb{R}^4$,

$$\langle \varphi^-, A(J^{\varphi^-} X)\rangle = i \langle \varphi^-, A(X)\rangle$$

holds, i.e. the functional $\mathbb{R}^4 \ni X \longrightarrow \langle \varphi^-, A(X)\rangle \in \mathbb{C}$ is J^{φ^-}-complex linear.

(iii) The trace $\sum\limits_{\alpha=1}^{4} e_\alpha \cdot A(e_\alpha)$ vanishes in Δ_4^+,

$$\sum_{\alpha=1}^{4} e_\alpha \cdot A(e_\alpha) = 0.$$

Then A is trivial, $A \equiv 0$.

Proof: In Δ_4^- and Δ_4^+ we fix the basis $u(1,-1)$, $u(-1,1)$ and $u(1,1)$, $u(-1,-1)$, respectively. The Clifford multiplication is given by

$$\begin{aligned}
e_1 \cdot u(1,-1) &= i\, u(1,1) & \qquad e_1 \cdot u(-1,1) &= i\, u(-1,-1)\\
e_2 \cdot u(1,-1) &= -u(1,1) & e_2 \cdot u(-1,1) &= u(-1,-1)\\
e_3 \cdot u(1,-1) &= i\, u(-1,-1) & e_3 \cdot u(-1,1) &= -i\, u(1,1)\\
e_4 \cdot u(1,-1) &= u(-1,-1) & e_4 \cdot u(-1,1) &= u(1,1).
\end{aligned}$$

Using these formulas we can calculate the complex structure $J^{\varphi^-}: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ for any spinor $0 \neq \varphi^- \in \Delta_4^-$. Suppose now that $A \neq 0$ is a map with the properties (i) - (iii). Without loss of generality we may assume that $A(e_1) = u(1,-1)$. Consider the spinor $\varphi^- = u(1,-1) = A(e_1)$. Then $J^{\varphi^-}(e_1) = e_2$ and we obtain

$$\begin{aligned}
\langle A(e_1), A(e_2)\rangle &= \langle \varphi^-, A(J^{\varphi^-}(e_1))\rangle =\\
&= i \langle \varphi^-, A(e_1)\rangle = i.
\end{aligned}$$

Next we consider the spinor $\eta^- = u(-1,1)$. The corresponding complex structure is given by

$$J^{\eta^-}(e_1) = -e_2 \qquad J^{\eta^-}(e_3) = e_4.$$

Thus, we have

$$\langle \eta^-, A(e_2)\rangle = -\langle \eta^-, A(J^{\eta^-} e_1)\rangle = -i\langle \eta^-, A(e_1)\rangle = 0.$$

Since $\langle \varphi^-, A(e_2)\rangle = i$ and $\langle \eta^-, A(e_2)\rangle = 0$, we conclude

$$A(e_2) = -i\ u(1,-1).$$

We write $A(e_3)$ and $A(e_4)$ in the form

$$A(e_3) = \alpha\, u(1,-1) + \beta\ u(-1,1)$$

$$A(e_4) = \gamma u(1,-1) + \delta\, u(-1,1).$$

With respect to $J^{\varphi^-}(e_3) = -e_4$ it follows that

$$\langle \varphi^-, A(e_4)\rangle = -\langle \varphi^-, A(J^{\varphi^-} e_3)\rangle = -i\langle \varphi^-, A(e_3)\rangle$$

i.e.

$$\gamma = i\alpha .$$

Moreover, using the complex structure J^{η^-} we deduce

$$\langle \eta^-, A(e_4)\rangle = \langle \eta^-, A(J^{\eta^-} e_3)\rangle = i\ \langle \eta^-, A(e_3)\rangle$$

and

$$\delta = -i\beta.$$

Consequently, the map $A\colon \mathbb{R}^4 \longrightarrow \Delta_4^-$ is defined by

$$\begin{aligned} A(e_1) &= u(1,-1)\\ A(e_2) &= -i\ u(1,-1)\\ A(e_3) &= \alpha\ u(1,-1) + \beta\ u(-1,1)\\ A(e_4) &= i\alpha\, u(1,-1) - i\beta\ u(-1,1)\ . \end{aligned}$$

Finally, we consider the spinor $\xi^- = u(1,-1) + i\ u(-1,1)$. An elementary calculation provides the formulas

$$J^{\xi^-}(e_1) = -e_3, \qquad J^{\xi^-}(e_2) = -e_4.$$

Then we have

$$\langle \xi^-, A(e_3)\rangle = -i\ \langle \xi^-, A(e_1)\rangle$$

and $\langle \xi^-, A(e_4)\rangle = -i\ \langle \xi^-, A(e_2)\rangle$.

We calculate the products and obtain

$$\alpha - i\ \beta = i$$
$$-\alpha - i\ \beta = i$$

and consequently $\alpha = 0$, $\beta = -1$.

The map $A\colon \mathbb{R}^4 \longrightarrow \Delta_4^-$ is therefore given by

$$A(e_1) = u(1,-1), \quad A(e_2) = -i\, u(1,-1)$$
$$A(e_3) = -u(-1,1), \quad A(e_4) = i\, u(-1,1).$$

This yields

$$\sum_{\alpha=1}^{4} e_\alpha \cdot A(e_\alpha) = 2\, i\, u(1,1) \neq 0,$$

a contradiction.

<u>Lemma 2:</u> Let $A: \mathbb{R}^4 \longrightarrow \Delta_4^-$ be a map with the properties:

(i) A is $\mathbb{R}$-linear.

(ii) For any spinor $\varphi^- \in \Delta_4^-$ and any vector $X \in \mathbb{R}^4$,

$$\langle \varphi^-, A(J^{\varphi^-} X)\rangle = i\langle \varphi^-, A(X)\rangle$$

holds.

Then there exists a spinor $\varphi^+ \in \Delta_4^+$ such that $A(X) = X \cdot \varphi^+$.

<u>Proof:</u> Consider $\varphi^+ = -\frac{1}{4} \sum_{\alpha=1}^{4} e_\alpha \cdot A(e_\alpha)$ as well as the map $A^* : \mathbb{R}^4 \to \Delta_4^-$, $A^*(X) = A(X) - X \cdot \varphi^+$. We calculate the trace

$$\sum_{\alpha=1}^{4} e_\alpha A^*(e_\alpha) = \sum_{\alpha=1}^{4} e_\alpha A(e_\alpha) + 4\varphi^+ = 0.$$

Moreover, if $\eta^- \in \Delta_4^-$, we have

$$\langle \eta^-, (J^{\eta^-} X)\varphi^+\rangle = -\langle (J^{\eta^-} X)\cdot \eta^-, \varphi^+\rangle =$$
$$= -i\langle X\cdot \varphi^-, \varphi^+\rangle = i\langle \eta^-, X\cdot\varphi^+\rangle .$$

A^* satisfies the conditions (i)-(iii) of Lemma 1 and we conclude $A^* \equiv 0$, i.e. $A(X) = X\cdot\varphi^+$.

<u>Theorem 6</u> (see [63]): A section $\psi^- \in \Gamma(S^-)$ is a twistor spinor, $\mathcal{D}\psi^- = 0$, if and only if the section $\hat{\psi}^- \in \Gamma(Z;H)$ is holomorphic.

<u>Proof:</u> Since $\hat{\psi}^-$ is linear on any fibre, the function $\hat{\psi}^-: (S^-)^* \setminus 0 \longrightarrow \mathbb{C}$ is holomorphic if and only if

$$d\hat{\psi}^-(IY) = i\, d\hat{\psi}(Y) \qquad \langle 1\rangle$$

holds for any horizontal vector $Y \in T^h((S^-)^* \setminus 0)$. Fix a point $m_0 \in M^4$ and locally an orthonormal frame in the tangent bundle such that $\nabla s_i(m_0) = 0$. The section ψ^- is locally given by a function $\psi^-: U \longrightarrow \Delta_4^-$ and $\hat{\psi}^-: U \times (\Delta_4^- \setminus 0)^* \longrightarrow \mathbb{C}$ has the form

$$\hat{\psi}^-(m,\xi^-) = \xi^-(\psi^-(m)).$$

The equation $\langle 1\rangle$ is equivalent to

$$\xi^-(d\psi^-(J^{\xi^-}X)) = i\,\xi^-(d\psi^-(X))$$

for any vector $X \in T_{m_o}M^4$ and any dual spinor $\xi^- \in (S^-_m)^*$. Since $\nabla s_i(m_o) = 0$, we see that $\hat{\psi}^-$ is a holomorphic function if and only if

$$\xi^-(\nabla_{J^{\xi^-}X}\psi^-) = i\,\xi^-(\nabla_X\psi^-) \qquad \langle 2\rangle$$

holds for any $X \in TM^4$ and $\xi^- \in (S^-)^*$. We represent the dual spinor ξ^- by a spinor φ^-, $\xi^-(\psi^-) = \langle\psi^-,\varphi^-\rangle$.
Then $J^{\xi^-} = -J^{\varphi^-}$ and the equation $\langle 2\rangle$ can be written as

$$-\langle\nabla_{J^{\varphi^-}X}\psi^-,\varphi^-\rangle = i\langle\nabla_X\psi^-,\varphi^-\rangle . \qquad \langle 3\rangle$$

Suppose now that ψ^- is a twistor spinor, $\mathcal{D}\psi^- = 0$. Then we have, for any vector $X \in TM^4$ and any spinor $\varphi^- \in S^-$,

$$-\langle\nabla_{J^{\varphi^-}X}\psi^-,\varphi^-\rangle = \langle\tfrac{1}{4}(J^{\varphi^-}X)\cdot D\psi^-,\varphi^-\rangle =$$

$$= -\langle\tfrac{1}{4}D\psi^-,(J^{\varphi^-}X)\cdot\varphi^-\rangle = -\langle\tfrac{1}{4}D\psi^-, i\,X\varphi^-\rangle =$$

$$= -i\langle\tfrac{1}{4}X\cdot D\psi^-,\varphi^-\rangle = i\langle\nabla_X\psi^-,\varphi^-\rangle .$$

Consequently, if ψ^- is a twistor spinor, the corresponding section $\hat{\psi}^-$ is holomorphic. Conversely, if $\hat{\psi}^-$ is holomorphic, the map

$$TM^4 \ni X \longrightarrow \nabla_X\psi^- \in S^-$$

satisfies the assumptions of Lemma 2.

We conclude that there exists an spinor field $\varphi^+ \in \Gamma(S^+)$ such that

$$\nabla_X\varphi^- = X\cdot\varphi^+,$$

i.e. φ^- is a twistor spinor.

Chapter 4: Odd-dimensional Riemannian Manifolds with Real Killing Spinors

In odd dimensions real Killing spinors are related to special contact structures (in addition to the Einstein condition). After explaining some special properties of contact forms and Sasakian manifolds we will discuss this relationship.
First we will prove a general existence theorem: Any simply connected Einstein-Sasakian manifold with spin structure admits two linearly independent Killing spinors. In every odd dimension we have a series of examples for such manifolds, namely starting with an arbitrary compact Kähler-Einstein manifold X^{2m} of positive scalar curvature we find a certain principal S^1-bundle M^{2m+1} over X^{2m} which has an Einstein-Sasakian and a spin structure. This yields a construction method for manifolds admitting Killing spinors in any odd dimension.
It turns out that in the dimensions 5 and 7 the converse of the above mentioned fact is true. Roughly speaking, there is a one-to-one correspondence between Killing spinors and Einstein-Sasakian structures on spin manifolds of these dimensions. In dimension 5 one can verify that under an additional regularity assumption on the associated Sasakian structure our construction method yields all possible manifolds admitting a Killing spinor. Well-known classification results concerning 4-dimensional Kähler-Einstein manifolds with positive scalar curvature now imply a classifiaction of 5-dimensional manifolds admitting one Killing spinor with a regular associated Sasakian structure. 5-Manifolds with two real Killing spinors with the same Killing number are conformally flat.

Analogously we can describe 7-dimensional manifolds admitting two real Killing spinors with regular structure. All of them, in particular the well-known homogeneous spaces with a Killing spinor as described in [26] can be obtained by our construction method. The existence of three independent Killing spinors on a simply connected 7-dimensional spin manifold M^7 is equivalent to a Sasakian 3-structure on M^7. This structure induces a Spin(3)-action on M^7. If the corresponding orbit space is a smooth closed manifold X^4, one can verify that M^7 is the manifold obtained by the construction method starting with the twistor space of X^4. Results of twistor theory now imply a classification of 7-manifolds with three "regular" Killing spinors. 7-Manifolds with more than three independent Killing spinors are conformally flat. Finally, we discuss some properties of 7-dimensional manifolds with one Killing spinor and give examples of such manifolds.

4.1. Contact Structures, Sasakian Manifolds

Let (M^{2m+1},g) be a Riemannian manifold of dimension 2m+1.

Definition 1: A contact metric structure on M^{2m+1} consists of a tensor field φ of type (1,1), a vector field ξ and a 1-form η on M^{2m+1} such that

(1) $\eta \wedge (d\eta)^m \neq 0$

(2) $\eta(\xi) = 1$

(3) $\varphi^2 = -Id + \eta \otimes \xi$

(4) $g(\varphi(X),\varphi(Y)) = g(X,Y) - \eta(X)\eta(Y)$

(5) $d\eta(X,Y) = 2g(X,\varphi(Y))$, where $d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y])$.

We call η a contact form and ξ the characteristic vector field. In particular, we have

$$\varphi(\xi) = 0 \qquad \eta \circ \varphi = 0$$
$$\eta(X) = g(\xi,X) \qquad d\eta(X,\xi) = 0.$$

in such a structure.

Definition 2: A contact metric structure (φ,ξ,η,g) is called a K-contact structure if ξ is a Killing vector field.

Lemma 1 ([15]): A contact metric structure (φ,ξ,η,g) is a K-contact structure if and only if

$$\nabla_X \xi = -\varphi(X)$$

holds.

Lemma 2 ([15]): If (φ,ξ,η,g) is a K-contact structure, then

1) $\mathcal{L}_\xi g = 0$

2) $\mathcal{L}_\xi \eta = 0$

3) $\mathcal{L}_\xi d\eta = 0$

4) $\mathcal{L}_\xi \varphi = 0.$

Definition 3: A manifold M^{2m+1} with K-contact structure (φ,ξ,η,g) is called a Sasakian manifold if

$$[\varphi,\varphi](X,Y) + d\eta(X,Y)\,\xi = 0,$$

where $[\varphi,\varphi](X,Y) := \varphi^2[X,Y] + [\varphi(X),\varphi(Y)] - \varphi[\varphi(X),Y] - \varphi[X,\varphi(Y)]$.

Then (φ, ξ, η, g) is called a Sasakian structure.

Lemma 3 ([15]): A K-contact structure (φ, ξ, η, g) is a Sasakian structure if and only if

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X.$$

Corollary 1: A tensor field φ of type (1,1), a vector field ξ and a 1-form η on (M^{2m+1}, g) constitute a Sasakian structure if and only if ξ is a Killing vector field of length 1 and

a) $\eta(X) = g(\xi, X)$

b) $-\nabla_X \xi = \varphi(X)$

c) $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$

d) $(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X$

hold.

On Sasakian manifolds we have curvature conditions [66]:

Lemma 4: If $\mathcal{R}$ is the curvature tensor on a Sasakian manifold, then

$$\mathcal{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

In particular, the scalar curvature R on an Einstein-Sasakian manifold of dimension 2m+1 equals $R = 2m(2m+1)$.

If $(M^{2m+1}; \varphi, \xi, \eta, g)$ is a Sasakian manifold and $s_1, \ldots, s_{2m+1}$ is an orthonormal frame on M^{2m+1}, then

$$\mathrm{Ric}(s_i, \varphi(s_j)) - \sum_{k=1}^{2m+1} \mathcal{R}(s_i, s_k, s_j, \varphi(s_k)) = (1-2m)g(\varphi(s_i), s_j) \quad (4.1)$$

for the Ricci curvature.

In particular, on an Einstein-Sasakian manifold we have

$$2g(s_i, \varphi(s_j)) = \sum_{k=1}^{2m+1} \mathcal{R}(s_i, s_j, s_k, \varphi(s_k)). \quad (4.2)$$

Remark 1: Let (M^{2m+1}, g) be an Einstein-Sasakian manifold. The curvature tensor $\mathcal{R}$ is a map

$$\mathcal{R} : \Lambda^2(TM^{2m+1}) \longrightarrow \Lambda^2(TM^{2m+1}).$$

Let Th denote the bundle of all vectors that are orthogonal to ξ. R maps $\Lambda^2(Th)$ into $\Lambda^2(Th)$. On the orthogonal complement of $\Lambda^2(Th)$ in $\Lambda^2(TM^{2m+1})$, $\mathcal{R}$ is equal to the $(-1)\cdot$identity. Since $W = \mathcal{R} + \frac{R}{n(n-1)}$ holds for the Weyl tensor on an n-dimensional Einstein manifold, $W(\Lambda^2(Th))$ is contained in $\Lambda^2(Th)$, and W vanishes on the orthogonal complement of $\Lambda^2(Th)$. Hence, we may consider $\mathcal{R}$ and W to be maps

$$\mathcal{R}, W : \Lambda^2(Th) \longrightarrow \Lambda^2(Th).$$

Definition 4: A triple (ξ_1, ξ_2, ξ_3) of Killing vector fields consists of three orthogonal Killing vector fields ξ_1, ξ_2, ξ_3 of length 1 satisfying

$$[\xi_1, \xi_2] = 2\xi_3, \quad [\xi_2, \xi_3] = 2\xi_1, \quad [\xi_3, \xi_1] = 2\xi_2.$$

Every integral manifold of a triple (ξ_1, ξ_2, ξ_3) of Killing vector fields is totally geodesic and of constant sectional curvature $K = 1$.

Definition 5: Three Sasakian structures $(\varphi_i, \xi_i, \eta_i, g)$ on a Riemannian manifold (M^{2m+1}, g) constitute a Sasakian-3-structure if (ξ_1, ξ_2, ξ_3) is a triple of Killing vector fields and if the relations

$$\begin{aligned}
\varphi_3\varphi_2 &= -\varphi_1 + \eta_2 \otimes \eta_3 & \qquad \varphi_2\varphi_3 &= \varphi_1 + \eta_3 \otimes \xi_2 \\
\varphi_1\varphi_3 &= -\varphi_2 + \eta_3 \otimes \xi_1 & \qquad \varphi_3\varphi_1 &= \varphi_2 + \eta_1 \otimes \xi_3 \\
\varphi_2\varphi_1 &= -\varphi_3 + \eta_1 \otimes \xi_2 & \qquad \varphi_1\varphi_2 &= \varphi_3 + \eta_2 \otimes \xi_1
\end{aligned}$$

are satisfied.

Using equation (4.1) one proves

Lemma 5: Any n-dimensional Riemannian manifold with Sasakian 3-structure is an Einstein manifold of scalar curvature $n(n-1)$.

Furthermore, one can show the following

Lemma 6: Let two Sasakian structures $(\varphi_i, \xi_i, \eta_i, g)$ $(i=1,2)$ with orthogonal characteristic vector fields ξ_1, ξ_2 be given on (M^{2m+1}, g). Then we have $\nabla_{\xi_2}\xi_1 = -\nabla_{\xi_1}\xi_2$, and by

$$\xi_3 := \nabla_{\xi_1}\xi_2, \quad \eta_3 := g(\xi_3, .), \quad \varphi_3 := -\nabla\xi_3$$

we can define a third Sasakian structure such that $(\varphi_i, \xi_i, \eta_i, g)$ $(i=1,2,3)$ constitute a Sasakian 3-structure.

Proof: First we show $\nabla_{\xi_2}\xi_1 = -\nabla_{\xi_1}\xi_2$. For any vector field Z it holds

$$\begin{aligned}
g(\nabla_{\xi_2}\xi_1, Z) &= g(-\nabla_Z\xi_1, \xi_2) = -Zg(\xi_1, \xi_2) + g(\xi_1, \nabla_Z\xi_2) \\
&= -g(Z, \nabla_{\xi_1}\xi_2),
\end{aligned}$$

since ξ_1 and ξ_2 are orthogonal Killing vector fields.
In particular, we have $\varphi_1(\xi_2) = -\varphi_2(\xi_1) = \xi_3$.
Furthermore, one deduces $[\xi_1, \xi_2] = \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 = 2\xi_3$.
Thus, ξ_3 is a Killing vector field, too. The length of ξ_3 equals one, since

$$g(\xi_3,\xi_3) = g(\varphi_1(\xi_2),\varphi_1(\xi_2)) = g(\xi_2,\xi_2) - \eta_1(\xi_2)\eta_1(\xi_2)$$
$$= g(\xi_2,\xi_2) = 1.$$

Using $\varphi_1^2 = -\mathrm{Id} + \eta_1 \otimes \xi_1$ and $(\nabla_X \xi_1)(Y) = g(X,Y)\,\xi_1 - \eta_1(Y)X$ we obtain

$$[\xi_1,\xi_3] = \nabla_{\xi_1}\xi_3 - \nabla_{\xi_3}\xi_1 = \nabla_{\xi_1}(\varphi_1(\xi_2)) + \varphi_1(\xi_3)$$
$$= (\nabla_{\xi_1}\varphi_1)(\xi_2) + \varphi_1(\nabla_{\xi_1}\xi_2) + \varphi_1(\xi_3)$$
$$= g(\xi_1,\xi_2)\xi_1 - \eta_1(\xi_2)\xi_1 + 2\varphi_1(\xi_3)$$
$$= 2\varphi_1(\xi_3) = -2\varphi_1(\nabla_{\xi_2}\xi_1) = 2\varphi_1^2(\xi_2)$$
$$= 2(-\xi_2 + \eta_1(\xi_2)\xi_1) = -2\xi_2.$$

Analogously one verifies $[\xi_2, \xi_3] = 2\xi_1$.
Using again $\varphi_1^2 = -\mathrm{Id} + \eta_1 \otimes \xi_1$ we deduce

$$\varphi_3(X) = -\nabla_X \xi_3 = -\nabla_X(\varphi_1(\xi_2)) = -(\nabla_X\varphi_1)(\xi_2) - \varphi_1(\nabla_X\xi_2)$$
$$= -g(X,\xi_2)\xi_1 - \eta_1(\xi_2)X - \varphi_1(\nabla_X\xi_2)$$
$$= -\eta_2(X) \otimes \xi_1 + \varphi_1\varphi_2(X)$$

and, in the same way,

$$\varphi_2\varphi_1 = -\varphi_3 + \eta_1 \otimes \xi_2.$$

Furthermore, one calculates

$$\varphi_3\varphi_2(X) = \varphi_3(-\nabla_X\xi_2) = \nabla_{\nabla_X\xi_2}\xi_3 = \nabla_{\nabla_X\xi_2}(\varphi_1(\xi_2))$$
$$= (\nabla_{\nabla_X\xi_2}\varphi_1)(\xi_2) + \varphi_1(-\nabla_{\nabla_X\xi_2}\xi_2)$$
$$= g(\nabla_X\xi_2,\xi_2)\xi_1 - \eta_1(\xi_2)\nabla_X\xi_2 + \varphi_1\varphi_2^2(X)$$
$$= \varphi_1\varphi_2^2(X) = \varphi_1(-X + \eta_2(X)\xi_2)$$
$$= -\varphi_1(X) + \eta_2(X)\xi_3,$$

i.e. $\varphi_3\varphi_2 = -\varphi_1 + \eta_2 \otimes \xi_3$ and analogously
$\varphi_3\varphi_1 = \varphi_2 + \eta_1 \otimes \xi_3$. Because of $\eta_1\varphi_2(X) = g(\xi_1, -\nabla_X\xi_2) =$
$= g(X, \nabla_{\xi_1}\xi_2) = g(X,\xi_3) = \eta_3(X)$ it holds

$$\begin{aligned}\varphi_1\varphi_3(X) &= \varphi_1(-\nabla_X\xi_3) = \varphi_1(-\nabla_X\varphi_1(\xi_2))\\ &= \varphi_1(-(\nabla_X\varphi_1)(\xi_2) - \varphi_1(\nabla_X\xi_2))\\ &= \varphi_1(-g(X,\xi_2)\xi_1 + \eta_1(\xi_2)X - \varphi_1(\nabla_X\xi_2))\\ &= \varphi_1^2\varphi_2(X) = -\varphi_2(X) + \eta_1(\varphi_2(X))\xi_1\\ &= -\varphi_2(X) + \eta_3(X)\xi_1.\end{aligned}$$

From $\eta_2\varphi_1(X) = -\eta_3(X)$ it follows in the same way that $\varphi_2\varphi_3 = \varphi_1 + \varphi_3 \otimes \xi_2$.

In order to prove that $(\varphi_3, \xi_3, \eta_3, g)$ is a Sasakian structure it remains to show that

$$\varphi_3^2 = -Id + \eta_3 \otimes \xi_3 \quad\text{and}\quad (\nabla_X\varphi_3)(Y) = g(X,Y)\xi_3 - \eta_3(Y)X.$$

Because of $\varphi_2(\xi_1) = -\xi_3$ and $\eta_2\varphi_1 = -\eta_3$ we have

$$\begin{aligned}\varphi_3^2 &= (\varphi_1\varphi_2 - \eta_2 \otimes \xi_1)(\varphi_1\varphi_2 - \eta_2 \otimes \xi_1)\\ &= \eta_2 \otimes \varphi_1(\xi_3) + \eta_3\varphi_2 \otimes \xi_1 + \varphi_1\varphi_2\varphi_1\varphi_2.\end{aligned}$$

From $\varphi_2\varphi_1 = -\varphi_1\varphi_2 + \eta_2 \otimes \xi_1 + \eta_1 \otimes \xi_2$ it follows now that

$$\varphi_3^2 = \eta_2 \otimes \varphi_1(\xi_3) + \eta_3\varphi_2 \otimes \xi_1 - \varphi_1\varphi_1\varphi_2\varphi_2 + \eta_1\varphi_2 \otimes \varphi_1(\xi_2).$$

Finally, the equations $\varphi_1(\xi_3) = \varphi_2\varphi_3(\xi_3) - \eta_3(\xi_3) \otimes \xi_2 = -\xi_2$, $\eta_3\varphi_2 = \eta_3\varphi_3\varphi_1 - \eta_1 \otimes \eta_3(\xi_3) = -\eta_1$ and $\eta_1\varphi_2 = \eta_3$ imply

$$\begin{aligned}\varphi_3^2 &= -\eta_2 \otimes \xi_2 - \eta_1 \otimes \xi_1 + (-Id + \eta_2 \otimes \xi_2 + \eta_1 \otimes \xi_1) + \eta_3 \otimes \xi_3\\ &= -Id + \eta_3 \otimes \xi_3.\end{aligned}$$

The Sasakian condition for φ_3 is a consequence of the corresponding ones for φ_1 and φ_2:

$$\begin{aligned}(\nabla_X\varphi_3)(Y) &= \nabla_X(\varphi_1\varphi_2)(Y) - \nabla_X(\eta_2 \otimes \xi_1)(Y)\\ &= (\nabla_X\varphi_1)\varphi_2(Y) + \varphi_1(\nabla_X\varphi_2)(Y) - X\eta_2(Y)\xi_1 +\\ &\quad + \eta_2(\nabla_X Y)\xi_1 - \eta_2(Y)\nabla_X\xi_1.\\ &= g(X,\varphi_2(Y)) - \eta_1(\varphi_2(Y))X + \varphi_1(g(X,Y)\xi_2 -\\ &\quad - \eta_2(Y)X) - Xg(Y,\xi_2)\xi_1 + g(\nabla_X Y,\xi_2)\xi_1 +\\ &\quad + \eta_2(Y)\varphi_1(X)\\ &= -\eta_1(\varphi_2(Y))X + g(X,Y)\varphi_1(\xi_2).\end{aligned}$$

Since we have already proved $\eta_1\varphi_2 = \eta_3$ and $\varphi_1(\xi_2) = \xi_3$, we obtain $(\nabla_X\varphi_3)(Y) = g(X,Y)\xi_3 - \eta_3(Y)X$.

<u>Remark 2:</u> If $(\varphi_i, \xi_i, \eta_i, g)$ $(i=1,2,3)$ is a Sasakian-3-structure, then the volume forms $\eta_i \wedge (d\eta_i)^m$ induce the same orientation.

4.2. An Existence Theorem for Killing Spinors on Odd-dimensional Manifolds

Theorem 1 ([44]): Let $(M^{2m+1};\varphi,\xi,\eta,g)$ be a simply connected Einstein-Sasakian manifold with spin structure.

a) If $m \equiv 0 \bmod 2$, then M^{2m+1} admits at least one Killing spinor for each of the values $\lambda = \pm\frac{1}{2}$.

b) If $m \equiv 1 \bmod 2$, then M^{2m+1} admits at least two Killing spinors for one of the values $\lambda = \pm\frac{1}{2}$.

Proof: We define two subbundles $E_\pm$ of the spinor bundle S by

$$E_\pm = \{\psi \in S : (\pm 2\varphi(X) + \xi X - X\xi)\psi = 0\}.$$

Furthermore, we introduce the covariant derivatives

$$\nabla^\pm_X \psi := \nabla_X \psi \pm \frac{1}{2} X\psi .$$

First of all we show that ∇^+, ∇^- are connections in E_+ and E_-, respectively. We differentiate the equations

$(\pm 2\varphi(X) + \xi X - X\xi)\psi = 0$ with respect to Y:

$$(\pm 2(\nabla_Y\varphi)(X) - \varphi(Y)X + X\varphi(Y)\}\psi + \{\pm 2\varphi(X) + \xi X - X\xi\}\nabla_Y\psi = 0.$$

This equation is equivalent to

$$\{\pm 2(\nabla_Y\varphi)(X) - \varphi(Y)X + X\varphi(Y) - \varphi(X)Y \mp \frac{1}{2}\xi XY \pm \frac{1}{2} X\xi Y\}\psi +$$
$$+ \{\pm 2\varphi(X) + \xi X - X\xi\}\nabla^\pm_Y\psi = 0$$

and we have to show that the first term of the last equation vanishes. A direct calculation yields this result by using the properties of the Sasakian manifold and the equation defining the bundle $E_\pm$:

$$\{\pm 2(\nabla_Y\varphi)(X) - \varphi(Y)X + X\varphi(Y) - \varphi(X)Y \mp \frac{1}{2}\xi XY \pm \frac{1}{2} X\xi Y\}\psi =$$
$$= \{\pm 2g(X,Y)\xi \mp 2\eta(X)Y - \varphi(Y)X + X\varphi(Y) - \varphi(X)Y \mp \frac{1}{2}\xi XY \pm \frac{1}{2} X\xi Y\}\psi$$
$$= \{\pm 2g(X,Y)\xi \mp 2\eta(X)Y + 2X\varphi(Y) + Y\varphi(X) \mp \frac{1}{2}\xi XY \pm \frac{1}{2} X\xi Y\}\psi$$
$$= \{\pm 2g(X,Y)\xi \mp 2\eta(X)Y \mp X\xi Y \pm XY\xi \mp \frac{1}{2} Y\xi X \pm \frac{1}{2} YX\xi \mp \frac{1}{2}\xi XY \pm \frac{1}{2} X\xi Y\}\psi$$
$$= \{\pm 2g(X,Y)\xi \mp 2\eta(X)Y \mp \frac{1}{2} X\xi Y \pm XY\xi \pm g(\xi,X)Y \pm \frac{1}{2}YX\xi \pm \frac{1}{2} YX\xi$$
$$\pm g(X,\xi)Y \pm X\xi Y\}\psi = 0.$$

The curvature $\mathcal{R}^\pm$ of the connections $\nabla^\pm$ is given by

$$\mathcal{R}^\pm(X,Y)\psi = \nabla^\pm_X\nabla^\pm_Y\psi - \nabla^\pm_Y\nabla^\pm_X\psi - \nabla^\pm_{[X,Y]}\psi$$
$$= \nabla_X\nabla_Y\psi \pm \frac{1}{2} X\nabla_Y\psi \pm \frac{1}{2}\nabla_X(Y\psi) + \frac{1}{4} XY\psi - \nabla_Y\nabla_X\psi$$
$$\mp \frac{1}{2} Y\nabla_X\psi \mp \frac{1}{2}\nabla_Y(X\psi) - \frac{1}{4} XY\psi - \nabla_{[X,Y]}\psi \mp \frac{1}{2}[X,Y]\psi$$

$$= R(X,Y)\psi + \frac{1}{4}(XY-YX)\psi .$$

where $\mathcal{R}$ denotes the curvature of ∇ .

Now we prove that $\mathcal{R}^{\pm}$ vanishes on $E_{\pm}$, i.e. $(E_{\pm},\nabla^{\pm})$ are flat bundles. Fix locally an orthonormal frame

$$s_1, s_2 = \varphi(s_1),\ s_3,\ s_4 = \varphi(s_3), \ldots, s_{2m} = \varphi(s_{2m-1}), \xi .$$

In particular, this fixes an orientation of M^{2m+1}.

This frame has the following properties.

a) If $\psi \in E_{\pm}$ and $s_i \neq s_j, \varphi(s_j)$, then $s_i s_j \psi$ is orthogonal to $E_{\pm}$.

Indeed, suppose $\psi_1, \psi_2 \in E_{\pm}$. Then we obtain

$$\begin{aligned}\langle s_i s_j \psi_1, \psi_2\rangle &= \mp\langle \xi s_i \varphi(s_j)\psi_1, \psi_2\rangle = \pm\langle\psi_1, \varphi(s_j)\, s_i \xi \psi_2\rangle \\ &= \pm\langle\psi_1, \xi\varphi(s_j)s_i\psi_2\rangle = \mp\langle\psi_1, \xi s_i \varphi(s_j)\psi_2\rangle \\ &= \langle\psi_1, s_i s_j \psi_2\rangle = \langle s_j s_i \psi_1, \psi_2\rangle \\ &= -\langle s_i s_j \psi_1, \psi_2\rangle\end{aligned}$$

b) If $\psi \in E_{\pm}$, then $s_i \xi\psi$ is orthogonal to $E_{\pm}$ since

$$\begin{aligned}\langle s_i\xi\psi_1, \psi_2\rangle &= \pm\langle s_i s_j \varphi(s_j)\psi_1, \psi_2\rangle = \mp\langle\psi_1, \varphi(s_j)s_j s_i\psi_2\rangle \\ &= \pm\langle\psi_1, s_i s_j \varphi(s_j)\psi_2\rangle = \langle\psi_1, s_i\xi\psi_2\rangle \\ &= -\langle s_i\xi\psi_1, \psi_2\rangle\end{aligned}$$

holds for $\psi_1, \psi_2 \in E_{\pm}$.

c) If $\psi \in E_{\pm}$, then $s_i\varphi(s_i)\psi = \mp\xi\psi$.

Now we can calculate $\mathcal{R}^{\pm}$. Let $\psi \in E_{\pm}$. $\mathcal{R}^{\pm}(X,Y)\psi$ equals the $E_{\pm}$-part of $\mathcal{R}^{\pm}(X,Y)\psi = \mathcal{R}(X,Y)\psi + \frac{1}{4}(XY-YX)\psi$. Because of the mentioned properties of the frame, the $E_{\pm}$-part of $\mathcal{R}(X,Y)\psi = \frac{1}{4}\sum_{i,j}\mathcal{R}(X,Y,s_i,s_j)s_i\cdot s_j\psi$ is given by

$$\frac{1}{4}\sum_{i=1}^{2m}\mathcal{R}(X,Y,s_i,\varphi(s_i))s_i\cdot\varphi(s_i)\psi = \mp\frac{1}{4}\Big(\sum_{i=1}^{2m}\mathcal{R}(X,Y,s_i,\varphi(s_i)\Big)\xi\psi .$$

Since M^{2m+1} is an Einstein-Sasakian manifold, formula (4.2) implies that this equals $\mp\frac{1}{2}g(X,\varphi(Y))\xi\psi$.

On the other hand, $\pm\frac{1}{2}g(X,\varphi(Y))\xi\psi$ is precisely the $E_{\pm}$-part of $\frac{1}{4}(YX-XY)\psi$.

It remains to calculate the dimension of the bundles $E_{\pm}$. Let $\psi \in S$ be equal to $u \in \Delta_{2m+1}$ relative to our frame. ψ is an element of $E_{\pm}$ if and only if it satisfies the equation $\mp X\varphi(X)\psi = \xi\psi$ for all vectors X orthogonal to ξ, i.e. if and only if $\mp e_{2j-1}e_{2j}u = e_{2m+1}u$ holds for all $j \leq m$. Relative to the basis $u(\varepsilon(1),\dots,\varepsilon(m))$ of Δ_{2m+1}, $e_{2j-1}e_{2j}$ is given by

$$\mp\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}i&0\\0&-i\end{pmatrix}\otimes\underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{[(j-1)/2]\ \times}$$

and e_{2m+1} by

$$i\begin{pmatrix}-1&0\\0&1\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

Consequently, $\sum c_{\varepsilon(1)\dots\varepsilon(m)}\,u(\varepsilon(1),\dots,\varepsilon(m))$ is an element of $K_j := \operatorname{Ker}(\mp e_{2j-1}e_{2j}-e_{2m+1})$ if and only if $u(\varepsilon(1),\dots,\varepsilon(m))\in K_j$ for all $\varepsilon(1),\dots,\varepsilon(m)$ such that $c_{\varepsilon(1)\dots\varepsilon(m)} \neq 0$. On the other hand, $u(\varepsilon(1),\dots,\varepsilon(m))$ is an element of K_j if and only if

$\mp e_{2j-1}e_{2j}\,u(\varepsilon(1),\dots,\varepsilon(m))-e_{2m+1}u(\varepsilon(1),\dots,\varepsilon(m)) =$

$= i(\mp\varepsilon(j)\,u(\varepsilon(1),\dots,\varepsilon(m))-(-1)^m\varepsilon(1)\cdot\ldots\cdot\varepsilon(m)u(\varepsilon(1),\dots,\varepsilon(m))$

$= i(\mp\varepsilon(j)-(-1)^m\varepsilon(1)\cdot\ldots\cdot\varepsilon(m))\,u(\varepsilon(1),\dots,\varepsilon(m))$

$= 0$

i.e. $(-1)^m\varepsilon(1)\cdot\ldots\cdot\varepsilon(m) = \mp\varepsilon(j)$ holds.

One obtains

$$\dim E_- = \begin{cases}0 & \text{if } m \text{ is odd}\\ 1 & \text{if } m \text{ is even}\end{cases}$$

$$\dim E_+ = \begin{cases}2 & \text{if } m \text{ is odd}\\ 1 & \text{if } m \text{ is even}\end{cases}.$$

This result is interesting since we can construct such Einstein-Sasakian manifolds as certain S^1-bundles over Kähler-Einstein manifolds of positive scalar curvature.

Example 1: Let $(X^{2m},\bar g,\bar J)$ be a Kähler-Einstein manifold of scalar curvature $\bar R = 4m(m+1)$ and $c_1(X^{2m})$ denote its first Chern class. Let A be the maximal integer such that $\frac{1}{A}c_1(X^{2m})$ is an integral cohomology class. Consider the S^1-bundle $(M^{2m+1},\pi,X^{2m};S^1)$ with the Chern class $c_1(M^{2m+1}\longrightarrow X^{2m}) = \frac{1}{A}c_1(X^{2m})$ as well as the connection form η' in this bundle, whose curvature form equals $d\eta' = \frac{2(m+1)}{A}i\bar\Omega$, where $\bar\Omega$ denotes the Kähler form of X^{2m}. If we define a metric g on M by $g = \pi^*\bar g - \frac{A^2}{(m+1)^2}\eta'\otimes\eta'$, then M^{2m+1} is an Einstein-Sasakian manifold.

M^{2m+1} is simply connected and admits a spin structure.

Proof: With the aid of the O'Neill formulas (s. [16]) one proves that g' is the unique Einstein metric on M^{2m+1} naturally defined by g and η', i.e. horizontally determined by g and vertically by the length of S^1.

Now we prove that M^{2m+1} is a Sasakian manifold. We define the contact form η, the characteristic vector field ξ and the endomorphism field φ by

$$\eta := \frac{A}{(m+1)!}\,\eta',\quad g(\xi,.) = \eta,\quad \varphi(X) = \begin{cases} \widetilde{\bar{J}(\pi_* X)}, & \text{if } X \perp \xi \\ 0, & \text{if } X \parallel \xi \end{cases}$$

where $\sim$ denotes the horizontal lift relative to η'.

Then $g = \bar{g} + \eta \otimes \eta$. Obviously, $\eta(\xi) = 1$ holds.

Relative to an orthonormal frame $s_1,\ldots,s_{2m}$ satisfying $\pi_* s_i = \bar{s}_i$, $\bar{s}_{2i} = \bar{J}\,\bar{s}_{2i-1}$ we have

$$\eta \wedge (d\eta)^m(\xi, s_1,\ldots,s_m) = \eta(\xi)2^m\,\bar{\Omega}^m(\bar{s}_1,\bar{s}_2,\ldots,\bar{s}_{2m}) = 2^m m!$$

The relations (3) and (4) of definition 1 hold because of the corresponding properties of $\bar{J}$. Furthermore, we have

$$d\eta(X,Y) = \frac{A}{(m+1)!}\,d\eta'(X,Y) = 2\bar{\Omega}(X,Y) = 2\bar{g}(X,\bar{J}Y) = 2g(X,\varphi Y),$$

i.e. (5) is satisfied. The vector field ξ is a Killing vector field since, because of $\mathcal{L}_\xi\, \pi^*\bar{g} = 0$ and $\mathcal{L}_\xi \eta = i(\xi)d\eta + d\{(i(\xi)\eta)\} = d\eta(\xi,.) = 0$, the equation $\mathcal{L}_\xi\, g = \mathcal{L}_\xi \pi^*\bar{g} + \mathcal{L}_\xi(\eta\otimes\eta) = 0$ holds. The Sasakian integrability condition is satisfied since X^{2m} is a complex manifold (s. [15]).

The Kähler-Einstein manifold X^{2m} is simply connected (s. [71]) and therefore the exact sequence

$$\longrightarrow \pi_2(X^{2m}) \xrightarrow{\partial} \pi_1(S^1) \longrightarrow \pi_1(M^{2m+1}) \longrightarrow \pi_1(X^{2m}) = 0 \quad \text{yields}$$

that $\pi_1(M^{2m+1})$ is trivial or a cyclic group. In particular, we conclude $\pi_1(M^{2m+1}) = H_1(M^{2m+1};\mathbb{Z})$.

Using the exact Thom-Gysin sequence of the S^1-bundle $(M^{2m+1},\pi,X^{2m};S^1)$

$$\ldots \longrightarrow H^{2m-2}(X^{2m};\mathbb{Z}) \xrightarrow{\cup c_1(M^{2m+1}\longrightarrow X^{2m})} H^{2m}(X^{2m};\mathbb{Z}) \xrightarrow{\pi^*}$$

$$\xrightarrow{\pi^*} H^{2m}(M^{2m+1};\mathbb{Z}) \longrightarrow H^{2m-1}(X^{2m};\mathbb{Z}) \longrightarrow \ldots$$

and the Poincaré duality for $H^{2m-1}(X^{2m};\mathbb{Z}) = H_1(X^{2m};\mathbb{Z}) = 0$ we obtain

$H^{2m}(M^{2m+1};\mathbb{Z}) = H^{2m}(X^{2m};\mathbb{Z})/c_1(M^{2m+1} \longrightarrow X^{2m}) \cup H^{2m-2}(X^{2m};\mathbb{Z})$.

Since $c_1(M^{2m+1} \longrightarrow X^{2m})$ is not a multiple of an integral cohomology class it turns out that the homomorphism

$$c_1(M^{2m+1} \longrightarrow X^{2m}) \cup : H^{2m-2}(X^{2m};Z) \to H^{2m}(X^{2m};Z)$$

is surjective. Finally we obtain $\pi_1(M^{2m+1}) = 0$.
It remains to prove that M^{2m+1} is a spin manifold. In case $\omega_2(X^{2m}) = 0$, this is obvious. Consider now the case $\omega_2(X^{2m}) \neq 0$. Then $c_1(X^{2m}) \equiv \omega_2(X^{2m}) \equiv 1 \bmod 2$ and, consequently, $c_1(M^{2m+1} \to X^{2m}) \equiv c_1(X^{2m}) \bmod 2$. We obtain

$$\omega_2(M^{2m+1}) = \pi^* \omega_2(X^{2m}) = \pi^* c_1(X^{2m}) = \pi^* c_1(M^{2m+1} \longrightarrow X^{2m}) \bmod 2.$$

On the other hand, from the exact Thom-Gysin sequence

$$H^0(X^{2m};\mathbb{Z}) \xrightarrow{\cup c_1(M^{2m+1} \to X^{2m})} H^2(X^{2m};\mathbb{Z}) \xrightarrow{\pi^*} H^2(M^{2m+1};\mathbb{Z}) \longrightarrow \dots$$

it follows that $\pi^* c_1(M^{2m+1} \longrightarrow X^{2m}) = 0$.

We will now discuss what this construction method provides in dimension 5 and 7. There arises the question on which 4-dimensional manifolds Kähler-Einstein metrics of positive scalar curvature R exist. First we note that such a manifold must have a positive first Chern class, since this class is represented by the Ricci form, which equals $\frac{R}{4}\omega > 0$ on 4-dimensional Kähler-Einstein manifolds, where ω denotes the Kähler form. A compact 4-dimensional Kähler manifold admits a positive first Chern class if and only if it is analytically equivalent to $S^2 \times S^2$, $\mathbb{C}P^2$ or to one of the del Pezzo surfaces P_k (P_k is the surface obtained by blowing up k points in a general position in $\mathbb{C}P^2$, see [11]), where $1 \leq k \leq 8$ (see [11]). Using a theorem of Matsushima (see [73]) we prove in Section 4.3 that if g is a Kähler-Einstein metric on $S^2 \times S^2$, then the isometry group acts transitively on this space and, consequently, g is the standard metric. On $\mathbb{C}P^2$ the same is true. On P_1 and P_2, there do not exist any Kähler-Einstein metrics ([17]). The existence of families of Kähler-Einstein metrics on P_k $(3 \leq k \leq 8)$ was shown by Tian and Yau ([101], [102]).
We obtain the following possibilities for X^4 and M^5, respectively.

X^4	M^5
$S^2 \times S^2$	$V_{4,2}$
$\mathbb{C}P^2$	S^5
P_k $(3 \leq k \leq 8)$	M^5_k

$V_{4,2}$ denotes the Stiefel manifold of oriented orthonormal 2-frames in $\mathbb{R}^4$. M^5_k is diffeomorphic to the k-fold connected sum $(S^2 \times S^3) \# \dots \# (S^2 \times S^3)$ (see Section 4.3).

On M_k^5, there exists a family of Einstein metrics with a Killing spinor. Let us discuss now examples in dimension 7. If we apply the construction method to the Kähler manifolds $X^6 = \mathbb{C}P^3$, the flag manifold $F(1,2)$, $S^2 \times S^2 \times S^2$, $\mathbb{C}P^2 \times S^2$ or to the Graßmann manifold $G_{5,2}$, we obtain metrics with two independent Killing spinors on $M^7 = S^7$,
$SU(3)/S^1 = N(1,1), [SU(2) \times SU(2) \times SU(2)]/U(1) \times U(1) =$

$= Q(1,1,1), [SU(3) \times SU(2) \times U(1)]/SU(2) \times U(1) \times U(1) = M(3,2)$

and the Stiefel manifold $V_{5,2}$, respectively. These examples are well-known (see [26]). Let now X^6 be $P_k \times S^2$, where P_k is one of the del Pezzo surfaces P_k $(3 \leq k \leq 8)$. Then our method yields a family of Einstein metrics with two Killing spinors on the corresponding 7-dimensional manifold M_k^7. Summing up we get the following examples

X^6	M^7
$\mathbb{C}P^3$	S^7
$F(1,2)$	$N(1,1)$
$S^2 \times S^2 \times S^2$	$Q(1,1,1)$
$\mathbb{C}P^2 \times S^2$	$M(3,2)$
$G_{5,2}$	$V_{5,2}$
$P_k \times S^2$ $(3 \leq k \leq 8)$	M_k^7

Some of these examples can be generalized for arbitrary odd dimension. Consider the Klein quadric $Q_{n-1}(\mathbb{C})$, i.e. the hypersurface in $\mathbb{C}P^n$ given by the equation

$$(z^0)^2 + (z^1)^2 + \ldots + (z^n)^2 = 0.$$

The restriction of the Fubini metric of $\mathbb{C}P^n$ to $Q_{n-1}(\mathbb{C})$ is a Kähler-Einstein metric of positive scalar curvature.
On the other hand, the Klein quadric $Q_{n-1}(\mathbb{C})$ is diffeomorphic to the oriented Graßmann manifold $G_{n-1,2}$. Applying now our construction method we obtain the Stiefel manifold $V_{n-1,2}$ and a metric with two independent Killing spinors on it.

4.3. Compact 5-dimensional Riemannian Manifolds with Killing Spinors

There are some special properties of the Spin(5)-representation based on the classical group isomorphisms

$$\mathrm{Spin}(5) = \mathrm{Sp}(2) \quad \text{and} \quad \mathrm{SU}(2) = \mathrm{Sp}(1).$$

We have $Sp(2)/_{Sp(1)} = S^7$, i.e. $Spin(5)/_{SU(2)} = S^7$.

We discuss now carefully how these isomorphisms are realized by the Spin(5)-representation.
Relative to the basis $u(1,1)$, $u(1,-1)$, $u(-1,1)$, $u(-1,-1)$ the Clifford multiplication is given by

$$e_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad e_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

From this we conclude

Lemma 7:

(i) $\{\omega^2 \in \Lambda^2(\mathbb{R}^5) : \omega^2 \cdot u(1,1) = 0\} =$

$$= \Big\{\sum \omega_{ij} \cdot e_i \wedge e_j : \begin{array}{ll} \omega_{12} + \omega_{34} = 0 & \omega_{13} = \omega_{24} \\ \omega_{14} + \omega_{23} = 0 & \omega_{i5} = 0 \quad (i=1,\dots,4) \end{array}\Big\}$$

(ii) Let $\lambda_1, \lambda_2, \lambda_3$ be complex numbers. Then we have

$$\dim \{\omega^2 \in \Lambda^2(\mathbb{R}^5) : \omega^2 u(1,1) = 0,\ \omega^2(\lambda_1 u(1,-1) + \lambda_2 u(-1,1) + \lambda_3 u(-1,-1)) = 0\}$$

$$= \begin{cases} 0 & \text{if } \lambda_1 \neq 0 \text{ or } \lambda_2 \neq 0 \\ 3 & \text{if } \lambda_1 = \lambda_2 = 0 \text{ and } \lambda_3 \neq 0. \end{cases}$$

Spin(5) acts on the 7-sphere $S^7(\Delta_5) := \{\psi \in \Delta_5 : |\psi| = 1\}$. We denote the isotropy group of $u(1,1)$ relative to this action by H^o.
H^o projects one-to-one onto a subgroup $H \subset SO(5)$. The Lie algebra of H equals

$$\underline{h} = \Big\{\sum_{i<j} \omega_{ij} E_{ij} : \sum \omega_{ij} e_i e_j \ u(1,1) = 0\Big\},$$

where $\Big\{E_{ij} = \begin{array}{c} \begin{pmatrix} & & & & \\ & & & -1 & \\ & & 0 & & \\ & 1 & & & \\ & & & & \end{pmatrix} \begin{array}{l} \\ i \\ \\ j \\ \\ \end{array} \\ \quad i \quad\ j \end{array}, 1 \leq i < j \leq 5\Big\}$ is the standard basis of $\underline{so}(5)$.

By Lemma 7 we have $\underline{h} = \underline{su}(2)$.

Lemma 8: Spin(5) acts transitively on $S^7(\Delta_5)$. The isotropy group H^o is isomorphic to $SU(2)$.

Proof: The first assertion follows from

$$\dim \mathrm{Spin}(5) = 10, \ \dim S^7(\Delta_5) = 7, \ \dim H^o = 3,$$

i.e. $\dim \mathrm{Spin}(5) = \dim S^7(\Delta_5) + \dim H^o$.

Now we have to show that H^o is simply connected. Consider the exact homotopy sequence of the fibration

$$S^7(\Delta_5) = \mathrm{Spin}(5)/H^o.$$

$$\ldots \ \pi_2(S^7) \xrightarrow{\partial} \pi_1(H^o) \longrightarrow \pi_1(\mathrm{Spin}(5)) \longrightarrow \pi_1(S^7) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} \pi_o(H^o) \longrightarrow \pi_o(\mathrm{Spin}(5)) \longrightarrow \pi_o(S^7).$$

From $\pi_2(S^7) = \pi_1(S^7) = \pi_o(S^7) = 0$ it follows now that $\pi_o(H^o) = \pi_o(\mathrm{Spin}(5)) = 0$ and $\pi_1(H^o) = \pi_1(\mathrm{Spin}(5)) = 0$.

Corollary 2: If $\psi \neq 0$ is an element of Δ_5, then there is a unique vector ξ of length one such that

$$\xi\psi = i\psi \ .$$

Proof: We may assume $\psi \in S^7(\Delta_5)$. Since $\mathrm{Spin}(5)$ acts transitively on $S^7(\Delta_5)$, let without loss of generality ψ be $u(1,1)$. The equation $\xi u(1,1) = i\, u(1,1)$ has obviously the unique solution $\xi = e_5$.

Corollary 3: Let M^5 be a compact 5-dimensional Riemannian spin manifold, P its frame bundle and $f: Q \longrightarrow P$, $\pi: Q \longrightarrow M$ a fixed spin structure. Then any section ψ of length 1 in the spinor bundle $S = Q \times_{\mathrm{Spin}(5)} \Delta_5$ defines $SU(2)$-reductions $Q(\psi)$, $P(\psi)$ of Q and P by

$$Q(\psi) = \{q \in Q : \psi(\pi(q)) = [q, u(1,1)]\}$$
$$P(\psi) = f(Q(\psi)).$$

We will now give estimates for the maximal number of Killing spinors. Let (M^5, g) be a compact 5-dimensional Einstein manifold of positive scalar curvature R with spin structure. We denote by m_+ and m_- the dimensions of the spaces of Killing spinors:

$$m_{\pm} = \dim\left\{\psi \in \Gamma(S) : \nabla_X \psi = \pm\tfrac{1}{2}\sqrt{\tfrac{R}{20}}\, X\psi\right\} \ .$$

Theorem 2: ([41]) If the Weyl tensor W of M^5 does not vanish identically, then

$$m_+ \leq 1 \quad \text{and} \quad m_- \leq 1.$$

Proof: Let $m_0 \in M^5$ be fixed such that $W \neq 0$ in a neighbourhood V of m_0. Assume that $m_+ \geq 2$. Then we can choose orthonormal Killing spinors ψ_1, ψ_2 with $\nabla_X \psi_i = \frac{1}{2}\sqrt{\frac{R}{20}}\, X\psi_i$ $(i=1,2)$. Let $s_1,\dots,s_5$ be a local section $V \supset U \to P(\psi_1)$. Relative to this frame we have $\psi_2 = \lambda_1 u(1,-1) + \lambda_2 u(-1,1) + \lambda_3 u(-1,-1)$. Because of formula (1.38) and Lemma 7, $W \neq 0$ in U implies $\lambda_1 = \lambda_2 = 0$. Hence,

$$\nabla_X \psi_1 = \frac{1}{2} \sum_{i<j} \omega_{ij}(X) e_i e_j\, u(1,1)$$

$$\nabla_X \psi_2 = X(\lambda_3)\, u(-1,-1) + \frac{\lambda_3}{2} \sum_{i<j} \omega_{ij}(X) e_i e_j\, u(-1,-1)$$

relative to $s_1,\dots,s_5$, where $\omega_{ij} = (\nabla s_i, s_j)$. For $X = s_1$ we obtain, noting $\nabla_{s_1}\psi_1 = \frac{1}{2}\sqrt{\frac{R}{20}}\, e_1\, u(1,1)$ and $\nabla_{s_1}\psi_2 = \frac{1}{2}\sqrt{\frac{R}{20}}\,\lambda_3 e_1\, u(-1,-1)$

$$-\omega_{15}(s_1) + i\,\omega_{25}(s_1) = \frac{1}{2}\sqrt{\frac{R}{20}}\, i$$

$$\lambda_3(-\omega_{15}(s_1) - \omega_{25}(s_1)) = i\lambda_3 .$$

On account of $R \neq 0$ and $\lambda_3 \neq 0$ this is a contradiction.

Later on we will prove:

Theorem 3 ([41]): If M^5 is simply connected, then $m_+ = m_-$ holds.

We will now see that in the 5-dimensional compact case a Killing spinor defines an Einstein-Sasakian structure. Consider a compact 5-dimensional Riemannian spin manifold M^5 with a Killing spinor of length $|\psi| = 1$. Since M^5 is an Einstein space, let the scalar curvature R be $R = 20$. Furthermore, we may assume $\nabla_X \psi = \frac{1}{2} X\psi$. With the aid of ψ we define now a Sasakian structure on M^5. The real 1-form η let be given by $\eta(X) := -i\langle X\psi, \psi\rangle$. The vector field ξ we define by the equation $\xi\psi = i\psi$. This is correct, since in a local section $s_1,\dots,s_5$ of $Q(\psi)$ the equation is written as $\xi u(1,1) = i\, u(1,1)$ and admits therefore a unique solution $(\xi = s_5)$.

Lemma 9: ξ is a Killing vector field of length $|\xi| = 1$.

Proof: ξ is a Killing vector field if and only if the equation $0 = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi)$ holds for all vector fields Y,Z. We will prove $\langle (g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi))\psi, \psi\rangle = 0$. Since $g(\nabla_Y \xi, Z) = -\frac{1}{2}((\nabla_Y \xi)Z + Z\nabla_Y \xi)$ and $g(Y, \nabla_Z \xi) = -\frac{1}{2}(Y\nabla_Z \xi + (\nabla_Z \xi)Y)$, it suffices to show

$$0 = \langle ((\nabla_Y \xi)Z + Y\nabla_Z \xi)\psi, \psi\rangle \tag{4.3}$$

From the differential equation for ψ it follows that

$$\frac{i}{2} Y\psi = i\nabla_Y\psi = \nabla_Y(\xi\psi) = (\nabla_Y \xi)\psi + \xi\nabla_Y\psi + \frac{i}{2}\xi Y\psi$$

and, analogously,

$$\frac{i}{2} Z\psi = (\nabla_Z \xi)\psi + \frac{1}{2}\xi Z\psi .$$

Consequently we obtain

$$\frac{i}{2}\langle Y\psi, Z\psi\rangle = \langle(\nabla_Y \xi)\psi, Z\psi\rangle + \langle\frac{1}{2}\xi Y\psi, Z\psi\rangle$$

$$-\frac{i}{2}\langle Y\psi, Z\psi\rangle = \langle Y\psi, (\nabla_Z \xi)\psi\rangle + \langle Y\psi, \frac{1}{2}\xi Z\psi\rangle$$

$$= \langle Y\psi, (\nabla_Z \xi)\psi\rangle - \langle\frac{1}{2}\xi Y\psi, Z\psi\rangle .$$

Addition yields the equation (4.3).

Remark 3: Lemma 9 means $g(\varphi(X), Y) + g(X, \varphi(Y)) = 0$.

Lemma 10: (φ, ξ, η, g) is a Sasakian structure on M^5.

Proof: We have to show a) - d) of Corollary 1.

ad a) It holds
$$\begin{aligned} 2\eta(X) &= -i\langle X\psi, \psi\rangle + i\langle\psi, X\psi\rangle \\ &= \langle X\psi, \xi\psi\rangle + \langle\xi\psi, X\psi\rangle \\ &= 2g(X,\xi)\,|\psi|^2 = 2g(X,\xi). \end{aligned}$$

ad b) Differentiating the equation $\xi\psi = i\psi$ relative to X one obtains $(\nabla_X\xi)\psi + \xi\nabla_X\psi = i\nabla_X\psi$. Using the differential equation for ψ and the definition of φ we see that this is equivalent to

$$-\varphi(X)\psi + \frac{1}{2}\xi X\psi = \frac{i}{2} X\psi \tag{4.4}$$

In particular, we have

$$-\varphi^2(X)\psi + \frac{1}{2}\xi\varphi(X)\psi = \frac{i}{2}\varphi(X)\psi .$$

Hence, $\varphi^2(X)\psi + \frac{i}{2}\xi X\psi + \frac{1}{2} X\psi = 0$.

If X, ξ are orthogonal, it follows

$$\varphi^2(X)\psi - \frac{i}{2} X(i\psi) + \frac{1}{2} X\psi = 0,$$

i.e. $\varphi^2(X) = -X$.

In case $X = \xi$, we obtain

$$\varphi^2(X)\psi - \frac{i}{2}\psi + \frac{i}{2}\psi = 0,$$

hence $\varphi^2(X) = 0$.

On the other hand, we know

$$(-\mathrm{Id} + \eta \otimes \xi)(X) = \begin{cases} -X & , \text{ if } X \perp \xi \\ 0 & , \text{ if } X \parallel \xi . \end{cases}$$

This proves b).

Remark: In case ξ and X are orthogonal, (4.4) implies $\varphi(X)\psi = -iX\psi$.

c) follows directly from the construction.

ad d) Differentiating equation (4.4) with respect to Y one obtains

$$(\nabla_Y \varphi)(X)\psi = (-\varphi(\nabla_Y X) - \tfrac{1}{2}\varphi(X)Y - \tfrac{i}{2}\nabla_Y X - \tfrac{i}{4}XY - \tfrac{1}{2}\varphi(Y)X - \tfrac{1}{2}\xi\nabla_Y X + \tfrac{1}{4}\xi XY)\psi .$$

Thus

$$\begin{aligned}(\nabla_Y \varphi)(X)\psi &= \tfrac{1}{2}(-\varphi(X)Y - \varphi(Y)X + \tfrac{1}{2}(-iXY + \xi XY))\psi \\ &= \tfrac{1}{2}(Y\varphi(X) + 2g(\varphi(X),Y) + X\varphi(Y) + 2g(X,\varphi Y) \\ &\qquad + \tfrac{1}{2}(-iXY + \xi XY))\psi \\ &= \tfrac{1}{2}(Y(-\tfrac{i}{2}X + \tfrac{1}{2}\xi X) + X(-\tfrac{i}{2}Y + \tfrac{1}{2}\xi Y) \\ &\qquad + \tfrac{1}{2}(-iXY + \xi XY))\psi \\ &= \tfrac{1}{2}(ig(X,Y) - g(X,\xi)Y + \tfrac{1}{2}(Y\xi X - iXY))\psi \\ &= \tfrac{1}{2}(g(X,Y)\xi - g(X,\xi)Y + Y(-g(X,\xi) - \tfrac{1}{2}X\xi) - \tfrac{i}{2}XY)\psi \\ &= (g(X,Y)\xi - g(X,\xi)Y)\psi ,\end{aligned}$$

which yields $(\nabla_Y \varphi)(X) = g(X,Y)\xi - \eta(X)Y$ for all vector fields X,Y.

Together with Theorem 1 we obtain

Theorem 4 ([41]): Let (M^5,g) be a 5-dimensional Einstein manifold of scalar curvature $R = 20$ with a Killing spinor $\psi \neq 0$. Then M^5 is an Einstein-Sasaki manifold.
Conversely, any 5-dimensional simply connected Einstein-Sasaki manifold (M^5,g) with spin structure admits a Killing spinor.

As a corollary one obtains Theorem 3.

We now make an additional regularity assumption. We suppose that the constructed Sasakian structure is regular, i.e. that all integral curves of ξ are closed and have the same length L. Then the transformation group $\{\varphi_t\}_{0 \le t \le 2\pi}$ of the vector field $\xi' := \frac{L}{2\pi}\xi$ induces a free S^1-action on M^5 , where $S^1 = \{e^{it}, 0 \le t \le 2\pi\}$. The orbit space X^4 is a 4-dimensional smooth manifold. Thus, the projection $\pi: M^5 \to X^4$ is a principal S^1-bundle. We identify the Lie algebra $\underline{S}^1$ of S^1 with $i\mathbb{R}$. Then the exponential map is given by $\exp(it) = e^{it}$. Obviously, the 1-form $\eta' := \frac{2\pi i}{L}\eta$ is a connection in $\pi: M^5 \to X^4$. Its curvature form is $D\eta' = d\eta' + \frac{1}{2}[\eta',\eta'] = \frac{2\pi i}{L} d\eta$

We study now geometrical and analytical properties of the orbit space and the topological type of the fibration. Because of $\mathcal{L}_\xi\, g = 0$ we can project g onto X^4 and obtain a Riemannian metric $\bar{g}$. In $\underline{S}^1$ we choose the unique inner product k such that $k(i,i) = \frac{L^2}{4\pi^2}$. Then $g = \pi^* \bar{g} + k\eta'$ holds and we can make use of the O'Neill formulas [16].

<u>Lemma 11:</u> The orbit space $(X^4,\bar{g})$ is an Einstein manifold of scalar curvature $\bar{R} = \frac{6}{5} R = 24$.

The endomorphism field φ maps the horizontal bundle T^hM^5 onto itself, where $\varphi^2\big|_{T^hM^5} = -\,\mathrm{Id}$. Furthermore, we consider the 2-form Ω on M^5 defined by $\Omega(X,Y) := g(X,\varphi Y) = \frac{1}{2}\, d\eta$. Because of $\mathcal{L}_\xi \varphi = 0$ and $\mathcal{L}_\xi\, d\eta = 0$, φ and Ω define an almost complex structure $\bar{J}$ and a 2-form $\bar{\Omega}$ on X^4, respectively. Obviously, $\bar{\Omega} = \bar{g}(\,.\,,\bar{J}\,)$ holds, i.e. $\bar{\Omega}$ is the Kähler form of $\bar{J}$.

<u>Lemma 12:</u> $(X^4,\bar{J},\bar{g})$ is a Kähler manifold.

<u>Proof:</u> $\bar{\Omega}$ is closed since $\Omega = \frac{1}{2}\, d\eta$ is closed. Furthermore, the Nijenhuis tensor $[\bar{J},\bar{J}]$ of $\bar{J}$ vanishes because of the integrability condition $[\varphi,\varphi] + d\eta \otimes \xi = 0$ for φ.

Now we can make use of the classification of 4-dimensional compact Kähler-Einstein manifolds of positive scalar curvature ([11], [17], see Section 4.2). X^4 has to be analytically isomorphic to $S^2 \times S^2$, $\mathbb{C}P^2$ or to one of the del Pezzo surfaces P_k $(3 \le k \le 8)$. P_k is the surface obtained by blowing up k points in general position in CP^2. Next we study the topological type of the S^1-fibration $(M^5, \pi, X^4; S^1)$.

<u>Lemma 13:</u> Let $c_1(M^5 \to X^4)$ and $c_1(X^4)$ denote the first Chern class of $(M^5, \pi, X^4; S^1)$ and the first Chern class of X^4, respectively.
Then the relations

(i) $c_1(X^4) = \frac{3L}{2\pi}\; c_1(M^5 \longrightarrow X^4)$

(ii) $c_1(X^4) = A \cdot c_1(M^5 \longrightarrow X^4)$ for a certain integer A hold.

Proof: We have the connection η' in $(M^5, \pi, X^4; S^1)$ with the curvature form $d\eta' = \frac{2\pi i}{L} d\eta$. Hence, $c_1(M^5 \longrightarrow X^4)$ is given by

$$c_1(M^5 \longrightarrow X^4) = \left[-\frac{1}{2\pi i} d\eta' \right] = \left[-\frac{1}{L} d\eta \right].$$

On the other hand, since X^4 is an Einstein-Kähler manifold of scalar curvature 24, its Chern class is given by the Ricci form $\bar{\Omega}_{Ric} = Ric(.,\bar{J}\,) = 6\bar{\Omega}$:

$$c_1(X^4) = \left[-\frac{1}{2\pi} \bar{\Omega}_{Ric} \right] = \left[-\frac{3}{2\pi} d\eta \right],$$

which proves (i).

Furthermore, we have an isomorphism $\pi^* T_{\mathbb{C}}X^* = T^hM^5 = Q(\psi) \times_{SU(2)} \mathbb{C}^2$ of 2-dimensional complex vector bundles. This isomorphism yields $\pi^* c_1(X^4) = 0$ because the first chern class of any SU(2)-bundle vanishes. Assertion (ii) follows now from the exact Thom-Gysin sequence of $(M^5, \pi, X^4; S^1)$:

$$\ldots\; H^0(X^4;\mathbb{Z}) \xrightarrow{c_1(M^5 \longrightarrow X^4)} H^2(X^4;\mathbb{Z}) \xrightarrow{\pi^*} H^2(M^5;\mathbb{Z}) \longrightarrow \;\ldots$$

Lemma 14: $H^1(M^5;\mathbb{Z}) = H^4(X^4;\mathbb{Z})/c_1(M^5 \longrightarrow X^4) \cup H^2(X^4;\mathbb{Z})$

The fundamental group of M^5 is cyclic.

Proof: Since $\pi_1(X^4) = 0$ [71], $H^3(X^4;\mathbb{Z}) = H_1(X^4;\mathbb{Z}) = 0$ follows from the Poincaré duality. The exact Thom-Gysin sequence

$$\ldots \rightarrow H^2(X^4;\mathbb{Z}) \xrightarrow{c_1(M^5 \longrightarrow X^4)} H^4(X^4;\mathbb{Z}) \xrightarrow{\pi^*} H^4(H^5;\mathbb{Z}) \longrightarrow$$
$$\longrightarrow H^3(X^4;\mathbb{Z}) = 0 \longrightarrow \;\ldots$$

yields $H^4(M^5;\mathbb{Z}) = H^4(X^4;\mathbb{Z})/c_1(M^5 \longrightarrow X^4) \cup H^2(X^4;\mathbb{Z})$.

Using the Poincaré duality we obtain the first assertion. The second one follows from the exact homotopy sequence of the S^1-fibration and $\pi_1(X^4) = 0$.

We want to classify all possible Einstein spaces now.

First case: $X^4 = \mathbb{C}P^2$. If X^4 is analytically isomorphic to $\mathbb{C}P^2$ and admits a Kähler-Einstein metric, then X^4 is isometric to $\mathbb{C}P^2$ with the Fubini metric [80].

The cohomology algebra $H^*(\mathbb{C}P^2)$ is isomorphic to $\mathbb{Z}[\alpha]/\alpha^3$ and the first Chern class is given by $c_1(\mathbb{C}P^2) = 3\alpha$, $\alpha \in H^2(\mathbb{C}P^2;\mathbb{Z})$.

Using Lemma 12 we see that there are two possibilities for the first Chern class of the S^1-fibration:

$c_1(M^5 \longrightarrow \mathbb{C}P^2) = \alpha$ or $c_1(M^5 \longrightarrow \mathbb{C}P^2) = 3\alpha$.

In the first case we have $\pi_1(M^5) = 0$, $L = 2\pi$, and in the second one $\pi_1(H^5) = \mathbb{Z}_3$, $L = \frac{2\pi}{3}$ (see Lemma 13). Since we know the curvature tensor of $\mathbb{C}P^2$ as well as the curvature form $d\eta' = \frac{4\pi i}{L}\Omega$ of the Riemannian submersion $\pi : M^5 \rightarrow X^4$, we can apply the O'Neill formulas again and conclude that M^5 is conformally flat. Consequently, M^5 is isometric to S^5 in case $c_1(M^5 \rightarrow X^4) = \alpha$, and isometric to $S^5/\mathbb{Z}_3$ in case $c_1(M^5 \rightarrow X^4) = 3\alpha$. The group of analytical isometries of $\mathbb{C}P^2$ acts transitively on $\mathbb{C}P^2$. Each of these isometries can be lifted to an isometry of M^5. Hence, $M^5 = S^5/\mathbb{Z}_3$ is the homogeneous space of curvature one and the fundamental group $\pi_1(M^5) = \mathbb{Z}_3$.

Second case: Suppose that the orbit space X^4 is analytically isomorphic to $S^2 \times S^2$. We will show that X^4 is isometric to the product of 2-spheres. We use a result due to L. Berard Bergery, stating that any compact 4-dimensional Einstein manifold whose isometry group is at least 4-dimensional is either symmetric or isometric to $\mathbb{C}P^2 \# \mathbb{C}P^2$ with the Page metric (s. [8]).

The Lie algebra of the isometry group is the Lie algebra $\underline{i}$ of Killing vector fields. Since X^4 is a Kähler-Einstein manifold of positive scalar curvature, the Lie algebra $\underline{h}$ of all holomorphic vector fields on X^4 is the complexification of the Lie algebra of all Killing vector fields ([73]). As $S^2 \times S^2$ is analytically isomorphic to the Klein quadric Q_2 in $\mathbb{C}P^3$, we conclude that the dimension of the isometry group of X^4 equals the dimension of the isometry group of Q_2 in the standard metric, which is $SO(4)$. Since $S^2 \times S^2$ is not homeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$, X^4 is a symmetric space. The de Rham decomposition yields now that X^4 is isometric to the product of 2-spheres with radius $\frac{1}{\sqrt{6}}$. The cohomology algebra of $S^2 \times S^2$ is $\Lambda(a,b)$, i.e. a commutative algebra with generators a,b and relations $a^2 = b^2 = 0$. The first Chern class of $S^2 \times S^2$ is $c_1(S^2 \times S^2) = 2a + 2b$. Using Lemma 13 again we see that there are two possibilities for $c_1(M^5 \rightarrow S^2 \times S^2)$, namely $a + b$ or $2a+2b$. In the first case we obtain $\pi_1(M^5) = 0$, $L = 4\pi/3$ and in the second one $\pi_1(M^5) = \mathbb{Z}_2$, $L = 2\pi/3$.

On the other hand, we have the following two examples of S^1-fibrations over $S^2 \times S^2$. First consider the Stiefel manifold $V_{4,2}$. $V_{4,2}$ admits an Einstein metric of scalar curvature 20 (see [32]).

The calculation in [32] shows that nontrivial Killing spinors $\psi_+, \psi_- (\nabla_X \psi_\pm = \pm \frac{1}{2} X\cdot\psi)$ defining regular Sasakian structures exist on this space. The integral curves of the characteristic vector field are the usual fibres of $\pi: V_{4,2} \to S^2 \times S^2 = G_{4,2}$, where $G_{4,2}$ denotes the Graßmann manifold. The length of the fibres in this metric is $\frac{4\pi}{3}$.

The second example we get from the first one. $V_{4,2}$ is the manifold of all oriented orthonormal 2-frames (v_1, v_2) of $\mathbb{R}^4$. Consider the fixed-point-free involution

$$I: V_{4,2} \longrightarrow V_{4,2}$$
$$(v_1, v_2) \longmapsto (-v_1, -v_2),$$

which maps all fibres onto itself. I is an orientation preserving isometry. $V_{4,2}/I = V_{4,2}/\mathbb{Z}_2$ admits two spin structures (see [32]), which are defined by the two possible lifts of I into the spin bundle P of $V_{4,2}$. Relative to one of these lifts, ψ_1 and ψ_2 are invariant. Hence, they define Killing spinors with a regular Sasakian structure on $V_{4,2}/\mathbb{Z}_2$.

Hence, the S^1-bundle $M^5 \longrightarrow X^4$ is isomorphic to $V_{4,2} \to S^2 \times S^2$ in case $\pi_1(M^5) = 0$, and isomorphic to $V_{4,2}/\mathbb{Z}_2 \to S^2 \times S^2$ in case $\pi_1(M^5) = \mathbb{Z}_2$. Furthermore, the metric on $S^2 \times S^2$ and the length of the fibres define uniquely the metric of M^5.

Consequently, M^5 is isometric to $V_{4,2}$ or to $V_{4,2}/\mathbb{Z}_2$ in the mentioned metric.

<u>Third case: $X^4 = P_k$.</u> Now let X^4 be one of the del Pezzo surfaces with $3 \le k \le 8$. The cohomology algebra $H^*(P_k)$ is generated by the elements α, $E_1, \ldots, E_k \in H^2(P_k)$ with the relations $\alpha^3 = 0, \alpha E_i = 0$, $E_i^2 = -1$. The first Chern class of P_k is $3\alpha + E_1 + \ldots + E_k$ (s. [11]). Consequently, there is only one possibilitiy for the Chern class of the fibration $M^5 \longrightarrow P_k$, namely $c_1(M^5 \to P_k) = 3\alpha + E_1 + \ldots + E_k$. From $H^4(P_k;\mathbb{Z}) = \mathbb{Z}$ and $E_i^2 = -1$ it follows that $H_1(M^5;\mathbb{Z}) = H^4(P_k;\mathbb{Z})/(3\alpha + E_1 + \ldots + E_k) \cup H^2(P_k;\mathbb{Z}) = 0$. Thus, M^5 is diffeomorphic to a simply connected principal S^1-bundle over one of the del Pezzo surfaces P_k $(3 \le k \le 8)$. On the other hand, on P_k there exists a family of Kähler-Einstein metrics with positive scalar curvature. Consequently, by Theorem 1 we obtain a family of Einstein metrics with a Killing spinor on each of these S^1-bundles.

There is a result due to S. Smale on the structure of 5-manifolds, (s. [98]) which states that a simply connected closed spin manifold of dimension 5 whose second homology group $H_2(M^5;\mathbb{Z})$ is free and abelian is diffeomorphic to $S^5 \# \underbrace{(S^2 \times S^3) \#\ldots\# (S^2 \times S^3)}_{k\ \times}$.

Therefore, we obtain a one-to-one-correspondence between metrics with Killing spinors defining a regular Sasakian structure on the k-fold connected sum $(S^2 \times S^3) \#\ldots\# (S^2 \times S^3)$ and Kähler-Einstein metrics on P_k.
Summing up, we have proved the following

Theorem 5 ([41]): Let (M^5,g) be an Einstein space with a Killing spinor ψ and the scalar curvature $R = 20$. Suppose in addition that the associated Sasakian structure is regular. Then there are three possibilities:
(1) M^5 is isometric to S^5 or $S^5/\mathbb{Z}_3$ with the homogeneous metric of constant curvature.
(2) M^5 is isometric to the Stiefel manifold $V_{4,2}$ or to $V_{4,2}/\mathbb{Z}_2$ with the Einstein metric considered in [32], [67].
(3) M^5 is diffeomorphic to the simply connected S^1-bundle with the Chern class $c_1(M^5 \longrightarrow P_k) = c_1(P_k)$ over a del Pezzo surface P_k $(3 \leq k \leq 8)$.

4.4. Compact 7-dimensional Riemannian Manifolds with Killing Spinors

The complex Spin(7)-representation Δ_7 is the complexification of a real representation, since the real Clifford algebra Cliff(7) is isomorphic to $M_R(8) \oplus M_R(8)$. In all calculations we use the realization of that real spin representation which we obtain from

$$
\begin{aligned}
e_1 &= E_{18} + E_{27} - E_{36} - E_{45}\\
e_2 &= -E_{17} + E_{28} + E_{35} - E_{46}\\
e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}\\
e_4 &= -E_{15} - E_{26} - E_{37} - E_{48}\\
e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}\\
e_6 &= E_{14} - E_{23} - E_{58} + E_{67}\\
e_7 &= E_{12} - E_{34} - E_{56} + E_{78}\ ,
\end{aligned}
$$

where E_{ij} is the standard basis of the Lie algebra $\underline{so}(8)$:

$$E_{ij} = \begin{array}{c} \begin{array}{ccc} i & \qquad & j \end{array} \\ \begin{pmatrix} \vartheta & \cdots & -1 \\ \vdots & & \vdots \\ 1 & \cdots\ \vartheta\ \cdots & \cdots \end{pmatrix} \end{array} \begin{array}{c} \\ i \\ \\ j \end{array}$$

We denote this real representation also by Δ_7.
Let $u_1,\dots,u_8$ be the standard basis of $\Delta_7 \approx \mathbb{R}^8$. Spin(7) acts transitively on the Stiefel manifolds

$$V_k(\Delta_7) = \{(v_1,\dots,v_k) : v_i \in \Delta_7, \langle v_i, v_j\rangle = \delta_{ij}\}.$$

We consider now the isotropy groups $H^0(u_1,\dots,u_k) \subset \mathrm{Spin}(7)$:

$$H^0(u_1,\dots,u_k) = \{g \in \mathrm{Spin}(7) : gu_\alpha = u_\alpha, 1 \le \alpha \le k\}$$

and their Lie algebras $\underline{h}(u_1,\dots,u_k)$. It is well-known (see [26]) that $\underline{h}(u_1)$, $\underline{h}(u_1,u_2)$, $\underline{h}(u_1,u_2,u_3)$ are isomorphic to the Lie algebras $\underline{g}_2$, $\underline{su}(3)$, $\underline{su}(2)$, respectively.

More precisely, we have

<u>Lemma 15:</u> $$\underline{h}(u_1) = \Big\{ \sum_{i<j} \omega_{ij}\, e_i e_j : \omega_{12} + \omega_{34} + \omega_{56} = 0,$$
$$\begin{aligned} -\omega_{13} + \omega_{24} - \omega_{67} &= 0, & -\omega_{14} - \omega_{23} - \omega_{57} &= 0, \\ -\omega_{16} - \omega_{25} + \omega_{37} &= 0, & \omega_{15} - \omega_{26} - \omega_{47} &= 0, \\ \omega_{17} + \omega_{36} + \omega_{45} &= 0, & \omega_{27} + \omega_{35} - \omega_{46} &= 0 \Big\} \end{aligned}$$

$$\underline{h}(u_1,u_2) = \Big\{ \sum_{i<j} \omega_{ij}\, e_i e_j : \omega_{12} + \omega_{34} + \omega_{56} = 0,$$
$$\omega_{13} = \omega_{24},\ \omega_{14} + \omega_{23} = 0,\ \omega_{15} = \omega_{26},$$
$$\omega_{16} + \omega_{25} = 0,\ \omega_{35} = \omega_{46},\ \omega_{36} + \omega_{45} = 0,$$
$$\omega_{i7} = 0 \quad (i=1,\dots,7) \Big\}$$

$$\underline{h}(u_1,u_2,u_3) = \Big\{ \sum_{i<j} \omega_{ij}\, e_i e_j : \omega_{13} = \omega_{24},\ \omega_{14} + \omega_{23} = 0,$$
$$\omega_{12} + \omega_{34} = 0,\ \omega_{i5} = \omega_{i6} = \omega_{i7} = 0 \ (1 \le i \le 7) \Big\}.$$

<u>Lemma 16:</u> The Spin(7)-actions on S^7, $V_2(\Delta_7)$ and $V_3(\Delta_7)$ are transitive. The isotropy groups $H^0(u_1)$, $H^0(u_1,u_2)$, $H^0(u_1,u_2,u_3)$ are isomorphic to G_2, $SU(3)$ and $SU(2)$, respectively.

<u>Proof:</u> The first assertion follows from

$\dim V_1(\Delta_7) = 7 = \dim Spin(7) - \dim H(u_1)$
$\dim V_2(\Delta_7) = 13 = \dim Spin(7) - \dim H(u_1,u_2)$
$\dim V_3(\Delta_7) = 18 = \dim Spin(7) - \dim H(u_1,u_2,u_3)$.

The exact homotopy sequence of the fibration $V_k(\Delta_7) = Spin(7)/H^0(u_1,\dots,u_k)$ (k=1,2,3) implies the second assertion because of $\pi_1(V_k(\Delta_7)) = \pi_2(V_k(\Delta_7)) = 0$ (k=1,2,3).

Lemma 17: For any orthogonal elements $\psi_1 \neq 0$ and $\psi_2 \neq 0$ of $\Delta_7 = \mathbb{R}^8$ there exists a unique vector $\xi \in \mathbb{R}^8$ such that $\xi\psi_1 = \psi_2$.

Proof: For any vector $X \in \mathbb{R}^8$ it holds that $(X\psi_1,\psi_1) = 0$. Therefore we have a linear map from $\mathbb{R}^7$ into the orthogonal complement of ψ_1 defined by

$$\begin{aligned} \mathbb{R}^7 &\longrightarrow \psi_1^{\perp} \\ X &\longmapsto X\psi_1. \end{aligned}$$

Since $\dim \mathbb{R}^7 = 7 = \dim \psi_1^{\perp}$, this is an isomorphism.

Consider now a 7-dimensional Riemannian spin manifold with a real spinor bundle S. As a corollary of Lemma 16 we obtain:

Lemma 18: If $\psi_1,\dots,\psi_k$ (k=1,2,3) are orthonormal sections in S, then we can define topological reductions of the spin structure Q and the frame bundle P by

$$Q(\psi_1,\dots,\psi_k) = \bigcup_{m \in M^7} Q_m(\psi_1,\dots,\psi_k)$$
$$Q_m(\psi_1,\dots,\psi_k) = \{q \in Q_m : \psi_i(m) = [q,u_i],\ i=1,\dots,k\}$$

and $P(\psi_1,\dots,\psi_k) = f(Q(\psi_1,\dots,\psi_k))$.

Further, we consider only sections in the real spinor bundle. The spaces of all real Killing spinors in the complex bundle are the complexification of the corresponding spaces in the real spinor bundle.

We now give estimates for the maximal number of independent Killing spinors.

Let (M^7,g) be a compact Einstein spin manifold with scalar curvature R and spinor bundle S. We denote by $m_{\pm}$ the dimension of the spaces of Killing spinors:

$$m_{\pm} = \dim\{\psi \in \Gamma(S) : \nabla_X\psi = \pm \tfrac{1}{2}\sqrt{\tfrac{R}{42}}\, X\psi\}.$$

Theorem 6 (see [89] or [58]): If $m_+ > 0$ and $m_- > 0$, then M^7 is isometric to the sphere S^7.

Proof: Given two Killing spinors ψ_+, ψ_- satisfying $\nabla_X \psi_\pm = \pm \frac{1}{2}\sqrt{\frac{R}{42}}\, X\psi_\pm$, we consider the function $f = (\psi_+, \psi_-)$. An obvious calculation yields $\Delta f = \frac{R}{6} f$. In case $f \not\equiv 0$, M^7 must be isometric to the sphere S^7 by Obata's Theorem (see [9]). In case $f \equiv 0$, i.e. ψ_+ and ψ_- are orthogonal, we consider the 1-form defined by

$$\omega(X) = (X\psi_+, \psi_-).$$

Because of $\dim\{\eta \in S : (\eta, \psi_-) = 0\} = 7 = \dim TM^7$ the 1-form ω vanishes nowhere. On the other hand, ω is a parallel form:

$$\begin{aligned}
(\nabla_X \omega)(Y) &= X(Y\psi_+, \psi_-) - ((\nabla_X Y)\psi_+, \psi_-) \\
&= (Y\nabla_X \psi_+, \psi_-) + (Y\psi_+, \nabla_X \psi_-) \\
&= \frac{1}{2}\sqrt{\frac{R}{42}}\{(XY\psi_+, \psi_-) + (YX\psi_+, \psi_-)\} \\
&= -\sqrt{\frac{R}{42}}\, g(X,Y)(\psi_+, \psi_-) = -\sqrt{\frac{R}{42}}\, g(X,Y)\cdot f \\
&= 0,
\end{aligned}$$

since we have $f \equiv 0$ in the situation considered. Now the Weitzenböck formula for 1-forms yields $\mathrm{Ric}(\omega) = \frac{R}{7}\omega = 0$. This is a contradiction.

Theorem 7 ([43],[44]): Let (M^7, g) be a compact connected Riemannian spin manifold. If $m_+ > 3$ or $m_- > 3$, then M^7 is a space of constant sectional curvature.

Proof: Let ψ_1, ψ_2, ψ_3 be three orthonormal Killing spinors with the same Killing number and $s_1, \dots, s_7$ a local section in $P(\psi_1, \psi_2, \psi_3)$. For the Weyl tensor

$$W : \Lambda^2(TM^7) \longrightarrow \Lambda^2(TM^7)$$

$$W(s_i \wedge s_j) = \frac{1}{2}\sum_{k,l} W_{ijkl}\, s_k \wedge s_l$$

we obtain from

$$W(\eta)\psi_1 = W(\eta)\psi_2 = W(\eta)\psi_3 = 0 \quad \text{for all } \eta \in \Lambda^2 TM^7$$

the relations

$$W_{ij13} = W_{ij24} \quad W_{ij14} + W_{ij23} = 0 \quad W_{ij12} + W_{ij34} = 0$$

$$W_{ijk5} = W_{ijk6} = W_{ijk7} = 0 \quad 1 \leq i,j,k \leq 7$$

(see Lemma 15)

We consider again the real Spin(7)-representation $\Delta_7 \approx \mathbb{R}^8$. The Lie algebra $\underline{h}(u_1,u_2,u_3)$ of the isotropy group $H^0(u_1,u_2,u_3)$ is a subgroup of $\underline{so}(\Delta_7) = \underline{so}(8)$ and given by

$$\underline{h}(u_1,u_2,u_3) = \Big\{(a_{ij})_{(1\leq i,j\leq 8)} \; a_{ij} = 0 \quad (1\leq i\leq 4,\ 1\leq j\leq 7)$$
$$a_{56} = a_{78} \quad a_{57} = -a_{68}$$
$$a_{58} = a_{67} \Big\}.$$

This follows immediately from Lemma 15. Hence $H^0(u_1,u_2,u_3)$ acts trivially on u_4 and coincides with the usual SU(2)-action on span $\{u_5,u_6,u_7,u_8\} = \mathbb{R}^4 = \mathbb{C}^2$.

Suppose now that M^7 admits four orthonormal Killing spinors $\psi_1,\dots,\psi_4$. In the reduction $P(\psi_1,\psi_2,\psi_3)$ of the frame bundle we may choose a section $s_1,\dots,s_7$ such that $\psi_4 = \lambda_4 u_4 + \lambda_5 u_5$ holds. If $\lambda_5 \neq 0$, then, because of the above mentioned relations for the Weyl tensor,

$$\begin{aligned} W(X,Y)\psi_4 = \tfrac{1}{2}\ \lambda_5((W_{XY12} - W_{XY34})u_6 \\ + (-W_{XY13} - W_{XY24})u_7 \\ + (W_{XY14} - W_{XY23})u_8) \\ = 0 \end{aligned}$$

implies $W = 0$. Thus we still have to prove that λ_5 does not vanish on an open set $U \subset M^7$. Assume that $\psi_4 = u_4$ on $U \subset M^7$. Without loss of generality let the scalar curvature be $R = 42$. We denote by ω_{ij} the connection forms of the Levi-Civita connection relative to $s_1,\dots,s_7$. The equations

$$\nabla_{s_1} \psi_i = \tfrac{1}{2}\, s_1 \psi_i \qquad (1\leq i\leq 4)$$

provide the conditions

$$\sum_{i<j} \omega_{ij}(s_1) e_i e_j u_1 = e_1 u_1 = u_8$$
$$\sum_{i<j} \omega_{ij}(s_1) e_i e_j u_2 = e_1 u_2 = u_7$$
$$\sum_{i<j} \omega_{ij}(s_1) e_i e_j u_3 = e_1 u_3 = -u_6$$
$$\sum_{i<j} \omega_{ij}(s_1) e_i e_j u_4 = e_1 u_4 = -u_5.$$

Thus, we get

$$\omega_{27}(s_1) + \omega_{35}(s_1) - \omega_{46}(s_1) = 1$$
$$\omega_{27}(s_1) - \omega_{35}(s_1) + \omega_{46}(s_1) = 1$$
$$\omega_{27}(s_1) - \omega_{35}(s_1) - \omega_{46}(s_1) = -1$$
$$\omega_{27}(s_1) + \omega_{35}(s_1) + \omega_{46}(s_1) = -1,$$

which is a contradiction.

In dimension 7 we can also prove the converse of Theorem 1. Let (M^7,g) be a compact Einstein manifold of scalar curvature $R = 42$ with spin structure. Furthermore, let ψ_1 and ψ_2 be two orthonormal Killing spinors with the Killing number $\lambda = \frac{1}{2}$. We define a vector field ξ by the equation $\xi\psi_1 = \psi_2$. This is correct, see Lemma 17. If $s_1,\dots,s_7$ is a local section in $P(\psi_1,\psi_2)$, then $\xi = s_7$ holds. Moreover, we introduce the 1-form $\eta(X) := (X\psi_1,\psi_2)$ as well as the (1,1)-tensor $\varphi := -\nabla\xi$.

<u>Lemma 19:</u> (φ,ξ,η,g) is a Sasakian structure on M^7.

<u>Proof:</u> First we prove that ξ is a Killing vector field, i.e.

$$g(\nabla_Y\xi,X) + g(Y,\nabla_X\xi) = 0.$$

We differentiate the equation $\xi\psi_1 = \psi_2$ with respect to Y and multiply it by $X\psi_1$:

$$((\nabla_Y\xi)\psi_1,X\psi_1) + \tfrac{1}{2}(\xi Y\psi_1,X\psi_1) = \tfrac{1}{2}(Y\psi_2,X\psi_1).$$

Analogously we have

$$((\nabla_X\xi)\psi_1,Y\psi_1) + \tfrac{1}{2}(\xi X\psi_1,Y\psi_1) = \tfrac{1}{2}(X\psi_2,Y\psi_1).$$

Adding both equations and taking into account $(X\psi,Y\psi) = g(X,Y)|\psi|^2$ we obtain

$$\begin{aligned} g(\nabla_Y\xi,X)+g(Y,\nabla_X\xi) &= \tfrac{1}{2}((Y\psi_2,X\psi_1) + (X\psi_2,Y\psi_1)) \\ &\quad - \tfrac{1}{2}((\xi Y\psi_1,X\psi_1)+(\xi X\psi_1,Y\psi_1)) \\ &= g(X,Y)(\psi_1,\psi_2) \\ &\quad + \tfrac{1}{2}((Y\psi_2,X\psi_1) + (X\psi_2,Y\psi_1)) + \\ &\quad g(Y,\xi)(\psi_1,X\psi_1) + g(X,\xi)(\psi_1,Y\psi_1) \\ &= 0. \end{aligned}$$

Moreover, we have $\eta(X) = (X\psi_1,\psi_2) = (X\psi_1,\xi\psi_1) = g(X,\xi)$. By our definition, $\varphi = -\nabla\xi$ holds. Now we verify the condition $\varphi^2 = -\mathrm{Id} + \eta\otimes\xi$. Suppose that X is orthogonal to ξ - the remaining case is obvious. From $(\nabla_X \xi)\psi_1 + \xi\nabla_X\psi_1 = \nabla_X\psi_2$ it follows that $\varphi(X)\psi_1 = \xi X\psi_1$ and, therefore,

$$-\varphi^2(X)\psi_1 = -\xi\varphi(X)\psi_1 = -\xi\xi X\psi_1 = X\psi_1, \text{ i.e.}$$
$$\varphi^2(X) = -X.$$

It remains to prove $(\nabla_X\varphi)(Y) = g(X,Y)\xi - \eta(Y)X$. We use the equation $2\varphi(X)\psi_1 = (\xi X - X\xi)\psi_1$, which we obtain by covariant differentiation of $\xi\psi_1 = \psi_2$.

We start with
$$\nabla_X(\varphi(Y)\psi_1) = \nabla_X(\varphi(Y))\psi_1 + \varphi(Y)\nabla_X\psi_1 = \nabla_X(\varphi(Y))\psi_1 + \tfrac{1}{2}\varphi(Y)X\psi_1$$
and
$$\nabla_X(\varphi(Y)\psi_1) = \nabla_X(-\tfrac{1}{2}Y\psi_2 + \tfrac{1}{2}\xi Y\psi_1) =$$
$$= -\tfrac{1}{2}(\nabla_X Y)\psi_2 - \tfrac{1}{4}YX\psi_2 - \tfrac{1}{2}\varphi(X)Y\psi_1 + $$
$$+ \tfrac{1}{2}\xi(\nabla_X Y)\psi_1 + \tfrac{1}{4}\xi YX\psi_1.$$

Now we obtain

$$\begin{aligned}
(\nabla_X\varphi)(Y)\psi_1 &= (\nabla_X\varphi(Y))\psi_1 - \varphi(\nabla_X Y)\psi_1 = \\
&= -\tfrac{1}{2}(\nabla_X Y)\psi_2 - \tfrac{1}{4}YX\psi_2 - \tfrac{1}{2}\varphi(X)Y\psi_1 + \tfrac{1}{2}\xi(\nabla_X Y)\psi_1 \\
&\quad + \tfrac{1}{4}\xi YX\psi_1 - \tfrac{1}{2}\varphi(Y)X\psi_1 - \varphi(\nabla_X Y)\psi_1 \\
&= -\tfrac{1}{4}YX\psi_2 - \tfrac{1}{2}\varphi(X)Y\psi_1 + \tfrac{1}{4}\xi YX\psi_1 - \tfrac{1}{2}\varphi(Y)X\psi_1 \\
&= -\tfrac{1}{4}YX\psi_2 + \tfrac{1}{2}(Y\varphi(X) + X\varphi(Y))\psi_1 - \tfrac{1}{4}Y\xi X\psi_1 - \tfrac{1}{2}\eta(Y)X\psi_1 \\
&= \tfrac{1}{2}(Y\varphi(X) + X\varphi(Y))\psi_1 + \tfrac{1}{2}\eta(X)Y\psi_1 - \tfrac{1}{2}\eta(Y)X\psi_1 \\
&= \tfrac{1}{4}(Y\xi X - YX\xi + X\xi Y - XY\xi)\psi_1 + \tfrac{1}{2}\eta(X)Y\psi_1 - \tfrac{1}{2}\eta(Y)X\psi_1 \\
&= (g(X,Y)\xi - \eta(Y)X)\psi_1.
\end{aligned}$$

Thus calculation provides $(\nabla_X\varphi)(Y) = g(X,Y)\xi - \eta(Y)X$.
Theorem 1 and Lemma 19 yield the following

<u>Theorem 8</u> ([44]): Let M^7 be a simply connected 7-dimensional spin manifold. Then there is a one-to-one correspondence between pairs of Killing spinors and Einstein-Sasakian structures on M^7.

Suppose now that the constructed Sasakian structure (φ, ξ, η, g) on M^7 is regular, i.e. all integral curves of ξ are closed and have the same length L. We use again the method described in case of dimension 5 (Lemmas 11, 12, 13) and obtain

<u>Theorem 9</u> ([43],[44]): Let (M^7,g) be a compact Einstein manifold of scalar curvature $R = 42$ with two Killing spinors such that the induced Sasakian structure is regular. Then M^7 is a principal S^1-

bundle over a Kähler-Einstein manifold of scalar curvature $\bar{R} = 48$. For the first Chern class $c_1(X^6)$ of X^6 and the first Chern class $c_1(M^7 \to X^6)$ of the S^1-bundle the relations

(i) $c_1(X^6) = \frac{2L}{\pi} c_1(M^7 \to X^6)$

(ii) $c_1(X^6) = A \cdot c_1(M^7 \to X^6)$ for a certain $A \in \mathbb{Z}$

hold.

Analogously to Lemma 14 one verifies

<u>Lemma 20:</u> $H_1(M^7;\mathbb{Z}) = H^6(X^6;Z)/c_1(M^7 \to X^6) \cup H^4(X^6;\mathbb{Z})$.

From the exact homotopy sequence of the fibration $S^1 \to M^7 \to X^6$ it follows that $\pi_1(M^7)$ is a cyclic group. Because of Myers' Theorem $\pi_1(M^7)$ is finite.

Our next aim is the classification of 7-dimensional manifolds with three independent Killing spinors.

Let (M^7,g) be a compact 7-dimensional Einstein manifold of scalar curvature $R = 42$ admitting three orthonormal real Killing spinors ψ_1, ψ_2, ψ_3 with the Killing number $\frac{1}{2}$. Solving the equations

$$X_1\psi_1 = \psi_2, \qquad X_2\psi_1 = \psi_3, \qquad X_3\psi_2 = \psi_3$$

we obtain three orthogonal Killing vector fields of length one, for instance $g(X_1,X_2) = (X_1\psi_1, X_2\psi_1) = (\psi_2,\psi_3) = 0$.

Defining $\eta_i = g(X_i,.)$, $\varphi_i = -\nabla X_i$ we obtain three Sasakian structures $(\varphi_i, X_i, \eta_i, g)$, $i=1,2,3$.

<u>Lemma 21:</u> $(\varphi_i, X_i, \eta_i, g)$ $(i=1,2,3)$ constitute a Sasakian 3-structure.

<u>Proof:</u> We differentiate the equation defining X_1 relative to X_2:

$$(\nabla_{X_2}X_1)\psi_1 + \frac{1}{2} X_1X_2\psi_1 = \frac{1}{2} X_2\psi_2,$$

i.e. $(\nabla_{X_2}X_1)\psi_1 = - X_1\psi_3$.

On the other hand, X_1 and $\nabla_{X_2}X_1$ are orthogonal to each other, since X_1 is a Killing vector field of length one. Consequently, we have

$$(\nabla_{X_2}X_1)\psi_2 = (\nabla_{X_2}X_1)X_1\psi_1 = -X_1(\nabla_{X_2}X_1)\psi_1 = -\psi_3.$$

Hence, $\nabla_{X_2}X_1 = -X_3$. The assertion follows now from Lemma 6.

Conversely, given a simply connected Riemannian spin manifold (M^7,g) with a Sasakian 3-structure $(\varphi_i, \xi_i, \eta_i, g)$, i=1,2,3. Then (M^7,g) is automatically an Einstein manifold and we can apply the method used in Section 4.2.
We consider the bundles

$$E_i^{\pm} = \{\psi \in S: (\pm \varphi_i(X) + \xi_i X - X\xi_i)\psi = 0\} \quad (i=1,2).$$

Let M^7 be not isometric to the sphere S^7. E_1^+ as well as E_2^+, or E_1^- as well as E_2^- have the dimension 2. Assume, for example, the first case. Theorem 7 implies $E_1^+ \cap E_2^+ \neq 0$ since each of the bundles E_1^+ and E_2^+ yields two Killing spinors. Therefore, we may choose a spinor $\psi \neq 0$ in $E_1^+ \cap E_2^+$. Then, by the definition of E_i^+ the spinors $\xi_1\psi$ and $\xi_2\psi$ are elements of E_1^+ and E_2^+, respectively. These spinors are orthogonal to each other, since $(\xi_1\psi, \xi_2\psi) = g(\xi_1, \xi_2)|\psi|^2 = 0$. Hence $E_1^+ \neq E_2^+$ and, consequently, M^7 admits three independent Killing spinors with the Killing number $\frac{1}{2}$. On the other hand, if M^7 is isometric to the sphere, then there exist four independent Killing spinors for each of the values $\frac{1}{2}$, $-\frac{1}{2}$.
Finally, we obtain:

Theorem 10 ([43],[44]): Let (M^7,g) be a compact Riemannian spin manifold with three independent Killing spinors. Then (M^7,g) admits a Sasakian 3-structure. Conversely, every simply connected spin manifold with Sasakian 3-structure admits at least three independent Killing spinors.

We consider now again the Killing vector fields X_1, X_2, X_3 constructed by means of three Killing spinors with

$$[X_1,X_2] = 2X_3,\ [X_2,X_3] = 2X_1,\ [X_3,X_1] = 2X_2.$$

By the Frobenius theorem these vector fields define a foliation of M^7. The leaves F_α^3 are totally geodesic (s. Section 4.1.) and have constant sectional curvature $K = 1$, i.e. they are isometric to S^3/Γ_α.
We consider the induced Spin(3)-action on M^7.

Now we want to classify all 7-dimensional compact Riemannian manifolds with three Killing spinors under a certain regularity assumption on this action. We suppose that M^7 is a simply connected compact spin manifold admitting three Killing spinors with the Killing number $\frac{1}{2}$ such that $M^7/\mathrm{Spin}(3) =: X^4$ is a smooth closed manifold. In this case M^7 is an S^1-fibration over the twistor space Z^+ of X^4. Indeed, let $p: M^7 \to X^4$ be the projection and identify the tangent space $T_{p(m)}X^4$ with the orthogonal complement of $\mathrm{span}(X_1, X_2, X_3)$ in $T_m M^7$. We define the projection $\pi: M^7 \to Z^+$ by the formula

$$\pi(m) = dp\, \varphi_1(m) dp^{-1}.$$

Then the kernel of the differential $d\pi: T_m M^7 \to T_{\pi(m)} Z^+$ is generated by X_1 and, consequently, $\pi: M^7 \to Z^+$ is an S^1-fibration. This projection coincides with the corresponding projection for two Killing spinors ψ_1, ψ_2. Theorem 9 provides now

<u>Lemma 22:</u> $\pi: M^7 \to Z^+$ is an S^1-fibration and Z^+ is a compact Kähler-Einstein manifold of scalar curvature $\bar{R} = 48$. The canonical complex structure of Z^+ is given by φ_1.

Since the only Kählerian twistor spaces are $\mathbb{C}P^3$ and the flag manifold $F(1,2)$ (see [45],[63], see also Section 3), Z^+ is analytically equivalent to one of these spaces. Moreover, on $\mathbb{C}P^3$ and $F(1,2)$ there exists only one Kähler-Einstein structure (see [69]) and, consequently, Z^+ is analytically isometric to $\mathbb{C}P^3$ or $F(1,2)$. By q we denote the fibration $q: Z^+ \to X^4$.
Now we carefully investigate the action of Spin(3) on M^7. For a given point $m \in M^7$ we denote by

$$H(m) = \{\gamma \in \mathrm{Spin}(3) : \gamma m = m\}$$

the isotropy group of this point.

<u>Lemma 23:</u> For any point $m \in M^7$ the isotropy group $H(m)$ is trivial or isomorphic to $\mathbb{Z}_2$.

<u>Proof:</u> We consider the orbit $M_x = p^{-1}(x)$ relative to the Spin(3)-action of $x = p(m)$. For the map

$$\begin{aligned} \mathrm{Spin}(3) &\longrightarrow M_x \\ \gamma &\longmapsto \gamma m \end{aligned}$$

$\gamma m = \gamma' m$ holds if and only if there is an $h_m \in H(m)$ such that $\gamma' = \gamma h m$. Thus, we have $M_x = \mathrm{Spin}(3)/_{H(m)}$, where $H(m)$ acts on Spin(3) from the right.
Let H be the subgroup of Spin(3) generated by X_1. The action of H on $\mathrm{Spin}(3)/_{H(m)}$ is given by

$$\begin{aligned} H \times S^3/_{H(m)} &\longrightarrow S^3/_{H(m)} \\ (h, [\gamma]) &\longmapsto [h\gamma] \end{aligned} \quad .$$

Denote by S^2_x the sphere $S^2_x := q^{-1}(x)$. Then the diagramme

$$\begin{tikzcd}
\mathrm{Spin}(3) \arrow[r] \arrow[d] & \mathrm{Spin}(3)/_{H(m)} = M_x \arrow[r, "\pi"] & S^2_x = {}_H\backslash(S^3/_{H(m)}) \arrow[d, phantom, "\wr"] \\
{}_H\backslash\mathrm{Spin}(3) \arrow[rr, "\bar{\pi}"] & & ({}_H\backslash\mathrm{Spin}(3))\ /\ H(m)
\end{tikzcd}$$

is commutative. Since π is a submersion and $S^3 \longrightarrow S^3/_{H(m)}$ is a covering, $\bar{\pi}$ is a submersion, too, and therefore a covering.

Since $(H\backslash\mathrm{Spin}(3))/_{H(m)} = S^2_x = S^2$, we deduce that $\bar{\pi}$ is one-to-one, i.e. $H(m)$ acts trivially on ${}_H\backslash\mathrm{Spin}(3)$. Identifying $\mathrm{Spin}(3) \approx SU(2)$ we may assume that

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} ,\ z \in S^1 \right\}$$

holds. Then, the action of H on Spin(3) is given by

$$\begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z\alpha & z\beta \\ \frac{\gamma}{z} & \frac{\delta}{z} \end{pmatrix}$$

Hence, the projection $SU(2) \approx \mathrm{Spin}(3) \longrightarrow {}_H\backslash\mathrm{Spin}(3) = S^2$ maps $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ onto $\frac{\alpha}{\beta} \in S^2 \cong \mathbb{C} \cup \{\infty\}$ and the action of $H(m) \subset SU(2)$ on ${}_H\backslash\mathrm{Spin}(3) = S^2$ is given by

$$\begin{aligned} S^2 \times H(m) &\longrightarrow S^2 \\ \left(\omega, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) &\longmapsto \frac{\omega A + C}{\omega B + D} \end{aligned} .$$

Since $H(m)$ acts trivially on S^2, we obtain $H(m) = \{e\}$ or $\mathbb{Z}_2$.

Lemma 24: The orbit type of the Spin(3)-action on M^7 is constant, i.e. there are two possible cases: either $H(m) = \{e\}$ or $H(m) = \mathbb{Z}_2$ for all points $m \in M^7$.

Proof: Consider $\gamma = (-1) \in \mathrm{Spin}(3)$ and the corresponding isometric involution $\gamma = (-1)\colon M^7 \longrightarrow M^7$. The fixed point set of γ is the union of closed totally geodesic submanifolds N_α. The manifolds N_α are Spin(3) invariant and $SO(3) = \mathrm{Spin}(3)/_{\{\pm 1\}}$ acts freely on it. Since γ preserves the orientation, the dimension of N_α is odd. Hence, $\dim N_\alpha = 3$, $\dim N_\alpha = 5$ or $N_\alpha = M^7$.
Next we show that the case $\dim N_\alpha = 5$ is impossible. We assume that the fixed point set of γ has a component N_α of dimension 5 and consider the images of N_α in X^4 and Z^+. Then

$\Sigma^2 := p(N_\alpha) \subset X^4$ is a surface and $\pi(N_\alpha) \subset Z^+$ is a complex submanifold of the twistor space Z^+. In fact, the tangent space of N_α contains all vectors Y invariant under the differential $d\gamma$ of γ. Let Y be in TN_α. Then $d\gamma\, \varphi_1(Y) = -d\gamma \nabla_Y X_1 = -\nabla_{d\gamma(Y)} d\gamma(X_1)$, since γ is an isometry. On the other hand, Y and X_1 are invariant under $d\gamma$ and therefore $d\gamma\, \varphi_1(Y) = -\nabla_Y X_1 = \varphi_1(Y)$ holds, i.e. TN_α is φ_1-invariant. Now we regard the twistor projection $q: Z^+ \to X^4$. Then we have $q^{-1}(\Sigma^2) = \pi(N_\alpha)$. Since $\pi(N_\alpha)$ is a complex submanifold, the latter equation means, by definition of the twistor space, that $T_x\Sigma^2$ is invariant under all algebraic complex structures of T_xX^4, a contradiction.

Finally, we prove that the fixed point set of γ cannot contain a component of dimension 3. Assume $\dim N_\alpha = 3$ and take a tubular neighbourhood $U = \mathrm{Spin}(3) \times_{\mathbb{Z}_2} D^4$. The $\mathbb{Z}_2$-action on Spin(3) is given by γ. On D^4, $(-1) \in \mathbb{Z}_2$ is an involution with the unique fixed point $0 \in D^4$. U is homotopy equivalent to N_α, therefore $\pi_1(U) = \pi_1(N_\alpha) = \pi_1(\mathrm{Spin}(3)/_{\{\pm 1\}}) = \mathbb{Z}_2$. On the other hand, $U \setminus N_\alpha$ is a principal Spin(3)-bundle over $p(U \setminus N_\alpha)$. $U \setminus N_\alpha$ is diffeomorphic to $\mathrm{Spin}(3) \times (\mathbb{R}^4 \setminus \{0\})$ for small U. Since $\pi_1(U \cap (M^7 \setminus N_\alpha)) = \pi_1(U \setminus N_\alpha) = 0$, the van Kampen Theorem implies now that $\pi_1(M^7)$ is the free product of $\pi_1(U)$ and $\pi_1(M^7 \setminus N_\alpha)$. Because of $\pi_1(U) = \mathbb{Z}_2$ and $\pi_1(M^7) = 0$ this is a contradiction.

Consequently, the fixed point set of γ is $\emptyset$ or M^7. In the first case we obtain $H(m) = 0$, and in the second one $H(M) = \mathbb{Z}_2$ for all $m \in M^7$.

We explain now the classification of simply-connected Riemannian manifolds with three Killing spinors. First of all we remark that two such spaces are known, namely the 7-dimensional sphere S^7 and the space $SU(3)/S^1_{1,1} = N(1,1)$ described in [26] (see also Section 4.5). We prove that under the regularity assumption on the Spin(3)-action, these are the only possible spaces with three Killing spinors.

<u>Theorem 11</u> (see [43],[44]): Let M^7 be a compact simply connected Riemannian spin manifold of scalar curvature $R = 42$ with three Killing spinors such that $M^7/\mathrm{Spin}(3)$ becomes a smooth closed manifold for the induced Spin(3)-action.
Then M^7 is isometric to the sphere S^7 or to the space $SU(3)/S^1_{1,1} = N(1,1)$.

Proof: We consider the map $\pi : M^7 \to Z^+$. Since the isotropy group $H(m)$ is constant, this is a principal S^1-bundle. On the other hand, Z^+ is a Kähler-Einstein twistor space and, therefore, isometric to the complex projective space $\mathbb{C}P^3$ or to the flag manifold $F(1,2)$ ([45],[63],[69], see also Section 3.). In case $Z^+ = \mathbb{C}P^3$, we have $c_1(\mathbb{C}P^3) = 4\alpha$, where $\alpha \in H^2(\mathbb{C}P^3;\mathbb{Z})$ is the generator of the second cohomology group. Since $\pi_1(M^7) = 0$, the Chern class of the fibration $\pi : M^7 \to \mathbb{C}P^3$ has to be equal to $c_1(M^7 \to \mathbb{C}P^3) = \alpha$ and because of the relation $c_1(\mathbb{C}P^3) = \frac{2L}{\pi} c_1(M^7 \to \mathbb{C}P^3)$ we obtain $L = 2\pi$ for the length L of the circles of this fibration (see Theorem 9). These data determine M^7 up to an isometry and it turns out that M^7 is isometric to the sphere S^7.

We handle the second case similarly. Let Z^+ be analytically isometric to $F(1,2)$. It holds $H^1(F(1,2);\mathbb{Z}) = H^3(F(1,2);\mathbb{Z}) = H^5(F(1,2),\mathbb{Z}) = 0$. The group $H^2(F(1,2);\mathbb{Z})$ has two generators, α, γ. $H^4(F(1,2);\mathbb{Z})$ is generated by α^2 and $\alpha\gamma$, $H^6(F(1,2);\mathbb{Z})$ by $\gamma\alpha^2$. The first Chern class of $F(1,2)$ is $c_1(F(1,2)) = 2\gamma$. Thus, the Chern class of the fibration equals $c_1(M^7 \to F(1,2)) = \gamma$ and the length of the fibres is $L = \pi$. Again, these data describe M^7 uniquely and we obtain $M^7 = SU(3)/S^1_{1,1}$.

Corollary 4: Every compact simply connected 7-dimensional spin manifold with regular Sasakian 3-structure is isometric to S^7 or $SU(3)/S^1_{1,1} = N(1,1)$.

4.5. An Example

Now we investigate in particular metrics with Killing spinors on homogeneous spaces $N(k,l) = SU(3)/S^1_{k,l}$, where the inclusion $S^1_{k,l} \to SU(3)$ is given by

$$\theta \longmapsto \begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{il\theta} & 0 \\ 0 & 0 & e^{-(k+l)\theta} \end{pmatrix}$$

We can assume $k \geq l > 0$. The Lie algebra $\underline{su}(3)$ of $SU(3)$ splits as $\underline{su}(3) = \underline{S}^1 + \underline{m}$, where the Lie algebra of S^1 is given by

$$\underline{S}^1 = \text{span}\left\{ \begin{pmatrix} ki & 0 & 0 \\ 0 & li & 0 \\ 0 & 0 & -(k+l)i \end{pmatrix} \right\} \subset \underline{su}(3)$$

and $\underline{m}$ is the following subset of $\underline{su}(3)$:

$$\underline{m} = \underline{p}_0 + \underline{p}_1 + \underline{p}_2 + \underline{p}_3,$$

where

$$\underline{p}_0 := \mathrm{span}\left\{ L = \begin{pmatrix} (2l+k)i & 0 & 0 \\ 0 & -(2k+l)i & 0 \\ 0 & 0 & (k-l)i \end{pmatrix} \right\}$$

$$\underline{p}_1 := \mathrm{span}\left\{ A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{A}_{12} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\underline{p}_2 := \mathrm{span}\left\{ A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ \tilde{A}_{13} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \right\}$$

$$\underline{p}_3 := \mathrm{span}\left\{ A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \ \tilde{A}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \right\}$$

With the aid of the Killing form $B(X,Y) = \frac{1}{2}\,\mathrm{Re}\big(\mathrm{tr}(XY)\big)$ we define a family g_λ of inner products on $\underline{m}$:

$$g_\lambda(\cdot,\cdot) = \lambda(-B)\big|_{\underline{p}_0} + \frac{1}{x}(-B)\big|_{\underline{p}_1} + \frac{1}{y}(-B)\big|_{\underline{p}_2} + \frac{1}{z}(-B)\big|_{\underline{p}_3}$$

$$(\lambda, x, y, z > 0).$$

Let α denote the number $\alpha = (k^2+l^2+kl)$, furthermore let $s = \frac{1}{\sqrt{\lambda}}$.
We fix the following orthonormal basis of $\underline{m}$:

$$X_1 = \sqrt{x}\,A_{12}, \quad X_2 = \sqrt{x}\,\tilde{A}_{12}, \quad X_3 = \sqrt{y}\,A_{13}, \quad X_4 = \sqrt{y}\,\tilde{A}_{13},$$
$$X_5 = \sqrt{z}\,A_{23}, \quad X_6 = \sqrt{z}\,\tilde{A}_{23}, \quad X_7 = \frac{s}{\sqrt{3\alpha}}\,L.$$

Relative to this basis we identify $\underline{m}$ and $\mathbb{R}^7$.
The isotropy representation $\mathrm{Ad}\colon S^1 \to SO(\underline{m})$ is, with respect to this basis, given by

$\mathrm{Ad}(\theta) =$

$$\begin{pmatrix} \cos(k-l)\theta & -\sin(k-l)\theta & 0 & 0 & 0 & 0 & 0 \\ \sin(k-l)\theta & \cos(k-l)\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(2k+l)\theta & -\sin(2k+l)\theta & 0 & 0 & 0 \\ 0 & 0 & \sin(2k+l)\theta & \cos(2k+l)\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(k+2l)\theta & -\sin(k+2l)\theta & 0 \\ 0 & 0 & 0 & 0 & \sin(k+2l)\theta & \cos(k+2l)\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For $\widetilde{Ad}: S^1 \longrightarrow Spin(\underline{m}) = Spin(7)$ this implies

$$\widetilde{Ad}(\theta) = (\cos \frac{k-l}{2}\theta + \sin \frac{k-l}{2}\theta\, e_1e_2)(\cos \frac{2k+l}{2}\theta + \sin \frac{2k+l}{2}\theta\, e_3e_4)\cdot$$
$$\cdot(\cos \frac{k+2l}{2}\theta + \sin \frac{k+2l}{2}\theta\, e_5e_6).$$

Consequently, relative to the basis $u_1,\dots,u_8$:

$$\widetilde{Ad}(\theta) = \begin{pmatrix} B^2-A^2 & -2AB & 0 & 0 & 0 & 0 & 0 & 0 \\ 2AB & B^2-A^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & AC-BO & BC+AD & 0 & 0 & 0 & 0 \\ 0 & D & AD-BC & AC+BD & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B^2+A^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A^2+B^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & BD-AC & BC+AD \\ 0 & 0 & 0 & 0 & 0 & 0 & -BC-AD & -BD-AC \end{pmatrix}$$

where $A := \sin \frac{2k+l}{2}\theta$, $B := \cos \frac{2k+l}{2}\theta$, $C := \sin \frac{3l}{2}\theta$,
$D := \cos \frac{3l}{2}\theta$.

A section in the spinor bundle $S = SU(3) \times_{\widetilde{Ad}} \Delta_7$ is a map
$\psi : SU(3) \longrightarrow \Delta_7$ which satisfies

$$\psi(g\theta) = \widetilde{Ad}(\theta^{-1})\psi(g) \quad \text{for all } g \in SU(3),\ \theta \in S^1.$$

Thus, u_5 and u_6 and, for $k=l=1$, also u_3 and u_4 as constant maps are sections in S.
The Levi-Civita connection of the homogeneous space is described by the map

$$\Lambda : \underline{m} \longrightarrow so(\underline{m})$$
$$\Lambda(X)(Y) := \frac{1}{2}[X,Y]_{\underline{m}} + U(X,Y),$$

where $[X,Y]_{\underline{m}}$ is the $\underline{m}$-component of $[X,Y]$ and $U: \underline{m} \times \underline{m} \longrightarrow \underline{m}$ is given by

$$2g(U(X,Y),Z) = g(X,[Z,Y]_{\underline{m}}) + g([Z,X]_{\underline{m}},Y)$$

(see [65]).

One calculates

$$\Lambda(X_1) = 2\tilde{c}\, E_{27} - c(E_{35} + E_{46})$$
$$\Lambda(X_2) = -2\tilde{c}E_{17} - c(E_{45} - E_{36})$$
$$\Lambda(X_3) = 2\tilde{d}E_{47} - d(E_{26} - E_{15})$$

$$\Lambda(X_4) = -2\tilde{d}\, E_{37} + d(E_{16} + E_{25})$$
$$\Lambda(X_5) = -2\tilde{e}\, E_{67} - e(E_{13} + E_{24})$$
$$\Lambda(X_6) = 2\tilde{e}\, E_{57} - e(E_{14} - E_{23})$$
$$\Lambda(X_7) = \sqrt{\tfrac{3}{\alpha}}\,(-(1+k)(\tfrac{x}{2s} - s)E_{12} - e(\tfrac{y}{2s} - s)E_{34} + k(\tfrac{z}{2s} - s)E_{56})$$

where $c := \frac{1}{2}\left(\sqrt{\frac{xz}{y}} - \sqrt{\frac{yz}{x}} + \sqrt{\frac{xy}{z}}\right)$, $\tilde{c} := \frac{x}{4s}(k+1)\sqrt{\frac{3}{\alpha}}$,

$d := \frac{1}{2}\left(\sqrt{\frac{xy}{z}} + \sqrt{\frac{yz}{x}} + \sqrt{\frac{xz}{y}}\right)$, $\tilde{d} := \frac{y}{4s}\, 1\, \sqrt{\frac{3}{\alpha}}$,

$e := \frac{1}{2}\left(\sqrt{\frac{xz}{y}} + \sqrt{\frac{yz}{x}} - \sqrt{\frac{xy}{z}}\right)$, $\tilde{e} := -\frac{z}{4s}k\sqrt{\frac{3}{\alpha}}$,

Consequently, the lift $\tilde{\Lambda} : \underline{m} \longrightarrow \underline{\mathrm{spin}}(7)$ of Λ is given by

$$\tilde{\Lambda}(X_1) = \tilde{c}\, e_2e_7 - \tfrac{c}{2}(e_3e_5 + e_4e_6)$$
$$\tilde{\Lambda}(X_2) = -\tilde{c}\, e_1e_7 - \tfrac{c}{2}(e_4e_5 - e_3e_6)$$
$$\tilde{\Lambda}(X_3) = \tilde{d}\, e_4e_7 - \tfrac{d}{2}(e_2e_6 - e_1e_5)$$
$$\tilde{\Lambda}(X_4) = -\tilde{d}\, e_3e_7 + \tfrac{d}{2}(e_1e_6 + e_2e_5)$$
$$\tilde{\Lambda}(X_5) = -\tilde{e}\, e_6e_7 - \tfrac{e}{2}(e_1e_3 + e_2e_4)$$
$$\tilde{\Lambda}(X_6) = \tilde{e}\, e_5e_7 - \tfrac{e}{2}(e_1e_4 - e_2e_3)$$
$$\tilde{\Lambda}(X_7) = \tfrac{1}{2}\sqrt{\tfrac{3}{\alpha}}\,(-(1+h)(\tfrac{x}{2s} - s)e_1e_2 - 1(\tfrac{y}{2s} - s)e_3e_4 + k(\tfrac{z}{2s} - s)e_5e_6).$$

We obtain for the maps $\tilde{\Lambda}(X_i) - \mu X_i$ relative to the basis $u_1, \ldots, u_8$:

$$\tilde{\Lambda}(X_1)-\mu X_1 = (\tilde{c}-\mu)E_{18} + (\tilde{c}-\mu)E_{27} + (\tilde{c}+c+\mu)E_{36} + (\tilde{c}-c+\mu)E_{45}$$
$$\tilde{\Lambda}(X_2)-\mu X_2 = (-\tilde{c}+\mu)E_{17} + (\tilde{c}-\mu)E_{28} + (-\tilde{c}+c-\mu)E_{35} + (\tilde{c}+c+\mu)E_{46}$$
$$\tilde{\Lambda}(X_3)-\mu X_3 = (-\tilde{d}+d+\mu)E_{16} + (\tilde{d}+d-\mu)E_{25} + (\tilde{d}+\mu)E_{38} + (-\tilde{d}-\mu)E_{47}$$
$$\tilde{\Lambda}(X_4)-\mu X_4 = (-\tilde{d}-d+\mu)E_{15} + (-\tilde{d}+d+\mu)E_{26} + (\tilde{d}+\mu)E_{37} + (\tilde{d}+\mu)E_{48}$$
$$\tilde{\Lambda}(X_5)-\mu X_5 = (\tilde{e}+\mu)E_{13} + (\tilde{e}+\mu)E_{24} + (\tilde{e}+e-\mu)E_{57} + (\tilde{e}-e-\mu)E_{68}$$
$$\tilde{\Lambda}(X_6)-\mu X_6 = (-\tilde{e}-\mu)E_{14} + (\tilde{e}+\mu)E_{23} + (-\tilde{e}-e+\mu)E_{58} + (\tilde{e}-e-\mu)E_{67}$$
$$\begin{aligned}\tilde{\Lambda}(X_7)-\mu X_7 = {} & (\tfrac{1}{2}\sqrt{\tfrac{3}{\alpha}}(\frac{-(1+k)x-1y+kz}{2s} + 2\ \ 1s) - \mu)E_{12} \\ & + (\tfrac{1}{2}\sqrt{\tfrac{3}{\alpha}}(\frac{-(1+k)x-1y-kz}{2s} + 2(1+k)s) + \mu)E_{34} \\ & + (\tfrac{1}{2}\sqrt{\tfrac{3}{\alpha}}\ \frac{-(1+k)x+1y+kz}{2s} + \mu)E_{56} \\ & + (\tfrac{1}{2}\sqrt{\tfrac{3}{\alpha}}(\frac{-(1+k)x+1y-kz}{2s} + 2ks) - \mu)E_{78}.\end{aligned}$$

Let us first consider the homogeneous space $N(1,1)$, i.e. $k=l=1$. On $N(1,1)$ we have the metrics g_1 and g_2, which are given by

$x_1 = \frac{1}{2}$, $y_1 = z_1 = 1$, $\lambda_1 = 2$ and $x_2 = \frac{1}{16}$, $y_2 = z_2 = \frac{1}{40}$, $\lambda_2 = 16$, respectively.

For g_1 we obtain

$$\tilde{\Lambda}(X_1) + \frac{1}{2\sqrt{2}} X_1 = \frac{1}{\sqrt{2}} (E_{18} + E_{27})$$
$$\tilde{\Lambda}(X_2) + \frac{1}{2\sqrt{2}} X_2 = \frac{1}{\sqrt{2}} (-E_{17} + E_{28})$$
$$\tilde{\Lambda}(X_3) + \frac{1}{2\sqrt{2}} X_3 = \sqrt{2}\, E_{25}$$
$$\tilde{\Lambda}(X_4) + \frac{1}{2\sqrt{2}} X_4 = -\sqrt{2}\, E_{15}$$
$$\tilde{\Lambda}(X_5) + \frac{1}{2\sqrt{2}} X_5 = \sqrt{2}\, E_{57}$$
$$\tilde{\Lambda}(X_6) + \frac{1}{2\sqrt{2}} X_6 = -\sqrt{2}\, E_{58}$$
$$\tilde{\Lambda}(X_7) + \frac{1}{2\sqrt{2}} X_7 = \frac{1}{\sqrt{2}} (E_{12} + E_{78}).$$

We see that u_3, u_4 and u_6 are in the kernel of each of these operators. Thus, $u_3, u_4, u_6 \in \Gamma(S)$ are Killing spinors. Hence g_1 is an Einstein metric on $N(1,1)$ with three linear independent Killing spinors.

For g_2 it holds

$$\tilde{\Lambda}(X_1) - \frac{3}{40} X_1 = \frac{1}{20} (E_{18} + E_{27}) + \frac{2}{5} E_{36}$$
$$\tilde{\Lambda}(X_2) - \frac{3}{40} X_2 = \frac{1}{20} (-E_{17} + E_{28}) + \frac{2}{5} E_{46}$$
$$\tilde{\Lambda}(X_3) - \frac{3}{40} X_3 = \frac{1}{10} (E_{16} + E_{38} - E_{47})$$
$$\tilde{\Lambda}(X_4) - \frac{3}{40} X_4 = \frac{1}{10} (E_{26} + E_{37} + E_{48})$$
$$\tilde{\Lambda}(X_5) - \frac{3}{40} X_5 = \frac{1}{10} (E_{13} + E_{24} - E_{68})$$
$$\tilde{\Lambda}(X_6) - \frac{3}{40} X_6 = \frac{1}{10} (-E_{14} + E_{23} - E_{67})$$
$$\tilde{\Lambda}(X_7) - \frac{3}{40} X_7 = \frac{1}{20} (E_{12} + E_{78}) + \frac{2}{5} E_{34}.$$

The intersection of all kernels of these operators contains u_5, i.e. $u_5 \in \Gamma(S)$ is a Killing spinor. Thus, g_2 is an Einstein metric with one Killing spinor.

Now we study the case of $k > l > 0$. We need the following fact.

<u>Lemma 25:</u> The system

$$0 = 3(l+k)A(1-B) + l(3B-2) + kB(2B-3) \qquad (4.5)$$

$$0 = (l+k)A(2A-3B) + 3l(B-A) + kB(2B-3A) \qquad (4.6)$$

$A > 0 \quad B > 0$

admits two solutions (A_1,B_1), (A_2,B_2) such that

$$-3(1+k)A_1 + 2l + 3kB_1 > 0$$

and $-3(1+k)A_2 + 2l + 3kB_2 < 0.$

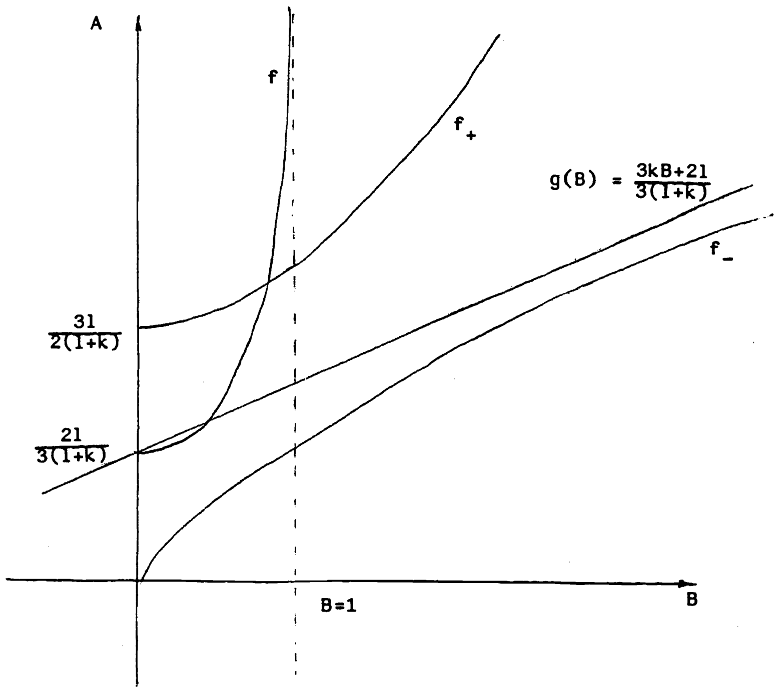
Proof: Discussing the curves of the functions

$$f(B) = \frac{l(3B-2) + kB(2B-3)}{3(1+k)(B-1)}$$

and

$$f_{\pm}(B) = \frac{3Bl + 6Bk + 3l}{4(1+k)} \pm \sqrt{B^2(9l^2+20k^2+20kl)+2B(-3l^2+6kl)+9l^2}$$

we obtain the fôllowing diagramme



Since the equations (4.5) and (4.6) are equivalent to $f(B) = A$ and $f_{\pm}(B) = A$, respectively, the intersections of the curves f_- and f as well as f_+ and f yield the solutions (A_1,B_1) and

(A_2, B_2).

Using these solutions (A_1, B_1), (A_2, B_2) we define positive real numbers y_1 and y_2 by

$$\begin{aligned}\sqrt{\tfrac{\alpha}{3}} &= (-3(1+k)A_1B_1 + 31B_1 + 2kB_1^2)\sqrt{y_1}\\ &= (-3(1+k)A_1+21+3kB_1)\sqrt{y_1}\\ &= (-2(1+k)A_1^2+31A_1+3kA_1B_1)\sqrt{y_1}\end{aligned}$$

and

$$\begin{aligned}\sqrt{\tfrac{\alpha}{3}} &= (3(1+k)A_2B_2 - 31B_2 - 2kB_2^2)\sqrt{y_2}\\ &= (3(1+k)A_2 - 21 - 3kB_2)\sqrt{y_2}\\ &= (2(1+k)A_2^2 - 31A_2 - 3kA_2B_2)\sqrt{y_2},\end{aligned}$$

respectively.

If we futhermore define $x_i = A_iy_i$, $z_i = B_iy_i$ $(i=1,2)$, each of the tuples (x_i, y_i, z_i) is a solution of one of the systems

$$\begin{aligned}\sqrt{\tfrac{\alpha}{3}}\sqrt{xyz} &= \mp 3(1+k)xz \pm 31yz \pm 3kz^2\\ \sqrt{\tfrac{\alpha}{3}}\sqrt{xyz} &= \mp 3(1+k)xy \pm 21y^2 \pm 3kzy\\ \sqrt{\tfrac{\alpha}{3}}\sqrt{xyz} &= \mp 2(1+k)x^2 \pm 31xy \pm 3kxz.\end{aligned}$$

Setting $\lambda = 2$ $(s = \frac{1}{4})$ and $\mu_i = \sqrt{\frac{3}{\alpha}}((1+k)x_i - 1y_i - kz_i)$ one obtains a solution (λ, x_1, y_1, z_1) of

$$\begin{aligned}c &= -\tilde{c} - \mu_1\\ d &= \tilde{d} - \mu_1\\ e &= \tilde{e} - \mu_1\\ \sqrt{\tfrac{\alpha}{3}}&(-(1+k)x + 1y + kz) + \mu_1 = 0\end{aligned}$$

and a solution (λ, x_2, y_2, z_2) of

$$\begin{aligned}c &= \tilde{c} + \mu_2\\ d &= -\tilde{d} + \mu_2\\ e &= -\tilde{e} + \mu_2\\ \sqrt{\tfrac{3}{\alpha}}&(-(1+k)x + 1y + kz) + \mu_2 = 0\,.\end{aligned}$$

For the metric defined by (λ, x_1, y_1, z_1) the spinor u_6 belongs to the kernel of each of the operators $\tilde{\lambda}(X_j) - \mu_1 X_j$ $(j=1,\ldots,7)$, i.e. it is a Killing spinor. For the metric defined by (λ, x_2, y_2, z_2) we calculate analogously that $u_5 \in \Gamma(S)$ is a Killing spinor.

Finally, we have

Theorem 12: For $k \geq l > 0$, $(k,l) = 1$, there exist two Einstein metrics with Killing spinors on $N(k,l) = SU(3)/S^1_{k,l}$. If $k=l=1$, then $N(k,l)$ admits three independent Killing spinors relative to one of these metrics and one Killing spinor relative to the other one. If $l \neq 1$ or $k \neq 1$, then there exists only one Killing spinor for each of these metrics.

4.6. 7-dimensional Riemannian Manifolds with one Real Killing Spinor

The above discussed example proves that there are 7-dimensional manifolds with exactly one Killing spinor. Unfortunately, we can not apply the methods described in Section 4.4 to these manifolds. However, there is an equivalence between metrics with one Killing spinor and certain vector cross products on 7-dimensional spin manifolds.

Let (M^7,g) be a compact 7-dimensional Riemannian spin manifold with a Killing spinor $\psi \neq 0$, i.e.

$$\nabla_X \psi = \lambda X \psi .$$

We define a (2,1)-tensor A by

$$YX\psi = -g(Y,X)\psi + A(Y,X)\psi .$$

This is correct because of Lemma 17.

Lemma 26: The above defined tensor A admits the following properties

1) $A(X,Y) = -A(Y,X)$
2) $g(Y,A(Y,X)) = 0$
3) $A(Y,A(Y,X)) = -|Y|^2 X + g(Y,X)Y$
4) $(\nabla_Z A)(Y,X) = 2\lambda\{g(Y,Z)X - g(X,Z)Y + A(Z,A(Y,X))\}$.

Furthermore, we obtain by polarization

5) $g(Y_1,A(Y_2,X)) + g(Y_2,A(Y_1,X)) = 0$
6) $A(Y_1,A(Y_2,X)) = -A(Y_2,A(Y_1,X)) - 2g(Y_1,Y_2)X + g(Y_1,X)Y_2 + g(Y_2,X)Y_1$.

Proof: 1) is obvious.

From the definition of A it follows

$$A(Y,A(Y,X))\psi = Y\cdot A(Y,X)\psi + g(Y,A(Y,X))\psi$$
$$= Y\cdot Y\cdot X\psi + g(Y,X)Y\psi + g(Y,A(Y,X))\psi$$
$$= -|Y|X\psi + g(Y,X)Y\psi + g(Y,A(Y,X))\psi .$$

This implies $A(Y,A(Y,X)) = -|Y|^2 X + g(Y,X)Y$ and $g(Y,A(Y,X)) = 0$.

In order to verify 4) we differentiate the equation defining A relative to Z:

$$(\nabla_Z Y)X\psi + Y(\nabla_Z X)\psi + \lambda YXZ\psi = -g(\nabla_Z Y,X)\psi - g(Y,\nabla_Z X)\psi$$
$$-g(Y,X)\lambda Z\psi + \nabla_Z(A(Y,X))\psi + A(Y,X)\lambda Z\psi$$
$$= -g(\nabla_Z Y,X)\psi - g(Y,\nabla_Z X)\psi - g(Y,X)\lambda Z\psi$$
$$+(\nabla_Z A)(Y,X)\psi + A(\nabla_Z Y,X)\psi + A(Y,\nabla_Z X)\psi$$
$$+ A(Y,X)\lambda Z\psi .$$

Thus we have

$$-g(Y,X)\lambda Z\psi + (\nabla_Z A)(Y,X)\psi + A(Y,X)\lambda Z\psi =$$
$$= \lambda YXZ\psi = \lambda ZYX\psi + 2g(Y,Z)\lambda X\psi - 2g(X,Z)\lambda Y\psi ,$$

i.e. $\lambda ZA(Y,X)\cdot\psi = -2g(Y,Z)\lambda X\psi + 2g(X,Z)\lambda Y\psi + A(Y,X)\lambda Z\psi$
$$+ (\nabla_Z A)(Y,X)\psi .$$

This implies

$$2A(Z,A(Y,X))\cdot\lambda\psi + 2\lambda g(Y,Z)X\psi - 2g(X,Z)\lambda Y\psi = (\nabla_Z A)(Y,X)\psi ,$$

which is equivalent to 4).

Remark 4: Because of Property 3), A is non-degenerate. Furthermore, we have by 3) and 5)

$$|A(X,Y)|^2 = g(A(X,Y),A(X,Y))$$
$$= |X|^2 |Y|^2 - g(X,Y)^2$$

and by 1) and 2)

$$g(A(X_1,X_2),X_i) = 0 \quad (i=1,2).$$

From 4) it follows in particular

$$(\nabla_{X_i} A)(X_1,X_2) = 0 \quad (i=1,2).$$

Thus A is a nearly parallel vector cross product in the sense of A. Gray (see [51]).

Recall that a Cayley multiplication on a real 8-dimensional Euclidean vector space $(W,\langle,\rangle)$ is a bilinear map $* : W \times W \to W$ such that
a) there exists an element e of W satisfying

$$x * e = e * x = x \quad \text{for all } x \in W.$$

b) $\| x * y\| = \| x\|\cdot\|y\|$, where $\|x\|^2 := \langle x,x\rangle$.

Consider now the bundle $TM^7 \oplus \xi^1$, where ξ^1 is the trivial line bundle as well as the metric h on $TM^7 \oplus \xi^1$ defined by $h((X,s),(Y,t)) = g(X,Y) + st$. Let m be a point of M^7 and set

$$(X,s) * (Y,t) = (A(X,Y) + tX + sY,\ st - g(X,Y))$$

for $X,Y \in T_mM^7$; $s,t \in \mathbb{R}$. Then, on account of the vector cross product properties of A, $*$ is a Cayley multiplication in each fibre of $(TM^7 \oplus \xi^1, h)$ that depends smoothly on $m \in M^7$.

Now we want to prove the converse, namely, if there exists a (2,1)-tensor A with the above mentioned properties 1) - 4) on a simply connected 7-dimensional spin manifold M^7, then M^7 admits a Killing spinor. Let A be such a (2,1)-tensor. First we note that A defines an orientation on M^7 in a canonical way. Indeed, consider the 3-form $\Omega_3(X,Y,Z) = g(X,A(Y,Z))$ and the following 4-form Ξ_4. Let $s_1,\ldots,s_7$ be an orthonormal frame and η_i defined by

$$\eta_i(X,Y) = g(A(X,Y),s_i).$$

Then $\Xi_4 = \sum_{i=1}^{7} \eta_i \wedge \eta_i$ is an invariantly defined 4-form. In order to show that $\Omega_3 \wedge \Xi_4$ does not vanish we choose the following local frame. We fix vector fields s_1,s_2 of length one which are orthogonal; let s_3 be a vector field of length one which is orthogonal to s_1,s_2 and $A(s_1,s_2)$. Furthermore, let $s_4,\ldots,s_7$ be

$$s_4 = A(s_3,A(s_1,s_2)),\ s_5 = A(s_1,s_3),\ s_6 = A(s_3,s_2),\ s_7 = A(s_2,s_1).$$

We obtain an orthonormal frame. One computes

$A(s_1,s_2) = -s_7$	$A(s_2,s_4) = s_5$	$A(s_3,s_7) = -s_4$
$A(s_1,s_3) = s_5$	$A(s_2,s_5) = -s_4$	$A(s_4,s_5) = s_2$
$A(s_1,s_4) = s_6$	$A(s_2,s_6) = s_3$	$A(s_4,s_6) = s_1$
$A(s_1,s_5) = -s_3$	$A(s_2,s_7) = -s_1$	$A(s_4,s_7) = s_3$
$A(s_1,s_6) = -s_4$	$A(s_3,s_4) = s_7$	$A(s_5,s_6) = -s_7$
$A(s_1,s_7) = s_2$	$A(s_3,s_5) = s_1$	$A(s_5,s_7) = s_6$
$A(s_2,s_3) = -s_6$	$A(s_3,s_6) = -s_2$	$A(s_6,s_7) = -s_5$

Now it can be checked easily that $\Omega_3 \wedge \Xi_4$ is a positive multiple of $s_1 \wedge \ldots \wedge s_7$. Hence, it defines an orientation. Consider now the subset

$$\tilde{E} = \{\psi \mid ZX\psi = -g(Z,X)\psi + A(Z,X)\psi\}$$

of the spinor bundle. Calculations in the above constructed special frame show that a spinor $\psi \in S$ belongs to $\tilde{E}$ if $\psi = t \cdot u_6$ $(t \in \mathbb{R})$ relative to $s_1, \ldots, s_7$. Consequently, $\tilde{E}$ is a 1-dimensional subbundle of S. With respect to the other orientation of M^7, $\tilde{E}$ has the dimension 0.

Next we show that we can define by

$$\tilde{\nabla}_X \psi = \nabla_X \psi - \lambda X \psi$$

a covariant derivative in $\tilde{E}$:
Let ψ be a local section in $\tilde{E}$; $X, Y, Z \in TM^7$. Differentiating

$$ZX\psi = -g(Z,X)\psi + A(Z,X)\psi$$

we obtain

$$(\nabla_Y Z)X\psi + Z(\nabla_Y X)\psi + ZX(\nabla_Y \psi) = -Yg(Z,X)\psi - g(Z,X)\nabla_Y \psi + \nabla_Y(A(Z,X)) + A(Z,X)\cdot \nabla_Y \psi,$$

i.e.

$$\begin{aligned}
ZX(\nabla_Y \psi - \lambda Y\psi) &= -Yg(Z,X)\psi - g(Z,X)\nabla_Y \psi + \nabla_Y(A(Z,X))\psi \\
&\quad + A(Z,X)\nabla_Y \psi - (\nabla_Y Z)X\psi - Z(\nabla_Y X)\psi - \lambda ZXY \\
&= -g(\nabla_Y Z, X)\psi - g(Z, \nabla_Y X)\psi - g(Z,X)\nabla_Y \psi \\
&\quad + (\nabla_Y A)(Z,X)\psi + A(\nabla_Y Z, X)\psi + A(Z, \nabla_Y X)\psi \\
&\quad + A(Z,X)\nabla_Y \psi - (\nabla_Y Z)X\psi - Z(\nabla_Y X)\psi \\
&\quad - \lambda ZXY\psi
\end{aligned}$$

since $\psi \in \Gamma(\tilde{E})$, this implies

$$\begin{aligned}
ZX(\nabla_Y \psi - \lambda Y\psi) &= -g(Z,X)\nabla_Y \psi + (\nabla_Y A)(Z,X)\psi + A(Z,X)\nabla_Y \psi - \lambda ZXY\psi \\
&= -g(Z,X)\nabla_Y \psi + (2\lambda g(Z,Y)X - 2\lambda g(X,Y)Z + \\
&\quad + 2\lambda A(Y, A(Z,X)))\psi + A(Z,X)\nabla_Y \psi - \lambda ZXY\psi \\
&= -\lambda YZX\psi - g(Z,X)\nabla_Y \psi + 2\lambda A(Y, A(Z,X))\psi + A(Z,X)\nabla_Y \psi \\
&= \lambda g(Z,X)Y\psi - \lambda YA(Z,X)\psi - g(Z,X)\nabla_Y \psi + 2\lambda \{YA(Z,X)\psi \\
&\quad + g(Y, A(t,X))\psi\} + A(Z,X)\nabla_Y \psi \\
&= -g(Z,X)(\nabla_Y \psi - \lambda Y\psi) + A(Z,X)\nabla_Y \psi + \lambda YA(Z,X)\psi + 2\lambda g(Y, A(Z,X))\psi \\
&= -g(Z,X)(\nabla_Y \psi - \lambda Y\psi) + A(Z,X)(\nabla_Y \psi - \lambda Y\psi).
\end{aligned}$$

Consequently, $\nabla_Y \psi - \lambda Y\psi$ is a section in $\Gamma(\tilde{E})$.
The curvature tensor $\tilde{\mathcal{R}}$ of this covariant derivative is given by $\tilde{\mathcal{R}}(X,Y)\psi = \mathcal{R}(X,Y)\psi + \lambda^2 (XY - YX)\psi$ $(\psi \in \Gamma(\tilde{E}))$, where $\mathcal{R}$ is the curvature tensor with respect to ∇. But

$$\tilde{R}(X,Y)\psi = \frac{1}{2}\sum_{i<j} R(X,Y,s_i,s_j)s_i s_j\psi + \lambda^2(XY-YX)\psi$$

has to be parallel to ψ, and by means of (1.5) and (1.7) we get

$$(\tilde{R}(X,Y)\psi,\psi) = -\frac{1}{2}\sum_{i<j} R(X,Y,s_i,s_j)g(s_i,s_j)|\psi|^2 = 0\ ,\ \text{hence}$$

$\tilde{R}$ vanishes in $\tilde{E}$. Thus, there exists a $\tilde{\nabla}$-parallel section in $\tilde{E}$ and we have proved

Theorem 13: Let (M^7,g) be a simply connected 7-dimensional Riemannian spin manifold and A a (2,1)-tensor on M^7 such that

1) $A(X,Y) = -A(Y,X)$
2) $g(Y,A(Y,X)) = 0$
3) $A(Y,A(Y,X)) = -|Y|^2 X + g(Y,X)Y$
4) $(\nabla_Z A)(Y,X) = 2\lambda\{g(Y,Z)X - g(X,Z)Y + A(Z,A(Y,X))\}$ $\lambda \in \mathbb{R}$, $\lambda > 0$.

Then M^7 admits a Killing spinor.

Chapter 5: Even-dimensional Riemannian Manifolds with Real Killing Spinors

By Theorem 13 and Corollary 4 of Chapter 1 it follows that the only complete, connected Riemannian Spin manifolds of dimension $n = 4$ and admitting real Killing Spinors are the standard spheres. An analogous result for $n = 8$ has been proved by O. Hijazi [12].

However, the conjecture that in any even dimension the classification of real Killing spinors leads to the standard spheres, fails already in dimension $n = 6$.
The first examples of 6-dimensional Riemannian manifolds admitting a real Killing spinor have been obtained by applying the twistor construction to the 4-manifolds S^4 and $\mathbb{C}P^2$. The corresponding twistor spaces are $\mathbb{C}P^3$ and the complex flag manifold $F(1,2)$, respectively, and these manifolds, endowed with a non-standard homogeneous Einstein metric, both have real Killing spinors [40]. Moreover, there is a series of other examples.
The main tool to describe the six-dimensional Riemannian spin manifolds (M^6,g) admitting real Killing spinors is an almost complex structure J which may be defined on M^6 by means of a non-trivial real Killing spinor. Although J turns out to be non-

integrable, it still satisfies $(\nabla_X J)(X) = 0$ for all vector fields X on M^6, hence the manifold is nearly Kähler. This class of manifolds was considered first by J. Koto [77] and studied in detail by A. Gray ([48]-[54]). The main result of this chapter is proved in § 3: A connected, simply connected six-dimensional almost hermitian manifold, which is nearly Kähler non-Kähler, admits a real Killing spinor.

However, the situation in higher even dimensions ($n \geq 10$) is widely unknown. The results for $n = 6$ possibly may be used to clarify the existence of real Killing spinors in the dimension $n = 10$, but also a procedure to obtain a list of examples in an arbitrarily high even dimension (similarly to that of the odd dimensional case) would be useful.

5.1. Real Killing Spinors on Even-dimensional Riemannian Spin Manifolds

Throughout this chapter, let (M^n,g) be a complete, connected Riemannian spin manifold of even dimension $n = 2m$ Suppose that there exists a non-trivial real Killing spinor $\varphi \in \Gamma(S)$ on M, i.e.

$$\nabla_X \varphi = BX \varphi \tag{5.1}$$

holds with a real number $B \in \mathbb{R} \setminus \{0\}$ and for any vector field X on M. According to Theorem 9 of Chapter 1, (M^n,g) is then a compact Einstein space of positive scalar curvature $R = 4B^2 n(n-1)$.

In case of an even-dimensional manifold, the spinor bundle S splits into two orthogonal subbundles $S = S^+ \oplus S^-$ corresponding to the irreducible components of the Spin(2m)-representation. Since the Clifford multiplication with vector fields exchanges the positive and negative part of S, we deduce from (5.1) that, for any Killing spinor $\varphi = \varphi^+ + \varphi^- \in \Gamma(S)$,

$$\begin{aligned} \nabla_X \varphi^+ &= BX \varphi^- \\ \nabla_X \varphi^- &= BX \varphi^+ \end{aligned} \tag{5.2}$$

holds for any vector field X on M. If $\varphi \in \Gamma(S)$ is a Killing spinor with (5.1), φ is also an eigenspinor of the Dirac operator, we have

$$D\varphi = -nB\varphi \tag{5.3}$$

and by (5.2) we derive the equations

$$\begin{aligned} D\varphi^+ &= -nB\varphi^- \\ D\varphi^- &= -nB\varphi^+ \end{aligned} \tag{5.4}$$

For a spinor field $\psi \in \Gamma(S)$, we consider the length function $u_\psi(x) = \langle \psi(x), \psi(x) \rangle$ and denote $|\psi| = \sqrt{u_\psi}$. If $\varphi \in \Gamma(S)$ is a real Killing spinor, it follows from the equations (1.5) and (1.9) of Chapter 1, that $|\varphi|$ is a positive constant function on M.

<u>Lemma 1:</u> Let (M^{2m},g) be not isometric to the standard sphere S^{2m}. If $\varphi = \varphi^+ + \varphi^-$ is a real Killing spinor on M, then $|\varphi^+| = |\varphi^-|$ is constant on M.

<u>Proof:</u> Consider the real functions

$$f^+(x) = \langle \varphi^+(x), \varphi^+(x) \rangle \quad \text{and} \quad f^-(x) = \langle \varphi^-(x), \varphi^-(x) \rangle$$

with $x \in M$ and let $(s_1, \ldots, s_n)$ be a local orthonormal frame on M. Then, from (1.9), we obtain

$$\Delta f^+ = - \sum_{i=1}^{n} \nabla_{s_i} \nabla_{s_i} f^+ - \sum_{i=1}^{n} \operatorname{div}(s_i) \nabla_{s_i} f^+$$

$$= \langle \Delta\varphi^+, \varphi^+ \rangle + \langle \varphi^+, \Delta\varphi^+ \rangle - 2 \sum_{i=1}^{n} \langle \nabla_{s_i} \varphi^+, \nabla_{s_i} \varphi^+ \rangle .$$

Since the Killing number B of φ can be expressed by means of the scalar curvature R of M, we conclude from (5.4) and the Lichnerowicz formula $D^2\varphi = \frac{R}{4}\varphi + \Delta\varphi$ (see § 1.3) that

$$\Delta f^+ = \frac{R}{2(n-1)} \langle \varphi^+, \varphi^+ \rangle - \frac{R}{2n(n-1)} \sum_{i=1}^{n} \langle s_i \varphi^-, s_i \varphi^- \rangle$$

$$= \frac{R}{2(n-1)} (f^+ - f^-).$$

Similar calculations lead to $\Delta f^- = \frac{R}{2(n-1)} (f^- - f^+)$.

Now, the Obata Theorem says that if there exists a positive constant c on an n-dimensional compact Riemannian manifold and a non-zero function f such that $\mathrm{Ric} \geq c \cdot \mathrm{Id}$ and $\Delta f = \frac{n}{n-1} c \cdot f$, then the manifold is isometric to the standard sphere (see [9]). Since in our case the manifold is Einstein, we can apply this theorem with $c = \frac{R}{n}$ and $f = f^+ - f^-$; the supposition $M^{2m} \not\cong S^{2m}$ then implies $f = 0$. With respect to $u_\varphi = f^+ + f^- \equiv$ constant, the assertion follows. □

As a generalization of the above proof we obtain the following result.

<u>Lemma 2:</u> Consider $M^{2m} \not\cong S^{2m}$ as in Lemma 1, let $\varphi_1, \ldots, \varphi_k \in \Gamma(S)$ be real Killing spinors with Killing number B on M and set $\varphi_i = \varphi_i^+ + \varphi_i^-$ for $i = 1, \ldots, k$. Then the functions

$f_{ij} = \langle\varphi_i^+,\varphi_j^+\rangle = \langle\varphi_i^-,\varphi_j^-\rangle$ are constant on M.

Proof: For $1\le i,j\le k$ define the functions $f_{ij}^+ = \langle\varphi_i^+,\varphi_j^+\rangle$ and $f_{ij}^- = \langle\varphi_i^-,\varphi_j^-\rangle$. The same considerations as in the proof of Lemma 1 then yield $f_{ij}^+ = f_{ij}^-$. In order to show that these functions are constant, we take a unitary vector field X on M and derive the equation

$0 = \langle\varphi_i^+,\varphi_j^+\rangle - \langle\varphi_i^-,\varphi_j^-\rangle$ in the direction of X. By (1.9), we obtain after some simplifications

$$\begin{aligned} 0 &= \langle X\varphi_i^-,\varphi_j^+\rangle + \langle\varphi_i^+,X\varphi_j^-\rangle - \langle X\varphi_i^+,\varphi_j^-\rangle - \langle\varphi_i^-,X\varphi_j^+\rangle \\ &= \langle X\varphi_i^-,\varphi_j^+\rangle + \langle\varphi_i^+,X\varphi_j^-\rangle + \langle\varphi_i^+,X\varphi_j^-\rangle + \langle X\varphi_i^-,\varphi_j^+\rangle \\ &= 2\langle X\varphi_i^-,\varphi_j^+\rangle + 2\langle\varphi_i^+,X\varphi_j^-\rangle . \end{aligned}$$

On the other hand, we have

$$X\langle\varphi_i^+,\varphi_j^+\rangle = B\{\langle X\varphi_i^-,\varphi_j^+\rangle + \langle\varphi_i^+,X\varphi_j^-\rangle\} = 0,$$

which completes the proof. □

As another application of the Obata Theorem, we still prove the following result (which is also valid in arbitrary dimension, not only for n even):

Lemma 3: Let (M^n,g) be a complete, connected Riemannian spin manifold which is not isometric to the standard sphere. If $\varphi,\psi\in\Gamma(S)$ are two non-trivial Killing spinors with real Killing numbers B and $(-B)$, respectively, then $\langle\psi,\varphi\rangle = 0$.

Proof: By assumption, we have $\nabla_X\varphi = BX\varphi$ and $\nabla_X\psi = -BX\psi$ for any vector field X on M. We consider the function $f = \langle\psi,\varphi\rangle$ on M, and similar calculations as in Lemma 1 then yield $\Delta f = 4B^2 n\langle\psi,\varphi\rangle$. Since, by Theorem 9 of chapter 1, (M^n,g) is a compact Einstein space of positive scalar curvature $R = 4B^2 n(n-1)$, we obtain $\Delta f = \frac{R}{n-1} f$. By the Obata Theorem and with respect to $M^n \neq S^n$, it follows that $f \equiv 0$ holds. □

Next, for a spin manifold M^{2m} of constant positive scalar curvature $R>0$, we set $\lambda = \frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$ and introduce the following subbundles of S:

$E_+ := \{\psi\in\Gamma(S): \nabla_X\psi + \lambda X\psi = 0$ for all $X\in\Gamma(TM)\}$,
$E_- := \{\psi\in\Gamma(S): \nabla_X\psi - \lambda X\psi = 0$ for all $X\in\Gamma(TM)\}$.

Clearly, if E_+ is non-empty, the manifold has a Killing spinor with Killing number $(-\lambda)$. Since, for any $\varphi \in E_+$, the mapping $\varphi = \varphi^+ + \varphi^- \longmapsto \psi = \varphi^+ - \varphi^-$ provides a bijection between E_+ and E_-, the complex ranks of these bundles are equal. Denote this common value by k.

By Lemma 2, it always holds that $k \leq \dim_{\mathbb{C}}(\Delta_n^+) = 2^{m-1}$, if $M^{2m} \neq S^{2m}$; for the standard spheres S^{2m} we have $k = 2^m$. By this property, the spheres are characterized uniquely (see [32]).

To conclude this section, we mention Hijazi's result concerning 8-dimensional spin manifolds with real Killing spinors [58].

<u>Theorem 1:</u> Let (M^8,g) be an 8-dimensional complete, connected Riemannian spin manifold with a real Killing spinor $\varphi \in \Gamma(S)$. Then (M^8,g) is isometric to the standard sphere S^8.

<u>Proof:</u> Note that, in the dimension $n = 8$, there exists a natural real structure in the spinor bundle S, i.e. an antilinear bundle map $j: S \to S$ with $j^2 = \mathrm{id}$, and having the properties

(i) $\nabla j = 0$,

(ii) $Xj = jX$ for vector fields $X \in \Gamma(TM)$,

(iii) $\langle j\varphi, j\psi\rangle = \langle \varphi, \psi\rangle$ for $\varphi, \psi \in \Gamma(S)$.

(cp. [1]). The corresponding real subbundle $\Sigma := \{\psi \in \Gamma(S): j(\psi) = \psi\}$ also splits into a positive and a negative part under the real Spin(8)-representation, $\Sigma = \Sigma^+ \oplus \Sigma^-$, with the fibre dimension $\dim_{\mathbb{R}} \Sigma^+_x = \dim_{\mathbb{R}} \Sigma^-_x = 8$ at any point $x \in M$. Given a real Killing spinor $\varphi \in \Gamma(S)$ with the Killing number $B \in \mathbb{R} \setminus \{0\}$, it corresponds, via $\varphi_R := \frac{1}{2}(\varphi + j\varphi)$, to an element $\varphi_R \in \Gamma(\Sigma)$ which is also a Killing spinor to the same Killing number. From (iii) and the property that $j(z\varphi) = \bar{z}j(\varphi)$ for any $z \in \mathbb{C}$ one easily sees that $\langle \varphi, j\varphi\rangle = 0$ and, therefore, $\langle \varphi_R, \varphi_R\rangle = \frac{1}{2}\langle \varphi, \varphi\rangle$ is also a non-zero constant. Writing $\varphi_R = \varphi_R^+ + \varphi_R^-$ according to the decomposition $\Sigma = \Sigma^+ \oplus \Sigma^-$, it follows from (5.2) that the relations $\nabla_X \varphi_R^{\pm} = BX\varphi^{\mp}$ hold for any vector field X on M.

Consequently, the same calculations as in the proof of Lemma 1 show that the real function

$$f = \langle \varphi_R^+, \varphi_R^+\rangle - \langle \varphi_R^-, \varphi_R^-\rangle \text{ satisfies } \Delta f = \frac{R}{n-1}\cdot f.$$

Now, let X be a unit vector field on M. A simple calculation involving (1.5) and (1.9) yields

$X(f) = 2B[\langle X\varphi_R^-, \varphi_R^+\rangle + \langle\varphi_R^+, X\varphi_R^-\rangle]$, hence

$X(f) = 2B\ \mathrm{Re}\langle X\varphi_R^-, \varphi_R^+\rangle = 2B\ (X\varphi_R^-, \varphi_R^+)$.

Since the mapping $\mu: T_xM \longrightarrow \Sigma_x^+$ given by $\mu(X) = X\cdot\varphi_R^-$ is injective, and $\dim_{\mathbb{R}} T_xM = \dim_{\mathbb{R}} \Sigma_x^+ = 8$, it follows that there exists at least one unit vector $X \in T_xM$ for which $X(f) \neq 0$. Consequently f does not vanish identically, and from Obata's Theorem (see proof of Lemma 1) the assertion follows. $\square$

5.2. The almost complex structure defined on a 6-dimensional manifold by a real Killing spinor

We now turn to the study of 6-dimensional spin manifolds with real Killing spinors. We start with a brief description of the Clifford multiplication in this dimension.

Since $\mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^6)$ is multiplicatively generated by the vectors $e_1,\dots,e_6 \in \mathbb{R}^6$, the $\mathbb{C}$-algebra isomorphism $\Phi : \mathrm{Cliff}^{\mathbb{C}}(\mathbb{R}^6) \longrightarrow \mathrm{End}(\Delta_6)$ will be completely described by its value on each of these generators. The restriction of Φ to $\mathrm{Spin}(6)$ splits into two irreducible complex 4-dimensional representations, which we denote by (Φ^+, Δ_6^+) and (Φ^-, Δ_6^-). According to § 1 of Chapter 1, we have

$$\left.\begin{aligned} \Delta_6^+ &= \mathrm{Lin}\{u(1,1,1), u(1,-1,-1), u(-1,1,-19, u(-1,-1,1)\} \\ \Delta_6^- &= \mathrm{Lin}\{u(1,1,-1), u(1,-1,1), u(-1,1,1), u(-1,-1,-1)\} \end{aligned}\right\} \quad (5.5)$$

and the mapping $\Phi^\pm(e_j) \in \mathrm{End}(\Delta_6)$ take each component to the other, $\Phi^\pm(e_j): \Delta_6^\pm \longrightarrow \Delta_6^\mp$, $1 \leq j \leq 6$.

Evaluating the general formulas of § 1.1 in this special ordering of the $u(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ as the basis of Δ_6^+, Δ_6^- given above, the $\Phi^\pm(e_j)$ are described by the matrices

$$\left.\begin{aligned}
&\Phi^\pm(e_1) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \qquad
\Phi^\pm(e_2) = \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\Phi^\pm(e_3) = \pm \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad
\Phi^\pm(e_4) = \pm \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&\Phi^\pm(e_5) = \pm \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad
\Phi^\pm(e_6) = \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned}\right\} (5.6)$$

In this way, we have also described the differential of Φ acting

on the Lie algebra $\underline{spin}(6)$, which we denote by Φ, too.
We remark that $\Phi : \mathrm{Cliff}^c(\mathbb{R}^6) \longrightarrow \mathrm{End}(\Delta_6)$ induces a group isomorphism $\mathrm{Spin}(6) \to SU(\Delta_6^+) \cong SU(4)$ (see [93]).

Now, recall the definition of the subbundles $E_+, E_- \subset S$ given at the end of Lemma 3.

Proposition 1: Suppose that M^6 is a six-dimensional complete, connected Riemannian spin manifold which is not isometric to the standard sphere S^6.
Then the complex rank of E_+ and E_- is not greater than 1.

Proof: Suppose that there exist two linearly independent spinor fields φ_1 and φ_2 in E_+ with $\varphi_i = \varphi_i^+ + \varphi_i^-$ (i=1,2) corresponding to the decomposition of S. By Lemma 2, we may assume that $|\varphi_1^-|^2 = |\varphi_2^-|^2 \equiv 1$, and $\langle \varphi_1^-, \varphi_2^- \rangle \equiv 0$ holds on M. For any vector field X on M, it follows by differentiating that

$$\mathrm{Re}\langle X\varphi_1^+, \varphi_1^- \rangle = 0 \tag{5.7}$$

$$\mathrm{Re}\langle X\varphi_2^+, \varphi_2^- \rangle = 0 \tag{5.8}$$

$$\langle \varphi_1^-, X\varphi_2^+ \rangle + \langle X\varphi_1^+, \varphi_2^- \rangle = 0 \tag{5.9}$$

Since SU(4) acts transitively on pairs of vectors $e_1, e_2 \in \mathbb{C}^4$ with $\langle e_i, e_j \rangle = \delta_{ij}$ and since $\mathrm{Spin}(6) \cong SU(4)$, there exists an element $q \in Q_x$ in the fibre of the Spin(6)-principal bundle $Q \to M$ at $x \in M$, such that the spinors $\varphi_1^+(x), \varphi_2^+(x) \in S_x^+ \cong \Delta_6^+$ can be expressed by

$$\varphi_1^+(x) = [q, u(1,1,1)] = (1,0,0,0) \quad \text{and}$$

$$\varphi_2^+(x) = [q, u(1,-1,-1)] = (0,1,0,0).$$

Representing here the spinors as 4-tuples of complex numbers, we have used the ordering of the $u(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ given in (5.5) to identify $\Delta_6^+ \cong \mathbb{C}^4$. With the same element $q \in Q_x$ and the basis of Δ_6^- fixed in (5.5) we can also write

$$\varphi_1^-(x) = [q, \alpha \cdot u(1,1,-1) + \ldots] = (\alpha, \beta, \gamma, \delta) \quad \text{and}$$

$$\varphi_2^-(x) = [q, x \cdot u(1,1,-1) + \ldots] = (x, y, z, w),$$

where $\alpha, \beta, \gamma, \delta, x, y, z, w \in \mathbb{C}$ are complex numbers.
Denote by $s = (s_1, \ldots, s_6)$ the orthonormal frame in T_xM given by $f(q) = s$ and let $(t_1, \ldots, t_6) \in \mathbb{R}^6$ be the components of a vector $t \in T_xM$ with respect to this frame. Then, in the above notation, (5.6) implies that the Clifford multiplication with t is given by

$$t\varphi_1^+(x) = (t_2+it_1, -t_4-it_3, t_6+it_5, 0)$$
$$t\varphi_2^+(x) = (-t_4+it_3, -t_2+it_1, 0, t_6+it_5). \tag{5.10}$$

According to (5.7) we thus obtain

$$\mathrm{Re}\{(t_2+it_1)\bar{\alpha} + (-t_4-t_3)\bar{\beta} + (t_6+it_5)\bar{\gamma}\} = 0,$$

and since $t \in T_xM$ can be varied, we have $\alpha = \beta = \gamma = 0$. Similarly, (5.8) yields $x = y = w = 0$, hence $\varphi_1^-(x) = (0,0,0,\delta)$ and $\varphi_2^-(x) = (0,0,z,0)$.

Now, (5.9) implies $\delta(t_6-it_5) + (t_6+it_5)\bar{z} = 0$ for any $t_5, t_6 \in \mathbb{R}$, so that $\delta = z = 0$ follows. Consequently, $\varphi_1^-(x) = \varphi_2^-(x) = 0$, and since $|\varphi_i^+| = |\varphi_i^-|$, also $\varphi_1 = \varphi_2 = 0$, which is a contradiction. □

Combining (5.7), (5.8) and (5.10) we obtain the following remark.

Corollary 1: On a 6-dimensional spin manifold M, a real Killing spinor $\varphi \in \Gamma(S)$ with the decomposition $\varphi = \varphi^+ + \varphi^-$ satisfies

$$\langle X\varphi^+, \varphi^-\rangle = \langle X\varphi^-, \varphi^+\rangle = 0$$

for all vector fields X on M.

Now, suppose that (M^6, g) is a complete, connected Riemannian spin manifold not isometric to the standard sphere, and that there exists a real Killing spinor $\varphi = \varphi^+ + \varphi^- \in \Gamma(S)$ with the Killing number $\beta \neq 0$.

Then, by Lemma 1, we may assume $|\varphi^+| = |\varphi^-| = 1$, and since $\mathrm{Spin}(6) \cong SU(4)$ acts transitively on $\mathbb{C}^4$, there exists such an element $q(x) \in Q_x$ for any $x \in M$ that $\varphi^+(x)$ is represented by $\varphi^+(x) = [q(x), u(1,1,1)]$.

Moreover, from the local triviality of the spinor bundle $Q \to M$ we conclude that, for a sufficiently small open set $U \subset M$, there also exists a smooth section $q: U \to Q|_U$ such that $\varphi^+(y) = [q(y), u(1,1,1)]$ holds for all $y \in U$.

By (5.10) and Corollary 1 we now obtain that, on $U \subset M$, the spinor φ^- takes the form $\varphi^-(y) = [q(y), z(y)\, u(-1,-1,-1)]$, $y \in U$, with a complex-valued function z on U.

At the point $x \in M$, consider the subspace

$$L_x := \{X \cdot \varphi^+(x);\ X \in T_xM\} \subset S_x^-.$$

Since the mapping $X \mapsto X \cdot \varphi^+(x)$ is injective, L_x forms a 6-dimensional real subspace of $S_x^- \cong \Delta_6^-$; by means of Corollary 1 we obtain an orthogonal decomposition

$S_x^- = \mathbb{C}\cdot\varphi^-(x) \oplus L_x$,

and, as the orthogonal complement of $\mathbb{C}\cdot\varphi^-(x)$, L_x is a complex subspace of S_x^-. Consequently, $iX\cdot\varphi^+(x)\in L_x$ for all $X\in T_xM$, a property that allows to define an almost complex structure J on M^6 by the relation

$$J(X)\cdot\varphi^+ := iX\cdot\varphi^+, \qquad X\in\Gamma(TM) \tag{5.11}$$

From the injectivity of the Clifford multiplication we conclude that J is well defined and satisfies $J^2 = -1$; the equation $X\cdot Y + Y\cdot X = -2g(X,Y)$, which is valid for vector fields X,Y regarded as endomorphisms of S, shows that J acts as an isometry.

<u>Remark 1:</u> Using the local description

$$\varphi^+(y) = [\,q(y),\, u(1,1,1)\,]\ , \quad y\in U\subset M^6$$

and denoting by $(t_1,\dots,t_6)$ the components of a tangent vector field

$$t = \sum_{i=1}^{6} t_i s_i \in\Gamma(TU)$$

with respect to the orthonormal frame $s = (s_1,\dots,s_6)$ determined by $f(q) = s$, we derive from (5.19) that

$$t\cdot\varphi^+ = (t_2+it_1,-t_4-it_3,t_6+it_5,0),\text{ hence}$$

$$it\cdot\varphi^+ = (-t_1+it_2,t_3-it_4,-t_5+it_6,0),$$

both spinor fields regarded as linear combinations of the basic sections $\eta(\varepsilon) = [q,u(\varepsilon)]\in\Gamma(S^-|_U)$.

Then, in the frame $s = (s_1,\dots,s_6)$ the local expression for J takes the form

$$J(t) = (t_2,-t_1,t_4,-t_3,t_6,-t_5) \tag{5.12}$$

hence, $J(t)$ is again a smooth vector field.

<u>Remark 2:</u> Analogously, if we represent $t\cdot\eta(-1,-1,-1) = [q,t\cdot u(-1,-1,-1)]$ as a linear combination of the corresponding basic sections $\eta(\varepsilon) = [q,u(\varepsilon)]\in\Gamma(S^+|_U)$, the formulas for $\Phi^-(e_j)$ yield

$$t\cdot\eta(-1,-1,-1) = (0,-t_6+it_5,-t_4+it_3,-t_2+it_1),$$

hence Remark 1 yields

$$\begin{aligned} J(t)\cdot\eta(-1,-1,-1) &= (0,t_5+it_6,t_3+it_4,t_1+it_2)\\ &= -it\,\eta(-1,-1,-1). \end{aligned}$$

Because of $\varphi^- = z\cdot\eta(-1,-1,-1) = [q, z\cdot u(-1,-1,-1)]$ we conclude

$$J(X)\varphi^- = -iX\varphi^-, \qquad X \in \Gamma(TM) \tag{5.13}$$

Consequently, the decomposition $S^+ = \mathbb{C}\cdot\varphi^+ \oplus L'$, with

$$L'_x = \{X\cdot\varphi^-(x);\ X \in T_xM\} \quad \text{for } x \in M^6,$$

induces the almost complex structure $(-J)$ on M^6.

<u>Lemma 4:</u> Let (M^6,g) be a complete, connected Riemannian spin manifold and $\varphi = \varphi^+ + \varphi^- \in \Gamma(S)$ a real Killing spinor with the Killing number $B \neq 0$. Suppose that (M^6,g) is not isometric to the standard sphere.
Then the almost complex structure defined by (5.11) satisfies

$$(\nabla_X J)(Y)\cdot\varphi^+ = 2iBYX\varphi^- + 2iBg(X,Y)\varphi^- + 2Bg(X,JY)\varphi^-$$

for vector fields X,Y on M.

<u>Proof:</u> By definition, we have

$$(\nabla_X J)(Y)\varphi^+ = [\nabla_X(JY)]\varphi^+ - J(\nabla_X Y)\varphi^+;$$

an application of (5.11) and (1.10) then yields

$$(\nabla_X J)(Y)\varphi^+ = \nabla_X(JY\varphi^+) - JY(\nabla_X\varphi^+) - i(\nabla_X Y)\varphi^+.$$

From (5.2) we now obtain

$$\begin{aligned}(\nabla_X J)(Y)\varphi^+ &= i\nabla_X(Y\varphi^+) - BJY\cdot X\cdot\varphi^- - i(\nabla_X Y)\varphi^+\\ &= i(\nabla_X Y)\varphi^+ + iY(\nabla_X\varphi^+) - BJY\cdot X\cdot\varphi^- - i(\nabla_X Y)\varphi^+\\ &= iBYX\varphi^- - BJY\cdot X\cdot\varphi^-.\end{aligned}$$

Because of (1.3) and (5.13) we conclude

$$\begin{aligned}(\nabla_X J)(Y)\varphi^+ &= iBYX\varphi^- + BX\cdot JY\cdot\varphi^- + 2Bg(JY,X)\varphi^-\\ &= iBYX\varphi^- - iBXY\varphi^- + 2Bg(JY,X)\varphi^-\\ &= 2iBYX\varphi^- + 2iBg(X,Y)\varphi^- + 2Bg(JY,X)\varphi^-. \quad \Box\end{aligned}$$

<u>Corollary 2:</u> For vector fields X,Y on M, we have
(a) $(\nabla_X J)(X) = 0$
(b) $(\nabla_X J)$ does not vanish identically.
(c) $\|(\nabla_X J)Y\|^2 = 4B^2\{\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(X,JY)^2\}$.

<u>Proof:</u> (a) and (b) are easy consequences of (c). To get (c), take the inner product at both sides of the equation in Lemma 4 by itself, and then use (1.7), (1.5) and (1.3) successively. $\Box$

Corollary 3: The almost complex structure defined by (5.11) is non-integrable.

Proof: We show that the Nijenhuis-Tensor of J defined by

$$N(X,Y) = \{[JX,JY]-[X,Y]- J[X,JY]- J[JX,Y]\}$$

does not vanish at any point $x \in M$. We have

$$N(X,Y) = (\nabla_X J)JY - (\nabla_{JY}J)X + (\nabla_{JX}J)Y - (\nabla_Y J)JX$$

by a simple calculation involving the definition of (∇J).

Now, Corollary 2(a) yields $(\nabla_X J)Y + (\nabla_Y J)X = 0$ for vector fields X and Y of M. On the other hand, since $J^2 = -1$, we have $0 = \nabla_X(J^2) = (\nabla_X J)J + J(\nabla_X J)$, which implies $(\nabla_X J)JY = -J(\nabla_X J)Y$.

Then we obtain $(\nabla_X J)JY = -J(\nabla_X J)Y = J(\nabla_Y J)X = -(\nabla_Y J)JX = (\nabla_{JX} J)Y$, hence $N(X,Y) = 4(\nabla_X J)JY$.

Now the assertion follows from Corollary 2(b). $\square$

Remark 3: The almost complex structure under consideration has been defined only for manifolds $M^6 \neq S^6$. However, using the algebraic properties of the Cayley numbers, an almost complex structure J can be defined also for the standard sphere S^6 so that S^6 is, in a canonical way, an almost hermitian manifold. Since the automorphism group G_2 of the Cayley numbers acts transitively on S^6 and leaves J invariant, we get the homogeneous space $S^6 = G_2/SU(3)$.

Although J is non-integrable, the same properties stated for the manifolds $M^6 \neq S^6$ in Corollary 2 are fullfilled in this case, too (cf. [50] and [74, ch. IX, § 2]).

5.3. Nearly Kähler Manifolds in the Dimension n = 6

Let (M^n,g,J) be an almost hermitian manifold with the Riemannian metric g and almost complex structure J, hence we have

$$g(JX,JY) = g(X,Y) \qquad \text{for all } X,Y \in \Gamma(TM) \tag{5.14}$$

Denote by ∇ the Levi-Civita connection of (M^n,g), the covariant derivative of the (1,1)-Tensor J is given by

$$(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y) \qquad X,\ Y \in \Gamma(TM) \tag{5.15}$$

(M^n,g,J) is called Kählerian iff $\nabla J = 0$.

Definition 1: An almost hermitian manifold (M^n, g, J) is said to be nearly Kählerian provided $(\nabla_X J)(X) = 0$ for all vector fields X on M.

Following [54], we introduce two additional notions. If M is nearly Kählerian, then M is called strict if we have $(\nabla_X J) \neq 0$ at any point $x \in M$ and for an arbitrary non-zero vector $X \in T_x M$. Secondly, M is said to be of (global) constant type if there is a real constant α such that for all vector fields X, Y on M

$$\|(\nabla_X J)(Y)\|^2 = \alpha\{\|X\|^2 \|Y\|^2 - g(X,Y)^2 - g(JX,Y)^2\}.$$

In this case, the number α is called the constant type of M.

Thus, in the previous section we have shown (compare Corollary 2 and Remark 3):

Corollary 4: Let (M^6, g) be a 6-dimensional complete, connected Riemannian spin manifold with a real Killing spinor. Then (M^6, g) has a natural almost hermitian structure, which is strictly nearly Kählerian of positive constant type.

First of all, we summarize some of the known identities valid for nearly Kähler manifolds and properties of such manifolds in lower dimensions. As usually, let $\mathfrak{S}$ denote the cyclic sum and define the iterated covariant derivative of the almost complex structure J for vector fields $X, Z \in \Gamma(TM)$ by

$$\nabla^2_{ZX} J = \nabla_Z(\nabla_X J) - \nabla_{(\nabla_Z X)} J \tag{5.16}$$

Lemma 5: Let (M^n, g, J) be a nearly Kähler manifold. Then for vector fields X, Y, Z on M we have

a) $(\nabla_X J)Y + (\nabla_Y J)X = 0$ (5.17)

b) $(\nabla_X J)JX = 0$, hence

$$(\nabla_{JX} J)Y = (\nabla_X J)JY \tag{5.18}$$

c) $J((\nabla_X J)Y) = -(\nabla_X J)JY = -(\nabla_{JX} J)Y$ (5.19)

d) $g((\nabla_X J)Y, Z) = -g((\nabla_X J)Z, Y)$ (5.20)

Proof: The Properties (a) and (b) are direct consequences of (5.15) and Definition 1. Further, since $0 = \nabla_X(J^2) = (\nabla_X J)\circ J + J\circ(\nabla_X J)$, we have $0 = (\nabla_X J)JY + J((\nabla_X J)Y)$, hence (c).
Finally, (d) follows from a more general formula valid for any

almost hermitian manifold (see[74], Chapter IX, § 4): Denote by $\Phi(X,Y) = g(X,JY)$ the fundamental form of J and by N the Nijenhuis Tensor as in Corollary 3, then

$$4g((\nabla_X J)Y,Z) = 6d\Phi(X,JY,JZ) - 6d\Phi(X,Y,Z) + g(N(Y,Z),JX). \quad \square$$

In particular, Lemma 5 implies that, for $X,Y \in T_xM$, the vector $(\nabla_X J)Y$ is always perpendicular to X, JX, Y and JY.

Although the proof of the following lemma is elementary, it requires more expense. The several parts are contained in [49], [52] and [53]. Notice that our sign convention for the curvature tensor is opposite to that of the papers quoted here.

<u>Lemma 6:</u> Let (M^n,g,J) be a nearly Kähler manifold. Then the relations

$$R(U,X,JY,JZ) - R(U,X,Y,Z) = g((\nabla_U J)X,(\nabla_Y J)Z) \tag{5.21}$$

$$2g((\nabla^2_{UX} J)Y,Z) = -\underset{XYZ}{\mathfrak{S}}\, g((\nabla_U J)X,(\nabla_Y J)JZ) \tag{5.22}$$

hold for arbitrary vector fields X,Y,Z and U on M.

In addition to the ordinary Ricci curvature tensor, we consider still another contraction of the curvature tensor on nearly Kähler manifolds. Let $(s_1,\dots,s_n)$ be a local frame field and choose two vector fields X,Y on M. Then, by setting

$$g(\mathrm{Ric}^*(X),Y) = -\frac{1}{2}\sum_{i=1}^{n} R(X,JY,s_i,Js_i), \tag{5.23}$$

a (1,1)-Tensor field Ric^* on M, which is called the Ricci $*$-curvature, is defined.

In the lower dimensions, the nearly Kähler manifolds are widely determined. In case of the dimension $n = 4$, a nearly Kähler manifold is also Kähler, since, for any vector $X \in T_xM^4$ and a vector $Y \in T_xM^4$ with $g(Y,X) = g(JY,X) = 0$, we have that $(\nabla_X J)Y$ is orthogonal to $\mathrm{Lin}\{X,JX,Y,JY\} \cong T_xM$, hence zero. Thus, $(\nabla_X J)Y = 0$ for all vector fields X,Y on M^4, and the manifold is Kähler [49]. For the dimension $n = 6$, A.Gray proved the following proposition ([54]):

<u>Proposition 2:</u> Let (M,g,J) be a 6-dimensional nearly Kähler manifold, and assume that M is not Kählerian. Then

(i) M is of constant type with a positive number α;

(ii) M is a strict nearly Kähler manifold;

(iii) (M,g) is an Einstein manifold;
(iv) the first Chern class of M vanishes;
(v) $\mathrm{Ric} = 5\,\mathrm{Ric}^* = 5\alpha I$ on M.

Since (M,g) is Einstein, $R\cdot I = 6\,\mathrm{Ric} = 30\alpha\, I$ for the scalar curvature R of M, hence $R = 30\cdot\alpha > 0$.
Throughout the rest of this section we shall assume that (M,g,J) is a connected 6-dimensional almost hermitian manifold which is nearly Kähler non-Kähler. Since $w_2(M) = c_1(M) \pmod 2$, by Proposition 2(iv) also the second Stiefel-Whitney class vanishes, hence (M,g) is known to be a spin manifold. Let $\alpha > 0$ be the constant type of M and set $\lambda = \frac{1}{2}\sqrt{\alpha}$. For local calculations we use an adapted orthonormal frame in general: Let $s_1, J(s_1), s_3, J(s_3)$ be orthonormal vector fields on an open subset V of M, then define a vector field s_5 on V by $(\nabla_{s_1}J)s_3 = 2\lambda s_5$.
Thus, s_5 is orthogonal to the vector fields already chosen and has constant length 1 so that $\{s_1, J(s_1), s_3, J(s_3), s_5, J(s_5)\}$ is an orthonormal frame on V. It satisfies $(\nabla_{s_i}J)s_j = 2\lambda s_k$ if (ijk) is an even permutation of (135). The other values of ∇J can easily be obtained from (5.17) - (5.20).

<u>Lemma 7:</u> For the vector fields $U,X,Y,Z \in \Gamma(TM)$, we have

(i) $$g((\nabla_U J)X, (\nabla_Y J)Z) = \alpha\cdot\{g(U,Y)g(X,Z) - g(U,Z)g(X,Y) - g(U,JY)g(X,JZ) + g(U,JZ)g(X,JY)\}$$

(ii) if $g(X,Y) = g(X,JY) = 0$, then
$$(\nabla_Y J)(\nabla_Y J)X = -\alpha\|Y\|^2 X.$$

<u>Proof:</u> Because of the linearity in each of the components, (i) is a direct checking on the elements of the local frame given above. Then (ii) follows from (i) and (5.20) by setting $U = Y$. □

Let X,Y,Z,V be vector fields on M. Since M is a 6-dimensional Einstein space and $\alpha = \frac{R}{30}$, the definition of the conformally Weyl tensor of M (compare § 1.4) reduces to

$$W(X,Y,Z,V) = R(X,Y,Z,V) + \alpha\{g(X,Z)g(Y,V) - g(X,V)g(Y,Z)\} \qquad (5.24)$$

By $W(X,Y,Z,V) = g(W(X,Y)Z,V)$ we regard W as a $(1,3)$-tensor on M, too.

Lemma 8: Let X,Y,Z be vector fields on M. Then

(i) $W(X,Y)JZ = J(W(X,Y)Z)$

(ii) $Tr(W(X,Y)\circ J) = 0$.

Proof: From (5.21) and (5.24), we have for $U\in\Gamma(TM)$

$$g((\nabla_X J)Y,(\bar\nabla_Z J)U) =$$
$$= W(X,Y,JZ,JU)-\alpha\{g(Y,JU)g(JZ,X)-g(X,JU)g(Y,JZ)\}-$$
$$-W(X,Y,Z,U)+\alpha\{g(Y,U)g(Z,X)-g(X,U)g(Y,Z)\}$$
$$= W(X,Y,JZ,JU)-W(X,Y,Z,U)+g((\nabla_X J)Y,(\nabla_Z)W)$$

by Lemma 7 (i). Therefore, $W(X,Y,Z,U) = W(X,Y,JZ,JU)$ or, equivalently, with $V = JU$,

$$g(W(X,Y)JZ,V) = -g(W(X,Y)Z,JV) = g(J(W(X,Y)Z),V).$$

This implies (i). For the proof of (ii) we notice that for a local frame $(s_1,\ldots,s_6)$ in $x\in M$, (5.23) and Proposition 2(v) imply

$$\sum_{i=1}^{6} R(X,Y,s_i,Js_i) = 2g(Ric^*(X),JY) = 2\alpha\, g(X,JY),$$

thus, by (5.24), we obtain

$$\sum_{i=1}^{6} W(X,Y,s_i,Js_i) = 2\alpha g(X,JY)+\alpha\sum_{i=1}^{6}\{g(X,s_i)g(Y,Js_i)-g(X,Js_i)g(Y,s_i)\}$$
$$= 2\alpha g(X,JY)-\alpha\sum_{i=1}^{6}\{g(X,s_i)g(JY,s_i)-g(JX,s_i)g(Y,s_i)\}$$
$$= 2\alpha g(X,JY)-\alpha g(X,JY)+\alpha g(JX,Y) = 0\ . \quad \square$$

Remark 4: Consider an adapted orthonormal frame $(s_1,\ldots,s_6)$ at $x\in M$ with the properties $(\nabla s_i J)s_j = 2\lambda s_k$ for any even permutation (ijk) of (135), and $s_{2i} = -J(s_{2i-1})$ for $i=1,2,3$.
Then, by Lemma 8, the coefficients $W_{ijkl} = g(W(s_i,s_j)s_k,s_l)$ of the Weyl tensor satisfy the equations

$$W_{ij12} + W_{ij34} + W_{ij56} = 0 \tag{5.25}$$

$$\left.\begin{array}{ll} W_{ij24} - W_{ij13} = 0 & W_{ij14} + W_{ij23} = 0 \\ W_{ij15} - W_{ij26} = 0 & W_{ij16} + W_{ij25} = 0 \\ W_{ij46} - W_{ij35} = 0 & W_{ij36} + W_{ij45} = 0 \end{array}\right\} \tag{5.26}$$

Lemma 9: If X,Y are vector fields on M with $\|X\|=\|Y\|= 1$ and $g(X,Y) = g(X,JY) = 0$, then

$$\alpha g(\nabla_Z X, JX) + \alpha g(\nabla_Z Y, JY) + g(\nabla_Z[(\nabla_X J)Y], J[(\nabla_X J)Y]) = 0$$

for any vector field Z on M.

Proof: First remark that

$$[\nabla_Z(\nabla_X J)](Y) = \nabla_Z[(\nabla_X J)Y] - (\nabla_X J)(\nabla_Z Y),$$

thus, from (5.16), we obtain

$$(\nabla^2_{ZX} J)(Y) = \nabla_Z[(\nabla_X J)Y] - (\nabla_X J)(\nabla_Z Y) - (\nabla_{(\nabla_Z X)} J)(Y).$$

To get simpler expressions, we write $A(X,Y) = (\nabla_X J)Y$ for the moment; then the above equation changes into

$$\nabla_Z[A(X,Y)] = [\nabla^2_{ZX} J](Y) + A(X, \nabla_Z Y) + A(\nabla_Z X, Y). \tag{5.27}$$

Then (5.22) yields

$$\begin{aligned} g([\nabla^2_{ZX} J](Y), J(A(X,Y))) &= -\tfrac{1}{2}\{g(A(Z,X), A(Y,A(Y,X))) + \\ &\quad + g(A(Z,JA(X,Y)), A(X,JY)) + g(A(Z,Y), A(JA(X,Y),JX))\} \\ &= -\tfrac{1}{2}\{g(A(Z,X), A(Y,A(Y,X))) - \\ &\quad - g(A(Z,A(X,JY)), A(X,JY)) - g(A(Z,Y), A(A(X,Y),X))\}. \end{aligned}$$

Now recall that $A(Y,A(Y,X)) = -\alpha X$ by Lemma 7; replacing $V = A(X,JY)$ in the second term, we get

$$\begin{aligned} g([\nabla^2_{ZX} J]Y, J(A(X,Y))) = \tfrac{1}{2}\{&\alpha g(A(Z,X),X) + \\ &+ g(A(Z,V),V) + \alpha g(A(Z,Y),Y)\} = 0, \end{aligned}$$

since $A(Z,X)$ is orthogonal to X. Now, from Lemma 7 (i) and the assumptions $\|X\| = \|Y\| = 1$, $g(X,Y) = g(X,JY) = 0$, it follows that

$$g(A(\nabla_Z Y, X), A(JY,X)) = \alpha g(\nabla_Z Y, JY) \quad \text{and}$$

$$g(A(\nabla_Z X, Y), A(JX,Y)) = \alpha g(\nabla_Z X, JX).$$

Consequently, we conclude from (5.27) that

$$\begin{aligned} &g(\nabla_Z[A(X,Y)], J(A(X,Y))) = \\ &\quad = g(A(X, \nabla_Z Y), J(A(X,Y))) + g(A(\nabla_Z X, Y), J(A(X,Y))) \\ &\quad = -g(A(X,\nabla_Z Y), A(X,JY)) - g(A(\nabla_Z X, Y), A(JX,Y)) \\ &\quad = -\alpha g(\nabla_Z Y, JY) - \alpha g(\nabla_Z X, JX). \end{aligned}$$

Hence the assertion follows. □

After these preparations, we are able to prove the central result of this chapter. [55]

Theorem 2: Let (M,g,J) be a 6-dimensional connected simply connected almost hermitian manifold, and assume that M is nearly Kähler non-Kähler. Then there exists a real Killing spinor on M.

Proof: Since $c_1(M) = 0$, and M is simply connected, there exists a unique spin structure $Q \to M$ over M. Denote by S the associated spinor bundle. Let $\alpha = \frac{R}{30}$ be the positive constant type of M, and set $\lambda = \frac{1}{2}\sqrt{\alpha}$. Consider locally an adapted orthonormal frame on M consisting of elements $\{s_i\}_{i=1,\dots,6}$ satisfying $(\nabla_{s_i}J)s_j = 2\lambda s_k$ for any even permutation (ijk) of (135), and

$$J(s_{2i}) = s_{2i-1} \quad \text{for} \quad i=1,2,3.$$

Then we choose the orientation of M so that $\{s_1,\dots,s_6\}$ is positively oriented.
Next, we form subbundles V_1, V_2 of S by

$$V_1 = \{\psi \in \Gamma(S) \; : \; J(X)\psi = iX\psi \text{ for all } X \in \Gamma(TM)\},$$

$$V_2 = \{\psi \in \Gamma(S) \; : \; J(X)\psi = -iX\psi \text{ for all } X \in \Gamma(TM)\}.$$

At a point $x \in M$, an element $\psi \in S_x \cong \Delta_6$ can be represented as a linear combination of the basis elements $u(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ of Δ_6, and the evaluation of $J(s_{2k})\psi = i\cdot s_{2k}\psi$ for $k=1,2,3$ by means of (5.6) yields, by some algebraic calculations,

$$V_1(x) = \mathbb{C}\cdot[s^*, u(1,1,1)] \;, \quad \text{and similarly}$$
$$V_2(x) = \mathbb{C}\cdot[s^*, u(-1,-1,-1)],$$

where s^* denotes a section in the spin structure Q corresponding to the frame $s = (s_1,\dots,s_6)$.
Consequently, $V_1 \subset S^+$, $V_2 \subset S^-$, and both are 1-dimensional complex subbundles of S. Now, consider the direct sum $V = V_1 \oplus V_2$; for any section $\psi \in \Gamma(V)$ we have the decomposition $\psi = \psi^+ + \psi^-$ according to $S = S^+ \oplus S^-$. Then we can define a subbundle E of V by

$$E = \{\psi \in \Gamma(V) : (\nabla_X J)(Y)\psi^+ = -2i\lambda\, YX\psi^- - 2i\lambda g(X,Y)\psi^- - 2\lambda g(X,JY)\psi^- \quad \text{for all } X,Y \in \Gamma(TM)\}.$$

Using again algebraic calculations with the $u(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ and the matrices given in (5.6), and exploiting $(\nabla_{s_i}J)s_j = 2\lambda s_k$ for even permutations (ijk) of (135) at a point $x \in M$ it turns out that we have

$$E_x = \mathbb{C}\cdot[s^*, u(1,1,1) - u(-1,-1,-1)] \;.$$

Thus, E is also a 1-dimensional complex subbundle of S. We introduce a covariant derivative $\tilde{\nabla}: \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ in E by the formula

$$\tilde{\nabla}_X \psi = \nabla_X \psi + \lambda X \psi, \qquad X \in \Gamma(TM), \psi \in \Gamma(E).$$

First we show that $\tilde{\nabla}$ is well-defined: Starting with $\psi \in \Gamma(E)$, the section $\tilde{\nabla}_X \psi$ also belongs to $\Gamma(E)$.

For this purpose, denote by $\{w_{ij}\}$ the family of local 1-forms on M defined by the Riemannian connection ∇ ; they are given by

$$w_{ij}(X) = g(\nabla_X s_i, s_j) \qquad \text{for} \quad X \in \Gamma(TM).$$

Recall from Chapter 1 that, for a section $\eta_\varepsilon \in \Gamma(S)$ of the form $\eta_\varepsilon = [s^*, u(\varepsilon_1, \varepsilon_2, \varepsilon_3)]$, we locally have

$$\nabla_X \eta_\varepsilon = \frac{1}{2} \sum_{i<j} w_{ij}(X) \cdot s_i \cdot s_j \cdot \eta_\varepsilon \tag{5.28}$$

Now, to investigate $\nabla_X \eta_\varepsilon$ for a fixed vector field X on M, let us consider the local endomorphisms $a(X)$, $b(X)$, $c(X)$ and $q(X)$ of S , which are defined by

$$a(X) = w_{13}(X)s_1s_3 + w_{24}(X)s_2s_4 + w_{14}(X)s_1s_4 + w_{23}(X)s_2s_3$$

$$b(X) = w_{15}(X)s_1s_5 + w_{26}(X)s_2s_6 + w_{16}(X)s_1s_6 + w_{25}(X)s_2s_5$$

$$c(X) = w_{35}(X)s_3s_5 + w_{46}(X)s_4s_6 + w_{36}(X)s_3s_6 + w_{45}(X)s_4s_5$$

$$q(X) = a(X) + b(X) + c(X).$$

We determine the value of $a(X)$ on the local sections $\eta_1 = [s^*, u(1,1,1)]$ and $\eta_{-1} = [s^*, u(-1,-1,-1)]$.

We have

$$\begin{aligned} 2\lambda &= g((\nabla_{s_1} J)s_3, s_5) = g((\nabla_{s_5} J)s_1, s_3) \\ &= g(\nabla_{s_5}(Js_1), s_3) - g(J(\nabla_{s_5} s_1), s_3) \\ &= -g(\nabla_{s_5} s_2, s_3) + g(\nabla_{s_5} s_1, Js_3) = -w_{23}(s_5) - w_{14}(s_5). \end{aligned}$$

Thus, $w_{23}(s_5) + w_{14}(s_5) = -2\lambda$ and, analogously

$$w_{23}(s_j) + w_{14}(s_j) = 0 \qquad \text{for} \quad j \neq 5.$$

In the same way we obtain

$$w_{13}(s_6) - w_{24}(s_6) = -2\lambda \quad , \text{ and}$$
$$w_{13}(s_j) - w_{24}(s_j) = 0 \quad \text{for} \quad j \neq 6.$$

Since $s_1 s_4 \eta_1 = s_2 s_3 \eta_1 = -s_5 \eta_{-1}$,

and $s_1 s_3 \eta_1 = -s_2 s_4 \eta_1 = -s_6 \eta_{-1}$, we obtain

$$a(X)\eta_1 = [w_{13}(X)-w_{24}(X)](-s_6\eta_{-1})+[w_{14}(X)+w_{23}(X)](-s_5\eta_{-1}).$$

Thus, $a(s_6)\eta_1 = 2\lambda s_6\eta_{-1}$,

$a(s_5)\eta_1 = 2\lambda s_5\eta_{-1}$

and $a(s_j)\eta_1 = 0$ for $j = 5,6$.

The same method provides

$$b(s_3)\eta_1 = 2\lambda s_3\eta_{-1}, \quad b(s_4)\eta_1 = 2\lambda s_4\eta_{-1},$$

$$c(s_1)\eta_1 = 2\lambda s_1\eta_{-1}, \quad c(s_2)\eta_1 = 2\lambda s_2\eta_{-1},$$

and $b(s_j)\eta_1 = 0$ for $j \neq 3,4$

$c(s_j)\eta_1 = 0$ for $j \neq 1,2$.

Summing up, we have $q(s_k)\eta_1 = 2\lambda s_k\eta_{-1}$ for $k=1,\dots,6$, and consequently $q(X)\eta_1 = 2\lambda X\eta_{-1}$ for any vector field X on M.

The same procedure also yields $q(X)\eta_{-1} = 2\lambda X\eta_1$.

Now, we use formula (5.28) to determine the covariant derivative of the local section η_1 and η_{-1}. A calculation shows that

$$s_1s_2\eta_1 = s_3s_4\eta_1 = s_5s_6\eta_1 = i\eta_1, \text{ and}$$

$$s_1s_2\eta_{-1} = s_3s_4\eta_{-1} = s_5s_6\eta_{-1} = -i\eta_{-1}.$$

Therefore,

$$\nabla_X\eta_1 = \tfrac{1}{2}[w_{12}(X)+w_{34}(X)+w_{56}(X)]\, i\eta_1 + \tfrac{1}{2}q(X)\eta_1.$$

However, setting $X = s_1$, $Y = s_3$ in Lemma 9 entails $w_{12}(X) + w_{34}(X) + w_{56}(X) = 0$ so that $\nabla_X\eta_1 = \lambda X\eta_{-1}$ holds.

Analogously, we obtain $\nabla_X\eta_{-1} = \lambda X\eta_1$.

Now it is easy to see that, for $\psi\in\Gamma(E)$, also $\tilde{\nabla}_X\psi$ is a section in E. Let $\psi = \theta\eta_1 - \theta\eta_{-1}$ be the local form of a section in E, where θ is a complex valued function. It follows that

$$\begin{aligned}\tilde{\nabla}_X\psi = \nabla_X\psi + \lambda X\psi &= \nabla_X(\theta\eta_1-\theta\eta_{-1})+\lambda X(\theta\eta_1-\theta\eta_{-1})\\ &= d\theta(X)\eta_1+\theta\nabla_X\eta_1-d\theta(X)\eta_{-1} - \theta\nabla_X\eta_{-1} + \lambda X(\theta\eta_1-\theta\eta_{-1})\\ &= d\theta(X)(\eta_1-\eta_{-1})+\lambda\theta X\eta_{-1}-\lambda\theta X\eta_1+\lambda X(\theta\eta_1-\theta\eta_{-1}).\end{aligned}$$

Hence, we obtain $\tilde{\nabla}_X\psi = d\theta(X)(\eta_1-\eta_{-1})$ and this is a section belonging to $\Gamma(E)$, too.

For the vector fields X,Y on M and a spinor field $\psi\in\Gamma(S)$, the curvature of the covariant derivative ∇ in $\Gamma(S)$ is given by

$$\tfrac{1}{2}R(X,Y)\psi = [\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}]\psi .$$

To get an expression for the curvature of $\tilde{\nabla}$, we take a $\psi \in \Gamma(E)$. Then

$$\tilde{\nabla}_Y \tilde{\nabla}_X \psi = \tilde{\nabla}_Y(\nabla_X \psi + \lambda X\psi) = \tilde{\nabla}_Y(\nabla_X \psi) + \tilde{\nabla}_Y(\lambda X\psi)$$
$$= \nabla_Y \nabla_X \psi + \lambda Y(\nabla_X \psi) + \lambda^2 YX\psi + \lambda \tilde{\nabla}_Y(X\psi),$$

and similarly,

$$\tilde{\nabla}_X \tilde{\nabla}_Y \psi = \nabla_X \nabla_Y \psi + \lambda X(\nabla_Y \psi) + \lambda^2 XY\psi + \lambda \nabla_X(Y\psi),$$
$$\tilde{\nabla}_{[X,Y]}\psi = \nabla_{[X,Y]}\psi + \lambda(\nabla_X Y - \nabla_Y X)\psi .$$

An addition, taking into account (1.10), yields

$$\frac{1}{2}\tilde{R}(X,Y) = [\tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}]\psi =$$
$$= \frac{1}{2} R(X,Y)\psi + \lambda^2(XY-YX)\psi .$$

Since $R(X,Y) = \frac{1}{2}\sum_{k,l} R(X,Y,s_k,s_l)\cdot s_k \cdot s_l$ holds (cf. Chapter 1), we have

$$2\tilde{R}(X,Y)\psi = \sum_{k,l} R(X,Y,s_k,s_l)\cdot s_k \cdot s_l \psi + 4\lambda^2(XY-YX)\psi .$$

Recall that $4\lambda^2 = \alpha = \frac{R}{30}$ so that, by formula (5.24), which defines the Weyl tensor on an Einstein space, we conclude

$$2\tilde{R}(X,Y)\psi = \sum_{k,l} W(X,Y,s_k,s_l) s_k \cdot s_l \psi \quad , \quad \psi \in \Gamma(E).$$

Now, using the local form $\psi = \theta\eta_1 - \theta\eta_{-1}$ for a section $\psi \in \Gamma(E)$ together with (5.6) to compute the right-hand side of this equation, Remark 4 shows that

$$\tilde{R}(X,Y)\psi = 0 \quad \text{for} \quad \psi \in \Gamma(E),\ X,Y \in \Gamma(TM).$$

Consequently, $(E,\tilde{\nabla})$ is a flat 1-dimensional bundle over a simply connected manifold M. Thus, there exists a $\tilde{\nabla}$-parallel section in E, i.e. a spinor field $\psi \in \Gamma(E)$ with $\nabla_X \psi + \lambda X\psi = 0$ for any vector field X on M.

Obviously, ψ is then a Killing spinor with the Killing number $\beta = -\lambda$, and writing $\varphi = \psi^+ - \psi^-$ we also obtain $\nabla_X \varphi = \lambda X \varphi$ for $X \in \Gamma(TM)$. □

<u>Remark 5:</u> As it was pointed out by A. Gray in [54], a nearly Kähler manifold (M^n,g,J) with

$$\|(\nabla_X J)Y\|^2 = \alpha\{\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(JX,Y)^2\}$$

for all vector fields X,Y on M and a positive constant α, necessarily has the dimension $n = 6$. Therefore, the general method of the above proof cannot work in higher dimensions.

5.4. Examples

The examples of 6-dimensional nearly Kähler non-Kähler manifolds which can be found in the literature are always reductive homogeneous spaces G/K with a Riemannian metric induced by an $Ad(G)$-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. Moreover, they carry the structure of Riemannian 3-symmetric spaces, i.e. the almost complex structure J comes from a Lie group automorphism σ on G of order 3 and with fixed point set K (see [53]). In [50], the following simply connected spaces are mentioned:

$$U(3)/_{U(1)\times U(1)\times U(1)}\,,\quad SO(5)/_{U(2)}\,,\quad SO(6)/_{U(3)}\,,\quad SO(5)/_{U(1)\times SO(3)}\,,$$

$$Sp(2)/_{U(2)} \quad \text{and} \quad S^6 = G_2/_{SU(3)}.$$

Furthermore, given a compact, connected non-abelian Lie group G, the structure of a Riemannian 3-symmetric space can be defined according to a construction of Ledger and Obata (see[78]) also on the product $G \times G$. Therefore, if $G = S^3$, we also obtain an almost hermitian structure on $S^3 \times S^3 = Spin(4)$, which is nearly Kähler and non-Kähler.

In the sequel, we more explicitly discuss some of the examples mentioned above. In particular, the existence of real Killing spinors will be shown by calculating the smallest eigenvalue of the Dirac operator.

First we consider the Levi-Civita connection of a homogeneous Riemannian manifold.

Let G be a connected compact Lie group and H a closed, connected subgroup of G. Consider the homogeneous space $M^n = G/_H$ with the isotropy representation $\alpha: H \longrightarrow SO(T_{x_0}M) \cong SO(n)$ at the point $x_0 = eH$ and suppose that there is a lifting $\tilde{\alpha} : H \longrightarrow Spin(n)$ of α such that the following diagramme commutes:

$$\begin{tikzcd} & Spin(n) \arrow[d, "\lambda"] \\ H \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & SO(n) \end{tikzcd}$$

The mapping $\tilde{\alpha}$ defines a natural spin structure $Q = G \times_{\tilde{\alpha}} Spin(n)$ over $G/_H$, and the associated spinor bundle $S = G \times_{\Phi\tilde{\alpha}} \Delta_n$ is a homogeneous vector bundle on $M = G/_H$.

Now, suppose that an $Ad(H)$-invariant, positive-definite symmetric bilinear form B is given on the Lie algebra $\mathfrak{g}$ of G, and choose a linear subspace $\mathfrak{m}$ in $\mathfrak{g}$ that is orthogonal to the Lie algebra

$\mathfrak{k}$ of H and satisfies $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since $\mathfrak{g} \perp \mathfrak{k}$, we have $[\mathfrak{k}, \mathfrak{g}] \subseteq \mathfrak{p}$ and $G/_H$ is a reductive homogeneous space.
Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be a decomposition of $\mathfrak{p}$ into linear subspaces that are orthogonal with respect to B and satisfy the relations

$$\left.\begin{array}{l} [\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}, \ [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k} \oplus \mathfrak{m}, \\ [\mathfrak{k}, \mathfrak{n}] \subseteq \mathfrak{n}, \ [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{k}, \ [\mathfrak{n}, \mathfrak{m}] \subseteq \mathfrak{m}. \end{array}\right\} \quad (5.29)$$

Then, for an arbitrary $t > 0$, the equation

$$B_t = B\restriction_{\mathfrak{m} \times \mathfrak{m}} + 2t\, B\restriction_{\mathfrak{n} \times \mathfrak{n}} \quad (5.30)$$

defines an Ad(H)-invariant scalar product on $\mathfrak{p}$, it yields a left invariant Riemannian metric on $G/_H$, which we shall denote by g_t.

<u>Lemma 10:</u> The Levi-Civita connection of g_t is given by the mapping $\Lambda_t : \mathfrak{p} \longrightarrow so(\mathfrak{p})$ defined by

$$\begin{aligned} \Lambda_t(X)Y &= \tfrac{1}{2}[X,Y]_{\mathfrak{n}} \\ \Lambda_t(X)B &= t[X,B] \\ \Lambda_t(A)Y &= (1-t)[A,Y] \\ \Lambda_t(A)B &= 0 \end{aligned}$$

for $X, Y \in \mathfrak{m}$ and $A, B \in \mathfrak{n}$.
(Here the index denotes the projection onto the corresponding component).

<u>Proof:</u> From Wang's Theorem it follows that the Levi-Civita connection induced by B_t is uniquely determined by a linear map $\Lambda_t : \mathfrak{p} \longrightarrow \mathrm{End}(\mathfrak{p})$ satisfying the conditions

(i) $\Lambda_t(X)Y - \Lambda_t(Y)X = [X,Y]_{\mathfrak{p}}$,

(ii) $B_t(\Lambda_t(X)Y, Z) + B_t(\Lambda_t(X)Z, Y) = 0$

for all vectors $X, Y, Z \in \mathfrak{p}$ (see [74], vol. II, Chapter X). Hence, it suffices to verify that the mapping defined in Lemma 10 satisfies the two conditions (i) and (ii), which is ensured by (5.29). □

The differential $\lambda_* : \underline{spin}(n) \longrightarrow \underline{so}(n)$ of λ is a Lie algebra isomorphism, hence $(\lambda_*)^{-1}$ exists, and we obtain a mapping

$$\Lambda_t = (\lambda_*)^{-1} \circ \Lambda_t : \mathfrak{p} \longrightarrow \underline{spin}(n).$$

To describe the action of the Dirac operator $D^t : \Gamma(S) \longrightarrow \Gamma(S)$ corresponding to B_t, we identify the sections of the spinor bundle $S = G \times_{\tilde{\Phi}} \Delta_n$ with the functions $\varphi : G \longrightarrow \Delta_n$ satisfying the relation

$$\varphi(gh) = \Phi\tilde{\alpha}(h^{-1})\varphi(g) \tag{5.31}$$

for all $g \in G$, $h \in H$.

For such a function $\varphi : G \to \Delta_n$, the action of the Dirac operator is then given by the formula

$$D^t\varphi = \sum_j \Phi(e_j)[e_j(\varphi) + \Phi\tilde{\Lambda}_t(e_j)\varphi] \quad , \tag{5.32}$$

where $\{e_j\}_{j=1,\dots,n}$ is a B_t-orthonormal basis of $\mathfrak{p}$, and $e_j(\varphi)$ denotes the derivative of φ in the direction of the vector field generated by e_j (see [65]).

We now consider several examples.

a) <u>The complex flag manifold F(1,2)</u>

The flag manifold $F(1,2)$ consists of pairs (l,v), where both l and v are linear subspaces of $\mathbb{C}^3$ of dimension 1 and 2, respectively, and $l \subset v$ holds. The $U(3)$-action in $\mathbb{C}^3$ is transitive on $F(1,2)$ with the isotropy subgroup $H = U(1) \times U(1) \times U(1)$, hence $F(1,2) = U(3)/_{U(1) \times U(1) \times U(1)}$. The Lie algebras of $U(3)$ and H are given by

$$\mathfrak{u}(3) = \{A \in M_3(\mathbb{C}) : \bar{A}^T + A = 0\},$$

$$\mathfrak{h} = \{A \in \mathfrak{u}(3) : A \text{ is diagonal}\}.$$

We decompose $\mathfrak{u}(3) = \mathfrak{h} \oplus \mathfrak{p}$ with $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$, where

$$\mathfrak{m} = \left\{\begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & 0 \\ -\bar{b} & 0 & 0 \end{pmatrix} ; a,b \in \mathbb{C}\right\} \quad , \quad \mathfrak{n} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\bar{c} & 0 \end{pmatrix} , c \in \mathbb{C}\right\},$$

and consider the inner product on $\mathfrak{u}(3)$ given by

$$B(A,B) = -\frac{1}{2}\operatorname{Re}(\operatorname{Tr}(AB)), \qquad A,B \in \mathfrak{u}(3).$$

Then $\mathfrak{m}$ and $\mathfrak{n}$ are orthogonal with respect to B, and the relations (5.29) are satisfied.

For $t > 0$, (5.30) thus determines a $U(3)$-invariant Riemannian metric g_t on $F(1,2)$. According to § 3 of Chapter 3, Einstein metrics are obtained for $t = \frac{1}{2}$ and $t = 1$; the parameter $t = 1$ corresponds to the standard Kähler-Einstein metric of $F(1,2)$ with scalar curvature $R = 24$, whereas $t = \frac{1}{2}$ yields a biinvariant Einstein metric of scalar curvature $R = 30$.

Denote by D_{ij} the $n \times n$-matrix consisting of a single 1 in the i-th row and j-th column, and zeros elsewhere. We set $E_{ij} = D_{ij} - D_{ji}$ for $i \neq j$, and $S_{ij} = \sqrt{-1}\,(D_{ij} + D_{ji})$. The matrices

E_{ij} $(i<j)$ then generate the Lie algebra $\underline{so}(n)$; notice that this notion differs from that of Chapter 1 by a sign, hence the Lie algebra isomorphism

$\lambda_*: \underline{spin}(n) \longrightarrow \underline{so}(n)$ is given by

$$\lambda_*(e_i e_j) = -2\, E_{ij} \tag{5.33}$$

To distinguish a basis of $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$, consider the matrices $e_1 = E_{12}$, $e_2 = S_{12}$, $e_3 = E_{13}$, $e_4 = S_{13}$ and $e_5 = \frac{1}{\sqrt{2t}} E_{23}$, $e_6 = \frac{1}{\sqrt{2t}} S_{23}$.

Then $\mathfrak{m} = \mathrm{Lin}\{e_1, e_2, e_3, e_4\}$, $\mathfrak{n} = \mathrm{Lin}\{e_5, e_6\}$ and the elements $e_1, \ldots, e_6$ form an orthonormal basis of $\mathfrak{p}$ with respect to B_t which we shall use to identify $\mathfrak{p}$ with R^6. A basis of $\mathfrak{h}$ is given by $\{H_1, H_2, H_3\}$ with $H_i = \frac{1}{2} S_{ii}$ for $i=1,2,3$.

Using the identification $\mathfrak{p} = \mathbb{R}^6$ we can also compute the isotropy representation $\alpha: H \longrightarrow SO(\mathfrak{p}) = SO(6)$ of $G/_H$. For an arbitrary element $h \in H$ with

$$h = \begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{is} & 0 \\ 0 & 0 & e^{ir} \end{pmatrix} \qquad t, r, s \in \mathbb{R}$$

it is defined by $\alpha(h) e_j = h \cdot e_j \cdot h^{-1}$; writing $\theta(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for $t \in [0, 2\pi]$, a calculation yields

$$\alpha(h) = \begin{pmatrix} \theta(t-s) & 0 & 0 \\ 0 & \theta(t-r) & 0 \\ 0 & 0 & \theta(s-r) \end{pmatrix} \in SO(6).$$

It follows that the differential $\alpha_*: \mathfrak{h} \longrightarrow \underline{so}(6)$ is given by the formulas

$$\left.\begin{aligned} \alpha_*(H_1) &= -E_{12} - E_{34} \\ \alpha_*(H_2) &= E_{12} - E_{56} \\ \alpha_*(H_3) &= E_{34} + E_{56} \end{aligned}\right\} \tag{5.34}$$

<u>Lemma 11:</u> There exists a lifting homomorphism $\tilde{\alpha}: H \longrightarrow Spin(6)$ of α with $\lambda \circ \tilde{\alpha} = \alpha$.

<u>Proof:</u> It suffices to show that $\alpha^*(\pi_1(H)) \subset \lambda^*(\pi_1(Spin(6))) = 0$, or, equivalently, that each generator of $\pi_1(H)$ vanishes under the superposition with α. Taking, for instance,

$$\gamma(t)=\begin{pmatrix} e^{it} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \; t\in[0,2\pi],$$

as an element of $\pi_1(H)$, the composition with the isotropy representation α yields

$$\alpha\circ\gamma(t)=\begin{pmatrix} \theta(t) & 0 & 0 \\ 0 & \theta(t) & 0 \\ 0 & 0 & \theta(0) \end{pmatrix} \quad ; \text{ now, since}$$

$$\begin{pmatrix} \theta(t) & 0 & 0 \\ 0 & \theta(0) & 0 \\ 0 & 0 & \theta(0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta(0) & 0 & 0 \\ 0 & \theta(t) & 0 \\ 0 & 0 & \theta(0) \end{pmatrix} \text{are}$$

homotopically equivalent in $SO(6)$ (they correspond to different rotations of the basis vectors of $\mathbb{R}^6$), we obtain for the homotopy classes in $\pi_1(SO(6))$

$$[\alpha\cdot\gamma(t)]=\left[\begin{pmatrix} \theta(t) & 0 & 0 \\ 0 & \theta(0) & 0 \\ 0 & 0 & \theta(0) \end{pmatrix}\right] + \left[\begin{pmatrix} \theta(0) & 0 & 0 \\ 0 & \theta(t) & 0 \\ 0 & 0 & \theta(0) \end{pmatrix}\right]=$$

$$= 2\left[\begin{pmatrix} \theta(t) & 0 & 0 \\ 0 & \theta(0) & 0 \\ 0 & 0 & \theta(0) \end{pmatrix}\right],$$

hence $[\alpha\circ\gamma(t)] = 0$ since $\pi_1(SO(6)) = \mathbb{Z}_2$. The other generating elements of $\pi_1(H)$ are treated analogously. $\square$

The map $\tilde{\alpha} : H \longrightarrow Spin(6)$ gives rise to a homogeneous spin structure $Q = U(3) \times_{\tilde{\alpha}} Spin(6)$ over $F(1,2)$. As $H^1(F(1,2);Z_2) = \{0\}$, this spin structure is the only possible one (see § 2 of Ch. 1). Using the identification $\mathfrak{X} \cong \mathbb{R}^6$ and the commutator relations that hold between the matrices $e_1,\dots,e_6$ we obtain by Lemma 10 the formulas for the Levi-Civita connection Λ_t corresponding to B_t. A computation shows that, with respect to (5.33), the mapping $\tilde{\Lambda}_t = \lambda_*^{-1}\Lambda_t : \mathbb{R}^6 \longrightarrow \underline{spin}(6)$ is given by

$$\tilde{\Lambda}_t(e_1) = -\frac{\sqrt{t}}{2\sqrt{2}}(e_3e_5 + e_4e_6)$$

$$\tilde{\Lambda}_t(e_2) = -\frac{\sqrt{t}}{2\sqrt{2}}(e_4e_5 - e_3e_6)$$

$$\tilde{\Lambda}_t(e_3) = -\frac{\sqrt{t}}{2\sqrt{2}}(e_2e_6 - e_1e_5)$$

$$\tilde{\Lambda}_t(e_4) = \frac{\sqrt{t}}{2\sqrt{2}}(e_1e_6 + e_2e_5)$$
$$\tilde{\Lambda}_t(e_5) = \frac{(t-1)}{2\sqrt{2t}}(e_1e_3 + e_2e_4)$$
$$\tilde{\Lambda}_t(e_6) = -\frac{(1-t)}{2\sqrt{2t}}(e_1e_4 - e_2e_3).$$

Since now the Dirac operator of $(F(1,2), g_t)$ is completely determined, we can state the following result. [40]

Proposition 3: Let g_t be the left-invariant Riemannian metric on $F(1,2)$ determined by (5.3D). Then:

(i) On the Einstein space $(F(1,2), g_{1/2})$ the Dirac operator has the eigenvalues $\pm \frac{1}{2}\sqrt{\frac{nR}{n-1}} = \pm 3$.

(ii) On the Kähler-Einstein space $(F(1,2), g_1)$ the Dirac operator has the eigenvalues $\pm \frac{1}{2}\sqrt{\frac{n+2}{n}R} = \pm 2\sqrt{2}$.

Proof: From (5.33), (5.34) and the formulas given in (5.6) we note that, for $i=1,2,3$, the homomorphism $\Phi\tilde{\alpha}_*(H_i) \in End(\Delta_6)$ annihilates both vectors $\varphi^+ = u(-1,1,-1)$ and $\varphi^- = u(1,-1,1)$. Hence, for arbitrary $z_1, z_2 \in \mathbb{C}$ and all $X \in \mathfrak{h}$,

$$\Phi\tilde{\alpha}_*(X)[z_1\varphi^+ + z_2\varphi^-] = 0$$

holds, and since H is connected, we conclude that, for any $h \in H$, we have

$$\Phi\tilde{\alpha}(h)[z_1\varphi^+ + z_2\varphi^-] = [z_1\varphi^+ + z_2\varphi^-].$$

Thus, the invariance property (5.31) is automatically satisfied for the constant function $\varphi: U(3) \longrightarrow \Delta_6$ given by $\varphi(g) = z_1\varphi^+ + z_2\varphi^-$, and φ defines a section in the spinor bundle $S = U(3) \times_{\Phi\tilde{\alpha}} \Delta_6$.
On this constant section φ the expression for the Dirac operator D^t corresponding to B_t, which is given in (5.32), simplifies to

$$D^t\varphi = \sum_{j=1}^{6} \Phi(e_j)[\Phi\tilde{\Lambda}_t(e_j)\varphi] .$$

For the particular terms we obtain

$$\Phi(e_j)(\Phi\tilde{\Lambda}_t(e_j)\varphi) = \frac{\sqrt{t}}{\sqrt{2}}(iz_2\varphi^+ - iz_1\varphi^-), \quad j=1,\dots,4$$

$$\Phi(e_j)(\Phi\tilde{\Lambda}_t(e_j)\varphi) = \frac{(1-t)}{\sqrt{2t}}(iz_2\varphi^+ - iz_1\varphi^-), \; j=5,6,$$

and after summation

for $t = \frac{1}{2}$: $\quad D^{1/2}\varphi = 3(iz_2\varphi^+ - iz_1\varphi^-)$,

for $t = 1$: $\quad D^1\varphi = 2\sqrt{2}(iz_2\varphi^+ - iz_1\varphi^-)$.

Consequently, eigenspinors of D^t which realize the desired eigenvalues are obtained by $z_1 = 1,\ z_2 = -i$ and $z_1 = 1,\ z_2 = +i$. □

Since in our calculations $\varphi^+ = u(-1,1,-1)$ and $\varphi^- = u(1,-1,1)$ hold, we easily check that the almost complex structure on $F(1,2)$, defined in (5.11), is explicitly given by

$$J(e_1) = e_2,\ J(e_3) = -e_4 \text{ and } J(e_5) = e_6.$$

Returning to our description of $\underline{\mathfrak{f}} = \underline{\mathfrak{m}} \oplus \underline{\mathfrak{n}}$, we identify an element of

$$\underline{\mathfrak{f}} = \left\{ \begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} \quad ; a,b,c \in \mathbb{C} \right\}$$

with the corresponding triple $(a,b,c) \in \mathbb{C}^3$. In this notation, the almost complex structure J becomes $J(a,b,c)) = (ia,-ib,ic)$.
From this, it can also be checked directly that the manifold $(F(1,2),g_{1/2}, J)$ is nearly Kähler non-Kähler. Furthermore, we set $z = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\cdot i$ and consider the diagonal matrix $C = (c_{kl})$ with the entries $c_{kl} = z^k \delta_{kl}$ $(1 \le k,l \le 3)$, which defines a Lie group automorphism $\vartheta: U(3) \longrightarrow U(3)$ by $\vartheta(A) := C^{-1}\cdot A \cdot C$ for any $A \in U(3)$. Then ϑ is of order 3 with a fixed point set equal to H, and the almost complex structure J is generated by ϑ in the following sense:
If we denote the natural projection by $\pi: U(3) \longrightarrow F(1,2)$ and define a transformation σ of $F(1,2)$ by the equation $\pi \circ \vartheta = \sigma \circ \pi$, then the differential $\sigma_*: \underline{\mathfrak{f}} \longrightarrow \underline{\mathfrak{f}}$ is related to J by $\sigma_* = -\frac{1}{2}\,\mathrm{id} + \frac{\sqrt{3}}{2} J$.
Therefore, $(F(1,2),g_{1/2})$ is also equipped with the structure of a 3-symmetric space.
Finally, we remark that in our notation the complex structure $I: \underline{\mathfrak{f}} \longrightarrow \underline{\mathfrak{f}}$ corresponding to the Kähler-Einstein structure of $(F(1,2),g_1)$ is given by $I((a,b,c)) = (ia,-ib,-ic)$.

b) <u>The complex projective space $\mathbb{C}P^3$</u>

According to example (a) of § 3 in Chapter 3, we choose the inner product in $\underline{so}(5)$ given by

$$B_1(X,Y) = -\frac{1}{2}\,\mathrm{Tr}(X\cdot Y)\ , \qquad X,Y \in \underline{so}(5),$$

decompose $\underline{so}(5)$ into $\underline{so}(5) = \underline{\check{\mathfrak{n}}}(2) \oplus \underline{\mathfrak{m}} + \underline{\mathfrak{n}}$ as described in § 3.3, and, for $t > 0$, consider the $\mathrm{Ad}(U(2))$-invariant bilinear

form B_t on $\mathfrak{m} \oplus \mathfrak{n}$ defined by $B_t = B_1|_{\mathfrak{m}\times\mathfrak{m}} + 2t\, B_1|_{\mathfrak{n}\times\mathfrak{n}}$.
If we denote $\mathfrak{k} = \check{\mathfrak{u}}(2)$, the commutator relations (5.29) are satisfied between $\mathfrak{k}$, $\mathfrak{n}$ and $\mathfrak{m}$, thus by Lemma 10 the Levi-Civita connection Λ_t of the left-invariant metric g_t on $\mathbb{C}P^3 = SO(5)/_{U(2)}$ corresponding to B_t can be determined. Moreover, considerations similar to that of Lemma 11 show that there exists a lifting homomorphism $\tilde{\alpha}: U(2) \longrightarrow Spin(6)$ of the isotropy representation $\alpha: U(2) \longrightarrow SO(6)$ of $\mathbb{C}P^3$, and by calculating the kernel of $\Phi\tilde{\alpha}_*(X) \in End(\Delta_6)$ for arbitrary $X \in \check{\mathfrak{u}}(2)$ we obtain a constant function $\varphi: SO(5) \longrightarrow \Delta_6$ describing a section in the spinor bundle S over $\mathbb{C}P^3$.
By analogous calculations as for the flag manifold $F(1,2)$ in example (a), the application of the Dirac operator D^t corresponding to g_t on this spinor field $\varphi \in \Gamma(S)$ can be determined.
According to § 3.3, the metric g_t on $\mathbb{C}P^3$ is an Einstein metric for the parameters $t = \frac{1}{2}$ and $t = 1$. The metric g_1 was shown to be the Kähler standard metric of $\mathbb{C}P^3$ and it has scalar curvature $R = 12$, whereas $g_{1/2}$ is normal homogeneous with respect to $SO(5)$ and has scalar curvature $R = 15$.
Similar to the case of $F(1,2)$, we can state the following result:

<u>Proposition 4:</u> Let g_t be the left-invariant Riemannian metric on $\mathbb{C}P^3$ described above.

(i) On the Einstein space $(\mathbb{C}P^3, g_{1/2})$ the Dirac operator has the eigenvalues $\pm \frac{1}{2}\sqrt{\frac{nR}{n-1}} = \pm \frac{3}{2}\sqrt{2}$.

(ii) On the Kähler-Einstein space $(\mathbb{C}P^3, g_1)$ the Dirac operator realizes the eigenvalues $\pm \frac{1}{2}\sqrt{\frac{n+2}{n}R} = \pm 2$.

c) <u>The Lie group $Spin(4) \cong S^3 \times S^3$</u>

We decompose the Lie algebra $\underline{spin}(4) \cong \underline{so}(4)$ into $\underline{so}(4) = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = span\{E_{12}, E_{13}, E_{23}\}$ and $\mathfrak{p} = span\{E_{14}, E_{24}, E_{34}\}$, where the matrices E_{ij} $(i < j)$ are the standard generators of $\underline{so}(4)$ as described in example (a) of § 5.4.
Then, by $B(X,Y) = -\frac{1}{2}Tr(X \circ Y)$, $X, Y \in \underline{so}(4)$ and $B_t = B|_{\mathfrak{p}\times\mathfrak{p}} + 2t\, B|_{\mathfrak{k}\times\mathfrak{k}}$ for $t > 0$, we get a family of left-invariant Riemannian metrics g_t on $S^3 \times S^3$. By Wang's Theorem, the Levi-Civita connection of g_t corresponds to a map $\Lambda_t: \underline{so}(4) \longrightarrow \underline{so}(\mathfrak{k} \oplus \mathfrak{p})$ which is given by

$$\Lambda_t(X)Y = \tfrac{1}{2}[X,Y]$$
$$\Lambda_t(X)A = (1-t)[X,A]$$
$$\Lambda_t(B)Y = t[B,Y]$$
$$\Lambda_t(A)B = \tfrac{1}{2}[A,B]$$

for $X,Y \in \mathfrak{k}$ and $A,B \in \mathfrak{g}$.
With $t = \frac{1}{2}$ and $t = \frac{1}{6}$ we obtain Einstein metrics on $G = S^3 \times S^3$, where $t = \frac{1}{2}$ corresponds to the usual product metric. The parameter $t = 1/6$ yields an Einstein metric with scalar curvature $R = 10$.
In the following, we fix $t = \frac{1}{6}$; an orthonormal basis of $\mathfrak{k} \oplus \mathfrak{g}$ with respect to $B_{1/6}$ is given by $e_1 = \sqrt{3}\,E_{12}$, $e_2 = \sqrt{3}\,E_{13}$, $e_3 = \sqrt{3}\,E_{23}$, $e_4 = E_{14}$, $e_5 = E_{24}$, $e_6 = E_{34}$ and we use these vectors $\{e_1,\dots,e_6\}$ to identify $\underline{so}(4)$ with R^6.
Consider the trivial spin structure $Q = G \times Spin(6)$ (which is the only existing one, since $G = S^3 \times S^3$ is simply connected); the associated spinor bundle is then the trivial vector bundle $S = G \times \Delta_6$, and hence $\Gamma(S)$ consists of all smooth functions $\varphi : G \to \Delta_6$. Using the description of Λ_t given above and Ikeda's formula (see example (a)) to express the Dirac operator D corresponding to $B_{1/6}$, we obtain several eigenspinors of D. They are given by

$$\varphi_1^{\pm} = u(1,1,1) \mp u(1,-1,1)$$

$$\varphi_2^{\pm} = u(-1,1,-1) \mp u(-1,-1,-1)$$

$$\varphi_3^{\pm} = [u(1,-1,-1) + u(-1,-1,1)] \mp [u(-1,1,1) + u(1,1,-1)]$$

$$\psi^{\pm} = [u(1,-1,-1) - u(-1,-1,1)] \mp [u(-1,1,1) - u(1,1,-1)].$$

In particular, we state

<u>Proposition 5:</u> On the Einstein space $(S^3 \times S^3, g_{1/6})$ the Dirac operator has the eigenvalues

$$\pm \frac{1}{2}\sqrt{\frac{nR}{n-1}} = \pm\sqrt{3} \quad \text{and} \quad \pm \frac{4}{3}\sqrt{3}\,. \quad \square$$

The eigenvalues $\pm\frac{4}{3}\sqrt{3}$ are realized by the spinors $\varphi_i^{\pm}$ $(i=1,2,3)$, whereas $\psi^{\pm}$ are eigenspinors for $\pm\sqrt{3}$, hence Killing spinors on $S^3 \times S^3$. Writing $\psi_1 = [u(1,-1,-1) - u(-1,-1,1)] \in \Gamma(S^+)$ and $\psi_2 = [u(-1,1,1) - u(1,1,-1)] \in \Gamma(S^-)$, we have $\psi^{\pm} = \psi_1 \mp \psi_2$, and the almost complex structure J, which makes $(S^3 \times S^3, g_{1/6})$ a nearly Kähler non-Kähler manifold, is then defined by $J(X)\psi_1 = iX\psi_1$, $X \in \underline{so}(4)$. In the above notation, it is described by $J(e_3) = e_4$, $J(e_1) = e_6$, $J(e_5) = e_2$.

Chapter 6: Manifolds with Parallel Spinor Fields

Let M^n be a Riemannian spin manifold. A spinor field $\psi \in \Gamma(S)$ is said to be parallel if $\nabla\psi = 0$. If a Riemannian spin manifold admits a parallel spinor field, its Ricci tensor vanishes, $Ric \equiv 0$. (see Chapter 1, Theorem 8). Consequently, a 3-dimensional Riemannian manifold with parallel spinor field is flat. We consider now the four-dimensional case. The bundle Λ^2M^4 decomposes into $\Lambda^2M^4 = \Lambda^2_+ \oplus \Lambda^2_-M^4$ and the curvature tensor $\mathcal{R} : \Lambda^2M^4 \longrightarrow \Lambda^2M^4$ is given by

$$\mathcal{R} = \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix} + \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} - \frac{R}{12}.$$

Suppose now that M^4 admits a parallel spinor ψ. Then $Ric = 0$ and, consequently, the curvature tensor $\mathcal{R}$ coincides with the Weyl tensor,

$$\mathcal{R} = \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix}.$$

Moreover, we have (see Chapter 1, Theorem 12)

$$W(\eta^2)\cdot\psi = 0.$$

The spinor bundle S decomposes into $S = S^+ \oplus S^-$. The condition

$$W(\eta^2)\psi^\pm = 0, \qquad 0 \neq \psi^\pm \in \Gamma(S^\pm)$$

implies $W_+ = 0$ by an algebraic computation (see Chapter 1, proof of Theorem 13$^-$). Therefore, a 4-dimensional Riemannian spin manifold M^4 with parallel spinors ψ^+, ψ^- in S^+, S^- is flat. We consider now the case that M^4 admits a parallel spinor $\psi^+ \in \Gamma(S^+)$. In this case we define an almost complex structure $J: TM^4 \longrightarrow TM^4$ by the formula

$$J(X)\cdot\psi^+ = i\,X\cdot\psi^+.$$

Since ψ^+ is parallel, J is parallel too. In particular, (M^4,g,J) is a Ricci flat Kähler manifold. In case M^4 is a compact manifold, its first Chern class $c_1(M^4)$ in the de-Rham-cohomology is given by the Ricci-form (see [105]). With respect to $Ric \equiv 0$ we conclude

$$c_1(M^4) = 0.$$

A compact, complex surface M^4 satisfying $c_1(M^4) = 0$ is said to be a K3-surface (see [18]). It follows from the solution of the Calabi conjecture that any K3-surface admits an (anti-)self-dual Ricci-flat Kähler metric (see [17]). Thus, we obtain

<u>Theorem 1</u> (see [60],[17]): A compact, non-flat 4-dimensional Riemannian spin manifold with a parallel spinor is isometric to a K3-surface with an (anti-) self-dual Ricci-flat Kähler metric. Any K3-surface admits two independent parallel spinors in S^+.

We describe now all non-flat compact 5-dimensional Riemannian manifolds M^5 with parallel spinors. In case of dimension 5 a parallel spinor ψ defines a 1-form η by $\eta(X) := -i \langle X\psi, \psi\rangle$. η is parallel and one obtaines a foliation of M^5. We will prove that this foliation is a fibration of M^5 over S^1. The fibres are totally geodesic K3-surfaces.

<u>Theorem 2</u> ([42]): If (M^5, g) is a non-flat compact Riemannian spin manifold with parallel spinor, then there exist a K3-surface F with an anti-selfdual Kähler-Einstein metric and a holomorphic isometry $\Phi : F \rightarrow F$ such that M^5 is isometric to $F_\Phi = F \times I/_\sim$ with the identification $(f,0) \sim (\Phi(f),1)$.
The two spin structures of M^5 correspond to the two possible lifts $\Phi_\pm$ of Φ into the unique spin structure of F. The parallel spinors ψ of M^5 with respect to the corresponding spin structure are given by the $\Phi_\pm$ -invariant parallel spinors ψ^* of F.

The bundle S^+ of a K3-surface is isomorphic to $\Lambda^{0,0} \oplus \Lambda^{0,2}$. The lifts $\Phi_\pm$ of a holomorphic isometry Φ into S^+ are given by $\Phi_\pm(f,\omega) = (\pm f\Phi^{-1}, \pm \Phi_*\omega)$. We call the spin structure of $M^5 = F_\Phi$ defined by Φ_+ a "positive spin structure", the other one a "negative spin structure".

<u>Theorem 3</u> ([42]): The space of parallel spinors of $M^5 = F_\Phi$ with respect to the positive spin-structure has dimension one or two. M^5 admits two linearly independent parallel spinors if and only if the holomorphic 2-form h^2 on the K3-surface is Φ-invariant.
The dimension of the space of all parallel spinors with respect to the negative spin structure is at most 1. In this spin structure a parallel spinor exists if and only if $\Phi^*(h^2) = -h^2$ holds.

Proof of Theorem 2 and Theorem 3: Let (M^5, g) be a compact non-flat Riemannian spin manifold with a parallel spinor field ψ of length $|\psi| = 1$. In particular, M^5 is Ricci-flat. By

$$\xi\psi = i\psi \quad \text{and} \quad \eta(X) := -i\langle X\psi, \psi\rangle$$

the spinor ξ defines a parallel vector field ξ and a parallel 1-form η, respectively, i.e. $\nabla\xi = 0$, $\nabla\eta = 0$. The 1-form η is closed and vanishes nowhere. By the Frobenius Theorem η defines a foliation of M^5, ξ is the normal vector field.

Because of $\nabla\xi = 0$, all leaves are totally geodesic submanifolds of M^5. We will now prove that this foliation is a fibration over S^1. Since η is closed, we can define a homomorphism

$$\int\eta : \pi_1(M^5) \longrightarrow \mathbb{R}$$

by $\gamma \mapsto \int_\gamma \eta$, where γ is a closed curve in M^5.

This homomorphism is non-trivial, since on account of $\mathrm{Hom}(\pi_1(M^5);\mathbb{R}) = \mathrm{Hom}(H_1(M^5;Z);\mathbb{R}) = H^1_{dR}(M^5;\mathbb{R})$, it would follows from $0 = \int\eta \in \mathrm{Hom}(\pi_1(M^5);\mathbb{R})$ that η is the differential of a smooth function on M^5. However, η vanishes nowhere.

Since M^5 is a compact Ricci-flat, non-flat Riemannian manifold, we have $b_1(M^5) \leq 1$ for the first Betti number $b_1(M^5)$ of M^5 (s. [104]). Hence, the image of this homomorphism is a discrete subgroup of $\mathbb{R}^1$, i.e. $C\cdot\mathbb{Z}$ for a positive number $C \in \mathbb{R}$.

We fix a point $m_0 \in M^5$ and define a function

$$f: M^5 \longrightarrow \mathbb{R}^1/C\cdot\mathbb{Z} = S^1$$

by $\quad f(m) := \int_\gamma \eta \bmod C\cdot\mathbb{Z},$

where γ is a curve from m_0 to m. Because of the above mentioned properties of $\int\eta \in \mathrm{Hom}(\pi_1(M^5);\mathbb{R})$ this definition is correct. We have $df = \eta$. Consequently f is a submersion and the leaves of the foliation $\eta = 0$ are the connected components of the fibres of f. If each fibre consists of k leaves, then $f_{\#}\pi_1(M^5)$ is a subgroup of index k in $\pi_1(S^1)$ because of the exactness of the homotopy sequence of the fibration f. In this case we can lift f into the k-fold covering of S^1 and we obtain a fibration $\hat{f}: M^5 \longrightarrow S^1$ with the property, that the fibres of $\hat{f}$ are the leaves of $\eta = 0$. $\nabla\xi = 0$ implies that the flow of ξ maps the fibres isometrically onto each other.

The fibres are anti-selfdual Ricci-flat Riemannian manifolds. This can be proved in the following way. We choose a local section in the SU(2)-reduction $Q(\psi)$ of the frame bundle (see [41]). Let $\omega_{i\gamma}$ $(1 \leq i, j \leq 5)$ be the coefficients of the Levi-Civita connection with respect to this frame. From $\nabla\Psi = 0$, i.e. $\sum_{i<j} \omega_{ij} e_i e_j u(1,1) = 0$, it follows that

$$\omega_{12} + \omega_{34} = 0, \quad \omega_{13} = \omega_{24}, \omega_{14} + \omega_{23} = 0$$
$$\omega_{i5} = 0 \quad (1 \leq i \leq 5).$$

However, the ω_{ij} $(1 \leq i,j \leq 4)$ are the 1-forms of the Levi-Civita connection of the fibres. Consequently, the Levi-Civita connection in the principal SO(4)-bundle is anti-selfdual. This is equivalent to the statement (s.[33]).

Thus each fibre is a compact, connected, anti-selfdual Ricci-flat Riemannian manifold. Consequently, we have only the following possibilities (s. [60]).

1) All fibres are flat.
2) All fibres are K3-surfaces.
3) The fibres are Enriques surfaces.
4) Each fibre is of the form N/T, where N is an Enriques surface and T is an antiholomorphic involution on N.

Case 1) is impossible, since it would imply M^5 to be flat. The Cases 3) and 4) are also impossible, because the fibres are spin manifolds, but Enriques surfaces do not admit a spin structure. Consequently M^5 fibres into K3-surfaces being isometrical to each other. Using the flow of ξ one can consider M^5 as the Riemannian product $F \times I$ of the fibre F and the interval $I = [0,1]$, where $F \times \{0\}$ and $F \times \{1\}$ are identified by an isometry $\Phi: F \to F$. We want to show that Φ is also holomorphic. The restriction ψ/F of ψ to any fibre is a parallel spinor, too. On the other hand, ψ/F is a section in S^+ since F is a K3-surface (see [60]). Hence, the equation $(J\vec{t})\psi = i\vec{t}\psi$ defines the complex structure $J: TF \to TF$ of the fibre F. Since ψ/F is invariant with respect to one of the lifts $\Phi_\pm$ of Φ, we obtain $d\Phi J = J d\Phi$, i.e. Φ is holomorphic. Finally, we see that the space of parallel spinors of M^5 is equal to the space of Φ_+- and Φ_--invariant parallel spinors on the K3-surface F, respectively. This proves Theorem 2.

Let h^2 denote the 'unique' holomorphic 2-form of the K3-surface F and let $\psi \in S^+$ be a parallel spinor. Because of the isomorphy $S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$, ψ corresponds to an element $(A, B\bar{h}^2) \in \Lambda^{0,0} \oplus \Lambda^{0,2}$, where $A,B \in \mathbb{C}$ are constant on F. Let $\Phi: F \to F$ be a holomorphic isometry. Then we have

$$\Phi_\pm(A, B\bar{h}^2) = (\pm A, \pm B \Phi^* \bar{h}^2).$$

Consequently there exists at most one parallel spinor with respect to the negative spin structure of F_Φ, namely $(0, \bar{h}^2)$. The spinor

$(0,\bar{h}^2)$ is Φ_--invariant if and only if $\Phi^* h^2 = -h^2$. With respect to Φ_+ we have at least one parallel spinor, namely $(1,0)$. Furthermore, the spinor $(0,\bar{h}^2)$ is invariant if and only if $\Phi^* h^2 = h^2$. This proves Theorem 3.

The existence of parallel spinors implies topological conditions on M^5. We consider $H^2(M^5;\mathbb{R})$. Let $M^5 = F_\Phi$.
Using

$$H^2(M^5;\mathbb{R}) = \{\varepsilon^2 \in H^2(F;\mathbb{R}) : \Phi^* \varepsilon^2 = \varepsilon^2\}$$

and the global Torelli Theorem [18] we obtain

Corollary 1: Let (M^5,g) be a 5-dimensional compact non-flat Riemannian manifold with parallel spinor. Then

$$2 \leq \dim H^2(M^5;\mathbb{R}) \leq 22.$$

Moreover, $\dim H^2(M^5;\mathbb{R}) = 22$ if and only if M^5 is isometric to the Riemannian product of a K3-surface by S^1.

Corollary 2: If there are two independent parallel spinors on M^5, then

$$4 \leq \dim H^2(M^5;\mathbb{R}) \leq 22.$$

Remark: Since the automorphism group of a K3-surface is finite all integral curves of the vector field ξ are closed.

Examples may be found in [42].
With the same method we now classify all 7-dimensional compact Riemannian manifolds with three or two parallel spinors.

Theorem 4 (see [44]): Let (M^7,g) be a non-flat compact 7-dimensional Riemannian spin manifold with at least three parallel spinors. Then there exists a K3-surface F with an anti-selfdual Ricci-flat Riemannian metric, a lattice $\Gamma \subset \mathbb{R}^3$ and a representation $\rho : \Gamma \longrightarrow \mathrm{Aut}_{h^2}(F)$ of Γ into the group of all automorphisms of F preserving the unique holomorphic 2-form h^2 such that M^7 is isometric to $(\mathbb{R}^3 \times F)/\Gamma$, where Γ acts on $\mathbb{R}^3 \times F$ by $\gamma\cdot(x,f) = (x+\gamma, \rho(\gamma)f)$.
Conversely, a 7-dimensional Riemannian manifold of this type admits at least four parallel spinors.

Proof: Let ψ_1, ψ_2, ψ_3 be three orthogonal parallel spinors of length 1. In the same way as in section 4.4 we define vector fields X_i and 1-forms η_i $(i=1,2,3)$ by $X_1\psi_1 = \psi_2$, $X_2\psi_2 = \psi_3$,

$X_3\psi_2 = \psi_3$, $\eta_i = g(\cdot, X_i)$, for which $\nabla\eta_i = 0$ and $\nabla X_i = 0$ holds. By the Frobenius Theorem we have a foliation $M^7 = \bigcup F^4_\alpha$ of M^7 into totally geodesic, connected, complete manifolds F^4_α. The 1-forms ω_{ij} of the Levi-Civita connection with respect to a frame of the SU(2)-reduction $Q(\psi_1, \psi_2, \psi_3)$ satisfy

$$\omega_{13} = \omega_{24}, \quad \omega_{14} + \omega_{23} = 0, \qquad \omega_{12} + \omega_{34} = 0$$
$$\omega_{i5} = \omega_{i6} = \omega_{i7} = 0 \quad (1 \le i \le 7).$$

So one can prove in the same way as above that the leaves F^4_α are antiselfdual and Ricci-flat.

We have $b_1(M^7) \le 3$ (see [30]). On the other hand, the 1-forms η_1, η_2, η_3 are linearly independent, and consequently $b_1(M^7) = 3$. We fix a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of the torsion-free part of $H_1(M^7; Z)$ and consider the homomorphism $L: \pi_1(M^7) \to R^3$ given by

$$L(\gamma) = \left(\int_\gamma \eta_1, \int_\gamma \eta_2, \int_\gamma \eta_3\right)$$

The vectors $L(\alpha_1), L(\alpha_2), L(\alpha_3)$ are linearly independent in $\mathbb{R}^3$, because $A_1L(\alpha_1) + A_2L(\alpha_2) + A_3L(\alpha_3) = 0$ implies

$$\int_{A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3} \eta_i = 0 \quad (i=1,2,3), \text{ and therefore}$$

$A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 = 0$ in $H_1(M^7; \mathbb{R})$. Let Γ be a lattice generated by $L(\alpha_1)$, $L(\alpha_2)$, $L(\alpha_3)$. Then we obtain a submersion $f: M^7 \to \mathbb{R}^3/\Gamma$ defined by $f(m) = \left(\int_c \eta_1, \int_c \eta_2, \int_c \eta_3\right) \bmod \Gamma$, where c is a curve from a fixed point m_0 to m.

Since $T_xF^4_\alpha = \{t \in T_xM^7 : df(t) = 0\}$, the leaves of the foliation $\bigcup F^4_\alpha$ are contained in the fibres of the submersion f. As in the case of dimension 5 we may assume that the fibres of f are connected and coincide with the leaves F^4_α. The parallel transport in a Riemannian submersion with totally geodesic fibres maps the fibres isometrically onto each other. Thus all fibres are isometric. They are anti-selfdual, Ricci-flat, compact Riemannian manifolds. They obtain a spin structure, since the normal bundle of any fibre is trivial. As in the proof of Theorem 2 one concludes, using again the Hitchin result (see [60]), that all fibres are isometric to a K3-surface F.

Consider the covering $\mathbb{R}^3 \to \mathbb{R}^3/\Gamma$ as well as the induced fibration $\tilde{M}^7 \to \mathbb{R}^3$ over $\mathbb{R}^3$. Then M^7 is isometric to $\tilde{M}^7/\Gamma$. On the other hand, the parallel transport defines an isometry from $\tilde{M}^7$ to $\mathbb{R}^3 \times F$. The action of Γ on F preserves the holomorphic structure as well as the unique holomorphic 2-form h^2. Indeed, the holomorphic structure of any K3-surface is given by the parallel spinors on it and h^2 is one of the two parallel spinors under the isomorphism $S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ of the spinor bundle S^+. We restrict the parallel spinors ψ_1, ψ_2, ψ_3 on M^7 to the fibre F. Since the restriction of the Spin(7)-representation to the subgroup Spin(4) is equivalent to $\Delta_4 \oplus \Delta_4$, each $\psi_i/_F$ (i=1,2,3) corresponds to a pair of parallel spinors on F. The Γ-action on Γ preserves $\psi_i/_F$ since the ψ_i are parallel on M^7. Consequently, Γ acts on F holomorphically and preserves h^2.

<u>Theorem 5</u> (see [44]): Let M^7 be a compact non-flat Riemannian manifold with two parallel spinors. Then either M^7 admits at least four parallel spinors and is isometric to $(\mathbb{R}^3 \times F)/\Gamma$ for a certain K3-surface F or there exists a Ricci-flat compact Kähler manifold N^6 and a holomorphic isometry $\Phi: N^6 \to N^6$ such that M^7 is isometric to $M^7 = N^6 \times [0,1]/_\sim$ with the identification $(x,0) \sim (\Phi(x),1)$.

<u>Proof:</u> Consider two parallel spinors ψ_1, ψ_2 as well as the parallel 1-form η defined by $\eta\psi_1 = \psi_2$. M^7 is a compact Ricci-flat Riemannian manifold and the first Betti number is at least 1.
In case $b_1(M^7) = 1$, we can prove in the same way as for dimension 5 that the leaves of the foliation given by η are fibres of a Riemannian submersion $f: M^7 \to S^1$ with totally geodesic Ricci-flat fibres N^6. Using the parallel transport defined by the vector field corresponding to η, M^7 becomes isometric to $N^6 \times I/\sim$ for some $\Phi: N^6 \to N^6$. Φ preserves $\psi_1|_{N^6}$ and $\psi_2|_{N^6}$. Consequently, Φ is holomorphic. If $b_1(M^7) \geq 2$, then there exists a harmonic 1-form η_0 orthogonal to η in L^2. The Weitzenböck formula $0 = \Delta\eta_1 = \nabla^*\nabla\eta_1 + Ric(\eta_1)$ yields that η_1 is a parallel 1-form orthogonal to η at any point of M^7. Then $\psi_1, \psi_2 := \eta\cdot\psi_1$ and $\psi_3 := \eta_1\cdot\psi_1$ are orthogonal and parallel spinors on M^7.

<u>Chapter 7: Riemannian Manifolds with Imaginary Killing Spinors</u>

According to Theorem 9 of Chapter 1 imaginary Killing spinors can only occur on non-compact manifolds.
In this chapter we will prove the following statement:
A complete, non-compact, connected Riemannian spin manifold admits non-trivial (imaginary) Killing spinors if and only if it is isometric to a warped product

$$(F^{n-1} \times \mathbb{R},\ e^{-4\mu t}h \oplus dt^2), \qquad \mu \in \mathbb{R} \setminus \{0\},$$

where (F^{n-1},h) is a complete, connected spin manifold with non-trivial parallel spinor fields. To prove this, we distinguish two types of imaginary Killing spinors, those where the constant Q_φ, defined in Chapter 2.3 for each twistor spinor, is zero and those, where Q_φ is greater than zero. These two types are characterized by a different behaviour of their length function. The length function of an imaginary Killing spinor is, in opposite to that of a real Killing spinor, non-constant and contains enough information to describe the above mentioned geometric structure of the underlying manifold.

7.1. Imaginary Killing Spinors of Type I and Type II

Let (M^n,g) be a connected spin manifold. First, we prove some properties of the length function

$$u_\varphi(x) := \langle \varphi(x), \varphi(x) \rangle$$

of an imaginary Killing spinor φ.

Lemma 1: Let φ be an imaginary Killing spinor to the Killing number μi. Then

1) $$X(u_\varphi) = 2\mu i \langle X\cdot\varphi, \varphi\rangle \tag{7.1}$$
$$YX(u_\varphi) = 2\mu i \langle \nabla_Y X\cdot\varphi, \varphi\rangle + 4\mu^2 g(X,Y)u_\varphi \tag{7.2}$$

2) $$\nabla_X \operatorname{grad} u_\varphi = 4\mu^2 u_\varphi X \tag{7.3}$$
$$\|\operatorname{grad} u_\varphi\|^2 = -4\mu^2 \sum_{j=1}^{n} \langle s_j\cdot\varphi, \varphi\rangle^2 \tag{7.4}$$

3) Let γ be a normal geodesic in (M,g).
Then
$$u_\varphi(\gamma(t)) = A\, e^{2\mu t} + B e^{-2\mu t}, \tag{7.5}$$
where A and B are real constants.

Proof: For a vector field X on M we have

$$X(u_\varphi) = \langle \nabla_X\varphi, \varphi\rangle + \langle \varphi, \nabla_X\varphi\rangle = i\mu\{\langle X\cdot\varphi, \varphi\rangle - \langle\varphi, X\cdot\varphi\rangle\} =$$
$$= 2\mu i\ \langle X\cdot\varphi, \varphi\rangle .$$

For the second derivative of u_φ this provides

$$\begin{aligned} XY(u_\varphi) &= 2\mu i\{\langle\nabla_Y X\cdot\varphi,\varphi\rangle + \langle X\cdot\nabla_Y\varphi,\varphi\rangle + \langle X\cdot\varphi,\nabla_Y\varphi\rangle\} \\ &= 2\mu i\{\langle\nabla_Y X\cdot\varphi,\varphi\rangle + \mu i\langle X\cdot Y\cdot\varphi,\varphi\rangle - \mu i\langle X\cdot\varphi, Y\cdot\varphi\rangle\} \\ &= 2\mu i\{\langle\nabla_Y X\cdot\varphi,\varphi\rangle + \mu i\langle (X\cdot Y+Y\cdot X)\varphi,\varphi\rangle\} \\ &= 2\mu i\{\langle\nabla_Y X\cdot\varphi,\varphi\rangle + 4\mu^2 g(X,Y)u_\varphi .\end{aligned}$$

(7.1) implies

$$\|\operatorname{grad} u_\varphi\|^2 = \sum_{j=1}^{n} s_j(u_\varphi)^2 = -4\mu^2 \sum_{j=1}^{n}\langle s_j\cdot\varphi,\varphi\rangle^2 .$$

Let $(s_1,\dots,s_n)$ be a local ON-basis arising from an ON-basis in x by parallel displacement along geodesics. Then, (7.2) implies

$$\begin{aligned} \nabla_X \operatorname{grad} u_\varphi &= \sum_{j=1}^{n}\nabla_X(s_j(u_\varphi)s_j) = \sum_{j=1}^{n} Xs_j(u_\varphi)\cdot s_j \\ &= 4\mu^2 u_\varphi \sum_{j=1}^{n} g(s_j,X)s_j \\ &= 4\mu^2 u_\varphi\cdot X .\end{aligned}$$

Let $\gamma(t)$ be a normal geodesic in (M,g) and let $u(t):= u_\varphi(\gamma(t))$. Using (7.2) we obtain

$$\begin{aligned} u''(t) &= 2\mu i\langle\nabla_{\gamma'}\gamma'\cdot\varphi,\varphi\rangle + 4\mu^2\|\gamma'\|^2 u(t) \\ &= 4\mu^2 u(t). \end{aligned}$$

The general solution of this equation is

$$u(t) = A\, e^{2\mu t} + Be^{-2\mu t},$$

where $A,B\in\mathbb{R}$.

In Chapter 2.3, a constant $Q_\varphi \geq 0$ was assigned to each twistor spinor. If φ is a Killing spinor to the Killing number μi, then, using (7.4), we obtain

$$Q_\varphi = n^2\mu^2\{u_\varphi^2(x) - \frac{1}{4\mu^2}\|\operatorname{grad} u_\varphi(x)\|^2\}.$$

Furthermore, according to Theorem 9, Chapter 2, we have

$$Q_\varphi = n^2\mu^2 u_\varphi\cdot \operatorname{dist}^2(V_\varphi, i\varphi),$$

where V_φ is the real subbundle $V_\varphi = \{t\cdot\varphi \mid t\in TM\}$ of the spinor bundle.

We call φ a <u>Killing spinor of type I</u> iff $Q_\varphi = 0$ and a <u>Killing spinor of type II</u> if $Q_\varphi > 0$.

Then, φ is of type I if and only if there exists a unit vector field ξ on M such that $\xi\cdot\varphi = i\varphi$. If φ is of type II, then the length function u_φ is bounded from below by the positive constant $\frac{1}{n|\mu|}\sqrt{Q_\varphi}$. As it was shown in Corollary 2, Chapter 4.3., all imaginary Killing spinors on 3- and 5-dimensional manifolds are of type I.

Now, we prove two lemmas which show how a Killing spinor to the Killing number $-\mu i$ can be constructed from a Killing spinor to the Killing number μi.

Lemma 2: Let φ be a Killing spinor of type II to the Killing number μi. Then the spinor field

$$\psi := (u_\varphi + \frac{1}{2\mu i} \text{ grad } u_\varphi \cdot)\varphi$$

is a Killing spinor of type II to the Killing number $-\mu i$.

Proof: By formula (7.3) we obtain

$$\begin{aligned}
\nabla_X\psi &= \nabla_X(u_\varphi\varphi) + \frac{1}{2\mu i}\cdot\nabla_X(\text{grad } u_\varphi\cdot\varphi)\\
&= X(u_\varphi)\varphi + i\mu u_\varphi X\cdot\varphi + \frac{1}{2\mu i}\nabla_X(\text{grad } u_\varphi)\cdot\varphi + \frac{1}{2}\text{ grad } u_\varphi\cdot X\cdot\varphi\\
&= X(u_\varphi)\varphi + i\mu u_\varphi X\cdot\varphi - 2\mu i\ u_\varphi\cdot X\cdot\varphi - \frac{1}{2} X\cdot\text{grad } u_\varphi\cdot\varphi\\
&\quad - g(\text{grad } u_\varphi, X)\varphi\\
&= -\mu i\ X\cdot\left\{u_\varphi + \frac{1}{2\mu i}\text{ grad } u_\varphi\right\}\cdot\varphi\\
&= -\mu i\ X\cdot\psi .
\end{aligned}$$

From (7.1) it follows

$$\begin{aligned}
u_\psi &= \| u_\varphi\cdot\varphi + \frac{1}{2\mu i}\text{ grad } u_\varphi\cdot\varphi\|^2\\
&= u_\varphi^3 + \frac{1}{4\mu^2}\|\text{grad } u_\varphi\|^2 u_\varphi + \frac{u_\varphi}{\mu i}\langle\text{grad } u_\varphi\cdot\varphi,\varphi\rangle\\
&= u_\varphi\left\{u_\varphi^2 - \frac{1}{4\mu^2}\|\text{grad } u_\varphi\|^2\right\}\\
&= u_\varphi\frac{Q_\varphi}{n^2\mu^2} .
\end{aligned}$$

This implies $Q_\psi = \frac{Q_\varphi^3}{n^4\mu^4} > 0$. Hence, ψ is of type II.

Lemma 3: Let (M^n,g) be an even-dimensional manifold with a Killing spinor $\varphi = \varphi^+ \oplus \varphi^-$ to the Killing number λ. Then $\varphi_1 := \varphi^+ - \varphi^-$ is a Killing spinor of the same type to the Killing number $-\lambda$.

Proof: Since the Clifford multiplication commutes the positive and negative part of S, we obtain

$$\nabla_X\varphi^+ = \lambda X\cdot\varphi^- \quad\text{ and }\quad \nabla_X\varphi^- = \lambda X\cdot\varphi^+.$$

Hence,

$$\nabla_X\varphi_1 = \nabla_X\varphi^+ - \nabla_X\varphi^- = \lambda X\cdot\varphi^- - \lambda X\cdot\varphi^+ = -\lambda X\cdot\varphi_1 .$$

Because of

$u_\varphi = \|\varphi\|^2 = \|\varphi^+\|^2 + \|\varphi^-\|^2 = \|\varphi_1\|^2 = u_{\varphi_1}$, it follows that $Q_\varphi = Q_{\varphi_1}$.

In Chapter 7.4 we will see that there exist odd-dimensional manifolds with non-trivial Killing spinors to the Killing number μi and no Killing spinors to the number $-\mu i$.
Finally, we want to study which Killing spinors of the hyperbolic space are of type I and which of type II.

In example 3, Chapter 1, we have seen that the Killing spinors on the hyperbolic space $H^n_{-4\mu^2}$ of constant sectional curvature $-4\mu^2$ (realized in the Poincaré-model) are the spinors

$$\mathcal{K}(H^n_{-4\mu^2})_{\mu i} = \left\{ \tilde{\varphi}_u \mid \varphi_u(x) := \sqrt{\frac{2}{1-4\mu^2\|x\|^2}}\,(1+2\mu i x\cdot)u,\ u \in \mathbb{C}^{2^m} \right\}$$

Using formula (7.4) we obtain for the constant $Q_{\tilde{\varphi}_u}$

$$\begin{aligned} \frac{1}{n^2\mu^2} Q_{\tilde{\varphi}_u} &= \|\varphi_u(0)\|^4 - \frac{1}{4\mu^2}\|\operatorname{grad}\langle\varphi_u,\varphi_u\rangle(0)\|^2 \\ &= 4\|u\|^4 + \sum_{j=1}^{n} \langle e_j\cdot\varphi_u(0),\varphi_u(0)\rangle^2 \\ &= 4\left\{\|u\|^4 + \sum_{j=1}^{n} \langle e_j\cdot u,u\rangle^2\right\}. \end{aligned}$$

Hence, in case $n \neq 3,5$, almost all Killing spinors on the hyperbolic space are of type II. Using formula (1.1) and (1.2) it is easy to verify that in case $n = 2m$ the space

$$\operatorname{span}\left\{ \tilde{\varphi}_u \mid u = u(\varepsilon_1,\dots,\varepsilon_m) + u(\varepsilon_1,\dots,\varepsilon_{m-1},-\varepsilon_m) \mid \varepsilon_j = \pm 1 \right\}$$

is a 2^{m-1}-dimensional subspace of Killing spinors of type I and the spinor fields $\tilde{\varphi}_u$ with $u = u(\varepsilon_1,\dots,\varepsilon_m)$ are of type II. In case $n = 2m+1$,

$$V_\pm := \operatorname{span}\left\{ \tilde{\varphi}_u \mid u = u(\varepsilon_1,\dots,\varepsilon_m) \mid \prod_{j=1}^{m} \varepsilon_j = \pm 1 \right\}$$

are 2^{m-1}-dimensional subspaces of Killing spinors of type I and the spinor fields $\tilde{\varphi}_u$ with $u = u(\varepsilon_1,\dots,\varepsilon_m) + u(\delta_1,\dots,\delta_m)$, where $(\delta_1,\dots,\delta_m)$ differs from $(\varepsilon_1,\dots,\varepsilon_m)$ in an odd and more than one number of elements, are Killing spinors of type II.

7.2. Complete Riemannian Manifolds with Imaginary Killing Spinors of Type II

In this chapter we will prove that the hyperbolic space is the only complete manifold admitting imaginary Killing spinors of type II.

Theorem 1 ([7]): Let (M^n,g) be a complete, non-compact, connected spin manifold with a non-trivial Killing spinor of type II to the Killing number μi. Then (M^n,g) is isometric to the hyperbolic space $H^n_{-4\mu^2}$ of constant sectional curvature $-4\mu^2$.

Proof: Acoording to Theorem 8, Chapter 1, and Theorem 7, Chapter 2, it is sufficient to prove that the length function u_φ of a non-trivial Killing spinor φ of type II attains a minimum. Because of $Q_{\alpha\varphi} = |\alpha|^4 Q_\varphi$ for $\alpha \in \mathbb{C}$ we can suppose that $Q_\varphi = n^2\mu^2$. Then u_φ is bounded from below by 1.
Let $c > 1$ be a real number such that there exists a point $y_0 \in M$ with $u_\varphi(y_0) < c$. We consider the subset $M_c := \{x \in M \mid u_\varphi(x) \leq c\} \subset M$. Let γ be a normal geodesic with $\gamma(0) = y_0$. According to (7.5) we have

$$u(t) = u_\varphi(\gamma(t)) = A\, e^{2\mu t} + B\, e^{-2\mu t} \geq 1.$$

Since (M,g) is complete, u is defined for all $t \in \mathbb{R}$, which implies that $A,B > 0$. The minimum of u is $2\sqrt{AB}$, which shows that

$$1 \leq 2\sqrt{AB} \leq u(0) = A + B < c \qquad \text{(i)}.$$

From (i) we obtain

$$A,B \in \left(\tfrac{1}{2}(c - \sqrt{c^2-1}),\ \tfrac{1}{2}(c + \sqrt{c^2-1})\right). \qquad \text{(ii)}$$

Let $d > 0$ be a parameter such that $\gamma(d) \in M_c$. Then

$$u(d) = A\, e^{2\mu d} + B\, e^{-2\mu d} \leq c.$$

The resulting quadratic equation for $e^{2|\mu| d}$ together with (i) and (ii) yields $e^{2|\mu| d} \leq 2c^2$. Thus, each point of M_c lies in the closed geodesic ball of radius $\frac{1}{|\mu|}\ln(2c)$ around c. Hence M_c is compact and u_φ has a minimum on M_c.

Remark: Let u_φ be the length function of an imaginary Killing spinor φ of type II with $Q_\varphi = n^2\mu^2$. Then (according to the proof of Theorem 7, Chapter 2, and Theorem 1) u_φ has exactly one critical point x_0. Let γ be a normal geodesic with $\gamma(0) = x_0$. For the constants A and B in

$$u(t) = u_\varphi(\gamma(t)) = A\, e^{2\mu t} + B\, e^{-2\mu t}$$

it follows

$$u(0) = u_\varphi(x_0) = \frac{\sqrt{Q_\varphi}}{n|\mu|} = 1 = A + B \quad \text{and}$$

$$u'(0) = 0 = 2\mu(A + B).$$

Hence $A = B = \frac{1}{2}$ and $u(t) = \cosh(2\mu t)$.
Consequently, the length function u_φ satisfies

$$u_\varphi(x) = \cosh(2\mu\, d(x,x_0)) \quad \text{for all} \quad x \in M,$$

where $d(x,x_0)$ denotes the geodesic distance of x and x_0.
We proved that the only <u>complete</u> connected manifold with imaginary Killing spinors of type II is the hyperbolic space. Finally, we want to remark that there exist <u>non-complete</u> manifolds of non-constant sectional curvature which admit imaginary Killing spinors of type II. Such manifolds can be constructed as follows:
Let (F^{n-1},h) be a compact connected spin manifold with real Killing spinors in $\mathcal{K}(F,h)_\mu$ as well as in $\mathcal{K}(F,h)_{-\mu}$. Examples of such manifolds can be found in Chapter 4.2, Theorem 1.
Let (M^n,g) be the warped product

$$(M,g) := (F \times (0,\infty),\ \sinh^2(2\mu t)\cdot h \oplus dt^2).$$

We show that (M,g) has imaginary Killing spinors of type II to the Killing number μi.

<u>Case 1:</u> $n = 2m+1$
Let $\varphi \in \mathcal{K}(F,h)_\mu$ be a real Killing spinor and denote by $\varphi = \varphi^+ \oplus \varphi^-$ the decomposition of φ with respect to $S_F^\pm$. Then $\nabla_X^{S_F}\varphi^+ = \mu X\cdot\varphi^-$ and $\nabla_X^{S_F}\varphi^- = \mu X\cdot\varphi^+$. We define

$$\psi(x,t) := e^{-\hat{\mu}t}\varphi^+(x) + (-1)^m i\, e^{\hat{\mu}t}\varphi^-(x), \quad \hat{\mu} := (-1)^m\mu.$$

Using the denotation of Chapter 1.2 for the spinor calculus on warped products and the formulas (1.20) and (1.21) we will prove that $\tilde{\psi} \in \mathcal{K}(M,g)_{\mu i}$:

$$\begin{aligned}
\nabla_{\tilde{X}}^S\tilde{\psi} &= \frac{1}{\sinh(2\mu t)}\,\widetilde{\left(e^{-\hat{\mu}t}\nabla_X^{S_F}\varphi^+ + (-1)^m i\, \nabla_X^{S_F}\varphi^-\right)} \\
&\quad - \mu\coth(2\mu t)\tilde{X}\cdot\xi\cdot\tilde{\psi} \\
&= \frac{1}{\sinh(2\mu t)}\Big\{ e^{-\hat{\mu}t}\mu\,\widetilde{X\cdot\varphi^-} + (-1)^m i\,\mu e^{\hat{\mu}t}\,\widetilde{X\cdot\varphi^+} - \\
&\qquad - \mu\cosh(2\mu t)\big[(-1)^m i\, \widetilde{e^{-\hat{\mu}t}X\cdot\varphi^+} - \widetilde{e^{\hat{\mu}t}X\cdot\varphi^-}\big]\Big\} \\
&= \mu i\,\big\{(-1)^m i\, \widetilde{e^{\hat{\mu}t}X\cdot\varphi^-} + \widetilde{e^{-\hat{\mu}t}X\cdot\varphi^+}\big\} \\
&= \mu i\,\tilde{X}\cdot\tilde{\psi} \qquad\qquad \text{if} \quad X \in T_xF
\end{aligned}$$

and

$$\begin{aligned}
\nabla_\xi^S\tilde{\psi} &= -\hat{\mu}\, e^{-\hat{\mu}t}\varphi^+ + i\mu\, e^{\hat{\mu}t}\varphi^- \\
&= \mu i\, e^{-\hat{\mu}t}\xi\cdot\tilde{\varphi}^+ - \mu(-1)^m e^{\hat{\mu}t}\xi\cdot\tilde{\varphi}^- \\
&= \mu i\,\xi\cdot\tilde{\psi}
\end{aligned}$$

Unless (F,h) is the standard sphere, $|\varphi^+|^2 = |\varphi^-|^2 = \text{const} =: c > 0$ (see Lemma 1, Chapter 5.1) is valid and it is easy to verify that $Q_{\tilde{\psi}} = 4c^2n^2\mu^2 > 0$.

Case 2: $n = 2m + 2$

Let $\varphi_1 \in \mathcal{K}(F,h)_\mu$ and $\varphi_2 \in \mathcal{K}(F,h)_{-\mu}$.
We define (using the denotations of Chapter 1.2 for the spinor calculus on warped products):

$$\psi(x,t) := [(ie^{\mu t} + e^{-\mu t})\varphi_1 + (-i\,e^{\mu t} + e^{-\mu t})\varphi_2] \oplus (-1)^m [(e^{-\mu t} - i\,e^{\mu t})\hat{\varphi}_1 + (e^{-\mu t} + i\,e^{\mu t})\hat{\varphi}_2]$$

Using

$$\hat{\nabla}^{S_F}_X \hat{\varphi}_1 = \widehat{\nabla^{S_F}_X \varphi_1} = \mu\, \widehat{X \cdot \varphi_1} = -\mu X \cdot \hat{\varphi}_1 \quad \text{and} \quad \nabla^{\hat{S}_F}_X \hat{\varphi}_2 = \mu X \cdot \hat{\varphi}_2$$

we obtain by applying (1.22) and (1.23) as in the first case

$\nabla^S_{\tilde{X}} \tilde{\psi} = i\,\mu \tilde{X} \cdot \tilde{\psi}$ for all $X \in T_xF$ and

$\nabla^S_{\xi} \tilde{\psi} = i\,\mu\, \xi \cdot \tilde{\psi}$.

Hence, $\tilde{\psi} \in \mathcal{K}(M,g)_{\mu i}$.
Unless (F,h) is isometric to the standard sphere, φ_1 and φ_2 have constant length $c_1 \equiv \|\varphi_1^2\|$ and $c_2 \equiv \|\varphi_2\|^2$, and are orthogonal to each other:

$\langle \varphi_1, \varphi_2 \rangle \equiv 0$ (see Chapter 5.1, Lemma 3).

Furthermore, $\|\varphi_i\|^2 = \|\hat{\varphi}_i\|^2$ and $\langle \varphi_1, \varphi_2 \rangle = \langle \hat{\varphi}_1, \hat{\varphi}_2 \rangle$.
It follows $Q_{\tilde{\psi}} = 16\,(c_1 + c_2)^2 \cdot n^2\mu^2 > 0$.

7.3. Complete Riemannian Manifolds with Imaginary Killing Spinors of Type I

In this chapter we will study the structure of complete spin manifolds admitting imaginary Killing spinors of type I. In fact, we consider a more general question. We derive the structure of complete manifolds admitting a twistor spinor φ such that $Q_\varphi = 0$ and

$$\nabla_X \varphi + \frac{ib}{n} X \cdot \varphi = 0$$

for a real function b on M without zeros (see also Chapter 2.5). If $b = -\mu n$ = constant, then φ is a Killing spinor of type I to the Killing number μi.

Theorem 2: ([6],[7],[85]): Let (M^n,g) be a complete, connected spin manifold admitting a twistor spinor φ such that $Q_\varphi = 0$ and

$\nabla_X\varphi + \frac{ib}{n} X\cdot\varphi = 0$ for a real function b on M without zeros. Then each level set F of the length function u_φ is an $(n-1)$-dimensional complete connected submanifold with parallel spinor fields. The function b is constant on F and for the normal geodesics $\gamma_t(x)$, orthogonal to F, the function $b(t) := b(\gamma_t(x))$ does not depend on $x \in F$. Moreover, (M^n, g) is isometric to the warped product

$$(F \times \mathbb{R},\ e^{\frac{4}{n}\int_0^t b(s)ds}\, g|_F \oplus dt^2).$$

<u>Proof:</u> Since $Q_\varphi = 0$ and b has no zeros, according to Theorem 9, Chapter 2, there exists a unit vector field ξ on M such that

$$\xi\cdot\varphi = i\varphi\ . \tag{7.6}$$

Using

$$\begin{aligned} X(u_\varphi) &= \langle\nabla_X\varphi, \varphi\rangle + \langle\varphi, \nabla_X\varphi\rangle \\ &= \langle -\tfrac{ib}{n} X\cdot\varphi, \varphi\rangle + \langle\varphi, -\tfrac{ib}{n} X\cdot\varphi\rangle \\ &= -\tfrac{2ib}{n}\langle X\cdot\varphi, \varphi\rangle \end{aligned} \tag{7.7}$$

we obtain

$$\xi(u_\varphi) = -\frac{2ib}{n}\langle\xi\cdot\varphi, \varphi\rangle = \frac{2b}{n} u_\varphi\ . \tag{7.8}$$

Hence, u_φ is regular in each point and the level sets $F(c) := u_\varphi^{-1}(c)$ of u_φ are $(n-1)$-dimensional complete submanifolds of M. Multiplying equation (2.14) by φ yields

$$\begin{aligned} &i\, s_\beta(b)u_\varphi + \left(\frac{nR}{4(n-1)} + b^2\right)\langle s_\beta\cdot\varphi, \varphi\rangle \ + \\ &+ \frac{1}{2}\, i \sum_{\alpha=1}^{n} s_\alpha(b)\langle(s_\alpha\cdot s_\beta - s_\beta\cdot s_\alpha)\cdot\varphi, \varphi\rangle = 0. \end{aligned}$$

Since $\langle s_\beta\cdot\varphi, \varphi\rangle$ and $(s_\alpha\cdot s_\beta - s_\beta\cdot s_\alpha)\cdot\varphi, \varphi\rangle$ are purely imaginary, the imaginary part of this equation yields:

$$i\, X(b)u_\varphi + \left(\frac{nR}{4(n-1)} + b^2\right)\langle X\cdot\varphi, \varphi\rangle = 0 \tag{7.9}$$

for all vectors X. Together with (7.7) this shows that b is constant on the level sets of u_φ. If X is tangent to a level set of u_φ, then (7.6) and (7.7) provide

$$0 = \langle(\xi\cdot X + X\cdot\xi)\varphi, \varphi\rangle = -2g(X, \xi)u_\varphi\ .$$

Hence, ξ is a normal vector field to the level sets of u_φ. Differentiating (7.6) we obtain

$$\begin{aligned} \frac{b}{n} X\cdot\varphi &= \nabla_X(\xi\cdot\varphi) = \nabla_X\xi\cdot\varphi - \frac{ib}{n}\xi\cdot X\cdot\varphi \\ &= \nabla_X\xi\cdot\varphi + \frac{ib}{n} X\cdot\xi\cdot\varphi + \frac{2ib}{n} g(\xi, X)\varphi \end{aligned}$$

and, therefore,

$$\nabla_X \xi \cdot \varphi - \frac{2b}{n} X \cdot \varphi + \frac{2b}{n} i\, g(\xi, X)\varphi = 0.$$

It follows

$$\nabla_\xi \xi = 0 \qquad (7.10)$$

$$\nabla_X \xi = \frac{2b}{n} X \qquad \text{for all } X \text{ orthogonal to } \xi. \qquad (7.11)$$

Using (7.10) and (7.11) we see that the 1-form η, which is dual to ξ, is closed. Hence, the level sets of u_φ are just the leaves of the foliation defined by η. For the Lie derivative of the metric g we obtain

$$\begin{aligned}(\mathcal{L}_\xi g)(W_1, W_2) &= g(\nabla_{W_1}\xi, W_2) + g(W_1, \nabla_{W_2}\xi) \\ &= \frac{4b}{n} g(W_1^\perp, W_2^\perp),\end{aligned} \qquad (7.12)$$

where $W_j^\perp$ denotes the component of W_j which is orthogonal to ξ. Let us denote by γ_t the integral curves of ξ. Because of (7.10) these curves are geodesics, which are defined for all $t \in \mathbb{R}$, since (M,g) is complete. Hence, $\{\gamma_t\}$ is a $\mathbb{R}$-parametric group of diffeomorphisms $\gamma_t : M \longrightarrow M$. From (7. 12) we obtain for each vector $X \in T_xM$

$$\begin{aligned}\frac{d}{dt}(\gamma_t^* g)_x(X, \xi) &= (\gamma_t^* \mathcal{L}_\xi g)_x(X, \xi) = \\ &= (\mathcal{L}_\xi g)_{\gamma_t(x)}(d\gamma_t(X), d\gamma_t(\xi)) \\ &= (\mathcal{L}_\xi g)_{\gamma_t(x)}(d\gamma_t(X)), \xi(\gamma_t(x))) \\ &= 0.\end{aligned}$$

Thus,

$$(\gamma_t^* g)_x(X, \xi) = g_{\gamma_t(x)}(d\gamma_t(X), \xi(\gamma_t(x))) = g_x(X, \xi(x))$$

for all $t \in \mathbb{R}$. In particular, for the level set $F(x)$ of u_φ containing the point x it follows

$$d\gamma_t(T_x F(x)) = \mathbb{R}\,\xi(\gamma_t(x))^\perp = T_{\gamma_t(x)} F(\gamma_t(x)) \qquad (7.13)$$

Hence, the diffeomorphism γ_t maps the connected components of the level sets of u_φ one upon the other: $\gamma_t(F(x_o)) = F(\gamma_t(x))_o$.

Let F be a fixed level set of u_φ, $x \in F$ and $c = u_\varphi(x)$. Denoting $u(t) := u_\varphi(\gamma_t(x))$ and $b(t) := b(\gamma_t(x))$ and applying (7.8) we obtain

$$u'(t) = \frac{2b(t)}{n} u(t), \qquad (7.14)$$

which provides

$$u(t) = c\, e^{\frac{2}{n}\int_0^t b(s)ds}$$

Since b has no zeros, $u(t)$ is strictly monoton. Therefore, the integral curve $\gamma_t(x)$ intersects F only for $t = 0$. This shows that the smooth map

$$\begin{aligned} \Phi : F \times \mathbb{R} &\longrightarrow M \\ (x,t) &\longrightarrow \gamma_t(x) \end{aligned}$$

is injective. We will prove that Φ is a diffeomorphism.
For the differential of Φ we obtain

$$d\Phi(X \oplus r\tfrac{\partial}{\partial t}) = d\gamma_t(X) + r\xi(\gamma_t(x)).$$

Because of (7.13), $d\Phi_{(x,t)}$ is an isomorphism and, therefore, Φ a local diffeomorphism. It remains to show that Φ is surjective.
By $U(z) \subset M$ we denote an open neighbourhood of $z \in M$ which is diffeomorphic to a product $V(z) \times (-\varepsilon,\varepsilon)$, where $V(z)$ is an open connected set in $F(z)$.
Let $x_o \in M \setminus F$. We fix a point $x \in F$, and consider a curve δ connecting x_o and x. δ can be covered by a finite number of sets $U(x_j)$, $j=0,\dots,p$, $x_p = x$.
Let $a_j = \gamma_{t_j}(x_{j-1}) = \gamma_{s_j}(\hat{x}_j) \in U(x_{j-1}) \cap U(x_j)$, where $\hat{x}_j$ lies in $V(x_j)$, hence in the connected component $F(x_j)_o$ of x_j in the level set $F(x_j)$.
It follows

$$\hat{x}_o = \gamma_t(\hat{x}_p), \quad \text{where} \quad t = \sum_j s_j - t_j,$$

and, therefore,

$$F(x_o)_o = \gamma_t(F(x))_o.$$

This shows that Φ is surjective, hence a diffeomorphism. In particular, the level sets of u_φ are connected and we have

$$\gamma_t(F(x)) = F(\gamma_t(x)).$$

It follows that $u(t) := u_\varphi(\gamma_t(x))$ do not depend on $x \in F$ and, because of (7.14), $b(t) := b(\gamma_t(x))$ has the same property.
Now, we want to prove that the induced metric $\Phi^* g$ is of the form

$$\Phi^* g = e^{\frac{4}{n}\int_0^t b(s)ds}\, g|_F \oplus dt^2.$$

Let $X,Y \in T_xF$. According to (7.12) and (7.13) we have

$$\begin{aligned} \frac{d}{dt}(\gamma_t^* g)_x(X,Y) &= (\gamma_t^* \mathcal{L}_\xi g)_x(X,Y) = \\ &= (\mathcal{L}_\xi g)_{\gamma_t(x)}(d\gamma_t(X), d\gamma_t(Y)) \end{aligned}$$

$$= \frac{4b(t)}{n} \quad (\gamma_t^* g)_x(X,Y).$$

This provides

$$(\gamma_t^* g)_x(X,Y) = g_x(X,Y) \quad e^{\frac{4}{n}\int_0^t b(s)ds} .$$

Then

$$(\Phi^* g)_{(x,t)}(X \oplus r \frac{\partial}{\partial t}, \hat{X} \oplus \hat{r} \frac{\partial}{\partial t}) =$$

$$= g_{\gamma_t(x)}(d\gamma_t(X) + r\,\xi(\gamma_t(x)), d\gamma_t(\hat{X}) + \hat{r}\,\xi(\gamma_t(x)))$$

$$= (\gamma_t^* g)_x(X,\hat{X}) + r\hat{r}$$

$$= e^{\frac{4}{n}\int_0^t b(s)ds} \quad g_x(X,\hat{X}) + dt^2(r\frac{\partial}{\partial t}, \hat{r}\frac{\partial}{\partial t}).$$

Finally, we show that F has a non-trivial parallel spinor field. We choose the orientation on F in such a way that Φ is orientation preserving.
Then we can apply the formulas (1.16)-(1.19) of Chapter 1.2 describing the spinor calculus on submanifolds of codimension one.

Consider first the case of $n = 2m+1$. Then, (1.17) implies for $X \in TF$

$$\nabla_X^S \varphi = -\frac{ib}{n} X\cdot\varphi = \nabla_X^{S_F}(\varphi|_F) - \frac{1}{2}\nabla_X \xi \cdot \xi \cdot \varphi$$

(7.6) and (7.11) imply $\nabla_X^{S_F}(\varphi|_F) = 0$, Consequently, the restriction of φ to F is a parallel spinor field on $(F, g|_F)$. From (1.16) it follows

$$\xi\cdot(\varphi|_F) = \xi\cdot(\varphi|_F^+ \oplus \varphi|_F^-) = i(-1)^m(\varphi|_F^+ - \varphi|_F^-)$$

$$= i(\varphi|_F^+ \oplus \varphi|_F^-).$$

Hence, the parallel spinor $\varphi|_F$ belongs to $\Gamma(S_F^+)$ if m is odd, and to $\Gamma(S_F^-)$ if m is even. In case of $n = 2m+2$, $\varphi|_F$ decomposes into $\varphi|_F = \varphi_1 \oplus \hat{\varphi}_2 \in \Gamma(S_F) \oplus \Gamma(\hat{S}_F)$. Using (1.18) we obtain

$$\xi\cdot\varphi|_F = \xi\cdot(\varphi_1 \oplus \hat{\varphi}_2) = (-1)^m\, i(\varphi_2 \oplus \hat{\varphi}_1) = i\varphi|_F$$

$$= i\varphi_1 \oplus i\hat{\varphi}_2,$$

which provides $\varphi_2 = (-1)^m \varphi_1$. Using (1.19) and (7.11) it follows for $X \in TF$

$$\nabla_X^S \varphi = -\frac{ib}{n} X\cdot(\varphi_1 \oplus (-1)^m \hat{\varphi}_1) =$$

$$= \nabla_X^{S_F}\varphi_1 \oplus (-1)^m \nabla_X^{\hat{S}_F}\hat{\varphi}_1 - \frac{1}{2}\nabla_X \xi\cdot(i(\varphi_1 \oplus (-1)^m\hat{\varphi}_1)$$

$$= (\nabla_X^{S_F}\varphi_1 - \frac{ib}{n} X\cdot\varphi_1) \oplus (-1)^m(\nabla_X^{\hat{S}_F}\hat{\varphi}_1 - \frac{ib}{n} X\cdot\hat{\varphi}_1).$$

This implies $\nabla_X^{S_F}\varphi_1 = 0$, hence, φ_1 is parallel on $(F, g|_F)$. This finishes the proof of Theorem 2.
In particular, Theorem 2 shows the following behaviour of the length function of an imaginary Killing spinor φ of type I to the Killing number μi: The level sets of u_φ are (n-1)-dimensional submanifolds and on the normal geodesics $\gamma_t(x)$ orthogonal to the level sets, u_φ has the form $u_\varphi(\gamma_t(x)) = e^{-2\mu t} u_\varphi(x)$. Moreover, Theorem 2 provides

Corollary 1: Let (M^n, g) be a complete, non-compact, connected spin manifold with a non-trivial Killing-spinor of type I to the Killing number μi, $\mu \in \mathbb{R} \setminus \{0\}$. Then (M^n, g) is isometric to a warped product $(F^{n-1} \times \mathbb{R},\ e^{-4\mu t} h \oplus dt^2)$, where (F^{n-1}, h) is a complete spin manifold with non-trivial parallel spinors.

Now, we prove that there really exist non-trivial Killing spinors on each warped product

$$(M^n, g) = (F^{n-1} \times \mathbb{R},\ e^{-4\mu t} h \oplus dt^2), \quad \mu \in \mathbb{R} \setminus \{0\},$$

where (F, h) is a Riemannian manifold with non-trivial parallel spinors.

Theorem 3: ([7], [85]): Let (F^{n-1}, h) be a spin manifold with non-trivial parallel spinor fields and let $c \in C^\infty(\mathbb{R})$ be a positive real function. Then, on the warped product $(M^n, g) := (F \times \mathbb{R}),\ c(t)h \oplus dt^2)$, there exists a twistor spinor $\varphi \neq 0$ satisfying $Q\varphi = 0$ and

$$\nabla_X \varphi + \frac{bi}{n} X \cdot \varphi = 0 \qquad (*),$$

where

$$b(x,t) = \begin{cases} \frac{n}{4} \frac{c'(t)}{c(t)} & \text{if } n \text{ is even} \\ \pm(-1)^m \frac{n}{4} \frac{c'(t)}{c(t)} & \text{if } n = 2m+1 \text{ and } (F,h) \text{ has parallel spinors in } \Gamma(S_F^\pm) \end{cases}$$

Proof: To prove the existence of the twistor spinors we use the formulas (1.20)-(1.23) of Chapter 1.2, which describe the spinor calculus on warped products.

1. case: $n = 2m+1$. Let $\psi^\pm \in \Gamma(S_F^\pm)$ be a parallel spinor in S_F^+ and S_F^-, respectively. We define a spinor field on (M, g) by

$$\varphi^\pm(x,t) := \sqrt[4]{c(t)}\ \psi^\pm(x).$$

According to (1.20) and (1.21) we have

$$\nabla^S_X \varphi^\pm = c^{-\frac{1}{2}} \widetilde{\nabla^{S_F}_X \psi^\pm} - \frac{1}{4} c^{-1} c' \tilde{X} \cdot \xi \cdot \varphi^\pm$$
$$= \mp \cdot \frac{1}{4} c^{-1} c' (-1)^m i \tilde{X} \cdot \varphi^\pm$$

for $X \in T_xF$ and

$$\nabla^S_\xi \varphi^\pm = \frac{1}{4} c' \cdot c^{-1} \cdot c^{\frac{1}{4}} \cdot \psi^\pm$$
$$= \frac{1}{4} c' c^{-1} \varphi^\pm$$
$$= \mp \frac{1}{4} c' c^{-1} (-1)^m i \xi \cdot \varphi^\pm .$$

Thus, $\varphi^\pm$ is a solution of (*) with $b(x,t) = \frac{n}{4} c'(t) c(t)^{-1} (-1)^m$. Assume that $Q_{\varphi^\pm} \neq 0$. Then, according to Corollary 3, Chapter 2, $b(x,t)$ is constant and $\varphi^\pm$ is a Killing spinor to the Killing number $\mu i \equiv \mp \frac{(-1)^m}{4} \frac{c'(t)}{c(t)} i$.

The length function of $\varphi^\pm$ is

$$u_{\varphi^\pm}(x,t) = \sqrt{c(t)}\, |\psi^\pm(x)|^2$$
$$= \sqrt{c(0)}\, e^{\mp(-1)^m 2\mu t} |\psi^\pm(x)|^2 .$$

$u_{\varphi^\pm}(x,t)$ tends to zero if $t \to \infty$ or $t \to -\infty$. Hence, $\varphi^\pm$ is a Killing spinor of type I. This contradicts $Q_\varphi \neq 0$.

<u>2. case:</u> $n = 2m+2$.

Let $\psi \in \Gamma(S_F)$ be parallel. Then $\hat{\psi} \in \Gamma(\hat{S}_F)$ is parallel, too. We consider the spinor field

$$\varphi(x,t) := \sqrt[4]{c(t)}\, \widetilde{(\psi(x) \oplus (-1)^m \hat{\psi}(x))}$$

on (M,g). According to (1.22) and (1.23) we have

$$\nabla^S_X \varphi = c^{-\frac{1}{2}} \{ \nabla^{S_F}_X \psi \oplus (-1)^m \nabla^{\hat{S}_F}_X \hat{\psi} \} - \frac{1}{4} c^{-1} c' X \cdot \xi \cdot \varphi$$
$$= - \frac{1}{4} c' c^{-1} c^{\frac{1}{4}} ((-1)^m i \widetilde{X \cdot ((-1)^m \psi \oplus \hat{\psi}})$$
$$= - \frac{1}{4} c' c^{-1} i X \cdot \varphi$$

for $X \in T_xF$ and

$$\nabla^S_\xi \varphi = \frac{1}{4} c(t)^{\frac{1}{4}-1} \cdot c'(t) \cdot \widetilde{(\psi \oplus (-1)^m \hat{\psi})}$$
$$= - \frac{1}{4} c^{-1} c' i \xi \cdot \varphi .$$

Therefore, φ is a solution of (*) with $b(x,t) = \frac{n}{4} \frac{c'(t)}{c(t)}$.
$Q_\varphi = 0$ follows as above.

Theorem 3 shows that there are <u>compact</u> manifolds with non-trivial solutions of

$$\nabla_X\varphi + \frac{ib}{n} X\cdot\varphi \equiv 0 \qquad (*)$$

for certain non-constant functions b, whereas for constant functions b $(*)$ is only solvable for <u>non-compact</u> manifolds. Consider for example a K3-surface (F^4,h) with a Yau-metric and a 2π-periodic, positive function $c \in C^\infty(\mathbb{R})$. Then, on the warped product $(F^4 \times S^1,\ c(t)h \oplus dt^2)$, there exist two linearly independent solutions of the equation

$$\nabla_X\varphi + \frac{c'(t)}{c(t)}\,\frac{i}{n}\, X\cdot\varphi = 0.$$

Furthermore, for the Killing spinor problem Theorem 3 yields

<u>Corollary 2:</u> Let (F^{n-1},h) be a complete spin manifold with non-trivial parallel spinor fields. Then the warped product

$$(M^n,g) := (F^{n-1} \times \mathbb{R},\ e^{-4\mu t}h \oplus dt^2), \quad \mu \in \mathbb{R} \setminus \{0\},$$

is a complete manifold with imaginary Killing spinors of type I. Moreover, in case $n = 2m+1$

$$\dim \mathcal{K}(M,g)_{\hat{\mu}i} \geq \dim \mathcal{K}(F,h)_0^+$$
$$\dim \mathcal{K}(M,g)_{-\hat{\mu}i} \geq \dim \mathcal{K}(F,h)_0^-,$$

where $\mathcal{K}(F,h)_0^\pm$ is the space of all parallel spinors in S_F^+ and S_F^-, respectively, and $\hat{\mu} = (-1)^m\mu$. In case $n = 2m+2$, it is

$$\dim \mathcal{K}(M,g)_{\mu i} = \dim \mathcal{K}(M,g)_{-\mu i} \geq \dim \mathcal{K}(F,h)_0.$$

The hyperbolic space $H^n_{-4\mu^2}$ is isometric to the warped product $(\mathbb{R}^{n-1} \times \mathbb{R},\ e^{-4\mu t}\, g_{\mathbb{R}^{n-1}} \oplus dt^2)$.

Hence, summing up the results of Theorem 1, Corollary 1 and Corollary 2 we obtain that a complete, non-compact connected spin manifold (M^n,g) admits non-trivial (imaginary) Killing spinors if and only if (M^n,g) is isometric to a warped product

$$(F^{n-1} \times \mathbb{R},\ e^{-4\mu t}\, h \oplus dt^2), \qquad \mu \in \mathbb{R} \setminus \{0\},$$

where (F,h) is a complete, connected spin manifold with non-trivial parallel spinor fields.

7.4. Killing Spinors on 5-dimensional, Complete, non-Compact Manifolds

Finally we will study the space of imaginary Killing spinors in dimension five in more detail. We will give a construction principle for <u>all</u> Killing spinors on a 5-dimensional complete, non-compact manifold. According to the results of Chapter 7.2 and Chapter 7.3 such a manifold has the form

$$(F^4 \times \mathbb{R},\ e^{-4\mu t}h \oplus dt^2),\ \mu \in \mathbb{R} \setminus \{0\},$$

where (F^4,h) is a complete manifold with parallel spinors.

<u>Theorem 4 ([6])</u>: Let (F^4,h) be a 4-dimensional complete spin manifold with a parallel spinor field in $\Gamma(S_F^+)$, and consider the warped product

$$(M^5,g) := (F \times \mathbb{R},\ e^{-4\mu t}h \oplus dt^2),\quad \mu \in \mathbb{R} \setminus \{0\}.$$

Then

1.) $$\dim \mathcal{K}(M,g)_{\mu i} = \begin{cases} 4 & \text{if } (F,h) \text{ is isometric to the Euclidean space} \\ \dim \mathcal{K}(F,h)_0^+ \leq 2 & \text{otherwise} \end{cases}$$

2.) If (F,h) is compact, then

$$\dim \mathcal{K}(M,g)_{-\mu i} = \dim \mathcal{K}(F,h)_0^- \leq 2.$$

If this dimension is greater than zero, (F,h) is flat, hence (M^5,g) is a space of constant sectional curvature $-4\mu^2$.

<u>Proof:</u> According to Corollary 2 we have $\dim \mathcal{K}(M,g)_{\mu i} \geq \dim \mathcal{K}(F,h)_0^+$. Let $\varphi \in \mathcal{K}(M,g)_{\mu i}$. Then, using the denotation of Chapter 1.2 for the spinor calculus on warped products, φ is described by $\varphi = \widetilde{\psi}$, where $\psi \in \Gamma(\pi^* S_F)$. ψ decomposes into $\psi = \psi^+ \oplus \psi^- \in \Gamma(S_F^+) \oplus \Gamma(S_F^-)$. Let $\varphi^{\pm} := \widetilde{\psi}^{\pm}$. From (1.20) and (1.21) it follows

$$\frac{\partial}{\partial t}(\varphi) = i\mu\, \xi \cdot \varphi = -\mu \varphi^+ + \mu \varphi^-.$$

Hence,

$$\frac{\partial}{\partial t}(\psi^+) = -\mu\psi^+ \quad \text{and} \quad \frac{\partial}{\partial t}(\psi^-) = \mu\psi^-.$$

Therefore, there exists spinors $\psi_0^{\pm} \in \Gamma(S_F^{\pm})$ satisfying

$$\psi^+(x,t) = e^{-\mu t}\psi_0^+(x) \quad \text{and} \quad \psi^-(x,t) = e^{\mu t}\psi_0^-(x).$$

According to (1.20) and (1.21) for $X \in T_xF$ we have

$$\begin{aligned} e^{2\mu t}\nabla_X^{S_F}\psi_t &= \mu i\, \widetilde{X} \cdot \varphi - \mu \widetilde{X} \cdot \xi \cdot \varphi \\ &= 2\mu i\, \widetilde{X \cdot \psi_t^-}\ . \end{aligned}$$

Since the Clifford multiplication commutes the components S_F^+ and S_F^- of S_F, it follows

$$\nabla^{S_F}\psi_o^- \equiv 0 \qquad (7.15)$$

$$\nabla_X^{S_F}\psi_o^+ = 2\mu i\, X\cdot\psi_o^- \qquad (7.16)$$

Suppose that $\varphi^- \equiv 0$. Then $\psi_o^- \equiv 0$ and ψ_o^+ is parallel. In this case, we have

$$\varphi(x,t) = \widetilde{e^{-\mu t}\psi_o^+(x)}. \qquad (7.17)$$

In particular, if $\dim \mathcal{K}(M,g)_{\mu i} > \dim \mathcal{K}(F,h)_o^+$, then there exists a Killing spinor $\varphi \in \mathcal{K}(M,g)_{\mu i}$ such that $\varphi^- \not\equiv 0$ and, according to (7.15) and (7.16) we have a non-trivial parallel spinor $\psi_o^- \in \Gamma(S_F^-)$ and a spinor $\psi_o^+ \in \Gamma(S_F^+)$ with

$$\nabla_X^{S_F}\psi_o^+ = 2\mu i X\cdot\psi_o^-$$

for all vectors X on F. A 4-dimensional manifold with non-trivial parallel spinors in $\Gamma(S_F^+)$ as well as in $\Gamma(S_F^-)$ is flat (see Chapter 6). Hence, if $\dim \mathcal{K}(M,g)_{\mu i} > \dim \mathcal{K}(F,h)_o^+$, (F,h) is flat (and complete), therefore isometric to a factor space $\mathbb{R}^4|_\Gamma$, where Γ is a discrete group of isometries of $\mathbb{R}^4$.

Now, we will prove that (F,h) is in fact isometric to $\mathbb{R}^4$. Since ψ_o^- is non-trivial and parallel, ψ_o^- has no zeros. Hence, the map

$$TF^4 \longrightarrow S_F^+$$
$$X \longmapsto X\cdot(i\psi_o^-)$$

is an isomorphism of the real 4-dimensional vector bundles. Let Z be the vector field defined by

$$\psi_o^+ = Z\cdot(i\psi_o^-).$$

Then (7.16) implies

$$2\mu i\, X\cdot\psi_o^- = \nabla_X^{S_F}\psi_o^+ = \nabla_X^F Z\cdot(i\psi_o^-) + iZ\cdot\nabla_X^{S_F}\psi_o^-$$
$$= i\,\nabla_X Z\cdot\psi_o^-.$$

This provides

$$\nabla_X^F Z = 2\mu X \qquad (7.18)$$

for all vector fields X on F.

In particular, each Killing spinor $\varphi \in \mathcal{K}(M,g)_{\mu i}$ that is not of the form (7.17) is described by

$$\varphi = i\widetilde{e^{-\mu t} Z\cdot\psi_o^-} + \widetilde{e^{\mu t}\psi_o^-}, \qquad (7.19)$$

where $\psi_o^- \in \Gamma(S_F^-)$ is a non-trivial parallel spinor field and Z a vector field satisfying (7.18). It is easy to verify that each spinor of the form (7.19) is a Killing spinor to the Killing number μi.
In the flat, complete, connected Riemannian manifold (F^4,h) there exists a closed, totally geodesic submanifold $N^k \subset F^4$ with the same homotopy type as F^4 (see [106], Theorem 3.3.3). Since N is totally geodesic, we have

(i) $\nabla_X^F Y = \nabla_X^N Y$ for all vector fields X,Y on N,

(ii) for a vector field H normal to N and any $X \in T_x N$, the derivative $\nabla_X^F H$ is normal to N, too.

Let $\hat{Z}$ denote the vector field

$$\hat{Z} := \mathrm{proj}_{TN} Z$$

on N. Then the divergence of the vector field $\hat{Z}$ on $(N,h|_N)$ is

$$\begin{aligned}\mathrm{div}^N(\hat{Z}) &= \sum_{j=1}^{k} h(\nabla_{a_j}^N \hat{Z}, a_j) = \sum_{j=1}^{k} h(\nabla_{a_j}^F \hat{Z}, a_j) = \\ &= \sum_{j=1}^{k} \{ h(\nabla_{a_j}^F Z, a_j) - h(\nabla_{a_j}^F (\mathrm{proj}_{norm} Z), a_j) \} \\ &= \sum_{j=1}^{k} h(\nabla_{a_j}^F Z, a_j),\end{aligned}$$

where $(a_1,\dots,a_k)$ is a local ON-basis of $(N,h|_N)$. From (7.18) it follows that $\mathrm{div}^N(\hat{Z}) = 2\mu \dim N$, and because of $\int_N \mathrm{div}^N(\hat{Z})dN = 0$, we obtain $\dim N = 0$. Hence, F is simply-connected and therefore isometric to the Euclidean space $\mathbb{R}^4$. In this case (M^5,g) is isometric to the hyperbolic space $H^5_{-4\mu^2}$ and we have $\dim \mathcal{K}(M,g)_{\mu i} = 4$.
This proves the first part of Theorem 4. According to Corollary 2 we have $\dim \mathcal{K}(M,g)_{-\mu i} \geq \dim \mathcal{K}(F,h)_o^-$. If $\dim \mathcal{K}(F,h)_o^- > 0$, then (F,h) is flat and (M^5,g) is a space of constant sectional curvature. In the same way as above it can be shown that $\dim \mathcal{K}(M,g)_{-\mu i} > \dim \mathcal{K}(F,h)_o^-$ if and only if there exists a vector field Z on F satisfying the equation

$$\nabla_X^F Z = -2\mu X \tag{7.20}$$

for all vector fields X on F. Each Killing spinor in $\mathcal{K}(M,g)_{-\mu i}$ that is not of the form

$$\varphi = e^{-\mu t} \psi_o^-, \tag{7.21}$$

where $\psi_o^- \in \Gamma(S_F^-)$ is parallel, can be described as follows:

$$\varphi = e^{\mu t} \psi_o^+ + i e^{-\mu t} Z \cdot \psi_o^+, \tag{7.22}$$

where $\psi_o^+ \in \Gamma(S_F^+)$ is parallel and Z is a vector field satisfying (7.20). Equation (7.20) in particular shows that $\mathrm{div}^F(Z) = -8\mu \neq 0$, which is impossible on a compact manifold. Hence, on compact manifolds we have $\dim \mathcal{K}(M,g)_{-\mu i} = \dim \mathcal{K}(F,h)_o^-$.

All Killing spinors on a 5-dimensional complete manifold $(M^5,g) = (F^4 \times \mathbb{R},\ e^{-4\mu t}h \oplus dt^2)$ different from the hyperbolic space, where (F^4,h) is a manifold with non-trivial parallel spinor fields in $\Gamma(S_F^+)$, are described by the formulas (7.17), (7.21) and (7.22). In particular, from Theorem 7.4 it follows:

1) If (F^4,h) is a K3-surface with the Yau-metric, then
   $\dim \mathcal{K}(M,g)_{\mu i} = 2$ and $\dim \mathcal{K}(M,g)_{-\mu i} = 0$.
2) If (F^4,h) is the flat torus with the canonical spinor structure, then
   $\dim \mathcal{K}(M,g)_{\mu i} = \dim \mathcal{K}(M,g)_{-\mu i} = 2$.

If (M^5,g) is the hyperbolic space, then we obtain all Killing spinors in the Poincaré model (see example 3, Chapter 1) or in the model

$$H^5_{-4\mu^2} = (\mathbb{R}^4 \times \mathbb{R},\ e^{-4\mu t}\, g_{\mathbb{R}^4} \oplus dt^2)$$

using the formulas (7.17), (7.19), (7.21) and (7.22) in the form

$$\left.\begin{aligned}\mathcal{K}(H^5_{-4\mu^2})_{\mu i} = \{ \tilde{\varphi}_{u,v} \mid \varphi_{u,v}(x,t) = e^{-\mu t}u + (e^{\mu t} + 2\mu i e^{-\mu t} x\cdot)v,\\ \text{where } u \in \Delta^2_+,\ v \in \Delta^2_- \end{aligned}\right\}$$

$$\left.\begin{aligned}\mathcal{K}(H^5_{-4\mu^2})_{-\mu i} = \{ \tilde{\varphi}_{u,v} \mid \varphi_{u,v}(x,t) = (e^{\mu t} - 2\mu i e^{-\mu t} x\cdot)u + e^{-\mu t}v,\\ \text{where } u \in \Delta^2_+,\ v \in \Delta^2_-. \end{aligned}\right\}$$

References

[1] M.F. Atiyah, R. Bott, A. Shapiro: Clifford modules. Topology 3 (1964), 3-38.

[2] M. F. Atiyah, N.J. Hitchin, I.M. Singer: Self-duality in four dimensional Riemannian geometry. Proc. R. Soc. London Serie A, 362(1978), 425-461.

[3] D'Auria, R.P. Fré, P. van Nieuwenhuizen: N = 2 matter coupled supergravity on a coset G/H possessing an additional Killing vector. Utrecht preprint 1983.

[4] H. Baum: Spin-Strukturen und Dirac-Operatoren über pseudo-Riemannschen Mannigfaltigkeiten. Teubner-Texte zur Mathematik Band 41, Teubner-Verlag Leipzig 1981.

[5] H. Baum: Variétés riemanniennes admettant des spineurs de Killing imaginaires. C.R. Acad. Sci. Paris Serie I, t. 309, (1989), 47-49.

[6] H. Baum: Odd-dimensional Riemannian manifolds with imaginary Killing spinors. Ann. Global Anal. Geom. 7, (1989), 141-154.

[7] H. Baum: Complete Riemannian manifolds with imaginary Killing spinors. Ann. Global Anal. Geom. 7, (1989), 205-226.

[8] L. Bérard Bergery: Sur de nouvelles variétés riemanniennes d'Einstein. Institut Elie Cartan, Preprint 1981.

[9] M. Berger, P. Gauduchon, E. Mazet: Le spectre d'une variété riemannienne. Lect. Notes Math. 194, Springer-Verlag 1970.

[10] E. Bergshoeff, M. de Roo, B. de Wit, P. van Nieuwenhuizen: Ten-dimensional Maxwell-Einstein supergravity, its currents and the issue of its auxiliary fields. Nucl. Phys. B 195(1982), 97-136.

[11] A. Besse: Géométrie Riemannienne en Dimension 4. Paris 1981.

[12] A. Besse: Einstein manifolds, Springer-Verlag 1988.

[13] B. Biran, F. Englert, 8. de Wit, H. Nicolai: Gauges N = 8 supergravity and its breaking from spontaneous compactification. Phys. Lett. 124 8 (1983), 45-50.

[14] B. Biran, Ph. Spindel: New compactifications of N = 1, d = 11 supergravity. Nucl. Phys. B 271(1986), 603-619.

[15] O. Blair: Contact manifolds in Riemannian Geometry. Lect. Notes Math. 509, Springer-Verlag 1976.

[16] O. Bleecker: Gauge Theory and variational Principles. London 1981.

[17] J. P. 8ourguignon (Ed.): Seminaire sur la preuve de la conjecture de Calabi. Palaiseau 1978.

[18] J. P. Bourguignon (Ed.): Séminaire Palaiseau, Géométrie des surfaces K 3: modules et périodes. Asterisque 126(1985).

[19] M. Cahen, S. Gutt, L. Lemaire, P. Spindel: Killing spinors. Bull. Soc. Math. Belgique 38 A(1986), 75-102.

[20] A.P. Calderon: Uniqueness in the Cauchy problem for partial differential equations. Amer. J. Math. 80(1958), 16-36.

[21] P. Candelas, G.T. Horowitz, A. Strominger, E. Witten: Vacuum configurations for superstrings. Nucl. Phys. B 258, 1(1985), 46-74.

[22] A.H. Chamsedine: N = 4 supergravity coupled to N = 4 matter and hidden symmetries. Nucl. Phys. B 185(1981), 403-415.

[23] C.F. Chapline, N.S. Mauton: Unification of Yang-Mills theory and supergravity in ten dimensions. Phys. Lett. 120 B(1983), 105-109.

[24] E. Cremmer, B. Julia, J. Scherk: Supergravity theory in 11 dimensions. Phys. Lett. 76 B(1978), 409-412.

[25] M.J. Duff. B.E.W. Nilsson and C.N. Pope: The criterion for vacuum stability in Kaluza-Klein supergravity. Phys. Lett. 139 B(1984), 154-158.

[26] M.J. Duff, B. Nilsson and C.N. Pope: Kaluza-Klein supergravity. Phys. Rep. 130(1986), 1-142.

[27] M.J. Duff, C.N. Pope: Kaluza-Klein supergravity and the 7-sphere. In: Supergravity 82 (ed. J. Ferrara, J. Taylor and P. van Nieuwenhuizen). World Scient. Publ. (1983).

[28] M.J. Duff, F.J. Toms: Kaluza-Klein Counterterms. In: Unification of the fundamental interactions II. (ed. S. Ferrara, J. Ellis) Plenum N.Y. (1982).

[29] F. Englert, M. Rooman, P. Spindel: Supersymmetry breaking by torsion and the Ricci-flat squashed seven-sphere. Phys. Lett. 127 B (1983), 47-50.

[30] A.E. Fischer, J.A. Wolf: The structure of compact Ricci-flat Riemannian manifolds. J. Diff. Geom. 10(1975), 277-288.

[31] A. Franc: Spin structures and Killing spinors on lens spaces. J. Geom. and Phys. 4(1987), 277-287.

[32] Th. Friedrich: Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung. Math. Nachr. 97(1980), 117-146.

[33] Th. Friedrich: Self-duality of Riemannian Manifolds and Connections. In: Self-dual Riemannian Geometry and Instantons. Teubner-Texte zur Math., 34, Teubner-Verlag 1981.

[34] Th. Friedrich: A remark on the first eigenvalue of the Dirac operator on 4-dimensional manifolds. Math. Nachr. 102(1981), 53-56.

[35] Th. Friedrich: Zur Existenz paralleler Spinorfelder über Riemannschen Mannigfaltigkeiten. Colloq. Math. 44(1981), 277-290.

[36] Th. Friedrich: Zur Abhängigkeit des Dirac-Operators von der Spin-Struktur. Colloq. Math. 48(1984), 57-62.

[37] Th. Friedrich: Riemannian manifolds with small eigenvalue of the Dirac operator. Proceedings of the "27. Mathematische Arbeitstagung", Bonn 12.-19. Juli 1987 (Preprint MPI).

[38] Th. Friedrich: On the conformal relations between twistor and Killing spinors. Suppl. Rend. Circ. Mat. Palermo (1989),59-75.

[39] Th. Friedrich, R. Grunewald: On Einstein metrics on the twistor space of a four-dimensional Riemannian manifold. Math. Nachr. 123(1985), 55-60.

[40] Th. Friedrich, R. Grunewald: On the first eigenvalue of the Dirac operator on 6-dimensional manifolds. Ann. Global Anal. Geom. 3(1985), 265-273.

[41] Th. Friedrich, I. Kath: Einstein manifolds of dimension five with small eigenvalue of the Dirac operator. J. Diff. Geom. 29(1989), 263-279.

[42] Th. Friedrich, I. Kath: Compact 5-dimensional Riemannian manifolds with parallel spinors. Math. Nachr. 147(1990),161-165.

[43] Th. Friedrich, I. Kath: Variétés riemanniennes compactes de dimension 7 admettant des spineurs de Killing. C.R. Acad. Sci. Paris 307 Serie I (1988), 967-969.

[44] Th. Friedrich, I. Kath: Compact seven-dimensional manifolds with Killing spinors. Comm. Math. Phys. 133(1990),543-561.

[45] Th. Friedrich, H. Kurke: Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature. Math. Nachr. 106(1982), 271-299.

[46] Th. Friedrich, O. Pokorna: Twistor spinors and solutions of the equation (E) on Riemannian manifolds. to appear in Suppl. Rend. Circ. Mat. Palermo.

[47] A. Futaki: Kähler-Einstein metrics and integral invariants. Lect. Notes Math. 1314, Springer-Verlag Berlin 1988.

[48] A. Gray: Some examples of almost hermitian manifolds. Ill. J. Math. 10(1966), 353-366.

[49] A. Gray: Almost complex submanifolds of the six-sphere. Proc. of the AMS 20(1969), 277-279.

[50] A. Gray: Kähler submanifolds of homogeneous almost hermitian manifolds. Tohoku Math. Journ. 21(1969), 190-194.

[51] A. Gray: Vector cross products on manifolds. Trans. Am. Math. Soc. 141(1969), 465-504.

[52] A. Gray: Nearly Kähler manifolds. Journ. Diff. Geom. 4(1970), 283-309.

[53] A. Gray: Riemannian manifolds with geodesic symmetries of order 3. Journ. Diff. Geom. (1972), 343-369.

[54] A. Gray: The structure of nearly Kähler manifolds. Math. Ann. 223(1976), 233-248.

[55] R. Grunewald. Six-dimensional Riemannian manifolds with a real Killing spinor. Ann. Glob. Anal. Geom. 8(1990), 43-59.

[56] S. Gutt: Killing spinors on spheres and projective spaces. in: "Spinors in Physics and Geometry" World. Sc. Publ. Co. Singapore 1988.

[57] O. Hijazi: A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors. Comm. Math. Phys. 104(1986), 151-162.

[58] O. Hijazi: Caractérisation de la sphére par les premières valeurs propres de l'opérateur de Dirac en dimension 3, 4, 7 et 8. C. R. Acad. Sc. Paris 303, Serie I, (1986), 417-419.

[59] O. Hijazi, A. Lichnerowicz: Spineurs harmonique, spineurs-twisteurs et géométrie conforme. C.R. Acad. Sci. Paris 307, Serie I (1988), 833-838.

[60] N. Hitchin: Compact four-dimensional Einstein manifolds. J. Diff. Geom. 9(1974), 435-442.

[61] N. Hitchin: Harmonic spinors. Adv. in Math. 14(1974), 1-55.

[62] N. Hitchin: Linear field equations on self-dual spaces. Proc. R. Soc. London Serie A, 370(1980), 173-191.

[63] N. Hitchin: Kählerian twistor spaces. Proc. Lond. Math. Soc., III Ser., 43 (1981), 133-150.

[64] L.P. Hugston, R. Penrose, P. Sommers and M. Walker: On a quadratic first integral for charged particle orbits in the charged Kerr solution. Comm. Math. Phys. 27(1972), 303-308.

[65] A. Ikeda: Formally self adjointness for the Dirac operator on homogeneous spaces. Osaka Journ. of Math. 12(1975), 173-185.

[66] S. Ishihara, M. Konishi: Differential Geometry of fibred spaces. Publications of the study group of geometry. Institute of Mathematics, Yoshida College, Kyoto University, Kyoto 1973.

[67] I. Kath: Killing-Spinoren und Kontaktformen. Dissertation (A) Humboldt-Universität 1989.

[68] K.D. Kirchberg: An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds with positive scalar curvature. Ann. Glob. Anal. Geom. 4(1986), 291-326.

[69] K.D. Kirchberg: Compact six-dimensional Kähler spin manifolds of positive scalar curvature with the smallest positive first eigenvalue of the Dirac operator. Math. Ann. 282(1988), 157-176.

[70] K.D. Kirchberg: Twistor-spinors on Kähler manifolds and the first eigenvalue of the Dirac operator, in J. Geom. Phys.(1991).

[71] S. Kobayashi: On compact Kähler manifolds with positive definite Ricci tensor. Ann. of Math. 74(1961), 570-574.

[72] S. Kobayashi: Topology of positive pinched Kähler geometry. Tôhoku Math. J. 15(1963), 121-139.

[73] S. Kobayashi: Transformation Groups in Differential Geometry. Springer-Verlag 1972.

[74] S. Kobayashi, K. Nomizu: Foundations of Differential Geometry. Vol. I. and II. Intersc. Publ. J. Wiley & Sons, New York - London 1963, 1969.

[75] N. Koiso, Y. Sakane: Non-homogeneous Kähler-Einstein metrics on compact complex manifolds. Preprint MPI Bonn (1988).

[76] B. Kostant: Quantization and unitary representations. Lect. Notes Math. 170, Springer-Verlag 1970.

[77] J. Kotō: Some theorems on almost Kählerian spaces. Journ. Math. Soc. Japan 12(1960), 422-433.

[78] O. Kowalski: Existence of generalized symmetric Riemannian spaces of arbitrary order. J. Diff. Geom. 12(1977), 203-208.

[79] M. Kreck, St. Stolz: A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with SU(3) x SU(2) x U(1)-symmetry. Ann. Math. 127(1988), 373-388.

[80] A. Lascoux, M. Berger: Variétés Kähleriennes compactes. Lect. Notes Math. 154, Springer-Verlag 1970.

[81] A. Lichnerowicz: Spineurs harmoniques. C. R. Acad. Sci. Paris 257, Serie I (1963), 7-9.

[82] A. Lichnerowicz: Variétés spinorielles et universalité de l'inegalité de Hijazi. C.R. Acad. Sc. Paris, Ser. I 304(1987), 227-231.

[83] A. Lichnerowicz: Spin manifolds, Killing spinors and universality of the Hijazi inequality. Lett. Math. Phys. 13(1987), 331-344.

[84] A. Lichnerowicz: Les spineurs-twisteurs sur une variété spinorielle compacte. C.R. Acad. Sci. Paris, t. 306 Serie I (1988), 381-385.

[85] A. Lichnerowicz: Sur les résultates de H. Baum et Th. Friedrich concernant les spineurs de Killing à valeur propre imaginaire. C.R. Acad. Sci. Paris 306, Serie I, (1989), 41-45.

[86] K. Neitzke: Die Twistorgleichung auf Riemannschen Mannigfaltigkeiten. Diplomarbeit Humboldt-Universität 1989.

[87] K. Neitzke-Habermann: The Twistor Equation on Riemannian Spin Manifolds. To appear in Journal of Geom. and Physics, 1991.

[88] P. van Nieuwenhuizen: An introduction to simple supergravity and the Kaluza-Klein program. In: Relativity and Topology II, Les Houches, 1983 North Holland,825-932.

[89] P. van Nieuwenhuizen and N.P. Warner: Integrability Conditions for Killing Spinors. Comm. Math. Phys. 93(1984), 227-284.

[90] B.E. Nilson, C.N. Pope: Scalar and Dirac eigenfunctions on the squashed seven-sphere. Phys. Lett. B 133(1983), 67-71.

[91] R.S. Palais: Seminar on the Atiyah-Singer Index Theorem. Princeton 1965.

[92] P. Penrose, W. Rindler: Spinors and Space Time. Vol. 2. Cambr. Mono. in Math. Physics (1986).

[93] M.M. Postnikov: Lectures on Geometry V, Lie groups and algebras, Moscow, 1982 (russ.).

[94] P.K. Raschewski: Riemannsche Geometrie und Tensoranalysis, Berlin 1959.

[95] S. Salamon: Topics in four-dimensional Riemannian geometry. In: Geometry Seminar L. Bianchi, Lect. Notes Math. 1022, Springer-Verlag 1983.

[96] E. Sezgin, P. Spindel: Compactification of N = 1, D = 10 supergravity to 2 and 3 dimensions. Nucl. Phys. B 261(1985), 28-40.

[97] I.M. Singer, J.A. Thorpe: The curvature of a 4-dimensional Einstein space. Global Analysis, Papers in honor of K. Kodaira, Princeton 1969.

[98] S. Smale: On the structure of 5-manifolds. Ann. of Math. 75 (1962), 38-46.

[99] S. Sulanke: Berechnung des Spektrums des Quadrates des Dirac-Operators D^2 auf der Sphäre und Untersuchungen zum ersten Eigenwert von D auf 5-dimensionalen Räumen konstanter positiver Schnittkrümmung. Humboldt-Universität zu Berlin 1980.

[100] S. Sulanke: Der erste Eigenwert des Dirac-Operators of S^5/Γ. Math. Nachr. 99(1980), 259-271.

[101] G. Tian: On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. Invent. Math. 89(1987), 225-246.

[102] G. Tian, S.T. Yau: Kähler-Einstein metrics on complex surfaces with $c_1 > 0$. Comm. Math. Phys. 112(1987), 175-203.

[103] McKenzie, Y. Wang: Parallel spinors and parallel forms. Ann. Global. Anal. Geom. 7(1989), 59-68.

[104] F. Watanabe: A family of homogeneous Sasakian structures on $S^2 \times S^3$. Math. Reports Acad. Sci. Can. 10(1988), 57-61.

[105] R.O. Wells: Differential analysis on complex manifolds. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1973.

[106] J.A. Wolf: Spaces of constant curvature. Berkeley, Calif. 1972.

[107] K. Yano, S. Bochner: Curvature and Betti numbers. Princeton, 1953.