Philipps Universität Marburg

How to classify homogeneous spaces. . . and why we should care about them

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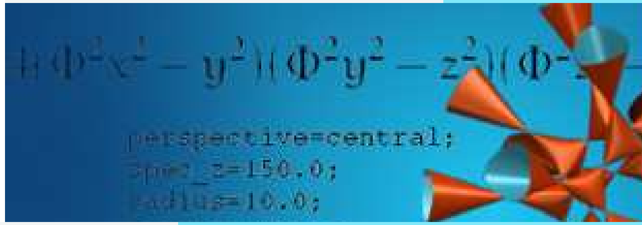
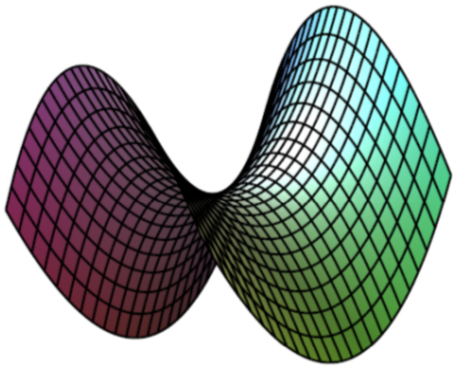
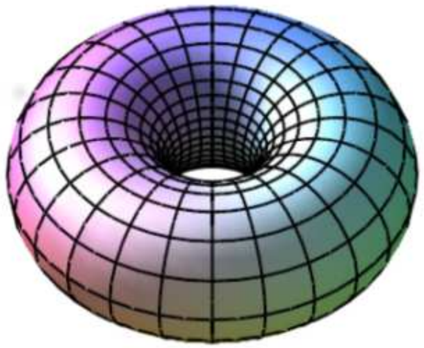
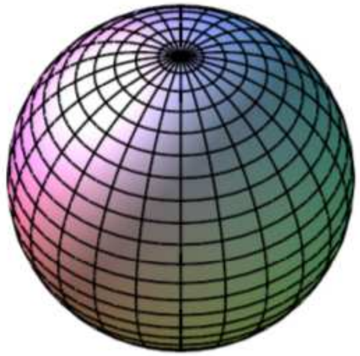


Durham, June 2020

– joint work with Ana Ferreira and Thomas Friedrich (†) –

A few keywords. . .

space: oldfashioned term for *manifold*, i. e. a smooth object that looks locally like $\mathbb{R}^n$ — for example, a surface without sharp edges or corners

homogeneous: "many symmetries" — but which and how?

Similar notions: symmetric, isotropic. . .

Solid state physics: Homogeneous vs. isotropic media

Idea: homogeneous – a solid which looks the same *at all points*

isotropic — a solid which looks the same *in all directions*

isotropic $\Rightarrow$ homogeneous, but homogeneous $\not\Rightarrow$ isotropic

Example: Calcite ($CaCO_3$) is perfectly crystalline, so it's homogeneous from a physics point of view; but it is not isotropic (phenomenon of 'birefringence', *Doppelbrechung* – a ray of light is split by polarization into two rays taking slightly different paths)

Mathematical problems with these concepts

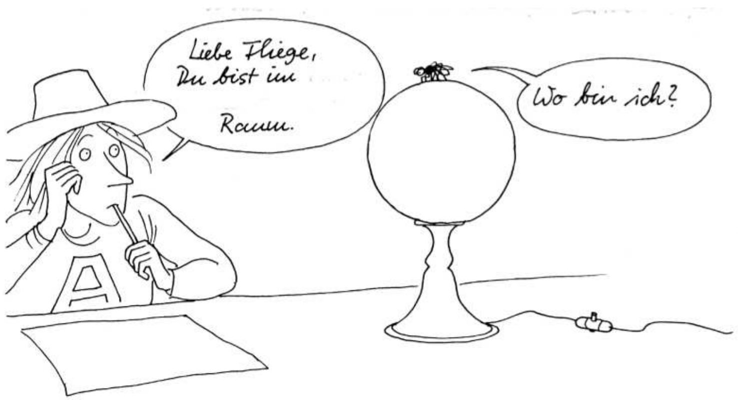
• dependent on the physical properties of light (polarization), not covered by the canonical concept of 'line'

• notion of direction or line not suitable for bounded objects (for example, compact manifolds) $\rightarrow$ replacement?

• requires an observer outside the material, i. e. not an intrinsic property

• The optical properties of Calcite can be described by a non-constant, axis-dependent refractive index: what are the 'right' functions to look at?

$\Rightarrow$ material properties are too complex to be covered by a notion of 'space', but give useful hints what to look at



Intrinsic vs. extrinsic symmetry I

Properties requiring an external observer may depend on the embedding in outer space:

sphere

pseudosphere

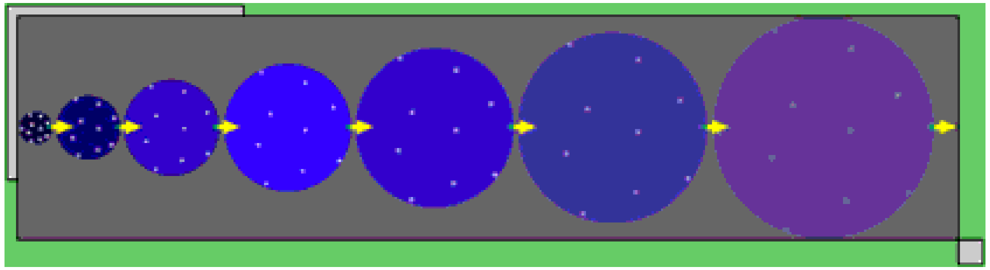
Embedding shows different amount of symmetry (1 vs. 2 rotational symmetries), but this cannot be seen in intrinsic geometric terms:

- Gaussian curvature const $= \pm 1$ in all points, 'standard' models of an elliptic and hyperbolic plane
- Later: together with $\mathbb{R}^2$, only 2-dim. 'symmetric' spaces of rank 1

Intrinsic vs. extrinsic symmetry II

Sometimes, an external observer is just not available:

First approximation: Universe is isotropic & homogeneous in space at large scales (*Copernican principle*).



expanding Friedman universe

Cosmology requires an **intrinsic notion of isotropy / homogeneity** for time slices $t =$ const. (assuming a notion of time exists. . .)

Mathematical prerequisites

Take a manifold M^n, $T_pM \cong \mathbb{R}^n$ its tangent space in $p \in M$

Assume: pos. definite scalar product $g(-,-)$ in each tangent space T_pM (metric; mnfd $+$ metric $=$ Riemannian mnfd)

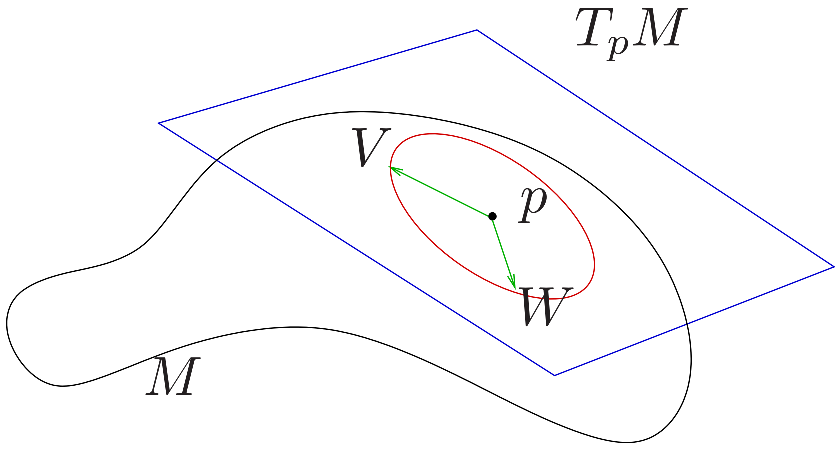
$\Rightarrow$
- notion of angle between vectors in p
- notion of shortest curves (geodesics)
- notion of isometry:

bijection $f : M \to M$ s. t. $df : T_pM \to T_{f(p)}M$ preserves scalar products

Mathematical definition of isotropy

Idea. direction $\rightsquigarrow$ vector in tangent space,

'looking the same' $\rightsquigarrow$ can be transformed into each other by an isometry

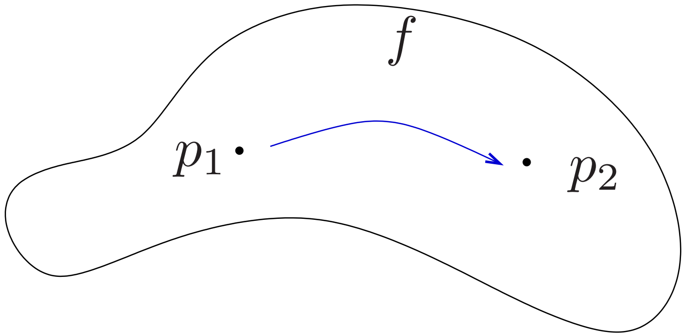


Dfn. Consider any $V, W \in T_pM$ of equal length. M is *isotropic* (in p), if $\exists$ isometry f with $f(p) = p$ s. t. $df(V) = W$.

Cosmology: isotropy is indicated by the observations of the microwave background radiation.

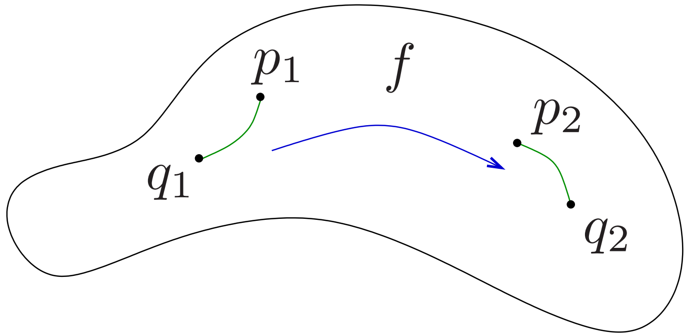
Mathematical definition of homogeneity

Homogeneous:



To any two points, $\exists$ isometry f mapping one to the other

2-point homogeneous:



To any two pairs of equidistant points, $\exists$ isometry f mapping one pair to the other

Obviously: 2-point homogeneous $\Rightarrow$ homogeneous, but not vice versa

Exa. $G_{2,n} = \{\mathbb{R}^2 \subset \mathbb{R}^n\}$, so points $=$ planes. Given two pairs of planes:

- distance is meas. by ≥ 2 'principal angles' between them, one if $n = 3$
- $\exists$ an isometry mapping one pair to the other iff all angles coincide

$\Rightarrow G_{2,3}$ is 2-point homogeneous, $G_{2,n} (n \geq 4)$ is only homogeneous

Thm. A space is 2-point homogeneous iff it is isotropic, and the only such spaces are:

$\mathbb{R}^n$, S^n, projective space over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, hyperbolic space over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (. . . and two exceptional spaces).

$=$ 'symmetric spaces of rank 1' (one number suffices to characterize whether pairs of points can be transformed into each other)

Élie Cartan, 1926: Classification of all symmetric spaces

. . . classification of all homogeneous spaces: impossible!

To put more structure on M: Need to do calculus, so an intrinsic notion of derivative $=$: 'connection'

Élie Cartan (1869–1951)

Given a manifold embedded in affine (or projective or conformal etc.) space, attribute to this manifold the affine (or projective or conformal etc.) connection that reflects in the simplest possible way the relations of this manifold with the ambient space.

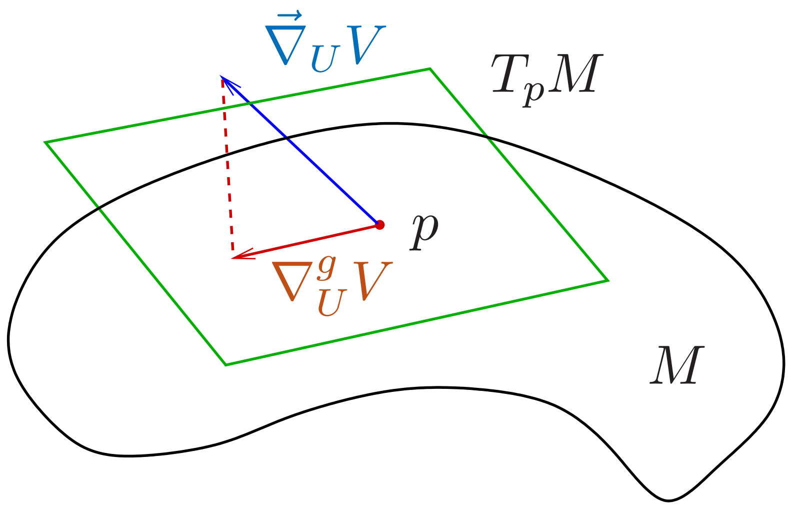
[Étant donné une variété plongée dans l'espace affine (ou projectif, ou conforme etc.), attribuer à cette variété la connexion affine (ou projective, ou conforme etc.) qui rende le plus simplement compte des relations de cette variété avec l'espace ambiant.].

Connections

Connection ∇: abstract derivation rule on mnfd satisfying all formal properties of the directional derivative

different name: "'covariant derivative"'

Exa. Projection $\nabla^g_U V$ of dir. derivative $\vec{\nabla}_U V$ to tangent plane
$=$ 'Levi-Civita connection' ∇^g



But: not only possibility $\longrightarrow$ connection with torsion [Dfn: Cartan, 1925]

Exa. Electrodynamics: $\nabla_U V := \vec{\nabla}_U V + \frac{ie}{\hbar} A(U) V$ ($\Leftrightarrow$ $\nabla_\mu = \partial_\mu + \frac{ie}{\hbar} A_\mu$)

A: gauge potential $=$ electromagnetic potential

Exa. If $n = 3$: $\nabla_U V := \vec{\nabla}_U V + U \times V$ additional term gives space an 'internal angular momentum', a *torsion*

Fact: $\exists$ 3 types of torsion: vectorial, skew symmetric, and [something else].

Why torsion?

• **General relativity**:

a) Cartan (1929): torsion $\sim$ intrinsic angular momentum, derived a set of gravitational field eqs., but postulated that the energy-momentum tensor should still be divergence-free $\rightarrow$ too restrictive

b) Einstein-Cartan theory ($\geq$ 1950): variation of the scalar curvature <u>and</u> of an additional Lagrangian generating the energy-momentum and the spin tensors: allowed *any torsion and not nec. metric*

• **Superstring theory**:

Classical Yang-Mills theory: curvature $\cong$ field strength,

in superstring theories: torsion $\cong$ higher order field strength

($+$ extra differential eqs.)

• **Differential geometry**: Connections adapted to the geometry useful for 'non-integrable' geometries, like: Hermitian non Kähler mnfds, contact manifolds. . .

Set-up: (M, g) Riemannian mnfd, ∇ metric conn., ∇^g Levi-Civita conn.

- Torsion as a $(2, 1)$-tensor: $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$
- Transform torsion into a $(3, 0)$-tensor via the metric:

$$T(X, Y, Z) \ := \ g(T(X, Y), Z)$$

- A metric connection is uniquely determined by its torsion

skew torsion implies:

- ∇-geodesics $= \nabla^g$-geodesics, so trajectories of test particles are not altered

$\longrightarrow$ explains relevance in physics

- ∇ may be written as $\quad \nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y, -)$

The Ambrose–Singer homogeneity theorem

Thm. A complete Riemannian manifold (M, g) equipped with a homogeneous structure, i. e. a metric connection ∇ with torsion T and curvature $\mathcal{R}$ such that $\nabla\mathcal{R} = 0$ and $\nabla T = 0$, is locally isometric to a Riemannian homogeneous space. [Ambrose–Singer, 1958]

• Symmetric spaces: Correspond to $T = 0$, the "integrable" case, $\nabla^g\mathcal{R}^g = 0$; intuitively this follows because $\nabla^g\mathcal{R}^g$ would be a $(5, 0)$-tensor, the invariance under reflections then forces it to vanish.

• Homogeneous non-symmetric spaces: $T \neq 0$.

Q. Is the connection unique?

– yes! This is a non-trivial consequence of the skew torsion holonomy theorem (except on spheres, Lie groups, and their coverings).

$\Rightarrow$ 3 classes of homogeneous spaces according to type of this torsion!

• Empirical fact: In the non-homogeneous case, metric connections with parallel torsion turn out to be very useful (and natural) as well.

Example: Compact Lie groups

Consider a compact Lie group G, $\mathfrak{g} = T_eG$. A metric g on G is called *biinvariant* if left and right translations are always isometries $\Leftrightarrow$

$$g([V,X],Y) + g(X,[V,Y]) \;=\; 0. \qquad (*)$$

Easy: $\nabla^g_X Y \;=\; \frac{1}{2}[X,Y]\ \forall X,Y \in \mathfrak{g}$. We make the Ansatz that T is proportional to $[\,,\,]$, i. e.

$$\nabla^s_X Y \;:=\; s[X,Y], \quad \forall\ s \in \mathbb{R}, \qquad \text{hence } T^s(X,Y) = (2s-1)[X,Y],$$

This defines an element $T \in \Lambda^3(G)$ iff the metric satisfies $(*)$. The curvature of this connection is

$$\mathcal{R}^s(X,Y)Z = s(1-s)[Z,[X,Y]] = \begin{cases} \frac{1}{4}[Z,[X,Y]] & \text{for the LC conn. } (s=\frac{1}{2}) \\ 0 & \text{for } s = 0,1 \end{cases}$$

The two flat connections are called the $\pm$-connection and were first decribed by Cartan and Schouten (1926).

One class: Naturally reductive homogeneous spaces

Traditional approach: $(M = G/H, g)$ a homogeneous space

Dfn. $M = G/H$ is *naturally reductive* if $\mathfrak{h}$ admits a reductive complement $\mathfrak{m}$ in $\mathfrak{g}$ s. t.

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle \;=\; 0 \;\text{ for all } X, Y, Z \in \mathfrak{m}, \qquad (*)$$

where $\langle -, -\rangle$ denotes the inner product on $\mathfrak{m}$ induced from g.

The PFB $G \to G/H$ induces a metric connection ∇ with torsion

$$T(X,Y,Z) = -\langle [X,Y]_{\mathfrak{m}}, Z\rangle,$$

the *canonical connection*. It satisfies $\nabla T = \nabla \mathcal{R} = 0$, so it's just the connection from the AS thm!

- If G/H is symmetric, then $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, hence $T = 0$ and $\nabla = \nabla^g$
- condition $(*)$ $\Leftrightarrow$ T is a 3-form, i. e. $T \in \Lambda^3(M)$.

Unfortunately, a classification in all dimensions is impossible!

Main pb: $\nexists$ invariant theory for $\Lambda^3(\mathbb{R}^n)$ under $\mathrm{SO}(n)$ for $n \geq 6$, i. e. normal forms for the $\mathrm{SO}(n)$-orbits of 3-forms!

- Use the recent progress on *metric connections with [parallel] skew torsion*
- Use *torsion* (instead of curvature) as basic geometric quantity, *find a G-structure (contact str., almost hermitian str. etc.)* inducing the nat. red. structure

In this talk: General strategy, some general results, classification for $n \leq 6$

small comments on proofs, details etc. in BLUE. Not in this talk: applications of the classification

Obviously: $\boxed{\text{nat.red.homog. Riemannian mnfds}} \subset \boxed{\text{(homogeneous) Riemannian mnfds with parallel skew torsion}}$

Important group associated with T:

$$\mathrm{Stab}(T) = \{A \in \mathrm{GL}(n, \mathbb{R}) \,|\, T(X, Y, Z) = T(AX, AY, AZ)\ \forall\ X, Y, Z\}$$

Review of some classical results

• all isotropy irreducible homogeneous manifolds are naturally reductive

• the $\pm$-connections on any Lie group with a biinvariant metric are naturally reductive (and, by the way, flat) [Cartan-Schouten, 1926]

• construction / classification (under some assumptions) of left-invariant naturally reductive metrics on compact Lie groups [D'Atri-Ziller, 1979]

• All 6-dim. homog. nearly Kähler mnfds (w. r. t. their canonical almost Hermitian structure) are naturally reductive. These are precisely: $S^3 \times S^3$, $\mathbb{CP}^3$, the flag manifold $F(1,2) = \mathrm{U}(3)/\mathrm{U}(1)^3$, and $S^6 = G_2/\mathrm{SU}(3)$.

• Known classifications:

- dimension 3 [Tricerri-Vanhecke, 1983], dimension 4 [Kowalski-Vanhecke, 1983], dimension 5 [Kowalski-Vanhecke, 1985]

These proceeded by finding normal forms for the curvature operator.

An important tool: the 4-form σ_T

Dfn. For any $T \in \Lambda^3(M)$, define ($e_1, \ldots, e_n$ a local ONF)

$$\sigma_T := \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T) \quad (= 0 \text{ if } n \leq 4)$$

Exa: For $T = \alpha\, e_{123} + \beta\, e_{456}$, $\sigma_T = 0$;
for $T = (e_{12} + e_{34})e_5$, $\sigma_T = -e_{1234}$

• σ_T **measures the 'degeneracy' of T and, if non degenerate, induces the geometric structure on M**

[σ_T appears in many important relations:

* 1st Bianchi identity: $\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}(X, Y, Z, V) = \sigma_T(X, Y, Z, V)$
* $T^2 = -2\sigma_T + \|T\|^2$ in the Clifford algebra
* If $\nabla T = 0$: $dT = 2\sigma_T$ and $\nabla^g T = \frac{1}{2}\sigma_T$]

σ_T and the Nomizu construction

Idea: for $M = G/H$, reconstruct $\mathfrak{g}$ from $\mathfrak{h}$, T, $\mathcal{R}$ and $V \cong T_xM$

Set-up: $\mathfrak{h}$ a real Lie algebra, V a real f. d. $\mathfrak{h}$-module with $\mathfrak{h}$-invariant pos. def. scalar product $\langle , \rangle$, i. e. $\mathfrak{h} \subset \mathfrak{so}(V) \cong \Lambda^2 V$

$\mathcal{R} : \Lambda^2 V \to \mathfrak{h}$ an $\mathfrak{h}$-equivariant map, $T \in (\Lambda^3 V)^{\mathfrak{h}}$ an $\mathfrak{h}$-invariant 3-form,

Define a Lie algebra structure on $\mathfrak{g} := \mathfrak{h} \oplus V$ by $(A, B \in \mathfrak{h}, X, Y \in V)$:

$$[A + X, B + Y] := ([A, B]_{\mathfrak{h}} - \mathcal{R}(X, Y)) + (AY - BX - T(X, Y))$$

Jacobi identity for $\mathfrak{g} \Leftrightarrow$

- $\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(X, Y, Z, V) = \sigma_T(X, Y, Z, V)$ (1st Bianchi condition)
- $\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(T(X, Y), Z) = 0$ (2nd Bianchi condition)

Observation: If (M, g, T) satisfies $\nabla T = 0$, then $\mathcal{R} : \Lambda^2(M) \to \Lambda^2(M)$ is symmetric (as in the Riemannian case).

Consider $\mathcal{C}(V) := \mathcal{C}(V, -\langle , \rangle)$: Clifford algebra, (recall: $T^2 = -2\sigma_T + \|T\|^2$)

Thm. If $\mathcal{R} : \Lambda^2 V \to \mathfrak{h} \subset \Lambda^2 V$ is symmetric, the first Bianchi condition is equivalent to $T^2 + \mathcal{R} \in \mathbb{R} \subset \mathcal{C}(V)$ ($\Leftrightarrow$ $2\sigma_T = \mathcal{R} \subset \mathcal{C}(V)$) , and the second Bianchi condition holds automatically.

Exists in the literature in various formulations: based on an algebraic identity (Kostant); crucial step in a formula of Parthasarathy type for the square of the Dirac operator (A, '03); previously used by Schoemann 2007 and Fr. 2007, but without a clear statement nor a proof.

Practical relevance: allows to evaluate the 1st Bianchi identity in one condition, good for implementation on a computer!

Splitting theorems

Dfn. For T 3-form, define [introduced in AFr, 2004]

- kernel: $\ker T = \{X \in TM \mid X \lrcorner T = 0\}$
- Lie algebra generated by its image: $\mathfrak{g}_T := \mathrm{Lie}\langle X \lrcorner T \mid X \in V\rangle$

  $\mathfrak{g}_T$ is *not* related in any obvious way to the isotropy algebra of T!

Thm 1. Let (M, g, T) be a c. s. c. Riemannian mfld with parallel skew torsion T. Then $\ker T$ and $(\ker T)^\perp$ are ∇-parallel and ∇^g-parallel integrable distributions, M is a Riemannian product s. t.

$$(M, g, T) = (M_1, g_1, T_1 = 0) \times (M_2, g_2, T_2), \quad \ker T_2 = \{0\}$$

Thm 2. Let (M, g, T) be a c. s. c. Riemannian mfld with parallel skew torsion T s. t. $\sigma_T = 0$, $TM = \mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_q$ the decomposition of TM in $\mathfrak{g}_T$-irreducible, ∇-par. distributions. Then all $\mathcal{T}_i$ are ∇^g-par. and integrable, M is a Riemannian product, and the torsion T splits accordingly

$$(M, g, T) = (M_1, g_1, T_1) \times \ldots \times (M_q, g_q, T_q)$$

The skew torsion holonomy theorem

Dfn. Let $0 \neq T \in \Lambda^3(V)$, $\mathfrak{g}_T$ as before, $G_T \subset \mathrm{SO}(n)$ its Lie group. Hence, $X \lrcorner T \in \mathfrak{g}_T \subset \mathfrak{so}(V) \cong \Lambda^2(V)\ \forall\ X \in V$. Then (G_T, V, T) is called a *skew-torsion holonomy system (STHS)*. It is said to be

- *irreducible* if G_T acts irreducibly on V,
- *transitive* if G_T acts transitively on the unit sphere of V,
- and *symmetric* if T is G_T-invariant.

Recall: The only transitive sphere actions are:

$\mathrm{SO}(n)$ on $S^{n-1} \subset \mathbb{R}^n$, $[\mathrm{S}]\mathrm{U}(n)$ on $S^{2n-1} \subset \mathbb{C}^n$, $\mathrm{Sp}(n)[\mathrm{Sp}(1)]$ on $S^{4n-1} \subset \mathbb{H}^n$, G_2 on S^6, $\mathrm{Spin}(7)$ on S^7, $\mathrm{Spin}(9)$ on S^{15}. [Montgomery-Samelson, 1943]

Thm (STHT). Let (G_T, V, T) be an irreducible STHS. If it is transitive, $G_T = \mathrm{SO}(n)$. If it is not transitive, it is symmetric, and

- V is a simple Lie algebra of rank ≥ 2 w. r. t. the bracket $[X, Y] = T(X, Y)$, and G_T acts on V by its adjoint representation,
- T is unique up to a scalar multiple.

[transitive: AFr 2004, general: Olmos-Reggiani, 2012; Nagy 2013]

A structure theorem for vanishing σ_T

Thm. Let (M^n, g) be an *irreducible*, c. s. c. Riemannian mnfld with parallel skew torsion $T \neq 0$ s. t. $\sigma_T = 0$, $n \geq 5$. Then M^n is a simple compact Lie group with biinvariant metric or its dual noncompact symmetric space.

Key ideas: $\sigma_T = 0 \Rightarrow$ Nomizu construction yields Lie algebra structure on TM

use $\mathfrak{g}_T$; use the Skew Torsion Holonomy Theorem to show that G_T is simple and acts on TM by its adjoint rep.

prove that $\mathfrak{g}_T = \mathfrak{iso}(T) = \mathfrak{hol}^g$, hence acts irreducibly on TM, hence M is an irred. symmetric space by Berger's Thm

Classification of nat. red. spaces in $n = 3$

[Tricerri-Vanhecke, 1983]

Then $\sigma_T = 0$, and the Nomizu construction can be applied directly to obtain in a few lines:

Thm. Let $(M^3, g, T \neq 0)$ be a 3-dim. c. s. c. Riemannian mnfld with a naturally reductive structure. Then (M^3, g) is one of the following:

- $\mathbb{R}^3, S^3$ or $\mathbb{H}^3$;

- isometric to one of the following Lie groups with a suitable left-invariant metric:

$$SU(2), \quad \widetilde{SL}(2,\mathbb{R}), \quad \text{or the 3-dim. Heisenberg group } H^3$$

N.B. A general classification of mnfds with par. skew torsion is meaningless – any 3-dim. volume form of a metric connection is parallel.

Classification of nat. red. spaces in $n = 4$

Thm. $(M^4, g, T \neq 0)$ a c. s. c. Riem. 4-mnfld with parallel skew torsion. Then

1) $V := *T$ is a ∇^g-parallel vector field.

2) $\mathrm{Hol}(\nabla^g) \subset \mathrm{SO}(3)$, hence M^4 is isometric to a product $N^3 \times \mathbb{R}$, where (N^3, g) is a 3-manifold with a parallel 3-form T.

• T has normal form $T = e_{123}$, so $\dim \ker T = 1$ and 2) follows at once from our 1st splitting thm: but the existence of V explains directly & *geometrically* the result in a few lines.

• Thm shows that the next result does not rely on the curvature or the homogeneity

Since a R. product is is nat. red. iff both factors are nar. red., we conclude:

Cor. A 4-dim. naturally reductive Riemannian manifold with $T \neq 0$ is locally isometric to a Riemannian product $N^3 \times \mathbb{R}$, where N^3 is a 3-dimensional naturally reductive Riemannian manifold. [Kowalski-Vanhecke, 1983]

Classification of nat. red. spaces in $n = 5$

Assume $(M^5, g, T \neq 0)$ is Riemannian mnfd with parallel skew torsion

- $\exists$ a local frame s. t (for constants $\lambda, \varrho \in \mathbb{R}$)

$$T = -(\varrho e_{125} + \lambda e_{345}), \quad *T = -(\varrho e_{34} + \lambda e_{12}), \quad \sigma_T = \varrho\lambda e_{1234}$$

- **Case A:** $\sigma_T = 0$ ($\Leftrightarrow \varrho\lambda = 0$): apply 2nd splitting thm, M^5 is then loc. a product $N^3 \times N^2$

- **Case B:** $\sigma_T \neq 0$, two subcases:

  * Case B.1: $\lambda \neq \varrho$, $\mathrm{Stab}(T) = \mathrm{SO}(2) \times \mathrm{SO}(2)$

  * Case B.2: $\lambda = \varrho$, $\mathrm{Stab}(T) = \mathrm{U}(2)$

Recall: Given a G-structure on (M, g), a *characteristic connection* is a metric connection with skew torsion preserving the G-structure (if existent, it's unique)

$n=5$: The induced contact structure

Case B: $\sigma_T \neq 0$

Dfn. A metric almost contact structure (φ, η) on (M^{2n+1}, g) is called (N: Nijenhuis tensor, $F(X,Y) := g(X, \varphi Y)$)

- quasi-Sasakian if $N = 0$ and $dF = 0$
- α-Sasakian if $N = 0$ and $d\eta = \alpha F$ (Sasaki: $\alpha = 2$)

Thm. Let (M^5, g, T) be a Riemannian 5-mnfld with parallel skew torsion T such that $\sigma_T \neq 0$. Then M is a quasi-Sasakian manifold and ∇ is its characteristic connection.

The structure is α-Sasakian iff $\lambda = \varrho$ (case B.2), and it is Sasakian if $\lambda = \varrho = 2$.

Construction: $V := *\sigma_T \neq 0$ is a ∇-parallel Killing vector field of constant length
$\equiv$ contact direction $\eta = e_5$ (up to normalisation)
Check: $T = \eta \wedge d\eta$, define $F = -(e_{12} + e_{34})$, then prove that this works.

$n = 5$: Classification I

For $\lambda = \varrho$ (case B.2), no classification for parallel skew torsion is possible (many non-homogeneous Sasakian mnfds are known). But for

Case B.1: $\lambda \neq \varrho$

Thm. Let (M^5, g, T) be Riemannian 5-manifold with parallel skew torsion s. t. T has the normal form

$$T = -(\varrho e_{125} + \lambda e_{345}), \quad \varrho\lambda \neq 0 \text{ and } \varrho \neq \lambda.$$

Then $\nabla\mathcal{R} = 0$, i. e. M is locally naturally reductive, and the family of admissible torsion forms and curvature operators depends on 4 parameters.

[Use Clifford criterion to relate $\mathcal{R}$ and σ_T]

Now one can apply the Nomizu construction to obtain the classification:

$n = 5$: Classification II

Thm. A c. s. c. Riemannian 5-mnfld (M^5, g, T) with parallel skew torsion $T = -(\varrho e_{125} + \lambda e_{345})$ with $\varrho\lambda \neq 0$ is isometric to one of the following naturally reductive homogeneous spaces:

If $\lambda \neq \varrho$ (B.1):

a) The 5-dimensional Heisenberg group H^5 with a two-parameter family of left-invariant metrics,

b) A manifold of type $(G_1 \times G_2)/\mathrm{SO}(2)$ where G_1 and G_2 are either $\mathrm{SU}(2)$, $\mathrm{SL}(2, \mathbb{R})$, or H^3, but not both equal to H^3 with one parameter $r \in \mathbb{Q}$ classifying the embedding of $\mathrm{SO}(2)$ and a two-parameter family of homogeneous metrics.

If $\lambda = \varrho$ (B.2): One of the spaces above or $\mathrm{SU}(3)/\mathrm{SU}(2)$ or $\mathrm{SU}(2, 1)/\mathrm{SU}(2)$ (the family of metrics depends on two parameters).

[Kowalski-Vanhecke, 1985]

The case $n = 6$ I

Assume $\ker T = 0$ from beginning. Distinction $\sigma_T =, \neq 0$ is too crude.

$*\sigma_T$: a 2-form $\equiv$ skew-symm. endomorphism, classify by its **rank!** (=0,2,4,6 / Case A, B, C, D)

Geometry: Can $*\sigma_T$ be interpreted as an almost complex structure, i. e. $\Omega = e_{12} + e_{34} + e_{56}$?

Exa. On $S^3 \times S^3$, there exist 3-forms with the following subcases:

$\mathrm{rk}\,(*\sigma_T)$	6	6	2	0
$\mathfrak{stab}(T)$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	T^2	$\mathfrak{so}(3) \times \mathfrak{so}(3)$

Case A: $\sigma_T = 0$

This covers, for example, torsions of form $\mu\, e_{123} + \nu\, e_{456}$. This is basically all by our 2nd splitting thm:

Thm. A c. s. c. Riemannian 6-mnfld with parallel skew torsion T s. t. $\sigma_T = 0$ and $\ker T = 0$ splits into two 3-dimensional manifolds with parallel skew torsion,

$$(M^6, g, T) = (N_1^3, g_1, T_1) \times (N_2^3, g_2, T_2)$$

Cor. Any 6-dim. nat. red. homog. space with $\sigma_T = 0$ and $\ker T = 0$ is locally isometric to a product of two 3-dimensional nat. red. homog. spaces.

The case $n = 6$ II

Case B: $\mathrm{rk}\,(*\sigma_T) = 2$

A priori, it is not possible to define an almost complex structure.

Use Nomizu construction to conclude:

Thm. A c. s. c. Riemannian 6-mnfd with parallel skew torsion T and $\mathrm{rk}\,(*\sigma_T) = 2$ is the product $G_1 \times G_2$ of two Lie groups equipped with a family of left invariant metrics. G_1 and G_2 are either $S^3 = \mathrm{SU}(2)$, $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, or H^3.

The case $n = 6$ III

Case B: $\mathrm{rk}\,(*\sigma_T) = 4$

Thm. For the torsion form of a metric connection with parallel skew torsion ($\ker T = 0$), the case $\mathrm{rk}\,(*\sigma_T) = 4$ cannot occur.

[but: such forms exist if $\nabla T \neq 0$! – these results explain why a classification is possible without knowing the orbit class. of $\Lambda^3(\mathbb{R}^6)$ under $\mathrm{SO}(6)$]

The case $n = 6$ IV

Case C: $\mathrm{rk}\,(*\sigma_T) = 6$

Thm. Such a 6-mnfd with parallel skew torsion admits an almost complex structure J.

All three eigenvalues of $*\sigma_T$ are equal, hence $*\sigma_T$ is proportional to Ω, the fundamental form of J. It's either nearly Kähler or it is naturally reductive and $\mathfrak{hol}^\nabla = \mathfrak{so}(3)$.

N.B. If M^6 nearly Kähler: the only homogeneous ones are $S^6, S^3 \times S^3, \mathbb{CP}^3, F(1,2)$. [Butruille, 2005]

Non-homogeneous nearly Kähler mnfds exist [Foscolo-Haskins, 2017]

Again, we have an explicit formula for torsion and curvature, then perform the Nomizu construction (. . . and survive).

The case $n = 6$ V

Final result of Nomizu construction:

Thm. A c. s. c. Riemannian 6-mnfd with parallel skew torsion T, $\mathrm{rk}\,(*\sigma_T) = 6$ and $\ker T = 0$ that is *not* isometric to a nearly Kähler manifold is one of the following Lie groups with a suitable family of left-invariant metrics:

- The nilpotent Lie group with Lie algebra $\mathbb{R}^3 \times \mathbb{R}^3$ with commutator $[(v_1, w_1), (v_2, w_2)] = (0, v_1 \times v_2)$,
- the direct or the semidirect product of S^3 with $\mathbb{R}^3$,
- the product $S^3 \times S^3$,
- the Lie group $\mathrm{SL}(2, \mathbb{C})$, viewed as a real mnfld (with a deformed complex str.!)

- prove that manifold is indeed a Lie group,
- identify its abstract Lie algebra by degeneracy / EV of its Killing form,
- find 3-dim. subalgebra defining a 3-dim. quotient and prove that the 6-dim. Lie alg. is its isometry algebra;

for example, $\mathrm{SL}(2, \mathbb{C})$ appears because it's the isometry group of hyperbolic space $\mathbb{H}^3$

Outlook: The higher dimensional case

Thm. Every naturally reductive space is in a unique way an extension of a space with its transvection algebra $\mathfrak{k}$ of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathbb{R}^n,$$

where $\mathfrak{h} \oplus \mathfrak{m}$ is semisimple and $\mathfrak{h}$ is the isotropy algebra of the torsion.

Every such extension is a fiber bundle of naturally reductive spaces; more specifically, the fibers are orbits of an abelian group of isometries. This means the fiber distribution is spanned by Killing vectors of constant length.

[Storm, 2018/19]

The Nomizu construction the allows the reconstruction of the full naturally reductive space

$\Rightarrow$ algorithm for construction in all dimensions (done for dimensions $7,\ 8$). Both lists are surprisingly short.

Homework. Identify the 6-dimensional Lie algebra $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{h} = \mathrm{span}(\Omega_1, \Omega_3, \Omega_5), \mathfrak{m} := \mathrm{span}(e_2, e_4, e_6)$ defined by $(\alpha, \alpha', \beta \in \mathbb{R})$

$$[\Omega_1, \Omega_3] = (\alpha - 2\beta)\Omega_5, \quad [\Omega_1, \Omega_5] = (2\beta - \alpha)\Omega_3, \quad [\Omega_3, \Omega_5] = (\alpha - 2\beta)\Omega_1$$

$$[\Omega_1, e_4] = [e_2, \Omega_3] = (\alpha - 2\beta)e_6, \; [\Omega_1, e_6] = [e_2, \Omega_5] = (2\beta - \alpha)e_4,$$

$$[\Omega_3, e_6] = [e_4, \Omega_5] = (\alpha - 2\beta)e_2.$$

$$[e_2, e_4] = -\beta\Omega_5 - \alpha' e_6, \quad [e_2, e_6] = \beta\Omega_3 + \alpha' e_4, \quad [e_4, e_6] = -\beta\Omega_1 - \alpha' e_2.$$

and use it to deduce the previous theorem.

Hint: Prove first that $\mathfrak{g}$ is not semisimple iff $\alpha = 2\beta$ or $4\beta(\alpha - 2\beta) = \alpha'^2$.

Example: The $(2n+1)$-dimensional Heisenberg group

$$H^{2n+1} = \left\{ \begin{bmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} ; x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \cong \mathbb{R}^{2n+1},$$ local coordinates $x_1, \dots, x_n, y_1, \dots, y_n, z$

- Metric: described by parameters $\lambda = (\lambda_1, \dots, \lambda_n)$, all $\lambda_i > 0$

$$g_\lambda = \sum_{i=1}^{n} \frac{1}{\lambda_i}(dx_i^2 + dy_i^2) + \left[dz - \sum_{j=1}^{n} x_j dy_j \right]^2$$

- Contact str.: $\eta = dz - \sum_{i=1}^{n} x_i dy_i$, $F = -\sum_{i=1}^{n} \frac{1}{\lambda_i} dx_i \wedge dy_i$

- Characteristic connection ∇: torsion: $T = \eta \wedge d\eta = -\sum_{i=1}^{n} \eta \wedge dx_i \wedge dy_i$

Nice property: For $n \geq 2$, H^{2n+1} admits Killing spinors with torsion, i. e. solutions of $\nabla_X \psi = \alpha \psi$ (but no Riemannian Killing spinors, i. e. no sol. for $\nabla = \nabla^g$ / $\nexists$ Einstein metric) [A-Becker-Bender, 2012]

Example: $\mathrm{SL}(2,\mathbb{C})$ viewed as a 6-dimensional real mnfd

- Write $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus i\,\mathfrak{su}(2)$;

Killing form $\beta(X,Y)$ is neg. def. on $\mathfrak{su}(2)$, pos. def.on $i\,\mathfrak{su}(2)$

- $M^6 = G/H = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)/\mathrm{SU}(2)$ with $H = \mathrm{SU}(2)$ embedded diag

(recall that $\mathfrak{hol}^{\nabla} = \mathfrak{so}(3)$; want that isotropy rep. $=$ holonomy rep.)

- $\mathfrak{m}_\alpha$ red. compl. of $\mathfrak{h}$ inside $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{su}(2)$ depending on $\alpha \in \mathbb{R} - \{1\}$,

$$\mathfrak{h} = \{(B,B) : B \in \mathfrak{su}(2)\}, \quad \mathfrak{m}_\alpha := \{(A+\alpha B, B) : A \in i\,\mathfrak{su}(2),\ B \in \mathfrak{su}(2)\}.$$

- Riemannian metric:

$$g_\lambda((A_1 + \alpha B_1, B_1), (A_2 + \alpha B_2, B_2)) := \beta(A_1, A_2) - \frac{1}{\lambda^2}\beta(B_1, B_2), \quad \lambda > 0$$

- In suitable ONB: almost hermitian str.: $\Omega := x_{12} + x_{34} + x_{56}$ with torsion

$$T = N + d\Omega \circ J = \left[2\lambda(1-\alpha) + \frac{4}{\lambda(1-\alpha)}\right] x_{135} + \frac{2}{\lambda(1-\alpha)}[x_{146} + x_{236} + x_{245}].$$

- Curvature: has to be a map $\mathcal{R} : \Lambda^2(M^6) \to \mathfrak{hol}^{\nabla} \subset \mathfrak{so}(6)$, here: mainly projection on $\mathfrak{hol}^{\nabla} = \mathfrak{so}(3)$.

- $\nabla T = \nabla \mathcal{R} = 0$, i. e. naturally reductive for all α, λ; type $W_1 \oplus W_3$ or W_3

Example: Quaternionic Heisenberg group

$N^7 = \mathbb{R}^7$ with basis elements z_1, z_2, z_3, and $\tau_1, \ldots, \tau_4$, metric depending on $\lambda > 0$ s. t. $\xi_i := \frac{z_i}{\lambda}$, τ_l are orthonormal, commutator relations

$$
\begin{array}{lll}
[\tau_r, \tau_{1+r}] = \lambda\, \xi_1 & [\tau_r, \tau_{2+r}] = \lambda\, \xi_2 & [\tau_r, \tau_{3+r}] = \lambda\, \xi_3 \\
[\tau_{2+r}, \tau_{3+r}] = \lambda\, \xi_1 & [\tau_{3+r}, \tau_{1+r}] = \lambda\, \xi_2 & [\tau_{1+r}, \tau_{2+r}] = \lambda\, \xi_3
\end{array}
$$

- ξ_1, ξ_2, ξ_3 are Killing vector fields; metric is never Einstein ($\Rightarrow \nexists$ Killing sp.)
η_i: dual form of ξ_i, θ_l: dual form of τ_l

- carries, in standard way, an almost 3-contact metric structure

Thm. The connection ∇ with skew torsion T satisfies

$$T = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3 - 4\lambda\eta_{123}$$

- $\nabla T = \nabla \mathcal{R} = 0$, hence it's naturally reductive [Tricerri-Vanhecke]

- Its holonomy algebra is isomorphic to $\mathfrak{su}(2)$, acting irreducibly on $T^v = \mathrm{span}(\xi_1, \xi_2, \xi_3)$ and on T^h.

- ∇ *is* the characteristic connection of the cocalibrated G_2 structure

$$\omega = -\eta_1 \wedge (\theta_{12} + \theta_{34}) - \eta_2 \wedge (\theta_{13} + \theta_{42}) - \eta_3 \wedge (\theta_{14} + \theta_{23}) + \eta_{123}.$$

As such, it admits a parallel spinor field ψ_0, $\nabla \psi_0 = 0$. What about $\xi_i \cdot \psi_0$?

Thm. The spinor fields $\psi_i := \xi_i \cdot \psi_0$, $i = 1, 2, 3$, are 'generalised Killing spinors' satisfying the differential equation

$$\nabla^g_{\xi_i} \psi_i = \frac{\lambda}{2} \xi_i \cdot \psi_i, \quad \nabla^g_{\xi_j} \psi_i = -\frac{\lambda}{2} \xi_j \cdot \psi_i \ (i \neq j), \quad \nabla^g_X \psi_i = \frac{5\lambda}{4} X \cdot \psi_i \ \text{ for } X \in T^h.$$

[A-Ferreira-Storm, 12/2014]

(gen. KS: $\nabla^g_X \psi = S(X) \cdot \psi$ with symm. endom. S, but not multiple of identity)

Observe: Only known example where S has three different eigenvalues