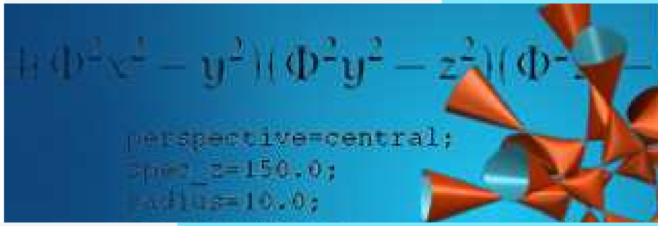
Philipps Universität Marburg

Non-integrable geometries, torsion, and holonomy
I: Mathematical tools – geometry of metric connections



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Outline

At border line between pure mathematics and theoretical physics

formation of mathematical concepts	$\Leftrightarrow$	building of physical models
differential geometry, analysis, group theory		general relativity, unified field theories, string theory

Lecture I: Mathematical tools – geometry of metric connections

Lecture II: Geometric structures and connections

Lecture III: Curvature properties of metric connections with skew torsion

Lecture IV: Classification of naturally reductive homogeneous spaces

Symmetry I

• Classical mechanics: Symmetry considerations can simplify study of geometric problems (i.e., Noether's theorem)

• Felix Klein at his inaugural lecture at Erlangen University, 1872 ("Erlanger Programm"):

> "Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben; man soll die der Mannigfaltigkeit angehörigen Gebilde hinsichtlich solcher Eigenschaften untersuchen, die durch die Transformationen der Gruppe nicht geändert werden".

> "Let a manifold and in this a transformation group be given; the objects belonging to the manifold ought to be studied with respect to those properties which are not changed by the transformations of the group."

$\longrightarrow$ *Isometry group* of a Riemannian manifold (M, g)

Symmetry II

- Around 1940-1950: Second intrinsic Lie group associated with a Riemannian manifold (M, g) appeared, its *holonomy group*.

$\longrightarrow$ strongly related to curvature and parallel objects

Recall: For $f: \mathbb{R}^n \to \mathbb{R}$ smooth, the directional derivative in x_0 in direction $U \in \mathbb{R}^n$ is defined by

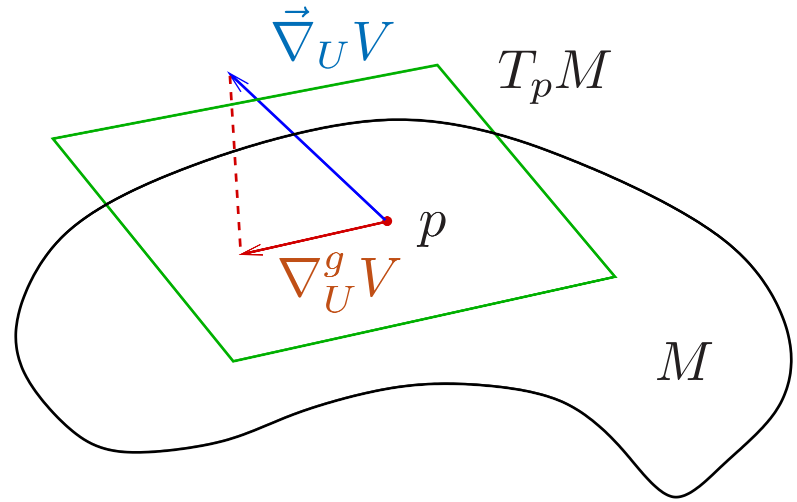
$$(\vec{\nabla}_U f)(x_0) := \lim_{t \to 0} \frac{f(x_0 + tU) - f(x_0)}{t} [= Df(x_0)(U)]$$

For a vector V : $\vec{\nabla}_U V$ is defined component wise

- For $V = \frac{\partial}{\partial x}$, $W = \frac{\partial}{\partial y}$: $\vec{\nabla}_V \vec{\nabla}_W = \vec{\nabla}_W \vec{\nabla}_V$ and $[V, W] = 0$

- compatible with scalar product: $V(\langle X, Y \rangle) = \langle \vec{\nabla}_V X, Y \rangle + \langle X, \vec{\nabla}_V Y \rangle$

Connections



Connection ∇: abstract derivation rule on mnfd satisfying all formal properties of the directional derivative

Exa. Projection $\nabla^g_U V$ of dir. derivative $\vec{\nabla}_U V$ to tangent plane
$=$ Levi-Civita connection ∇^g

But: not only possibility $\longrightarrow$ connection with torsion [Dfn: Cartan, 1925]

Exa. Electrodynamics: $\nabla_U V := \vec{\nabla}_U V + \frac{ie}{\hbar} A(U)V$ $(\Leftrightarrow \ \nabla_\mu = \partial_\mu + \frac{ie}{\hbar} A_\mu)$

A: gauge potential $=$ electromagnetic potential

Exa. If $n = 3$: $\nabla_U V := \vec{\nabla}_U V + U \times V$ additional term gives space an 'internal angular momentum', a *torsion*

A priori, the holonomy group is defined *for an arbitrary connection* ∇ on TM. For reasons to become clear later, we concentrate mainly on

$$\text{Metric connections } \nabla : \; Xg(V,W) \;=\; g(\nabla_X V, W) + g(V, \nabla_X W).$$

The torsion (viewed as $(2,1)$- or $(3,0)$-tensor)

$$T(X,Y) \;:=\; \nabla_X Y - \nabla_Y X - [X,Y], \quad T(X,Y,Z) \;:=\; g(T(X,Y),Z)$$

can (for the moment. . .) be arbitrary.

Why torsion?

• **General relativity**:

a) Cartan (1929): torsion $\sim$ intrinsic angular momentum, derived a set of gravitational field eqs., but postulated that the energy-momentum tensor should still be divergence-free $\rightarrow$ too restrictive

b) Einstein-Cartan theory ($\geq$ 1950): variation of the scalar curvature <u>and</u> of an additional Lagrangian generating the energy-momentum and the spin tensors: allowed *any torsion and not nec. metric*

• **Superstring theory**:

Classical Yang-Mills theory: curvature $\cong$ field strength,

in superstring theories: torsion $\cong$ higher order field strength

($+$ extra differential eqs.)

• **Differential geometry**: Connections adapted to the geometry useful for 'non-integrable' geometries, like: Hermitian non Kähler mnfds, contact manifolds. . .

Types of metric connections

(M^n, g) oriented Riemannian mnfd, ∇ any connection:

$$\nabla_X Y \;=\; \nabla^g_X Y \,+\, A(X,Y)\,.$$

Then: ∇ is metric $\Leftrightarrow$ $g(A(X,Y),Z) + g(A(X,Z),Y) = 0$

$$\Leftrightarrow \;\; A \in \mathcal{A}^g := \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)$$

For metric connections: difference tensor A $\Leftrightarrow$ torsion T via

$$2A(X,Y,Z) = T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y),$$

$$T(X,Y,Z) = A(X,Y,Z) - A(Y,X,Z)$$

So identify $\mathcal{A}^g$ with $\mathcal{T}$: space of possible torsion tensors,

$$\mathcal{A}^g \;\cong\; \mathcal{T} \;\cong\; \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n), \quad \dim = \frac{n^2(n-1)}{2}.$$

Decompose this space under $\mathrm{SO}(n)$ action (E. Cartan, 1925), $n \geq 3$:

$$\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n) \;\cong\; \mathbb{R}^n \oplus \Lambda^3(\mathbb{R}^n) \oplus \mathcal{T}'.$$

(For $n = 2$: $\mathbb{R}^2 \otimes \Lambda^2(\mathbb{R}^2) \;\cong\; \mathbb{R}^2$ is irreducible).

- $A \in \Lambda^3(\mathbb{R}^n)$: "Connections with skew (symmetric) torsion":

$$\nabla_X Y \;:=\; \nabla^g_X Y + \frac{1}{2}\mathrm{T}(X, Y, -)\,.$$

Lemma. ∇ is metric and geodesics preserving iff its torsion T lies in $\Lambda^3(TM)$. In this case, $2\,A = T$, and the ∇-Killing vector fields coincide with the Riemannian Killing vector fields.

$\longrightarrow$ Connections used in superstring theory

- $A \in \mathbb{R}^n$: "Connections with vectorial torsion", V a vector field:

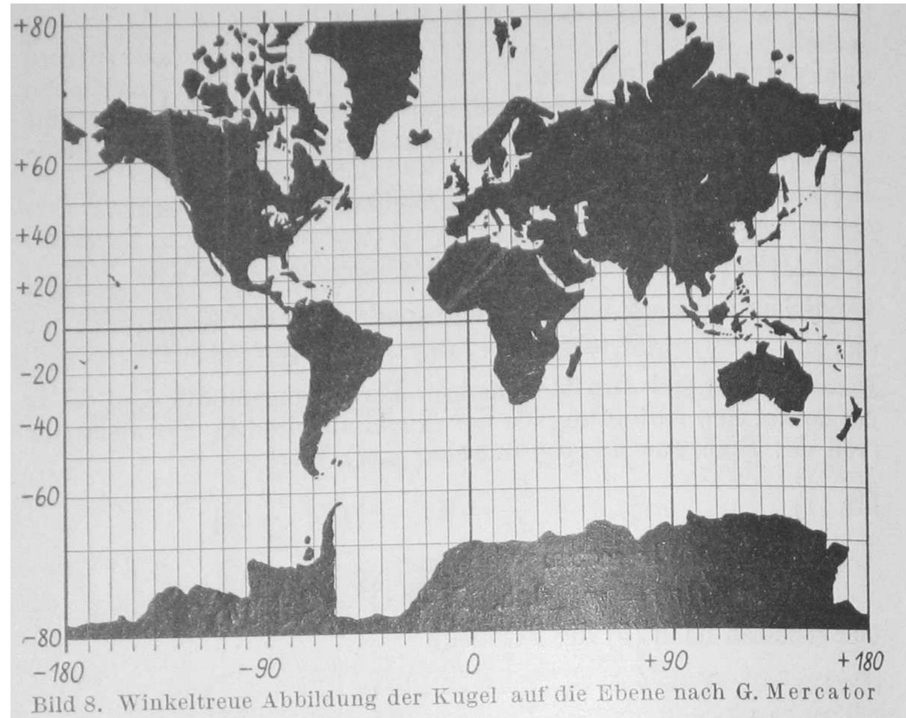
$$\nabla_X Y \;:=\; \nabla^g_X Y - g(X, Y) \cdot V + g(Y, V) \cdot X\,.$$

In particular, *any metric connection* on a surface is of this type!

Mercator map

- conformal (angle preserving), hence maps loxodromes to straight lines
- Cartan (1923):

"On this manifold, the straight lines [of the flat connection] are the *loxodromes*, which intersect the meridians at a constant angle. The only straight lines realizing shortest paths are those which are normal to the torsion in every point: *these are the meridians*.



$S^2 - \{N, S\} \to I \times \mathbb{R}$ (1569)

- Explanation & generalisation to arbitrary manifolds?
- Existence of a Clairaut style invariant?

Thm. (M,g) Riemannian manifold, $\sigma \in C^\infty(M)$ and $\tilde g = e^{2\sigma} g$ the conformally changed metric. Let

$\tilde\nabla^g$: metric connection with vectorial torsion $V = -\mathrm{grad}\,\sigma$ on (M,g),

$$\tilde\nabla^g_X Y \;=\; \nabla^g_X Y - g(X,Y)V + g(Y,V)X$$

$\nabla^{\tilde g}$: Levi-Civita connection of $(M,\tilde g)$. Then

(1) Every $\tilde\nabla^g$-geodesic $\gamma(t)$ is (up to reparametrisation) a $\nabla^{\tilde g}$-geodesic;

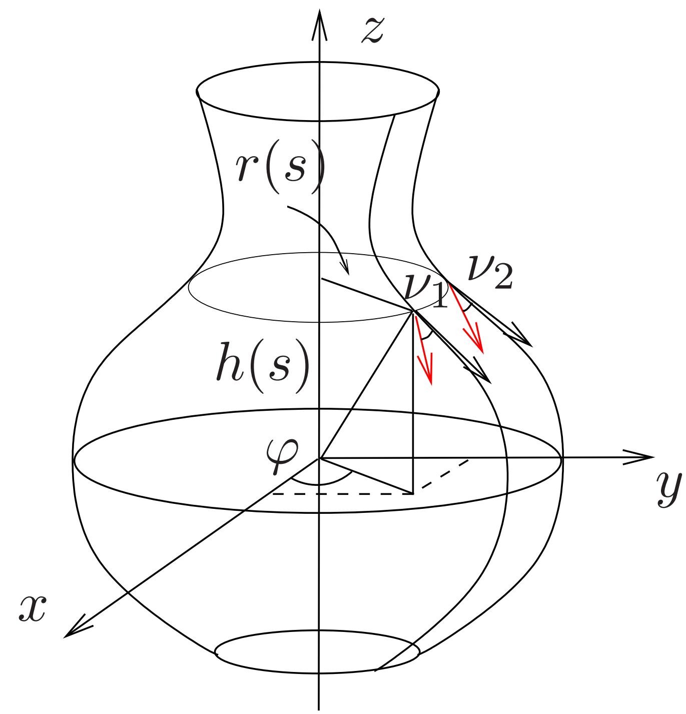
(2) If X is a Killing vector field of $\tilde g$, then $e^\sigma g(\gamma', X)$ is a constant of motion for every $\tilde\nabla^g$-geodesic $\gamma(t)$. [A-Thier, 2003]

N.B. The curvatures of $\tilde\nabla^g$ and $\nabla^{\tilde g}$ coincide, but the curvatures of ∇^g and $\nabla^{\tilde g}$ are unrelated.

$\longrightarrow$ Beltrami's theorem does not hold anymore [“If a portion of a surface S can be mapped LC-geodesically onto a portion of a surface S^* of constant Gaussian curvature, the Gaussian curvature of S must also be constant”]

Connections with vectorial torsion on surfaces

- Curve: $\alpha = \big(r(s), h(s)\big)$
- Surface of revolution:
  $\big(r(s)\cos\varphi, r(s)\sin\varphi, h(s)\big)$
- Riemannian metric:
  $g = \mathrm{diag}\big(r^2(s), 1\big)$
- Orthonormal frame:
  $e_1 = \frac{1}{r}\partial_\varphi,\ e_2 = \partial_s$

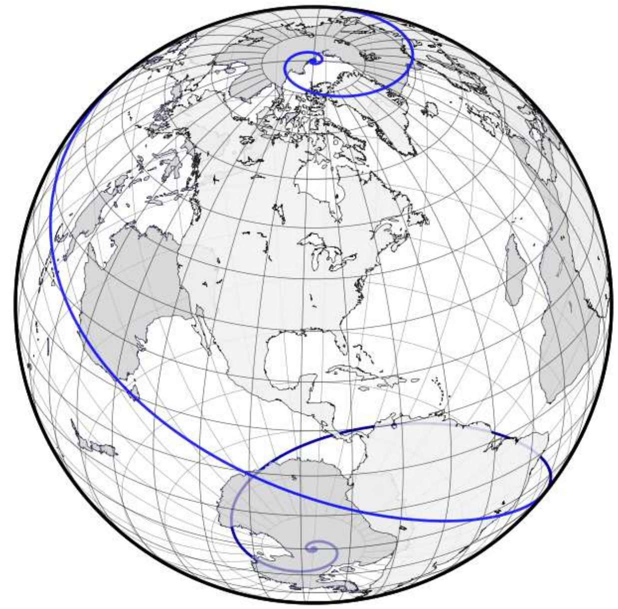


Dfn: Call two tangent vectors v_1 and v_2 of same length parallel if their angles ν_1 and ν_2 with the generating curves through their origins coincide

→ 'Mercator connection ∇'

- Hence $\nabla e_1 = \nabla e_2 = 0$
- Torsion: $T(e_1, e_2) = \frac{r'(s)}{r(s)}\, e_2$
- Corresponding vector field:

$V = \frac{r'(s)}{r(s)}\, e_1 = -\mathrm{grad}\big(- \ln r(s)\big)$



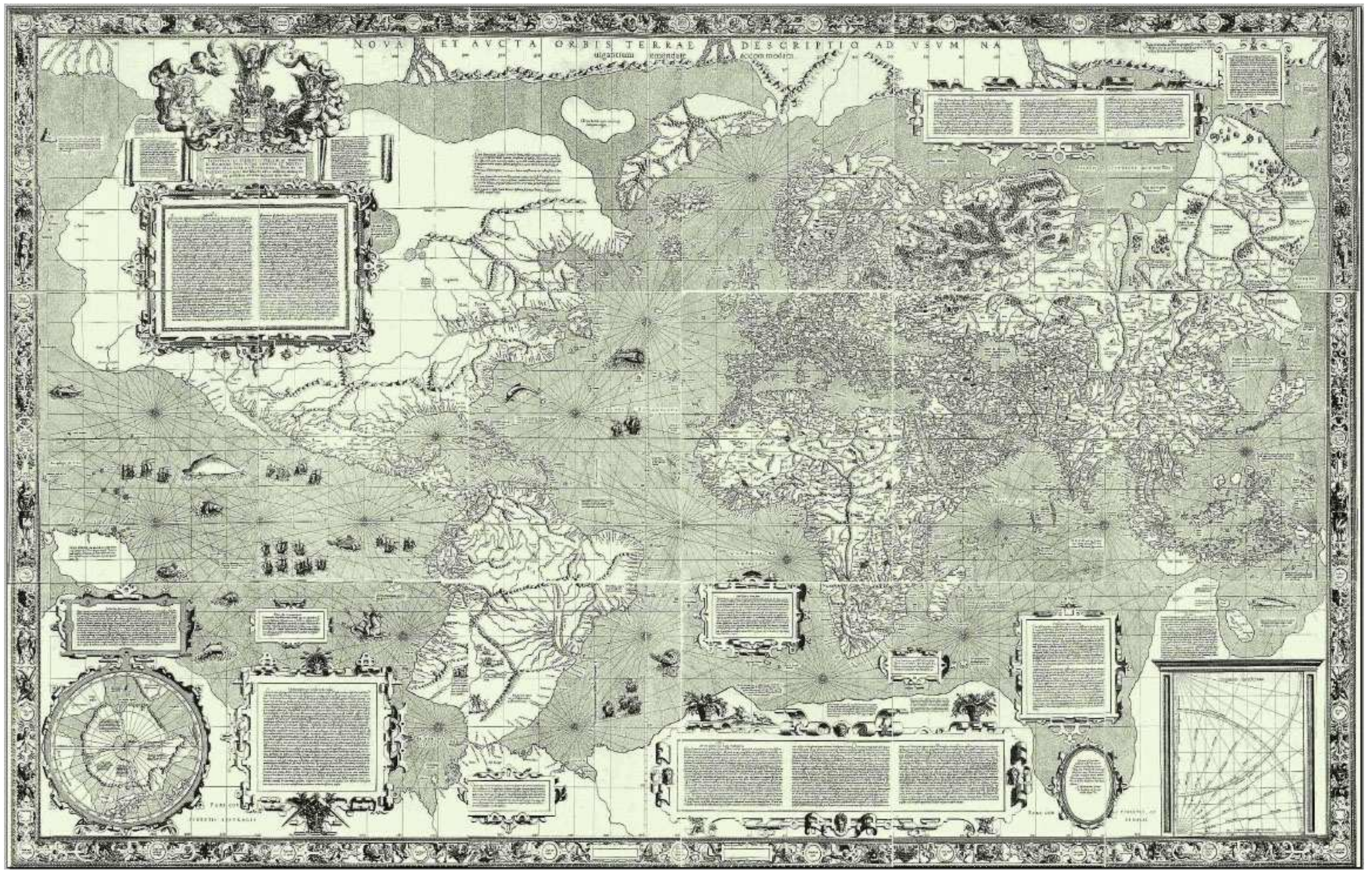
loxodrome with direction 292.5^o

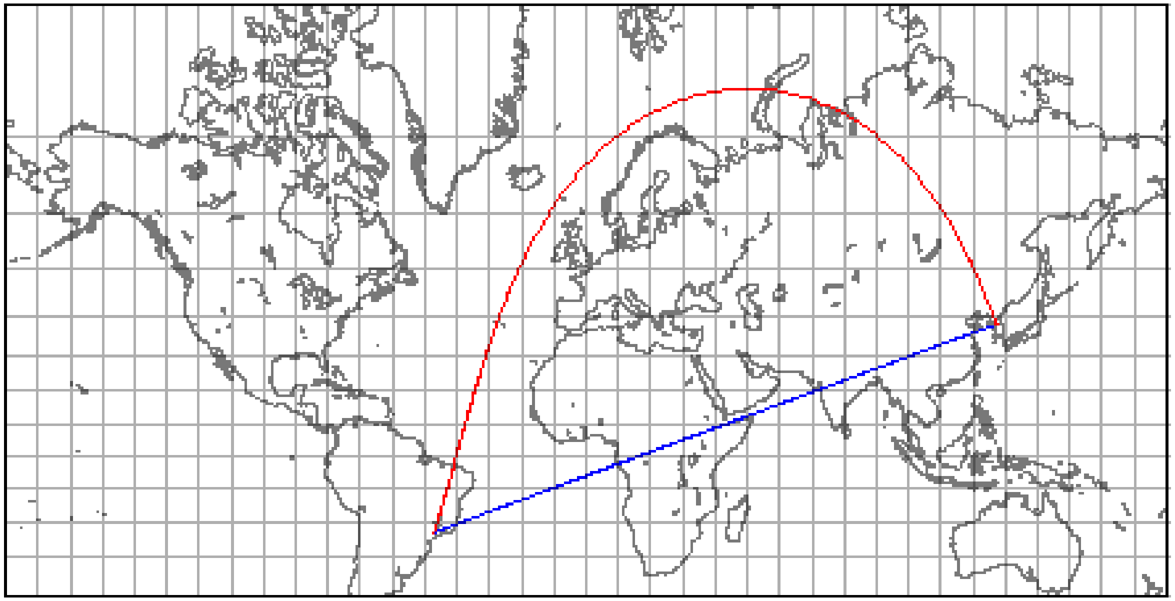
- geodesics are LC geodesics of the conformally equivalent metric $\tilde{g} = e^{2\sigma} g = \mathrm{diag}(1/r^2, 1)$ (coincides with euclidian metric under $x = \varphi$, $y = \int ds/r(s)$)

- $X = \partial_\varphi$ is Killing vector field for $\tilde{g}$, invariant of motion:

$\mathrm{const} = e^{\sigma} g(\dot{\gamma}, X) = \frac{1}{r(s)} g(\dot{\gamma}, \partial_\varphi) = g(\dot{\gamma}, e_2)$

Mercator’s world map (1569)





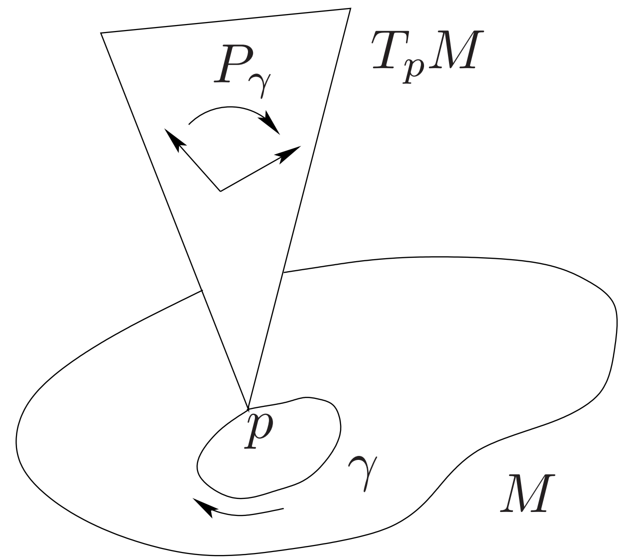
[www.progonos.com]

Comparison of a loxodrome and a great circle on the Mercator map – by habit, we take for granted that the straight line corresponds to the shortest distance!

N. B. It is conformal, but not area-preserving (compare former Sovjet Union with Africa or Europe with South America)

Holonomy of arbitrary connections

- γ from p to q, ∇ any connection
- $P_\gamma : T_pM \to T_qM$ is the unique map s. t. $V(q) := P_\gamma V(p)$ is parallel along γ, $\nabla V(s)/ds = \nabla_{\dot\gamma} V = 0$.
- $C(p)$: closed loops through p
  $\mathrm{Hol}(p; \nabla) = \{P_\gamma \mid \gamma \in C(p)\}$
- $C_0(p)$: null-homotopic el'ts in $C(p)$
  $\mathrm{Hol}_0(p; \nabla) = \{P_\gamma \mid \gamma \in C_0(p)\}$



Independent of p, so drop p in notation: $\mathrm{Hol}(M; \nabla)$, $\mathrm{Hol}_0(M; \nabla)$.

A priori:

(1) $\mathrm{Hol}(M; \nabla)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$,

(2) $\mathrm{Hol}_0(p)$ is the connected component of the identity of $\mathrm{Hol}(M; \nabla)$.

Holonomy of metric connections

Assume: M carries a Riemannian metric g, ∇ *metric*

$\Rightarrow$ parallel transport is an isometry:

$$\frac{d}{ds}g\big(V(s),W(s)\big) \;=\; g\Big(\frac{\nabla V(s)}{ds},W(s)\Big)+\Big(V(s),\frac{\nabla W(s)}{ds}\Big) \;=\; 0.$$

and $\mathrm{Hol}(M;\nabla)\subset \mathrm{O}(n,\mathbb{R})$, $\mathrm{Hol}_0(M;\nabla)\subset \mathrm{SO}(n,\mathbb{R})$.

Notation: $\mathrm{Hol}_{(0)}(M;\nabla^g) =$ "Riemannian (restricted) holonomy group"

N.B. (1) $\mathrm{Hol}_{(0)}(M;\nabla)$ needs not to be closed!

(2) The holonomy representation needs not to be irreducible on irreducible manifolds!

$\longrightarrow$ Larger variety of holonomy groups, but classification difficult

Curvature & Holonomy

Holonomy can be computed through curvature:

Thm (Ambrose-Singer, 1953). For any connection ∇ on (M, g), the Lie algebra $\mathfrak{hol}(p; \nabla)$ of $\mathrm{Hol}(p; \nabla)$ in $p \in M$ is exactly the subalgebra of $\mathfrak{so}(T_pM)$ generated by the elements

$$P_\gamma^{-1} \circ \mathcal{R}(P_\gamma V, P_\gamma W) \circ P_\gamma \quad V, W \in T_pM, \quad \gamma \in C(p).$$

But only of restricted use:

Thm (Bianchi I). (1) For a metric connection with vectorial torsion $V \in TM^n$: $\quad \overset{X,Y,Z}{\sigma} \mathcal{R}(X, Y)Z = \overset{X,Y,Z}{\sigma} dV(X, Y)Z.$

(2) For a metric connection with skew symmetric torsion $T \in \Lambda^3(M^n)$:

$$\overset{X,Y,Z}{\sigma} \mathcal{R}(X, Y, Z, V) = dT(X, Y, Z, V) - \sigma^T(X, Y, Z, V) + (\nabla_V T)(X, Y, Z),$$

$2\,\sigma^T := \sum_{i=1}^{n} (e_i \lrcorner T) \wedge (e_i \lrcorner T)$ for any orthonormal frame $e_1, \ldots, e_n$.

Thm (Berger, Simons, $>$ 1955). For a non symmetric Riemannian manifold (M, g) and the Levi-Civita connection ∇^g, the possible holonomy groups are $\mathrm{SO}(n)$ or

$4n$	$2n$	$2n$	$4n$	7	8	16
$\mathrm{Sp}_n\mathrm{Sp}_1$	$\mathrm{U}(n)$	$\mathrm{SU}(n)$	Sp_n	G_2	$\mathrm{Spin}(7)$	$(\mathrm{Spin}(9))$
quatern. Kähler	Kähler	Calabi-Yau	hyper-Kähler	par.	par.	par.
$\nabla J \neq 0$	$\nabla^g J = 0$	$\nabla^g J = 0$	$\nabla^g J = 0$	$\nabla^g \omega^3 = 0$	$\nabla^g \Omega^4 = 0$	$--$
$\mathrm{Ric} = \lambda g$	$--$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$\mathrm{Ric} = 0$	$--$

Existence of Ricci flat compact manifolds:

• Calabi-Yau, hyper-Kähler: Yau, 1980's.

• G_2, $\mathrm{Spin}(7)$: D. Joyce since $\sim$ 1995, Kovalev (2003). Both rely on heavy analysis and algebraic geometry !

Later: Reformulate Bianchi id. and get replacement for Berger's thm!

General Holonomy Principle

Thm (General Holonomy Principle). M a manifold, E a (real or complex) vector bundle over M with (any!) connection ∇. Then the following are equivalent:

(1) E has a global section α which is invariant under parallel transport, i. e. $\alpha(q) = P_\gamma(\alpha(p))$ for any path γ from p to q;

(2) E has a parallel global section α, i. e. $\nabla\alpha = 0$;

(3) In some point $p \in M$, there exists an algebraic vector $\alpha_0 \in E_p$ which is invariant under the holonomy representation on the fiber.

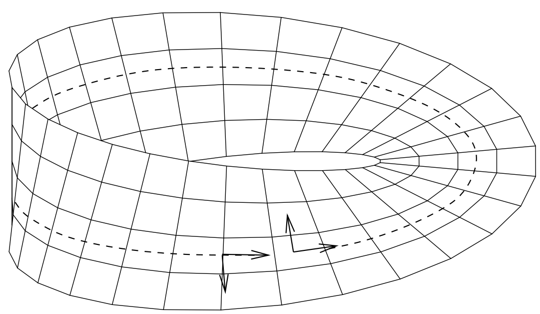
Corollary. The number of parallel global sections of E coincides with the number of trivial representations occuring in the holonomy representation on the fibers.

Example. Orientability from a holonomy point of view:

Lemma. The determinant ist an $\mathrm{SO}(n)$-invariant element in $\Lambda^n(\mathbb{R}^n)$ that is not $\mathrm{O}(n)$-invariant.

Corollary. (M^n, g) is orientable iff $\mathrm{Hol}(M; \nabla) \subset \mathrm{SO}(n)$ for *any metric connection* ∇, and the volume form is then ∇-parallel.

[Take $dM_p := \det = e_1 \wedge \ldots \wedge e_n$ in $p \in M$, then apply holonomy principle to $E = \Lambda^n(TM)$.]



An orthonormal frame that is parallel transported along the drawn curve reverses its orientation.

Geometric stabilizers

Philosophy: Invariants of geometric representations are candidates for parallel objects. Find these!

- Invariants for $G \subset \mathrm{SO}(m)$ in tensor bundles (as just seen)
- Assume that $G \subset \mathrm{SO}(m)$ can be lifted to a subgroup $G \subset \mathrm{Spin}(m)$

$\Rightarrow$ G acts on the spin representation Δ_m of $\mathrm{Spin}(m)$

Recall: • $m = 2k$ even: $\Delta_m = \Delta_m^+ \oplus \Delta_m^-$, both have dimension 2^{k-1}

• $m = 2k+1$ odd: Δ_m is irreducible, of dimension 2^k

Elements of Δ_m: "algebraic spinors" (in opposition to spinors on M that are sections of the spinor bundle)

Now decompose Δ_m under the action of G.

In particular: Are there invariant algebraic spinors?

$\mathrm{U}(n)$ in dimension $2n$

- Hermitian metric $h(V,W) = g(V,W) - ig(JV,W)$

- h is invariant under $A \in \mathrm{End}(\mathbb{R}^{2n})$ iff A leaves invariant g and the Kähler form $\Omega(V,W) := g(JV,W) \Rightarrow$

$$\mathrm{U}(n) \;=\; \{A \in \mathrm{SO}(2n) \mid A^*\Omega = \Omega\}.$$

Lemma. Under the restricted action of $\mathrm{U}(n)$, $\Lambda^{2k}(\mathbb{R}^{2n}), k = 1,\ldots,n$ contains the trivial representation once, namely, $\Omega, \Omega^2, \ldots, \Omega^n$.

Only the Lie algebra $\mathfrak{u}(n)$ can be lifted to a subgroup of $\mathfrak{spin}(2n)$, but it has no invariant algebraic spinors:

Ω generates the one-dimensional center of $\mathfrak{u}(n)$ (identify $\Lambda^2(\mathbb{R}^{2n}) \cong \mathfrak{so}(2n)$).

Set $S_r = \{\psi \in \Delta_{2n} \;:\; \Omega\psi = i(n-2r)\psi\}, \quad \dim S_r = \binom{n}{r}, \quad 0 \le r \le n.$

$S_r \cong (0,r)$-forms with values in S_0 and

$$\Delta^+_{2n}\big|_{\mathrm{U}(n)} \;\cong\; S_n \oplus S_{n-2} \oplus \ldots, \quad \Delta^-_{2n}\big|_{\mathrm{U}(n)} \;\cong\; S_{n-1} \oplus S_{n-3} \oplus \ldots$$

$\Rightarrow$

• no trivial $\mathfrak{u}(n)$-representation for n odd

• For $n = 2k$ even, Ω has eigenvalue zero on S_k, but this space is an irreducible representation of dimension $\binom{2k}{k} \neq 1$

• S_0 and S_n are one-dimensional, and they become trivial under $\mathrm{SU}(n)$ ($\mathrm{SU}(n)$ does lift into $\mathrm{Spin}(2n)$)

Lemma. $\Delta^{\pm}_{2n}$ contain no $\mathfrak{u}(n)$-invariant spinors. If one restricts further to $\mathrm{SU}(n)$, there are exactly two invariant spinors.

G_2 in dimension 7

• Geometry of 3-forms plays an exceptional role in Riemannian geometry, as it ocurs only in dimensions seven and eight:

n	$\dim \mathrm{GL}(n,\mathbb{R}) - \dim \Lambda^3\mathbb{R}^n$	$\dim \mathrm{SO}(n)$
3	$9-1=8$	3
4	$16-4=12$	6
5	$25-10=15$	10
6	$36-20=16$	15
7	$49-35=14$	21
8	$64-56=8$	28

$\Rightarrow$ stabilizer $G^n_{\omega^3} := \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \omega^3 = A^*\omega^3\}$ of a generic 3-form ω^3 cannot lie in $\mathrm{SO}(n)$ for $n \leq 6$ (for example: $G^3_{\omega^3} = \mathrm{SL}(3,\mathbb{R})$).

Engel had had this idea already in 1886. From a letter to Killing (8.4.1886):

> "There seem to be relatively few simple groups. Thus first of all, the two types mentioned by you [$\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$]. If I am not mistaken, the group of a linear complex in space of $2n-1$ dimensions $(n>1)$ with $(2n+1)2n/2$ parameters [$\mathrm{Sp}(n,\mathbb{C})$] is distinct from these. In 3-fold space [$\mathbb{CP}^3$] this group [$\mathrm{Sp}(4,\mathbb{C})$] is isomorphic to that [$\mathrm{SO}(5,\mathbb{C})$] of a surface of second degree in 4-fold space. I do not know whether a similar proposition holds in 5-fold space. The projective group of 4-fold space [$\mathbb{CP}^4$] that leaves invariant a trilinear expression of the form
>
> $$\sum_{ijk}^{1\ldots5} a_{ijk} \begin{vmatrix} x_i & y_i & z_i \\ x_k & y_k & z_k \\ x_j & y_j & z_j \end{vmatrix} = 0$$
>
> will probably also be simple. This group has 15 parameters, the corresponding group in 5-fold space has 16, in 6-fold space [$\mathbb{CP}^6$] has 14, in 7-fold space [$\mathbb{CP}^7$] has 8 parameters. In 8-fold space there is no such group. These numbers are already interesting. Are the corresponding groups simple? Probably this is worth investigating. But also Lie, who long ago thought about similar things, has not yet done so."

Reichel, 1907 (Ph D student of F. Engel in Greifswald):

- computed a system of invariants for a 3-form in seven variables

- showed that there are exactly two $\mathrm{GL}(7,\mathbb{R})$-open orbits of 3-forms

- showed that stabilizers of any representatives ω^3, $\tilde{\omega}^3$ of these orbits are 14-dimensional simple Lie groups of rank two, a compact and a non-compact one:

$$G^7_{\omega^3} \cong G_2 \subset \mathrm{SO}(7), \quad G^7_{\tilde{\omega}^3} \cong G_2^* \subset \mathrm{SO}(3,4)$$

- realized $\mathfrak{g}_2$ and $\mathfrak{g}_2^*$ as explicit subspaces of $\mathfrak{so}(7)$ and $\mathfrak{so}(3,4)$

As in the case of almost hermitian geometry, one has a favourite normal form for a 3-form with isotropy group G_2:

$$\omega^3 := e_{127} + e_{347} - e_{567} + e_{135} - e_{245} + e_{146} + e_{236}.$$

An element of the second orbit ($\rightarrow G_2^*$) may be obtained by reversing any of the signs in ω^3.

Lemma. Under G_2: $\Lambda^3(\mathbb{R}^7) \cong \mathbb{R} \oplus \mathbb{R}^7 \oplus S_0(\mathbb{R}^7)$, where

$\mathbb{R}^7$: 7-dimensional standard representation of $G_2 \subset \mathrm{SO}(7)$

$S_0(\mathbb{R}^7)$: traceless symmetric endomorphisms of $\mathbb{R}^7$ (has dimension 27).

• G_2 can be lifted to a subgroup of $\mathrm{Spin}(7)$. From a purely representation theoretic point of view, this case is trivial:

$\dim \Delta_7 = 8$ and the only irreducible representations of G_2 of dimension ≤ 8 are the trivial and the 7-dimensional representation $\Rightarrow$

Lemma. Under G_2: $\Delta_7 \cong \mathbb{R} \oplus \mathbb{R}^7$.

In fact, the invariant 3-form ω^3 and the invariant algebraic spinor ψ are equivalent data:

$$\omega^3(X, Y, Z) \ = \ \langle X \cdot Y \cdot Z \cdot \psi, \psi \rangle.$$

But $\dim \Delta_7 = 8 < \dim \Lambda^3(\mathbb{R}^7) = 35$, so the spinorial picture is easier to treat!

Assume now that $G \subset G_2$ fixes *a second spinor* $\Rightarrow G \cong \mathrm{SU}(3)$

• this is one of the three maximal Lie subgroups of G_2, $\mathrm{SU}(3)$, $\mathrm{SO}(4)$ and $\mathrm{SO}(3)$

• $\mathrm{SU}(3)$ has irreducible real representations in dimension 1, 6 and 8, so

Lemma. Under $\mathrm{SU}(3) \subset G_2$: $\Delta_7 \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^6$ and $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^6$.

This implies:

• If ∇^g on (M^7, g) has two parallel spinors, M has to be (locally) reducible, $M^7 = M^6 \times M^1$ and the situation reduces to the 6-dimensional case.

• If ∇ is some other metric connection on (M^7, g) with two parallel spinors, M^7 will, in general, not be a product manifold. Its Riemannian holonomy will typically be $\mathrm{SO}(7)$, so ∇^g does not measure this effect!

$\Rightarrow$ geometric situations not known from Riemannian holonomy will typically appear.

In a similar way, one treats the cases

$\mathrm{Spin}(7)$ in dimension 8. As just seen, $\mathrm{Spin}(7)$ has an 8-dimensional representation, hence it can be viewed as a subgroup of $\mathrm{SO}(8)$. Δ_8 has again one $\mathrm{Spin}(7)$-invariant spinor.

$\mathrm{Sp}(n)$ in dimension $4n$. Identifying quaternions with pairs $(z_1, z_2) \in \mathbb{C}^2$ yields $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$, and $\mathrm{SU}(2n)$ is then realized inside $\mathrm{SO}(4n)$ as before. It has $n+1$ invariant spinors.

The easiest case: ∇^g-parallel spinors

Thm (Wang, 1989).

(M^n, g): complete, simply connected, irreducible Riemannian manifold

N: dimension of the space of parallel spinors w. r. t. ∇^g

If (M^n, g) is non-flat and $N > 0$, then one of the following holds:

(1) $n = 2m$ $(m \geq 2)$, Riemannian holonomy repr.: $\mathrm{SU}(m)$ on $\mathbb{C}^m$, and $N = 2$ ("Calabi-Yau case"),

(2) $n = 4m$ $(m \geq 2)$, Riemannian holonomy repr.: $\mathrm{Sp}(m)$ on $\mathbb{C}^{2m}$, and $N = m + 1$ ("hyperkähler case"),

(3) $n = 7$, Riemannian holonomy repr.: 7-dimensional representation of G_2, and $N = 1$ ("parallel G_2 case"),

(4) $n = 8$, Riemannian holonomy repr.: spin representation of $\mathrm{Spin}(7)$, and $N = 1$ ("parallel $\mathrm{Spin}(7)$ case").

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