Dirac operators in Riemannian geometry

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General References

Th. Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics No. 25, AMS 2000.

This book contains 275 references up to the year 2000

N. Ginoux, *The Dirac Spectrum*, Lecture Notes No. 1976, Springer 2009.

This book contains 240 references on eigenvalues of the Dirac operator up to the year 2009

H. Baum, Th. Friedrich, R. Grunewald, I. Kath, *Twistor and Killing spinors on Riemannian manifolds*, Teubner-Verlag Leipzig/Stuttgart 1991.

This book contains 107 references on Twistor and Killing spinors up to the year 1991

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Motivation

a) From complex analysis: Consider the Cauchy-Riemann operators

$$\partial \;=\; \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} \;=\; \frac{1}{2}(\partial_x + i\partial_y)$$

Define a differential operator $P : \mathbb{C}^\infty(\mathbb{R}^2; \mathbb{C}^2) \to \mathbb{C}^\infty(\mathbb{R}^2; \mathbb{C}^2)$ by

$$P \begin{bmatrix} f \\ g \end{bmatrix} \;=\; 2i \begin{bmatrix} \partial g \\ \bar{\partial} f \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}_{\gamma_x} \partial_x \begin{bmatrix} f \\ g \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\gamma_y} \partial_y \begin{bmatrix} f \\ g \end{bmatrix}$$

Then γ_x, γ_y satisfy the *Clifford relations*

$$\gamma_x^2 \;=\; \gamma_y^2 \;=\; -\mathrm{Id}, \quad \gamma_x \cdot \gamma_y + \gamma_y \cdot \gamma_x \;=\; 0$$

and $4\,\partial\bar{\partial} = 4\,\bar{\partial}\partial = \Delta$ (Laplacian).

More generally: (M^{2n}, g, J) – Kähler manifold, $\Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}$ with

$$\Lambda^{1,0} \; = \; \{\eta : \eta(JX) = i\eta(X)\}, \quad \Lambda^{0,1} \; = \; \{\eta : \eta(JX) = -i\eta(X)\},$$

and $df = \mathrm{pr}_{\Lambda^{1,0}}(df) + \mathrm{pr}_{\Lambda^{0,1}}(df) =: \partial f + \bar{\partial} f$

Then: $\quad 2(\partial\bar{\partial} + \bar{\partial}\partial) = \Delta.$

Question: Does there exist a generalization of the Cauchy-Riemann operator on a more general class of manifolds?

b) From theoretical physics: Consider a free classical particle with

m : mass, $\quad p = \frac{vm}{\sqrt{1-v^2/c^2}}$: momentum, $\quad E$: Energy.

Then special relativity predicts the relation

$$E \; = \; \sqrt{c^2p^2 + m^2c^4}.$$

According to the quantization rules of quantum mechanics:

$E \to i\hbar\partial_t, \quad p \to -i\hbar\nabla$, both acting on some state function ψ

$$\Rightarrow i\hbar\partial_t\psi \;=\; \sqrt{c^2\hbar^2\Delta + m^2c^4} \qquad \text{"Dirac equation"}$$

Question: What is the meaning of the square root?

c) From topology:

Theorem (Freedman 1982):
Any unimodular quadratic form $\mathcal{L}$ over $\mathbb{Z}$ can be realized as the intersection form $\mathcal{L} = H^2(X^4;\mathbb{Z})$ of a 4-dimensional, compact and simply connected *topological* manifold X^4.

Theorem (Rochlin 1950): If M^4 is *smooth*, closed manifold s. t. $\omega_2(M^4) = 0$ then $\sigma(M^4) = 0 \bmod 16$.

Theorem (Hirzebruch) $\displaystyle \frac{1}{8}\sigma(M^4) = \frac{1}{24}\int_{M^4} p_1.$

Example:

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$E_8 \geq 0$ is of typ II and $\sigma(E_8) = \dim E_8 = 8$.

Question: Does there exist a vector bundle $S \to M^4$ and an elliptic differential operator $D : \Gamma(S) \to \Gamma(S)$ s. t.

t-index$(D) = \frac{1}{8}\sigma(M^4)$,

a-index$(D) = \dim \ker D - \dim \operatorname{coker} D = 0 \bmod 2$?

Clifford algebras

$(\mathbb{R}^n, g)$, $e_1, \ldots, e_n$ an orthonormal basis. Then the (finite dimensional!) associative algebra

$$Cl(\mathbb{R}^n) \;:=\; \bigotimes \mathbb{R}^n / \{e_i \cdot e_j + e_j \cdot e_i = 0,\, e_1^2 = -1\}$$

is called the *Clifford algebra* of $\mathbb{R}^n$. $Cl^{\mathbb{C}}(\mathbb{R}^n)$ denotes its *complexification*.

Example. $n = 2$, g : standard euclidean scalar product.

Then $e_1 \mapsto \gamma_x$, $e_2 \mapsto \gamma_y$ shows: $Cl^{\mathbb{C}}(\mathbb{R}^2) \cong \mathcal{M}_{\mathbb{C}}(2)$

$\Rightarrow$ $Cl^{\mathbb{C}}(\mathbb{R}^2)$ acts on $\mathbb{C}^2$ by endomorphisms. More generally:

Thm. There exists a unique representation of smallest dimension of the algebra $Cl^{\mathbb{C}}(\mathbb{R}^n)$ on a complex vector space Δ_n:

$$Cl^{\mathbb{C}}(\mathbb{R}^n) \longrightarrow \mathrm{End}(\Delta_n), \quad \dim \Delta_n = 2^{[n/2]}.$$

Δ_n : space of (Dirac) spinors

• The Spin(n) group is a two-fold covering of $\mathrm{SO}(n)$ and can be realized in $Cl(\mathbb{R}^n)$,

$$\mathrm{Spin}(n) \;=\; \{x_1 \cdot \;.... \cdot x_{2l},\; x_i \in \mathbb{R}^n \text{ and } |x_i| = 1\}\;.$$

• Every vector $x \in \mathbb{R}^n$ acts on Δ_n by an endomorphism:

$$\mathbb{R}^n \times \Delta_n \ni (x, \psi) \longmapsto x \cdot \psi \in \Delta_n : \qquad \text{"Clifford multiplication"}$$

$$\mu \;:\; \mathbb{R}^n \otimes \Delta_n \longrightarrow \Delta_n\;.$$

• The $\mathrm{Spin}(n)$-representation $\mathbb{R}^n \otimes \Delta_n$ splits into

$$\mathbb{R}^n \otimes \Delta_n \;=\; \Delta_n \oplus \ker(\mu)\;.$$

• There is a universal projection of $\mathbb{R}^n \otimes \Delta_n$ onto $\ker(\mu)$,

$$p(x \otimes \psi) \;=\; x \otimes \psi + \frac{1}{n} \sum_{i=1}^{n} e_i \otimes e_i \cdot x \cdot \psi \;.$$

• This splitting yields two differential operators of first order, the Dirac operator and the twistor operator .

• If $n = 2k$ is even, then the $\mathrm{Spin}(n)$ representation splits into two irreducible pieces,

$$\Delta_{2k} \;=\; \Delta_{2k}^{+} \oplus \Delta_{2k}^{-} \, , \quad x \,:\, \Delta_{2k}^{\pm} \longrightarrow \Delta_{2k}^{\mp} \, .$$

- Additional $\mathrm{Spin}(n)$-invariant structures in Δ_n:

α_n	real structures	quaternionic structures
commutes with Clifford multiplication	$n \equiv 6, 7 \; mod\, 8$	$n \equiv 2, 3 \; mod\, 8$
anti-commutes with Clifford multiplication	$n \equiv 0, 1 \; mod\, 8$	$n \equiv 4, 5 \; mod\, 8$

Proposition:
The representation $\Delta^{\pm}_{8k}$ admits a $\mathrm{Spin}(8k)$-invariant *real* structure.

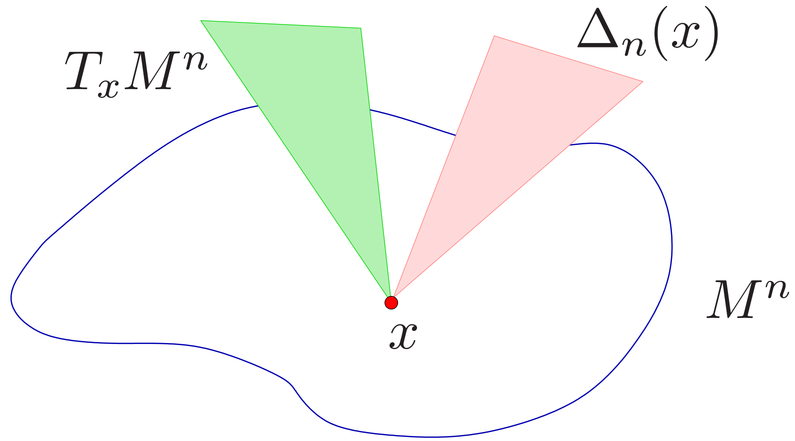
The representation $\Delta^{\pm}_{8k+4}$ admits a $\mathrm{Spin}(8k+4)$-invariant *quaternionic* structure.

Spin Structures.

Idea: Attach a copy of Δ_n to every point x of a Riemannian manifold (M^n, g):

Tangent bundle:
$$T(M^n) = \bigcup_{x \in M^n} T_x M^n$$

Spinor bundle:
$$S(M^n) = \bigcup_{x \in M^n} \Delta_n(x)$$



However:

• Denote by $\mathcal{F}(M^n, g)$ the oriented frame bundle. M^n admits a spin structure iff the $\mathrm{SO}(n)$-principal bundle $\mathcal{F}$ admits a reduction $\mathcal{P} \to \mathcal{F}$ to the group $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

$\longrightarrow$ notion of a *Riemannian spin manifold*

Different spin structures:

Suppose that a discrete group Γ acts properly discontinuous on a manifold $\tilde{M}^n$ and denote by $\pi : \tilde{M}^n \to M^n := \Gamma/\tilde{M}^n$ the projection onto the orbit space. Moreover, suppose that M^n admits a spin structure $\mathcal{P}$. The induced bundle

$$\pi^*(\mathcal{P}) = \left\{(\tilde{m}, p) \in \tilde{M}^n \times \mathcal{P} : \pi(\tilde{m}) = \pi_1(p)\right\}$$

is a $\mathrm{Spin}(n)$-principal bundle with the action of the spin group

$$(\tilde{m}\,,\, p) \cdot g \;=\; (\tilde{m}\,,\, p \cdot g)\;, \quad g \in \mathrm{Spin}(n)\;.$$

Moreover, Γ acts on $\pi^*(\mathcal{P})$ via

$$\gamma \cdot (\tilde{m}\,,\, p) \;=\; (\gamma \cdot \tilde{m}\,,\, p)$$

and the fixed spin bundle can be reconstructed,

$$\mathcal{P} \;=\; \Gamma/\pi^*(\mathcal{P})\;.$$

Consider a homomorphism $\epsilon : \Gamma \to \{1, -1\} \subset \mathrm{Spin}(n)$ and introduce a new Γ_ϵ-action via the formula

$$\gamma \cdot (\tilde{m}\,,\, p) \;=\; (\gamma \cdot \tilde{m}\,,\, p \cdot \epsilon(\gamma))\ .$$

The space

$$\mathcal{P}_\epsilon \;:=\; \Gamma_\epsilon / \pi^*(\mathcal{P})$$

is still a $\mathrm{Spin}(n)$-principal fiber bundle over M^n, a new spin structure of the manifold.

If $\tilde{M}^n$ is the universal covering of M^n , then the group Γ is isomorphic to the fundamental group $\pi_1(M^n)$ of M^n. In particular we proved

Theorem: If M^n admits at least one spin structure, then all spin structures correspond to the set

$$\mathrm{Hom}(\pi_1(M^n)\,,\,\mathbb{Z}_2) \;=\; H^1(M^n\,;\,\mathbb{Z}_2)\ .$$

Existence of a spin structure

Consider the classifying map $f : M^n \to B\mathrm{SO}(n)$ of the tangent bundle. M^n admits a spin structure iff f lifts into the classifying space $B\mathrm{Spin}(n)$. Since

$$\pi_2(B\mathrm{Spin}(n)) \;=\; \pi_1(\mathrm{Spin}(n)) \;=\; 0 \quad \text{and} \quad \pi_1(B\mathrm{Spin}(n)) \;=\; 0$$

we have $H^2(B\mathrm{Spin}(n); \mathbb{Z}_2) = H^1(B\mathrm{Spin}(n); \mathbb{Z}_2) = 0$. Consequently, the image of the second Stiefel-Whitney class $\omega_2 \in H^2(B\mathrm{SO}(n); \mathbb{Z}_2)$ under the map $H^2(B\mathrm{SO}(n); \mathbb{Z}_2) \to H^2(B\mathrm{Spin}(n); \mathbb{Z}_2)$ is zero. This argument yields a necessary condition for the existence of a spin structure, namely $\omega_2(M^n) = 0$. Indeed , the condition is sufficient, too.

Theorem: An oriented manifold admits a spin structure iff its second Stiefel-Whitney class vanishes, $\omega_2(M^n) = 0$.

Examples:

- S^n, $\mathbb{C}(P)^{2n+1}, \ldots$ are spin manifolds with a unique spin structure.
- T^n admits 2^n different spin structures.
- $\mathbb{C}(P)^{2n}, SU(3)/SO(3), \ldots$ are not spin manifolds.

Let $(M^n, g, \mathcal{P})$ be a Riemannian spin manifold with a fixed spin structure.

The associated bundle

$$S \;:=\; \mathcal{P} \times_{\mathrm{Spin}(n)} \Delta_n \ .$$

is the spinor bundle S .

The Levi-Civita connection ∇ can be lifted from the tangent bundle to the spinor bundle S in a unique way.

Dirac operator

In an orthonormal frame $e_1, \dots, e_n$

$$D: \; \Gamma(S) \longrightarrow \Gamma(S), \quad D\psi = \mu \circ \nabla\psi, \quad D\psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi.$$

Properties of D:

- D is an elliptic differential operator of first order
- $D^2 = \Delta^S + \frac{1}{4}\mathrm{Scal}$ (Schrödinger 1932, Lichnerowicz 1962)

For the Laplacian Δ_q on differential forms in $\Lambda^q(M^n)$, Hodge - de Rham theory implies that

$$\dim\ker(\Delta_q) \; =: \; b_q(M^n) \quad \text{is a topological invariant.}$$

For the Dirac operator, $\dim\ker(D)$ is *not a topological invariant*.

Basic Example: (see Hitchin 1974)

Consider the Lie group $\mathrm{Spin}(3) = S^3$ and the basis e_1, e_2, e_3 of its Lie algebra with the commutator relations

$$[e_1\,,\,e_2] = 2\cdot e_3\,, \quad [e_2\,,\,e_3] = 2\cdot e_1\,, \quad [e_3\,,\,e_1] = 2\cdot e_2\,.$$

We introduce a left invariant metric defined by the conditions

$$|e_1| = |e_2| = 1\,, \quad |e_3| = \lambda\,, \quad \langle e_i\,,\,e_j\rangle = 0 \quad \text{if} \quad i \neq j\ .$$

The eigenvalues of the Dirac operator are given by the formulas

$$\begin{aligned} \mu_p(\lambda) \quad &= \quad \frac{p}{\lambda} + \frac{\lambda}{2}\,, \quad p = 1,\,2\,\ldots \text{ with multiplicity } \quad 2p \\ \nu^{\pm}_{p,q}(\lambda) \quad &= \quad \frac{\lambda}{2} \pm \frac{1}{\lambda}\sqrt{4pq\lambda^2 + (p-q)^2}\,, \quad p = 1,\,2,\,\ldots,q = 0,\,1,\ldots \\ &\quad \text{with} \quad \text{multiplicity} \quad (p+q)\ . \end{aligned}$$

The kernel of the Dirac operator corresponds to $\nu^-_{p,q} = 0$, i.e.

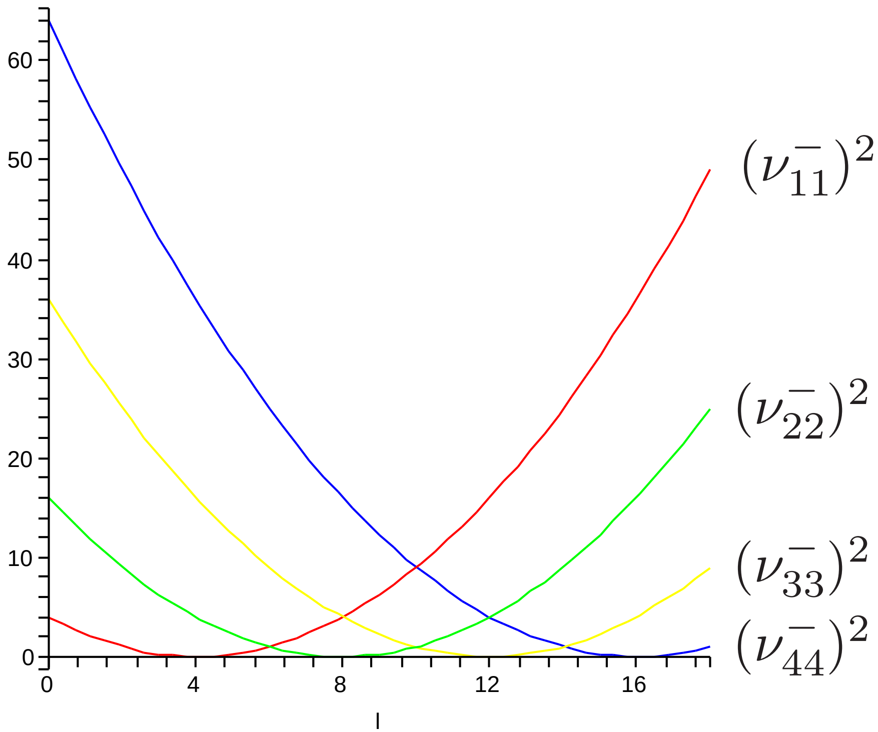
$$\lambda^4 = 4\left(4pq\lambda^2 + (p-q)^2\right) .$$

- If the parameter λ is a transcendent number, then the kernel of the Dirac operator is trivial .

- If $\lambda = 4p$ is an integer divisible by 4, then $p = q$ is a solution. In this case the dimension of the kernel of the Dirac operator is at least $2p$.

- Remark that

$$\lim_{\lambda\to 0} \mu_p(\lambda) = \infty , \quad \lim_{\lambda\to 0} \nu^{\pm}_{p,p}(\lambda) = \pm 2p , \quad \lim_{\lambda\to 0} \nu^{\pm}_{p,q}(\lambda) = \pm\infty .$$



60
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0
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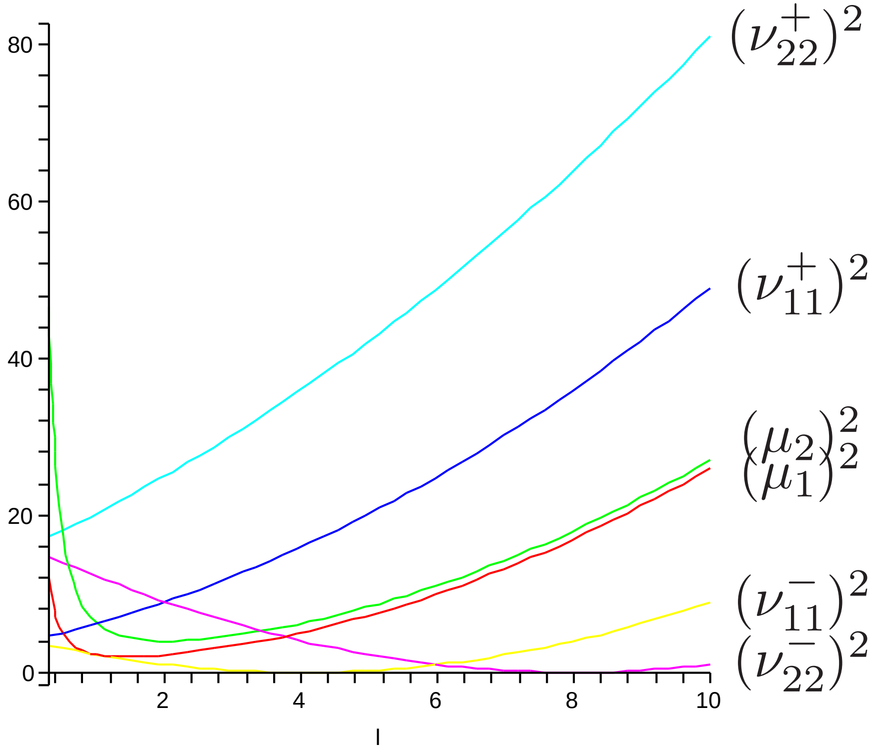
l
$(\nu_{11}^{-})^2$
$(\nu_{22}^{-})^2$
$(\nu_{33}^{-})^2$
$(\nu_{44}^{-})^2$



80
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$(\nu_{22}^{+})^2$
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$(\mu_2)^2$
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$(\nu_{11}^{-})^2$
$(\nu_{22}^{-})^2$

Conformal change of the metric

- $g_1 = \sigma \cdot g$ – two conformally equivalent metrics on M^n.
- D_1 and D – the corresponding Dirac operators.
- After a suitable identification of spinors we obtain the formula

$$D_1(\psi) \;=\; \sigma^{-\frac{n+1}{4}} D(\sigma^{\frac{n-1}{4}} \psi) \;.$$

Theorem: The dimension of the kernel of the Dirac operator is a conformal invariant.

Corollary: Let (M^n, g) be a compact Riemannian spin manifold. If the metric is conformally equivalent to a metric g_1 with positive scalar curvature, then the kernel of the Dirac operator is trivial.

Example: For any metric on S^2, the kernel of the Dirac operator is trivial.

The index of the Dirac operator

• If (M^n, g) is a complete Riemannian manifold, then the Dirac operator is essentially self-adjoint.

• If $n = 2k$ is even, then the representation $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ splits, the spin bundle $S = S^+ \oplus S^-$ splits and the Dirac operator splits, too,

$$D^+ : \Gamma(S^+) \longrightarrow \Gamma(S^-), \quad D^- : \Gamma(S^-) \longrightarrow \Gamma(S^+) .$$

• Let M^n be compact. Then the index of D^+ is given by the $\hat{A}$-genus,

$$\mathrm{index}(D^+) = \hat{A}(M^n) .$$

• If $n = 4$, then $\hat{A} = \frac{1}{24} p_1$. If $n = 8$, then $\hat{A} = \frac{1}{5760}(7p_1^2 - 4p_2)$.

• In case a a compact 4-manifold we have $\hat{A}(M^4) = \frac{1}{8}\sigma(M^4)$.

Theorem: The $\hat{\mathcal{A}}$-genus of any compact spin manifold is an integer, $\hat{\mathcal{A}}(M^n) \in \mathbb{Z}$. Moreover, in dimensions $8k+4$ the $\hat{\mathcal{A}}$-genus is an even number.

Example: $\hat{\mathcal{A}}(\mathbb{CP}^2) = 1/8$. $\mathbb{CP}^2$ is not spin.

Corollary: (Rochlin) The signature of a smooth, compact, 4-dimensional spin manifold is divisible by 16.

Theorem: Let M^n be compact and spin. If it admits a Riemannian metric with positive the scalar curvature, then the $\hat{\mathcal{A}}$-genus vanishes, $\hat{\mathcal{A}}(M^n) = 0$.

Example: The scalar curvature of the Kähler metric of $\mathbb{CP}^2$ is positive and $\hat{\mathcal{A}}(\mathbb{CP}^2) = 1/8 \neq 0$. But $\mathbb{CP}^2$ is not spin.

Eigenvalue estimates for the Dirac operator

(M^n, g): compact, Riemannian spin manifold

R_0: minimum of the scalar curvature

λ: eigenvalues of the Dirac operator

- The Schrödinger-Lichnerowicz formula implies immediately $\lambda^2 \geq \mathrm{R}_0/4$ $\longrightarrow$ not optimal.

Theorem: (The Riemannian case – Friedrich 1980):

$$\lambda^2 \;\geq\; \frac{n}{4(n-1)} \cdot \mathrm{R}_0 \;.$$

Idea of the proof: Fix a real-valued function $f : M^n \longrightarrow \mathbb{R}^1$ and introduce a new spinorial connection

$$\nabla^f_X \psi \;:=\; \nabla_X \psi + f \cdot X \cdot \psi \;.$$

Next generalize the Schrödinger-Lichnerowicz-formula

$$(D - f)^2\psi = \Delta^f\psi + \frac{1}{4}R \cdot \psi + (1 - n)f^2 \cdot \psi \ .$$

If $D\psi = \lambda \cdot \psi$, then use the latter formula with $f = \lambda/n$ and integrate. The estimate follows after some elementary computation.

Idea of a second proof:

Consider the Twistor Operator

$$P \ : \Gamma(\mathrm{S}) \ \to \ \Gamma(\mathrm{T} \otimes \mathrm{S}) \ , \quad P(\psi)(X) \ := \ \nabla^g_X \psi + \frac{1}{n} X \cdot D(\psi) \ .$$

and prove the formula (A. Lichnerowicz 1987):

$$||P(\psi)||^2_{L^2} = \frac{n-1}{n}||D(\psi)||^2_{L^2} - \frac{1}{4}\int_{M^n} R \cdot ||\psi||^2 \ .$$

Remark:

If $D^2\psi = \frac{n}{4(n-1)} R_0 \cdot \psi$,then the spinor field ψ satisfies a stronger equation, namely

$$\nabla_X \psi = \pm \frac{1}{2} \sqrt{\frac{R_0}{n(n-1)}} X \cdot \psi \ .$$

Riemannian Killing spinors

Example:

The lower bound is realized for example on all spheres. But there are other manifolds, too (see later – Killing spinors).

Further Results:

- The Kähler case (Kirchberg 1986-1990):

$$\lambda^2 \geq \frac{m}{4(m-1)} \cdot \mathrm{R}_0 \qquad \text{if } m := n/2 \text{ is even .}$$

$$\lambda^2 \geq \frac{m+1}{4 \cdot m} \cdot \mathrm{R}_0 \qquad \text{if } m := n/2 \text{ is odd .}$$

- The quaternionic-Kähler case (Kramer, Semmelmann, Weingard 1997):

$$\lambda^2 \geq \frac{k+3}{4(k+2)} \cdot \mathrm{R}_0 \qquad \text{where } k := n/4 \text{ .}$$

• Conformal estimate (Lott 1986):

Let $[g_0]$ be a conformal structure on a Riemannian spin manifold such that $\ker(D_{g_0}) = 0$. Then there exists a constant $C = C([g_0])$ such that for any metric $g \in [g_0]$ the inequality holds

$$\lambda^2(D_g) \ \geq \frac{C}{\mathrm{vol}(M^n, g)^{2/n}} \ .$$

• The case of S^2 (Hijazi 1986, Bär 1991):

S^2 has only one conformal structure. The corresponding Lott constant equals $C = 4 \cdot \pi$, i. e. for any metric g on S^2 the inequality holds :

$$\lambda^2(D_g) \ \geq \frac{4 \cdot \pi}{\mathrm{vol}(S^2, g)} \ .$$

• Eigenvalue estimates for the Dirac operator depending on the other components of the curvature tensor, i. e. depending on Ric or W. See for example

Th.Friedrich and K.-D.Kirchberg, Journ. Geom. Phys. 41 (2002), 196 - 207.
Th.Friedrich and K.-D.Kirchberg, Math. Ann. 324 (2002), 700-716.
Th. Friedrich and E.C. Kim, Journ. Geom. Phys. 37 (2001), 1-14.

Riemannian manifolds with Killing spinors

Killing Spinor on (M^n, g) :

$$\nabla_X \psi \;=\; \lambda \cdot X \cdot \psi\,, \quad X \in T(M^n) \quad \text{and} \quad \lambda \in \mathbb{R}^1\,.$$

Necessary conditions:

- (M^n, g) is an Einstein manifold with positive scalar curvature $R > 0$.
- The so-called Killing number λ is given by the scalar curvature,

$$\lambda \;=\; \pm\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\,.$$

Theorem: (Friedrich,Grunewald,Kath, Hijazi 1986-1989)

Let (M^n, g) be a simply-connected spin manifold. Then is admits a Killing spinor if and only if

- $n = 3, 4, 8$: M^n has positive constant curvature, i.e. $M^n = S^n$.
- $n = 5$: M^5 is an Einstein-Sasakian manifold.
- $n = 6$: M^6 is a nearly Kähler manifold.
- $n = 7$: M^7 is a nearly parallel G_2-manifold.
- Any Einstein-Sasakian manifold M^{2k+1} admits two Killing spinors.

Examples: S^1-fibrations $M^{2k+1} \to X^{2k}$ over Kähler-Einstein manifolds X^{2k}.

The twistor operator

Consider the kernel $\ker(\mu) \subset T(M^n) \otimes S$ of the Clifford multiplication as well as the projection onto this subbundle,

$$p : T(M^n) \otimes S \longrightarrow \ker(\mu) \; .$$

The covariant derivative $\nabla\psi$ of any spinor field is a section in $T^*(M^n) \otimes S = T(M^n) \otimes S$ and we can apply the projection. The operator

$$\mathcal{P}(\psi) \; := \; p \circ \nabla\psi \qquad \mathcal{P} \; : \; \Gamma(S) \longrightarrow \Gamma(\ker(\mu))$$

is the twistor operator. In a local frame we obtain

$$\mathcal{P}(\psi) \; = \; \sum_{i=1}^{n} e_i \otimes (\nabla_{e_i}\psi + \frac{1}{n} e_i \cdot D(\psi)) \; .$$

The twistor equation $\mathcal{P}(\psi) = 0$ reads as

$$\nabla_X \psi + \frac{1}{n} X \cdot D(\psi) \; = \; 0 \, , \quad X \in T(M^n) \; .$$

Proposition: Any Killing spinor is a solution of the twistor equation.

Proof: $\nabla_X \psi = \lambda \, X \cdot \psi$ implies $D(\psi) = - \, n \, \lambda \, \psi$ and then we obtain

$$\nabla_X \psi + \frac{1}{n} X \cdot D(\psi) \; = \; \nabla_X \psi + \frac{1}{n} X \cdot (-n \, \lambda \psi) \; = \; \nabla_X \psi - \lambda \, X \cdot \psi \; = \; 0 \; .$$

Proposition: The dimension of the kernel of the twistor operator is a conformal invariant. If M^n is connected, then it is bounded by

$$\ker(\mathcal{P}) \; \leq 2^{[n/2]+1} \; = \; 2 \dim(\Delta_n) \; .$$

Theorem: (Lichnerowicz 1987, Friedrich 1989)

Let ψ be a twistor spinor on a connected M^n. Then the functions

$$
\begin{aligned}
C(\psi) &:= \mathrm{Re}\big(\psi\,,\,D(\psi)\big)\ , \\
Q(\psi) &:= |\psi|^2|D(\psi)|^2 - C^2(\psi) - \sum_{i=1}^{n}\big(\mathrm{Re}(D(\psi)\,,\,e_i\cdot\psi)\big)^2
\end{aligned}
$$

are constant.

Proposition: (a local result – Friedrich 1989)

The zeros of a twistor spinor on a connected manifold are isolated. Outside the zero set there is a conformal change of the metric such that the twistor spinor becomes the sum of two Killing spinors.

Theorem: (a global result – Lichnerowicz 1989)

Let (M^n, g) be a compact Riemannian spin manifold with $\ker(\mathcal{P}) \neq 0$. Then there exists an Einstein metric g^* such that the space $\ker(\mathcal{P}) = \ker(\mathcal{P}^*)$ coincides with the space of Killing spinor on (M^n, g^*).

Proof: Use the solution of the Yamabe problem as well as the limiting case in the estimate of the Dirac operator.

Further results on twistor spinors with zeros:

- K. Habermann, J. Geom. Phys. 1990 and 1992
- W. Kühnel and H.-B. Rademacher – several papers since 1994.

Intrinsic upper bounds for metrics on S^2 and T^2

- If $n = 2$ and $g = e^{2\,u} g_0$, then

$$\Delta_g \;=\; e^{-\,2\,u}\Delta_{g_0}\,, \quad D_g \;=\; e^{-\,u}\Big(D_{g_0} + \frac{1}{2}\,\mathrm{grad}_{g_0}(u)\Big)\;.$$

- The Rayleigh quotient

$$\frac{|D_g(\psi)|^2_{L^2(M,g)}}{|\psi|^2_{L^2(M,g)}} \;=\; \frac{|D_{g_0}(\psi) \,+\, \frac{1}{2}\,\mathrm{grad}_{g_0}(u)\cdot\psi|^2_{L^2(M,g_0)}}{|e^u\,\psi|^2_{L^2(M,g_0)}}\;.$$

Suppose that on (M^2, g_0) there exists a spinor field ψ_0 such that

$$||\psi_0|| \;\equiv\; 1\,, \quad D_{g_0}\psi_0 \;=\; \Lambda\cdot\psi_0\,, \quad \Lambda : M^2 \to \mathbb{R}^1\;.$$

Now apply the Rayleigh quotient with a family of test spinor $\psi = f\cdot\psi_0$.

Theorem: For any metric $g = e^{2u}g_0$ on M^2 and any function $f : M^2 \to \mathbb{R}^1$ the following estimate holds:

$$\lambda_1^2(D_g) \int e^{2u} f^2 dM^2(g_0) \leq \int \{\Lambda^2 f^2 + ||\mathrm{grad}_{g_0}(f) + \frac{1}{2} f \mathrm{grad}_{g_0}(u)||^2\} dM^2(g_0)$$

- Consider $f \equiv 1$. Then

$$\lambda_1^2(D_g)\ \mathrm{vol}(M^2, g) \ \leq \ \int \Lambda^2 dM^2(g_0) \ + \ \frac{1}{4} \int ||\mathrm{grad}_{g_0}(u)||^2 dM^2(g_0) \ .$$

- Consider $f = e^{-u/2}$. Then $\mathrm{grad}_{g_0}(f) + \frac{1}{2} f \mathrm{grad}_{g_0}(u) = 0$ and we obtain

$$\lambda_1^2(D_g) \int e^{u} dM^2(g_0) \ \leq \ \int \Lambda^2 e^{-u} dM^2(g_0) \ .$$

These estimates can be used in the following cases:

- $(M^2, g_0) = (S^2, g_{can})$ and ψ_0 is the Killing spinor, $\Lambda = \lambda_1(D_{g_0})$.

- $(M^2, g_0) = (T^2, g_{flat})$ with a non-trivial spin structure and $\Lambda = \lambda_1(D_{g_0})$. Then we know that $\ker(D_{g_0}) = 0$ and the eigenspinors ψ_0 have constant length.

- A surface $M^2 \subset \mathbb{R}^3$. The restriction ψ_0 of a $\mathbb{R}^3$-parallel spinor to M^2 has constant length and satisfies the equation $D(\psi_0) = H \cdot \psi_0$, where $\Lambda = H$ is the mean curvature.

Application:

Consider the ellipsoid

$$x^2 \ + \ y^2 \ + \ \frac{z^2}{a^2} \ = \ 1$$

and denote by $\lambda_1^2(a)$ the first eigenvalue of the square of the Dirac operator. Then we obtain

$$2 \ \leq \ \limsup_{a \to 0} \lambda_1^2(a) \ \ \leq \ \ \frac{3}{2} + \ln(2) \ \simeq \ 2,2$$
$$\limsup_{a \to \infty} \lambda_1^2(a) \ \ \leq \ \ \frac{1}{4} \, .$$

A further result: (M. Kraus 1999) $\frac{1}{4} \leq \liminf_{a \to \infty} \lambda_1^2(a)$.

Consequently, the upper bound is in the asymptotic optimal.

The case of T^2 with a trivial spin structure:
g_0 – the flat matric on the torus $T^2 = \mathbb{R}^2/\Gamma$, $g = e^{2u}\, g_0$. The kernel $\ker(D_{g_o}) \simeq \ker(D_g)$ is 2-dimensional and coincides with the g_o-parallel spinors. Consequently

$$\lambda_1^2(D_{g_0}) \;=\; \lambda_1^2(D_g) \;=\; 0 \;.$$

We estimate $\lambda_2^2(D_g)$. Fix a g_0-parallel spinor ψ_0. Then $\psi_0^* := e^{-u/2}\psi_0$ belongs to the kernel of D_g. We use test spinors $\psi := f\, e^{-3u/2}\psi_0$ being orthogonal to the kernel,

$$(\psi\, ,\, \psi_0^*)_{L(T^2,g)} \;=\; \int_{T^2} f dT^2 \;=\; 0 \;.$$

Theorem: For any function such that $\int f dT^2 = 0$,

$$\lambda_2^2(D_g) \int_{T^2} |f|^2 e^{-u} dT^2 \;\leq\; \int_{T^2} e^{-3u} ||\mathrm{grad}(f) \;-\; f\,\mathrm{grad}(u)||^2 dT^2 \;.$$

We apply the inequality for eigenfunctions of the Laplace operator

$$f_{v^*}(x) \;=\; e^{i\langle v^*\,,\,x\rangle}\,, \quad v^* \in \Gamma^* \;.$$

Then

$$\mathrm{grad}(f) \;=\; i\,f\,v^*\,, \quad ||\mathrm{grad}(f) - f\,\mathrm{grad}(u)||^2 \;=\; ||v^*||^2 + ||\mathrm{grad}(u)||^2\;.$$

Minimizing with respect to $0 \neq v^* \in \Gamma^*$, we obtain

$$\lambda_2^2(D_g)\int_{T^2} e^{-u} dT^2 \;\leq\; \lambda_2^2(D_{g_0})\int_{T^2} e^{-3u} dT^2 + \int_{T^2} e^{-3u}||\mathrm{grad}(u)||^2 dT^2\;.$$

I. Agricola, Th. Friedrich, Journ. Geom. Phys. 30 (1999), 1-22.
I. Agricola, B. Ammann, Th. Friedrich, Manusc. Math. 100 (1999), 231-258.
M. Kraus, Journ. Geom. Phys. 31 and 32 (1999), 209-216 and 341-348.

Surfaces, mean curvature and the Dirac operator

- $M^n \subset \mathbb{R}^{n+1}$, $\quad S : TM^n \to TM^n$ – second fundamental form.
- ψ_0 – parallel spinor in $\mathbb{R}^{n+1}$, $\psi := \psi_0|M^n$ -spinor on M^n.
- D – the Dirac operator on M^n , H – the mean curvature.

Then we have

$$\nabla_X \psi \;=\; \frac{1}{2} S(X) \cdot \psi \,, \quad D\psi \;=\; \frac{n}{2} H \cdot \psi \,.$$

Theorem: If M^n is compact and oriented, then

$$\lambda_1^2(D)\,\mathrm{vol}(M^n, g) \;\leq\; \frac{n^2}{4} \int_{M^n} H^2 \, dM^n \,.$$

The construction of the immersion using the spinor

- Any immersion $M^n \subset \mathbb{R}^{n+1}$ induces on M^n a Riemannian metric g, a function $H : M^n \to \mathbb{R}^1$ and a spinor field ψ of length one such that $D(\psi) = \frac{n}{2} H \psi$.

Theorem:(The spin formulation of the fundamental theorem for surfaces)

Let (M^2, g, H, ψ) be a 4-tuple consisting of a simply-connected Riemannian 2-manifold, a function $H : M^2 \to \mathbb{R}^1$ and a spinor field ψ of length one such that $D\psi = H \psi$. Then there exits an isometric immersion $M^2 \subset \mathbb{R}^3$.

Idea of the proof: Define the endomorphism $S^* : TM^2 \to TM^2$ by

$$g(S^*(X), Y) = 2 \operatorname{Re}(\nabla_X \psi, Y \cdot \psi) .$$

Then S^* is a symmetric endomorphism and $\nabla_X \psi = \frac{1}{2} S^*(X) \cdot \psi$ holds. The integrability condition of the latter equation is equivalent to the Gauss- and Codazzi-equation.

The Weierstrass representation of a surface

- α – the quaternionic structure in the 2-dimensional spin representation.
- $\Omega^{\psi}(X) := (X \cdot \psi \,,\, \alpha(\psi))$ – a complex valued 1-form.
- $\omega^{\psi}(X) := \mathrm{Re}(X \cdot \psi \,,\, \psi)$ – a real valued 1-form.

These forms are closed,

$$d\omega^{\psi} \;=\; d\Omega^{\psi} \;=\; 0$$

and the isometric immersion $f : M^2 \to \mathbb{R}^1 \oplus \mathbb{C} = \mathbb{R}^3$ is given by

$$f \;=\; \int_{M^2} \left(\omega^{\psi} \,,\, \Omega^{\psi}\right) .$$

(Weierstrass representation of a surface in $\mathbb{R}^3$ – not only minimal ones)

Th. Friedrich, Journ. Geom. Phys. 28 (1998), 143-157.
Generalization by B. Morel, M.-A. Lawn and J. Roth between 2005-2012.

The Dirac operator depending on a connection with totally skew-symmetric torsion

- $(M^n\,,\,g\,,\,\nabla\,,\,\mathrm{T})$ – Riemannian manifold,
- The torsion T of ∇ is a 3-form.
- linear metric connection

$$\nabla_X Y \;:=\; \nabla^g_X Y \,+\, \frac{1}{2}\,\mathrm{T}(X, Y, -)\,.$$

- covariant derivative on spinors

$$\nabla_X \psi \;:=\; \nabla^g_X \psi \,+\, \frac{1}{4}\,(X \,\lrcorner\, \mathrm{T}) \cdot \psi\,.$$

- a first order differential operator

$$\mathcal{D}\psi \;:=\; \sum_{k=1}^{n} (e_k \,\lrcorner\, \mathrm{T}) \cdot \nabla_{e_k}\psi\,,$$

- a 4-form derived from T,

$$\sigma_{\mathrm{T}} \ := \ \frac{1}{2}\sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \wedge (e_k \lrcorner \mathrm{T})\,.$$

- D – Dirac operator of the connection ∇
- $\not{D}$ – Dirac operator related to $\mathrm{T}/3$.

First formula:

$$D^2 \ = \ \Delta_{\mathrm{T}} + \frac{3}{4}d\mathrm{T} - \frac{1}{2}\sigma_{\mathrm{T}} + \frac{1}{2}\delta\mathrm{T} - \mathcal{D} + \frac{1}{4}\mathrm{Scal}^g\,,$$

Second formula:

$$\not{D}^2 \ = \ \Delta_{\mathrm{T}} + \frac{1}{4}d\mathrm{T} + \frac{1}{4}\mathrm{Scal}^g - \frac{1}{8}\,||\mathrm{T}||^2\,.$$

History of the $1/3$-shift: Slebarski (1987), Bismut (1989), Kostant (1999), Agricola (2002).

A Vanishing Theorem. Let (M^n, g, T) be a compact, Riemannian spin manifold s.t. $\mathrm{Scal}^g \leq 0$. If there exists a spinor $\psi \neq 0, (d\mathrm{T} \cdot \psi, \psi) \leq 0$ in the kernel of Δ_{T}, then the 3-form and the scalar curvature vanish, $\mathrm{T} = 0 = \mathrm{Scal}^g$, and ψ is parallel with respect to the Levi-Civita connection.

Corollary. On a Calabi-Yau or Joyce manifold, a metric connection with 3-form T s.t. $d\mathrm{T} = 0$ can only admit parallel spinors if $\mathrm{T} = 0$.

The Casimir operator

(M^n, g) – Riemnannian spin manifold, D – Riemannian Dirac operator.

- Schrödinger-Lichnerowicz formula:

$$D^2 \;=\; \Delta + \frac{1}{4} R \;.$$

- If $M^n = \mathrm{G/H}$ is a symmetric space, then (Parthasarathy formula):

$$D^2 \;=\; \Omega + \frac{1}{8} R$$

$(M^n\,,\,g\,,\,\nabla\,,\,\mathrm{T})$ - Riemannian manifold with torsion.

Definition: The **Casimir operator** acting on spinor fields of the triple is defined by

$$\begin{aligned}\Omega \;&:=\; D\!\!\!/^2 + \frac{1}{8}(d\mathrm{T} - 2\,\sigma_{\mathrm{T}}) + \frac{1}{4}\delta(\mathrm{T}) \\ &-\; \frac{1}{8}\mathrm{Scal}^g - \frac{1}{16}||\mathrm{T}||^2 \\ &=\; \Delta_{\mathrm{T}} + \frac{1}{8}(3\,d\mathrm{T} - 2\,\sigma_{\mathrm{T}} + 2\,\delta(\mathrm{T}) + \mathrm{Scal}^g)\,.\end{aligned}$$

Motivation: For a naturally reductive space and its canonic connection, the operator Ω coincides with the usual Casimir operator (Parthasarathy, 1972; Kostant, 1999; Agricola, 2002).

I. Agricola and Th. Friedrich, Journ. Geom. Phys. 50 (2004), 188-204.

Example: For the Levi-Civita connection $(\mathrm{T} = 0)$, we obtain

$$\Omega \;=\; D^2 - \frac{1}{8}\mathrm{Scal}^g \;=\; \Delta + \frac{1}{8}\mathrm{Scal}^g$$

Proposition: The kernel of the Casimir operator contains all ∇-parallel spinor.

Corollary: Lower bounds for the eigenvalues of $\not{D}^2$ yield that the kernel of the Casimir operator is trivial. In particular, then there are no ∇-parallel spinors.

The case $\nabla T = 0$:

$$\begin{aligned}\Omega &= D\!\!\!/^2 - \frac{1}{16}\left(2\,\mathrm{Scal}^g + \|T\|^2\right) \\ &= \Delta_T + \frac{1}{16}\left(2\,\mathrm{Scal}^g + \|T\|^2\right) - \frac{1}{4}T^2 \\ &= \Delta_T + \frac{1}{8}\left(2\,dT + \mathrm{Scal}^g\right) .\end{aligned}$$

Proposition: If the torsion form is ∇-parallel, then Ω and $D\!\!\!/^2$ commute with the endomorphism T,

$$\Omega \circ T = T \circ \Omega \,, \quad D\!\!\!/^2 \circ T = T \circ D\!\!\!/^2 \,.$$

In the compact case, T preserves the kernel of $D\!\!\!/$.

5-Dimensional Sasakian Manifolds

- M^5 – a 5-dimensional Sasakian manifold.
- η – the contact structure.
- The characteristic connection, $\mathrm{T}^c := \mathrm{T}$:

$$\begin{aligned} \nabla \mathrm{T} &= 0\,, \quad \mathrm{T} = \eta \wedge d\eta = 2\,(e_{12} + e_{34}) \wedge e_5\,, \\ \mathrm{T}^2 &= 8 - 8\,e_{1234}\,, \quad \mathrm{T} = \mathrm{diag}(4\,, 0\,, 0\,, -4)\,. \end{aligned}$$

$\Rightarrow$ the Casimir operator splits into

$$\Omega = \Omega_0 \oplus \Omega_4 \oplus \Omega_{-4},$$

$$\begin{aligned} \Omega_0 &= \Delta_{\mathrm{T}} + \frac{1}{8}\mathrm{Scal}^g + \frac{1}{2} = D\!\!\!/^2 - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{2}, \\ \Omega_{\pm 4} &= \Delta_{\mathrm{T}} + \frac{1}{8}\mathrm{Scal}^g - \frac{7}{2} = D\!\!\!/^2 - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{2}. \end{aligned}$$

If $\mathrm{Scal}^g \neq -4$, $\mathrm{Ker}(\Omega_0) = 0$. If $\mathrm{Scal}^g < -4$ or $\mathrm{Scal}^g > 28$, $\mathrm{Ker}(\Omega_{\pm 4}) = 0$.

The interesting cases: $-4 \leq \mathrm{Scal}^g \leq 28$.

Case $\mathrm{Scal}^g = -4$: The kernel of Ω_0 coincides with the space of ∇-parallel spinors ψ such that $\mathrm{T} \cdot \psi = 0$. Examples: Friedrich/Ivanov, 2002.

Spinors in both kernels $\mathrm{Ker}(\Omega_0)$ and $\mathrm{Ker}(\Omega_{\pm 4})$ exist on the 5-dimensional Heisenberg group

$$e_1 = \frac{1}{2} dx_1, \; e_2 = \frac{1}{2} dy_1, \; e_3 = \frac{1}{2} dx_2, \; e_4 = \frac{1}{2} dy_2,$$

$$e_5 = \eta := \frac{1}{2} \big(dz - y_1 \cdot dx_1 - y_2 \cdot dx_2 \big) .$$

Spinors in the kernel of $\Omega_{\pm 4}$ occur on Sasakian η-Einstein manifolds of type $\mathrm{Ric}^g = -2 \cdot g + 6 \cdot \eta \otimes \eta$ (Friedrich/Kim, 2000).

Case $\mathrm{Scal}^g = 28$:

$$\Omega_0 = \Delta_{\mathrm{T}} + 4 = \not{D}^2 - 4\,, \quad \Omega_{\pm 4} = \Delta_{\mathrm{T}} = \not{D}^2 - 4\,.$$

• The kernel of $\Omega_{\pm 4}$ coincides with the space of ∇-parallel spinors ψ such that $\mathrm{T}\cdot\psi = \pm 4\psi$. Examples: Friedrich/Ivanov, 2002.

Sasakian-Einstein manifolds, $\mathrm{Scal}^g = 20$:

$$\Omega_0 = \Delta_{\mathrm{T}} + 3\,, \quad \Omega_{\pm 4} = \Delta_{\mathrm{T}} - 1 = \not{D}^2 - 3\,.$$

Theorem: The Casimir operator of a compact 5-dimensional Sasakian-Einstein manifold has trivial kernel.

6-Dimensional nearly Kähler manifolds

- $(M^6, g, \mathcal{J})$ – a 6-dimensional nearly Kähler manifold.
- M^6 is Einstein, $\mathrm{Ric}^g = \frac{5}{2} \cdot a \cdot g, \quad a > 0$.
- The characteristic connection, $\mathrm{T}^c := \mathrm{T}$:

$$\nabla \mathrm{T} \ = \ 0 \, , \quad 4\,\mathrm{T} \ = \ \mathrm{N} \, , \quad \mathrm{Ric}^{\nabla} \ = \ 2\, a\, g \, .$$

$$2\, \sigma_{\mathrm{T}} \ = \ d\mathrm{T} \ = \ a \left(\omega \wedge \omega \right) , \quad ||\mathrm{T}||^2 \ = \ 2\, a \, .$$

- If M^6 is compact, then

$$\mathrm{Ker}(\Omega) \ = \ \mathrm{Ker}(\nabla) \ = \ \{\text{Killing spinors}\} \, ,$$

$$D\!\!\!/^{\,2} \ \geq \ \frac{2}{15} \mathrm{Scal}^g \ = \ 2 \cdot a \ > \ 0 \, .$$

7-Dimensional G_2-Manifolds

- (M^7, g, ω) – cocalibrated G_2-manifold $(d * \omega = 0)$
- Suppose that $(d\omega\,,\,*\,\omega)$ is constant.
- The characteristic connection:

$$\mathrm{T} \;=\; -\,*\,d\omega + \frac{1}{6}(d\omega\,,\,*\omega)\cdot\omega\,, \quad \delta(\mathrm{T}) \;=\; 0\,.$$

- Main difference to the previous examples:

$$\nabla\mathrm{T} \;\neq\; 0\,, \quad d\mathrm{T} \;\neq\; 2\cdot\sigma_{\mathrm{T}}\,, \quad \mathrm{Scal}^g \;=\; 2\,(\mathrm{T}\,,\,\omega)^2 - \frac{1}{2}\,||\mathrm{T}||^2\,.$$

- The parallel spinor ψ_0 corresponding to ω satisfies

$$\nabla\psi_0 \;=\; 0\,, \quad \mathrm{T}\cdot\psi_0 \;=\; -\frac{1}{6}(d\omega\,,\,*\omega)\cdot\psi_0\,.$$

Nearly parallel G_2-structures: $d\omega = -a\,(*\omega)$.

$$\Omega \;=\; D\!\!\!/^{\,2} - \frac{49}{144}\,a^2 .$$

Theorem: Let (M^7, g, ω) be a compact, nearly parallel G_2-manifold and denote by ∇ its characteristic connection. The kernel of the Casimir operator of the triple (M^7, g, ∇) coincides with the space of ∇-parallel spinors,

$$\mathrm{Ker}(\Omega) = \{\psi \,:\, \nabla\psi \;=\; 0,\; \mathrm{T}\cdot\psi \;=\; \frac{7}{6}\,a\cdot\psi\} = \mathrm{Ker}(\nabla).$$

Remark: This case includes Sasakian-Einstein manifolds and 3-Sasakian manifolds in dimension $n = 7$.

G_2**-structure of type** $\mathcal{W}_3$: $d * \omega = 0\,,\ (d\omega\,,\, *\omega) = 0.$

- Torsion and parallel spinor:

$$\mathrm{T} \;=\; -\,*\,d\omega\,, \quad \mathrm{Scal}^g \;=\; -\frac{1}{2}\,||\mathrm{T}||^2\,, \quad \nabla\psi_0 \;=\; 0\,, \quad \mathrm{T}\cdot\psi_0 \;=\; 0\,.$$

- Casimir operator:

$$\Omega = D\!\!\!\!/^{\,2} + \frac{1}{8}\,(d\mathrm{T} - 2\,\sigma_{\mathrm{T}}) = \Delta_{\mathrm{T}} + \frac{1}{8}\,(3\,d\mathrm{T} - 2\,\sigma_{\mathrm{T}} - 2\,||\mathrm{T}||^2)\,.$$

Results: The metrics and 3-forms on $N(1,1)$ with parallel spinors described before yield examples of G_2-structures such that

$$\Omega - D\!\!\!\!/^{\,2}\,, \quad \Omega - \Delta_{\mathrm{T}}$$

are negative or positive (no general relation between these operators).

Some references

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Th. Friedrich, Asian Journ. Math. 5 (2001), 129-160.

Th. Friedrich, S. Ivanov, Asian Journ. Math. 6 (2002), 303-336.

I. Agricola , Th. Friedrich, Math. Ann. 328 (2004), 711-748.

I. Agricola, Arch.Math. 42 (2006), 5-84.

I. Agricola, Handbook of pseudo-Riemannian Geometry and Supersymmetry, EMS Publishing House 2010.

Eigenvalue estimates for $D\!\!\!/^2$ via deformations

Thm. Assume $\nabla T = 0$ und let $\Sigma = \oplus_\mu \Sigma_\mu$ be the splitting of the spinor bundle into eigenspaces of T. Then:

a) ∇ preserves the splitting of Σ, i. e. $\nabla\Sigma_\mu \subset \Sigma_\mu \quad \forall\mu$,

b) $D\!\!\!/^2 \circ T = T \circ D\!\!\!/^2$, i. e. $D\!\!\!/^2 \Sigma_\mu \subset \Sigma_\mu \quad \forall\mu$. [2004]

$\Rightarrow$ Estimate on every subbundle of Σ_μ

Idea: Deform the connection ∇ by a *symmetric and parallel* endomorphism $S : \Gamma(\Sigma) \to \Gamma(\Sigma)$, for example $S =$ polynomial in T,

$$\nabla^S_X \psi \ := \ \nabla_X \psi - \frac{1}{2}(X \cdot S + S \cdot X) \cdot \psi$$

The formula:

$$\langle (D\!\!\!/ + S)^2\psi, \psi\rangle \;=\; \|\nabla^S\psi\|^2 - \frac{1}{4}\sum_{i=1}^{n}\|(e_i\cdot S + S\cdot e_i)\psi\|^2 - \frac{1}{4}\|T\psi\|^2 + \frac{1}{8}\|T\|^2\cdot\|\psi\|^2 + \frac{1}{4}\int_{M^n}\mathrm{Scal}^g\,\|\psi\|^2\,dM^n + \|S\psi\|^2 - \langle TS\,\psi,\psi\rangle$$

For example l.h.s.: $= \underbrace{\langle D\!\!\!/^2\psi, \psi\rangle}_{\lambda^2\|\psi\|^2,\ \text{o.k.}} + \underbrace{\|S\psi\|^2}_{\text{r.h.s., o.k.}} + 2\,\underbrace{\langle D\!\!\!/\psi, S\psi\rangle}_{???}$

The last term needs to be estimated and leads in the equality case to an equation of twistor type ("$n\nabla^g_X\psi = -X\cdot D^g\psi$")

I. Agricola, Th. Friedrich and M. Kassuba, Diff. Geom. and its Appl. 26 (2008), 613-624.

The 5-dimensional Sasaki case

- T has EV $0, \pm 4$,

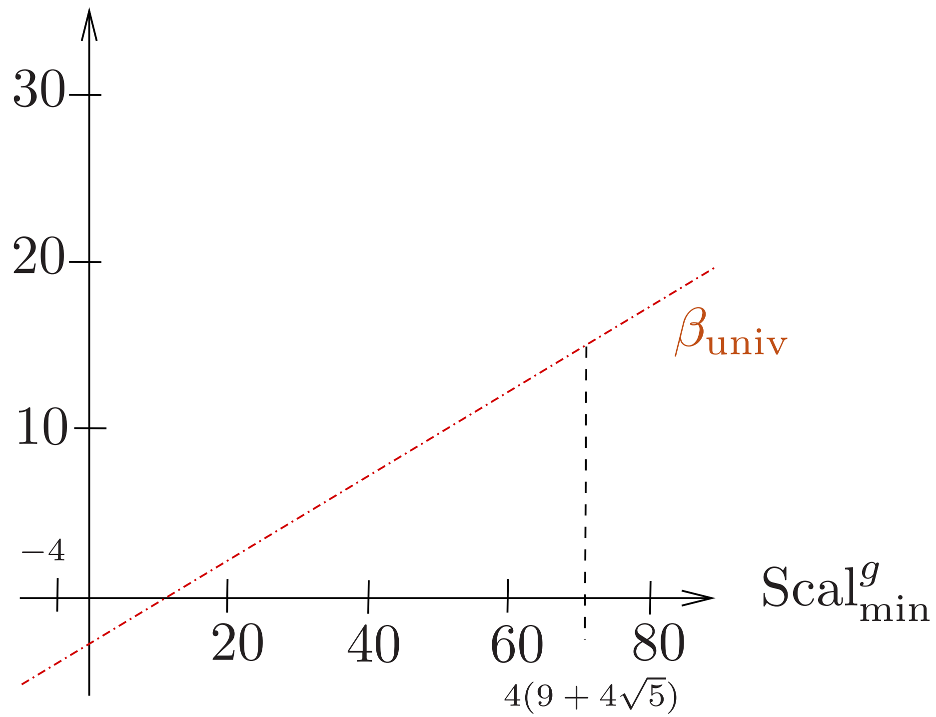
$\Sigma = \Sigma_4 \oplus \Sigma_0 \oplus \Sigma_{-4}$

- $\|T\|^2 = 8$ fixed

- $\mathrm{Scal}^g_{\min} > -4$

- Universal estimate:

$\lambda^2 \geq \frac{1}{4}\mathrm{Scal}^g_{\min} - 3 =: \beta_{\mathrm{univ}}$



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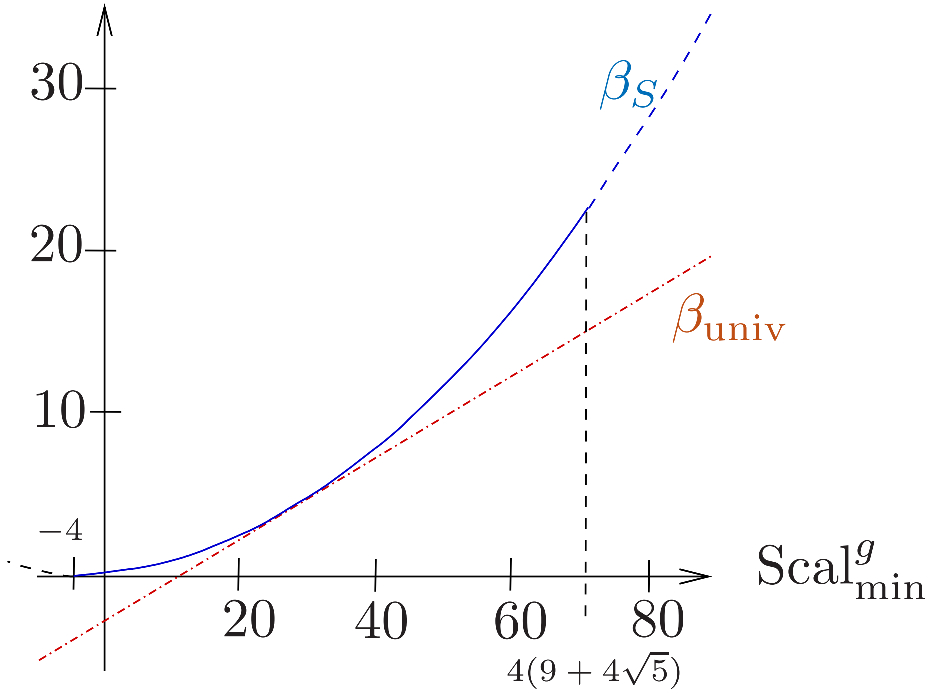
- S-deformed estimate:

$\lambda^2 \geq \frac{1}{16}\left[\frac{1}{4}\mathrm{Scal}^g_{\min} + 1\right]^2 =: \beta_S$



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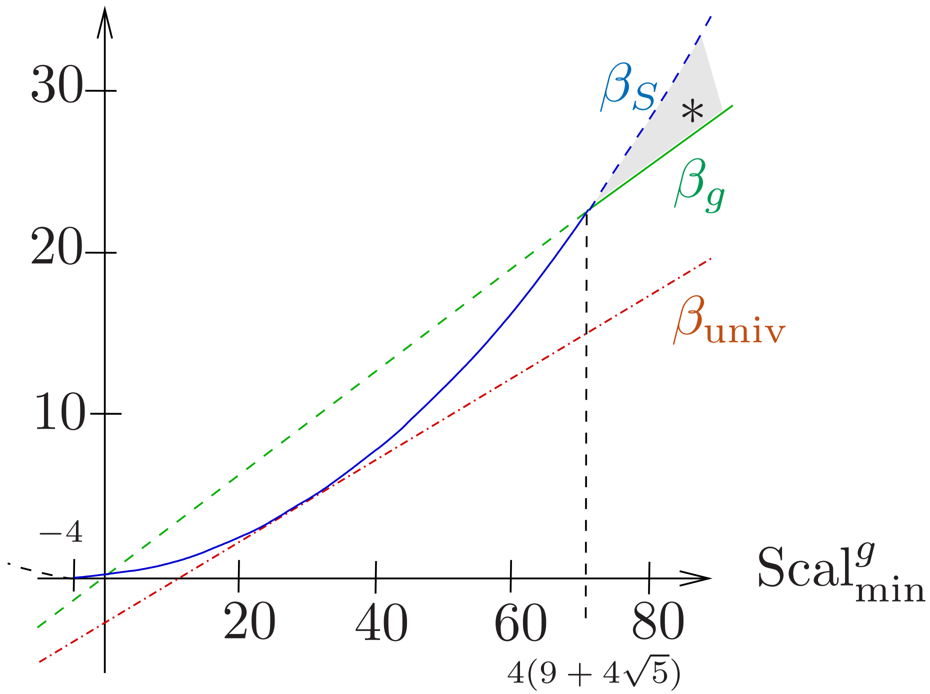


A subtle argument based on the fact that 0 is an EV of T shows:

$$\lambda^2 \geq \frac{5}{16}\mathrm{Scal}^g_{\min} = \frac{n}{4(n-1)}\mathrm{Scal}^g_{\min} =: \beta_g \qquad \text{for } \mathrm{Scal}^g_{\min} \geq 4(9+4\sqrt{5}) \cong 71,78$$

In the region $*$, we have in addition $\lambda^2_{\min}(D\!\!\!/^2_{|\Sigma_0}) = \lambda^2_{\min}(D\!\!\!/^2_{|\Sigma_{\pm 4}})$.

First known estimate with *quadratic* dependence on the scalar curvature!
[Sasaki condition is not scaling invariant]

Dfn. A Sasaki mnfd is called an *η-Einstein-Sasaki mnfd* if it is Einstein on $\eta^\perp$, i. e. $\mathrm{Ric} = (a, a, a, a, 4)$ for some $a \in \mathbb{R}$.

Thm. On a simply connected Sasaki mnfd (M^5, g, η), $\beta_S = \frac{1}{16}\left[\frac{1}{4}\mathrm{Scal}^g_{\min} + 1\right]^2$ is an EV of $\not{D}^2$ iff (M^5, g, η) is an η-Einstein-Sasaki mnfd.

Example. Regular compact 5-dimensional Sasaki mnfds are S^1-PFB over 4-dimensional Kähler mnfds; these are η-Einstein-Sasaki iff the base is a Kähler-Einstein mnfd.

Non regular compact 5-dim. Sasaki mnfds were constructed by Boyer / Galicki.

Open problem: Examples in the region $*$?

Eigenvalue estimates for $\not{D}^2$ via twistor operator

$m : TM \otimes \Sigma M \to \Sigma M$: Clifford multiplication

$p =$ projection on $\ker m$: $p(X \otimes \psi) = X \otimes \psi + \frac{1}{n}\sum_{i=1}^{n} e_i \otimes e_i X \psi$

∇^s: $\nabla^s_X Y := \nabla^g_X Y + 2sT(X, Y, -)$

($s = 1/4$ is the "standard" normalisation, $\nabla^{1/4} =$ char. conn.)

twistor operator: $P^s = p \circ \nabla^s$

Fundamental relation: $\|P^s\psi\|^2 + \frac{1}{n}\|D^s\psi\|^2 = \|\nabla^s\psi\|^2$

ψ is called s-twistor spinor $\Leftrightarrow$ $\psi \in \ker P^s$ $\Leftrightarrow$ $\nabla^s_X\psi + \frac{1}{n}XD^s\psi = 0$.

A priori, not clear what the **right value of** s might be:

different scaling in ∇ $\left[s = \frac{1}{4}\right]$ and $\not{D}$ $\left[s = \frac{1}{4\cdot 3}\right]$!

Idea: Use possible improvements of an eigenvalue estimate as a guide to the 'right' twistor spinor

Thm (twistor integral formula). Any spinor φ satisfies

$$\int_M \langle D\!\!\!/^2\varphi,\varphi\rangle dM = \frac{n}{n-1}\int_M \|P^s\varphi\|^2 dM + \frac{n}{4(n-1)}\int_M \mathrm{Scal}^g\|\varphi\|^2 dM + \frac{n(n-5)}{8(n-3)^2}\|T\|^2\int \|\varphi\|^2 dM - \frac{n(n-4)}{4(n-3)^2}\int_M \langle T^2\varphi,\varphi\rangle dM,$$

where $s = \frac{n-1}{4(n-3)}$.

Thm (twistor estimate). The first EV λ of $D\!\!\!/^2$ satisfies ($n > 3$)

$$\lambda \geq \frac{n}{4(n-1)}\mathrm{Scal}^g_{\min} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 - \frac{n(n-4)}{4(n-3)^2}\max(\mu_1^2,\dots,\mu_k^2),$$

where $\mu_1,\dots,\mu_k$ are the eigenvalues of T, and "=" iff

- Scal^g is constant,
- ψ is a twistor spinor for $s_n = \frac{n-1}{4(n-3)}$,
- ψ lies in Σ_μ corresponding to the largest eigenvalue of T^2.

- reduces to Friedrich's estimate for $T \to 0$
- estimate is good for $\mathrm{Scal}^g_{\min}$ dominant (compared to $\|T\|^2$)

Ex. (M^6, g) of class $\mathcal{W}_3$ ("balanced"), $\mathrm{Stab}(T)$ abelian

Known: $\mu = 0, \pm\sqrt{2}\|T\|$, no ∇^c-parallel spinors

twistor estimate: $$\lambda \ \geq \ \frac{3}{10}\mathrm{Scal}^g_{\min} - \frac{7}{12}\|T\|^2$$

universal estimate: $$\lambda \ \geq \ \frac{1}{4}\mathrm{Scal}^g_{\min} - \frac{3}{8}\|T\|^2$$

- better than anything obtained by deformation

On the other hand:

Ex. (M^5, g) Sasaki: deformation technique yielded better estimates.

I.Agricola, J. Becker-Bender, H. Kim, Adv. Math. 243 (2013), 296-329.

Killing and Twistor Spinors with Torsion

Thm (twistor eq). ψ is an s_n-twistor spinor ($P^{s_n}\psi = 0$) iff

$$\nabla^c_X \psi + \frac{1}{n} X \cdot D\!\!\!/ \psi + \frac{1}{2(n-3)} (X \wedge T) \cdot \psi \; = \; 0,$$

Dfn. ψ is a Killing spinor with torsion if $\nabla^{s_n}_X \psi = \kappa X \cdot \psi$ for $s_n = \frac{n-1}{4(n-3)}$.

$$\Leftrightarrow \;\; \nabla^c \psi - \left[\kappa + \frac{\mu}{2(n-3)} \right] X \cdot \psi + \frac{1}{2(n-3)} (X \wedge T) \psi \; = \; 0.$$

In particular:

- ψ is a twistor spinor with torsion for the same value s_n
- κ satisfies the quadratic eq.

$$n \left[\kappa + \frac{\mu}{2(n-3)} \right]^2 = \frac{1}{4(n-1)} \mathrm{Scal}^g + \frac{n-5}{8(n-3)^2} \|T\|^2 - \frac{n-4}{4(n-3)^2} \mu^2$$

- $\mathrm{Scal}^g =$ constant.

In general, this twistor equation cannot be reduced to a Killing equation.

... with one exception: $n = 6$

Thm. Assume ψ is a s_6-twistor spinor for some $\mu \neq 0$. Then:

- ψ is a $D\!\!\!/$ eigenspinor with eigenvalue

$$D\!\!\!/\psi = \frac{1}{3}\left[\mu - 4\frac{\|T\|^2}{\mu}\right]\psi$$

- the twistor equation for s_6 is equivalent to the Killing equation $\nabla^s\psi = \lambda X \cdot \psi$ for the same value of s.

Observation:

The Riemannian Killing / twistor eq. and their analogue with torsion behave very differently depending on the geometry!

Integrability conditions & Einstein-Sasaki manifolds

Thm (curvature in spin bundle). For any spinor field ψ:

$$\mathrm{Ric}^c(X)\cdot\psi \;=\; -2\sum_{k=1}^{n} e_k\mathcal{R}^c(X,e_k)\psi + \frac{1}{2}X\lrcorner dT\cdot\psi.$$

Thm (integrability condition). Let ψ be a Killing spinor with torsion with Killing number κ, set $\lambda := \frac{1}{2(n-3)}$. Then $\forall X$:

$$\begin{aligned}\mathrm{Ric}^c(X)\psi \;=\;& -16s\kappa(X\lrcorner T)\psi + 4(n-1)\kappa^2 X\psi + (1-12\lambda^2)(X\lrcorner\sigma_T)\psi + \\ & +2(2\lambda^2+\lambda)\sum e_k(T(X,e_k)\lrcorner T)\psi\,.\end{aligned}$$

Cor. A 5-dimensional Einstein-Sasaki mnfd with its characteristic connection cannot have Killing spinors with torsion.

Killing spinors on nearly Kähler manifolds

- (M^6, g, J) 6-dimensional nearly Kähler manifold

- ∇^c its characteristic connection, torsion is parallel

- Einstein, $\|T\|^2 = \frac{2}{15}\mathrm{Scal}^g$

- T has EV $\mu = 0, \pm 2\|T\|$

- $\exists$ 2 Riemannian KS $\varphi_\pm \in \Sigma_{\pm 2\|T\|}$, ∇^c-parallel

- univ. estimate = twistor estimate, $\lambda \geq \frac{2}{15}\mathrm{Scal}^g$

Thm. The following classes of spinors coincide:

- Riemannian Killing spinors
- ∇^c-parallel spinors
- Killing spinors with torsion
- Twistor spinors with torsion

There is exactly one such spinor $\varphi_\pm$ in each of the subbundles $\Sigma_{\pm 2\|T\|}$.

A 5-dimensional example with Killing spinors with torsion

- 5-dimensional Stiefel manifold $M = SO(4)/SO(2)$, $\mathfrak{so}(4) = \mathfrak{so}(2) \oplus \mathfrak{m}$

- Jensen metric: $\mathfrak{m} = \mathfrak{m}_4 \oplus \mathfrak{m}_1$ (irred. components of isotropy rep.),

$$\langle (X,a),(Y,b)\rangle_t = \tfrac{1}{2}\beta(X,Y) + 2t\cdot ab,\ \ t>0,\ \ \beta = \text{Killing form}\,\Big|_{\mathfrak{m}_4}$$

- $t = 1/2$: undeformed metric: 2 parallel spinors

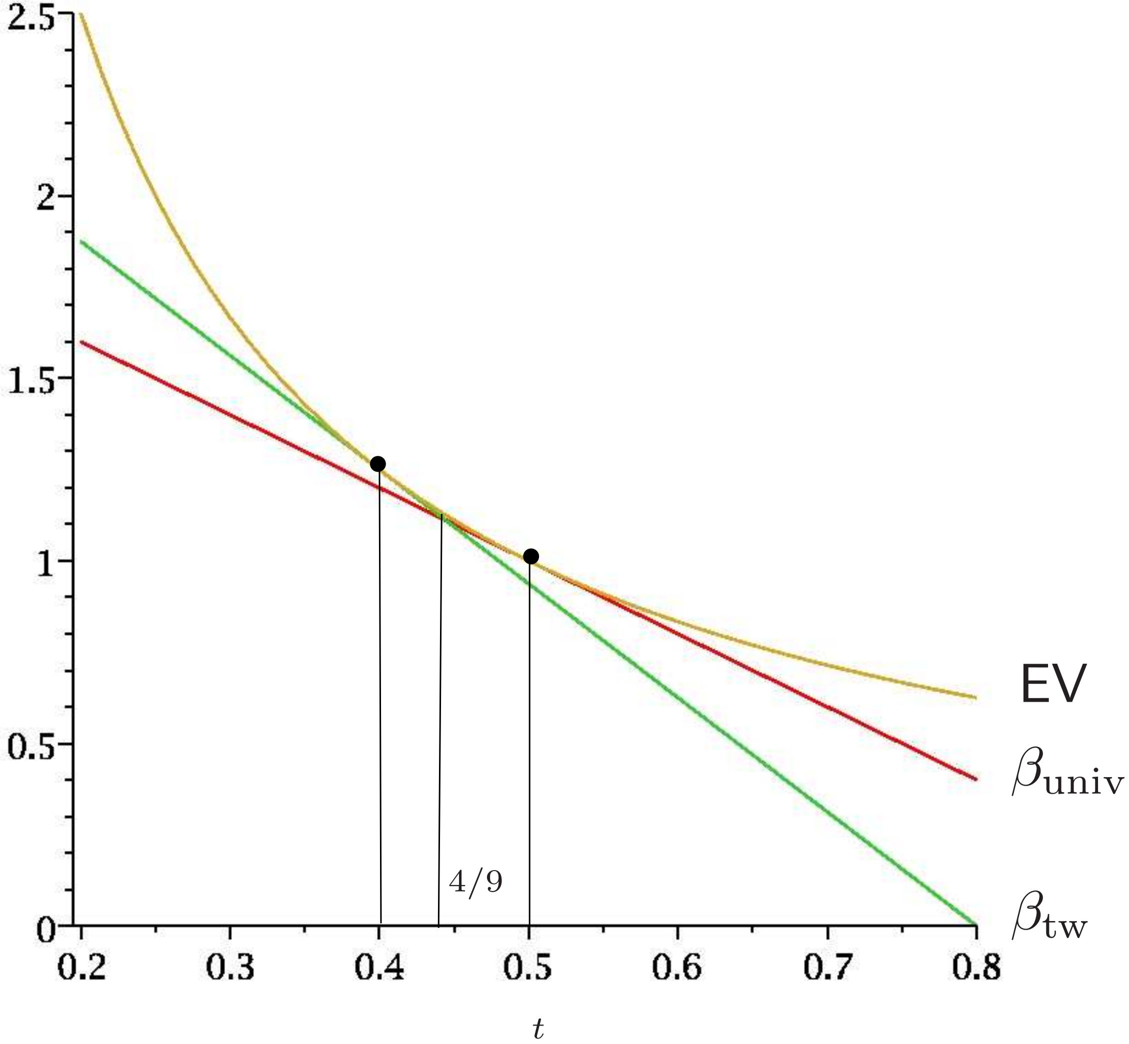
- $t = 2/3$: Einstein-Sasaki with 2 Riemannian Killing spinors

- For general t: metric contact structure in direction $\mathfrak{m}_1$ with characteristic connection ∇ satisfying $\nabla T = 0$

- $\|T\|^2 = 4t$, $\mathrm{Scal}^g = 8 - 2t$, $\mathrm{Ric}^g = \mathrm{diag}(2-t, 2-t, 2-t, 2-t, 2t)$.

- Universal estimate: $\lambda \geq 2(1-t) =: \beta_{\mathrm{univ}}$

- Twistor estimate: $\lambda \geq \frac{5}{2} - \frac{25}{8}t =: \beta_{\mathrm{tw}}$



Result: there exist 2 twistor spinors with torsion for $t = 2/5$, and these are even Killing spinors with torsion.

Generalisation: deformed Sasaki mnfds with Killing spinors with torsion

- (M, g, ξ, η): Sasaki mnfd, η: contact form, dimension $2n+1$

- Tanno deformation of metrics: $g_t := tg + (t^2 - t)\eta \otimes \eta$, again Sasaki with $\xi_t = \frac{1}{t}\xi$, $\eta_t = t\eta$ ($t \in \mathbb{R}^*$)

- If Einstein-Sasaki: admits two Riemannian Killing spinors

Thm. Let (M, g, ξ, η) be Einstein-Sasaki, g_t the Tanno deformation. Then there exists a t s.t. (M, g_t, ξ_t, η_t) has two Killing spinors with torsion.

– establishes existence of examples in all odd dimensions –

Remarks on the second Dirac eigenvalue

Th. Friedrich, Advances in Applied Clifford Algebras, 22 , (2012), 301-311.

(M^n, g) – Riemannian manifold, ψ – Killing spinor

$$\nabla_X \psi = a \cdot X \cdot \psi, \qquad n^2 a^2 = \mu_1(D^2) = \frac{n}{4(n-1)} R .$$

New test spinors for upper bounds of $\mu_2(D^2)$: $\psi^* = f \cdot \psi + \eta \cdot \psi$.

λ_1^0 – first eigenvalue of the Laplacian on functions. Lichnerowicz/Obata :

$$\text{If} \quad M^n \neq S^n, \quad \text{then} \quad \lambda_1^0 > \frac{R}{n-1} = 4na^2 .$$

A first family of test spinors : $\eta = df$.

Theorem: Let $M^n \neq S^n$ be a compact Riemannian spin manifold with a Killing spinor ψ, $\nabla_X \psi = a \cdot X \cdot \psi$. The numbers

$$\Big(\pm \sqrt{\lambda_1^0 + a^2(1-n)^2} \; - \; |a| \Big)^2$$

are eigenvalues of D^2, too. The second eigenvalue can be estimated by

$$a^2 n^2 \; = \; \mu_1(D^2) \; < \; \mu_2(D^2) \; \leq \; \Big(\sqrt{\lambda_1^0 + a^2(1-n)^2} \; - \; |a| \Big)^2$$

Finally, if

$$a^2 n^2 \; = \; \mu_1(D^2) \; < \; \mu(D^2) \; < \; \Big(\sqrt{\lambda_1^0 + a^2(1-n)^2} \; - \; |a| \Big)^2$$

is any "small" eigenvalue and ψ^* the eigenspinor, then the inner product $\langle \psi \, , \, \psi^* \rangle$ vanishes identically.

Estimates for small $\mu_2(D^2)$: $\quad \psi^* = \eta \cdot \psi$.

$0 < \Lambda_1 < \Lambda_2 < \ldots$ – eigenvalues of the problem

$$\Delta_1(\eta) \;=\; \Lambda\,\eta\,, \quad \delta\eta \;=\; 0\,, \quad \Lambda_1 \;\geq\; \frac{2R}{n} \;=\; 8(n-1)a^2\,.$$

Theorem: The spinor field $\psi^* = \eta \cdot \psi$ is an eigenspinor, $D(\psi^*) = m\,\psi^*$, if and only if

$$\Big\{\big((n-2)a \;-\; m\big)\,\eta \;+\; d\eta\Big\} \cdot \psi \;=\; 0\,.$$

In this case the 1-form η is a coclosed eigenform of the Laplace operator, and the eigenvalue can be estimated by

$$na \;\leq\; \sqrt{\Lambda_1 \,+\, a^2(n-3)^2} \;-\; |a| \;\leq\; |m|\,.$$

Corollary: If M^n is a 7-dimensional Riemannian manifold ($n = 7$), then

$$\min\left(\left(\sqrt{\lambda_1^0 + a^2(1-n)^2} - |a|\right)^2, \left(\sqrt{\Lambda_1 + a^2(n-3)^2} - |a|\right)^2\right) \leq \mu_2(D^2) .$$

Proof: Fix a Killing spinor ψ. In dimension $n = 7$ *any* spinor field ψ^* is given by a function f and a 1-form η, $\psi^* = f \cdot \psi + \eta \cdot \psi$.

Remark: The method applies also in some other small dimensions $n = 5, 6, 8$.