Pluricomplex geometry and quaternionic manifolds

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Geometric structures on manifolds and their applications
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Motivation

- Many gauge-theoretic moduli spaces have natural hyperkähler metrics: moduli spaces of instantons on hyperkähler 4-manifolds, magnetic monopoles on $\mathbb{R}^3$, Higgs bundles on Riemann surfaces.
- Monopoles exist also on other 3-manifolds, in particular the hyperbolic space H^3. The geometry of the moduli spaces has been a long-standing problem.
- The geometry should be a deformation of the hyperkähler geometry. A natural candidate is a quaternion-Kähler geometry.
- At the beginning of the 90's, Hitchin classified $SO(3)$-invariant self-dual Einstein 4-manifolds, and found a large class corresponding to moduli spaces of *centred* hyperbolic $SU(2)$-monopoles of charge 2.
- These metrics are deformations of the Atiyah-Hitchin metric on the moduli space of *centred* Eulidean $SU(2)$-monopoles of charge 2.

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- Around 2009, Hitchin also constructed quaternion-Kähler metrics on moduli spaces of *centred* hyperbolic $SU(3)$-monopoles with charge 2 and *minimal* symmetry breaking.
- These are 8-manifolds and the metrics are deformations of Dancer's hyperkähler metric on the moduli spaces of *centred* Euclidean $SU(3)$-monopoles with charge 2 and *minimal* symmetry breaking. Their symmetry group is $SO(3) \times SU(2)$.
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- In 2008, O. Nash showed, via twistor methods, that the *complexification* of the natural geometry of hyperbolic moduli space is the same as the *complexification* of a hyperkähler structure.
- This implies that the underlying real geometry is unlikely to be quaternion-Kähler.
- In 2011, Lorenz Schwachhöfer and I identified this real geometry. It's a new type of geometry, which we call "pluricomplex geometry".
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Pluricomplex structures

Let V be an $2n$-dimensional real vector space and $\mathcal{J}(V) \simeq GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ its twistor space, i.e. the space of (isomorphism classes) of complex structures on V.

A hypercomplex structure on V may be viewed as a very special $\mathbb{P}^1$ inside $\mathcal{J}(V)$.

We define a linear *pluricomplex structure* on V as a much less special $\mathbb{P}^1 \subset \mathcal{J}(V)$: a holomorphic embedding $K : \mathbb{P}^1 \to \mathcal{J}(V)$ such that the subspaces $V_\zeta^{1,0}$ corresponding to different $\zeta \in \mathbb{P}^1$ form a holomorphic vector bundle isomorphic to $O(-1) \otimes \mathbb{C}^n$ *and* the quotient bundle is isomorphic to $O(1) \otimes \mathbb{C}^n$. n must be even!

Given a pluricomplex structure $K : \mathbb{P}^1 \to \mathcal{J}(V)$, we obtain a second one $\widehat{K} = -K \circ \sigma$, where $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ is the antipodal map. It is also a pluricomplex structure and we write $\widehat{J}_\zeta = -J_{\sigma(\zeta)}$, $\widehat{V}_\zeta^{1,0}$ for the subspace of vectors of type $(1,0)$ for $\widehat{J}_\zeta$.

A pluricomplex structure is hypercomplex iff $\widehat{K} = K$.

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Characteristic curve and characteristic sheaf

Let W be a holomorphic bundle on $\mathbb{P}^1 \times \mathbb{P}^1$, the fibre of which at (ζ, η) is $V_\zeta^{1,0} \oplus \widehat{V}_\eta^{1,0}$.

We have a natural map

$$W \to \mathcal{O} \otimes V^{\mathbb{C}}$$

of vector bundles on $\mathbb{P}^1 \times \mathbb{P}^1$, which induces an injective map $\mathcal{W} \to \mathcal{O} \otimes V^{\mathbb{C}}$ on the sheaves of sections. We denote by $\mathcal{F}$ the cokernel of this map, so that we have an exact sequence

$$0 \to \mathcal{W} \to \mathcal{O} \otimes V^{\mathbb{C}} \to \mathcal{F} \to 0.$$

Definition

The sheaf $\mathcal{F}$ is called the *characteristic sheaf* and its support S the *characteristic curve* of a pluricomplex structure.

S (as a set) is the set of $(\zeta, \eta) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $V_\zeta^{1,0} \cap \widehat{V}_\eta^{1,0} \neq 0$.

In particular S does not intersect the anti-diagonal $\overline{\Delta} = \{(\zeta, \sigma(\zeta))\}$.

Also, S is invariant under σ, $\sigma(\zeta, \eta) = (-1/\bar{\eta}, -1/\bar{\zeta})$ (and so is $\mathcal{F}$).

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The short exact sequence defining $\mathcal{F}$ can be written as follows:

$$0 \to \mathcal{O}(-1,0)\otimes\mathbb{C}^n \oplus \mathcal{O}(0,-1)\otimes\mathbb{C}^n \xrightarrow{M} \mathcal{O}\otimes\mathbb{C}^{2n} \to \mathcal{F} \to 0.$$

It follows that $\mathcal{F}$ satisfies also the following cohomological conditions:

$$h^0(\mathcal{F}) = 2n, \;\; h^1(\mathcal{F}) = 0,$$

$$H^*(\mathcal{F}(-1,-1)) = H^*(\mathcal{F}(0,-2)) = H^*(\mathcal{F}(-2,0)) = 0.$$

There is a 1-1 correspondence between such sheaves on $\mathbb{P}^1\times\mathbb{P}^1$ and linear pluricomplex structures on $\mathbb{C}^{2n}$.
V is the space σ-invariant sections of $\mathcal{F}|_S$, and $V_\zeta^{1,0}$ consists of sections of $\mathcal{F}|_S$ vanishing on $\{\zeta\times\mathbb{P}^1\}\cap S$.

Remark: A pluricomplex structure is hypercomplex iff the characteristic curve S is the diagonal in $\mathbb{P}^1\times\mathbb{P}^1$ (and the sheaf $\mathcal{F}$ is $\bigoplus\mathcal{O}(1)$).

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Pluricomplex manifolds

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- A pluricomplex structure on M is said to be *integrable* if every $J_\zeta = K(\zeta)$ is integrable. Pluricomplex manifolds = manifolds with an integrable pluricomplex structure.
- Obtain a complexified hypercomplex structure on $M^{\mathbb{C}}$. Projecting the Obata connection from $T^{1,0}M^{\mathbb{C}}$ onto TM gives a canonical torsion-free connection.
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- A pluricomplex structure on M is said to be *integrable* if every $J_\zeta = K(\zeta)$ is integrable. Pluricomplex manifolds = manifolds with an integrable pluricomplex structure.
- Obtain a complexified hypercomplex structure on $M^{\mathbb{C}}$. Projecting the Obata connection from $T^{1,0}M^{\mathbb{C}}$ onto TM gives a canonical torsion-free connection.
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Strongly integrable pluricomplex structures

- For a pluricomplex structure on a manifold M, consider the pointwise sequence defining the characteristic curve, i.e.

$$0 \longrightarrow \mathcal{W}_m \longrightarrow \mathcal{O} \otimes T_m^{\mathbb{C}} M \longrightarrow \mathcal{F}_m \longrightarrow 0,$$

for every $m \in M$, where $W_m|_{(\zeta,\eta)} = T_\zeta^{1,0} \oplus \widehat{T}_\eta^{1,0}$. We denote the support of $\mathcal{F}_m$ by S_m.

- Consider the following fibration over M

$$Y = \{(m,\zeta,\eta) \in M \times \mathbb{P}^1 \times \mathbb{P}^1;\ (\zeta,\eta) \in S_m\} \stackrel{\nu}{\longrightarrow} M,$$

and the corresponding map $p : Y \to \mathbb{P}^1 \times \mathbb{P}^1$.

- We say that the pluricomplex structure is *strongly integrable* if this fibration can be extended to a double fibration

$$Z \stackrel{\mu}{\longleftarrow} Y \stackrel{\nu}{\longrightarrow} M,$$

so that M is then the parameter space of real curves $\tilde{S}_m \simeq S_m$ in Z; the normal sheaf of each $\tilde{S}_m$ in Z is $\mathcal{F}_m$. Z is also equipped with a map $\rho : Z \to \mathbb{P}^1 \times \mathbb{P}^1 - \overline{\Delta}$, such that $p = \rho \circ \mu$.

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Pluricomplex structures of degree 1

The characteristic curve $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree (k,k). We call k the degree of a pluricomplex structure.

Let M be equipped with a strongly integrable pluricomplex structure of degree 1. Then M is the parameter space of σ-invariant rational curves with normal bundle isomorphic to a direct sum of $O(1)$'s. Therefore M is a quaternionic manifold.

At every point $m \in M$ we have a $\mathbb{P}^1$ of almost complex structures, which behave algebraically as the $\mathbb{P}^1$ of unit imaginary quaternions.

The twistor spaces of the pluricomplex structure and of the quaternionic structure coincide. If we view Z as the twistor space of the quaternionic structure, then Z is equipped with a projection onto M, the fibres of which are the σ-invariant $\mathbb{P}^1$-s.

Having a pluricomplex structure as well means that this fibration is trivial, $Z \simeq M \times \mathbb{P}^1$, and the projection $\pi : Z \to \mathbb{P}^1$ is holomorphic (but $\pi \circ \sigma \neq \sigma \circ \pi$, unless M is hypercomplex). The map $\rho : Z \to \mathbb{P}^1 \times \mathbb{P}^1 - \overline{\Delta}$ is given by $\rho(z) = (\pi(z), \sigma \circ \pi \circ \sigma(z))$.

Example $H^3 \times S^1$ (a.k.a. moduli space of hyperbolic monopoles of charge 1).

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Pluricomplex structure of hyperbolic monopoles

- The moduli space $\mathcal{M}_{k,m}$ of hyperbolic $SU(2)$-monopoles of charge k and mass m has a natural strongly integrable pluricomplex structure (at least on an open dense subset; vanishing of $H^*(S, \mathcal{F}(-2,0))$ everywhere needed).
- Its twistor space Z is the total space of the line bundle $O(2m+k, -2m-k)$ over $\mathbb{P}^1 \times \mathbb{P}^1 - \overline{\Delta}$ with the zero section removed (a $\mathbb{C}^*$-principal bundle).
- Similarly, moduli spaces of hyperbolic $SU(3)$-monopoles with minimal symmetry breaking have a natural strongly integrable pluricomplex structure. This time Z is the total space of an $GL(2,\mathbb{C})$-principal bundle over $\mathbb{P}^1 \times \mathbb{P}^1 - \overline{\Delta}$.

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Hitchin's QK metric from pluricomplex structures

Let V be a vector space equipped with a pluricomplex structure of degree 2, i.e. its characteristic curve is elliptic. This is the case for hyperbolic monopoles of charge 2. We have $V^{\mathbb{C}} \simeq H^0(S, \mathcal{F})$. Suppose that $\mathcal{O}_S(1/2, -1/2) \not\simeq \mathcal{O}_S$, so that $h^0(\mathcal{O}_S(1/2, 1/2)) = 2$. Then the exact sequence

$$0 \to \mathcal{F}(-1,-1) \to \mathcal{F}(-1/2,-1/2) \otimes H^0(S, \mathcal{O}(1/2,1/2)) \to \mathcal{F} \to 0,$$

yields a canonical isomorphism

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i.e. $V^{\mathbb{C}} \simeq \mathbb{C}^{2r} \otimes \mathbb{C}^2$. Both factors have a quaternionic involution compatible with the real involution on $V^{\mathbb{C}}$.

If M is a pluricomplex manifold, so that each S_m is elliptic and $\mathcal{O}_{S_m}(1/2, -1/2) \not\simeq \mathcal{O}_{S_m}$, we obtain an almost quaternionic structure on M.

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To make a connection with the construction of Hitchin, consider the holomorphic involution τ on $\mathbb{P}^1 \times \mathbb{P}^1$, which interchanges the two factors. If τ can be lifted to an involution on M, then (under reasonable assumptions) the above almost quaternionic structure will give an integrable quaternionic structure on M^τ.

I'll show that this quaternionic structure is integrable: it arises from a twistor space $\tilde{Z}$, i.e. a complex manifold, equipped with a fixed-point-free antiholomorphic involution σ, such that M^τ is the parameter space of a family of σ-invariant holomorphic $\mathbb{P}^1$ with normal bundle $O(1) \otimes \mathbb{C}^{2n}$. If, additionaly, we want a quaternion-Kähler metric, then we need a complex contact structure on $\tilde{Z}$, i.e. a section θ of $K^{1/(n+1)}$ such that $\theta \wedge d\theta^n$ is nonzero on each twistor line (plus positivity).

Let M be a strongly pluricomplex manifold M with twistor space Z, $\rho : Z \to \mathbb{P}^1 \times \mathbb{P}^1 - \overline{\Delta}$, with each S_m an elliptic curve. The quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by τ is $\mathbb{P}^2$, and we have a double covering

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I'll show that this quaternionic structure is integrable: it arises from a twistor space $\tilde{Z}$, i.e. a complex manifold, equipped with a fixed-point-free antiholomorphic involution σ, such that M^τ is the parameter space of a family of σ-invariant holomorphic $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$. If, additionaly, we want a quaternion-Kähler metric, then we need a complex contact structure on $\tilde{Z}$, i.e. a section θ of $K^{1/(n+1)}$ such that $\theta \wedge d\theta^n$ is nonzero on each twistor line (plus positivity).

Let M be a strongly pluricomplex manifold M with twistor space Z, $\rho : Z \to \mathbb{P}^1 \times \mathbb{P}^1 - \overline{\Delta}$, with each S_m an elliptic curve. The quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by τ is $\mathbb{P}^2$, and we have a double covering

$$\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2.$$

We can define the "direct image" $\pi_* Z$ of Z.

Direct image of manifolds

Let X be the quotient of a complex manifold Y by a holomorphic involution τ, and let $p : Z \to Y$ be a holomorphic submersion. Consider the fibred product

$$Z_\pi^2 = Z \times_p \tau^* Z = \{(z, w) \in Z \times Z;\, p(z) = \tau(p(w))\},$$

with the induced submersion $\bar{p} : Z_\pi^2 \to Y$, $\bar{p}(z, w) = p(z)$. We have a $\mathbb{Z}_2$-action on Z_π^2 given by $t : (z, w) \mapsto (w, z)$. Let $Z_\pi^{[2]}$ denote the manifold obtained by blowing up the fixed point set of t and quotienting the result by the induced $\mathbb{Z}_2$, and let $\tilde{C} \subset Z_\pi^{[2]}$ be the proper transform of $C = \bar{p}^{-1}(Y^\tau)$. Then

$$\pi_* Z = Z_\pi^{[2]} - \tilde{C}.$$

Observe that $\pi_* Z$ is precisely the subset of $Z_\pi^{[2]}$ where the induced projection $Z_\pi^{[2]} \to X$ is a submersion.

Let us go back to the case of $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ and Z the twistor space of a strongly integrable pluricomplex structure of degree 2.

We obtain the direct image $\pi_* Z$. If the involution τ can be lifted to Z, we obtain a holomorphic involution $\tilde{\tau}$ on $\pi_* Z$.

Let $\tilde{Z}$ be the $\tilde{\tau}$-invariant part of $\pi_* Z$. A τ-invariant S_m in Z will descend to a rational curve in $\tilde{Z}$, and its normal bundle is a sum of $O(1)$'s. Moreover, if the lift of τ is compatible with the real structure σ on Z, then we obtain a real structure on $\tilde{Z}$.

Thus $\tilde{Z}$ is the twistor space of an integrable quaternionic structure on M^τ. To obtain a qK metric, we need a contact structure θ on $\tilde{Z}$. I do not know (yet) what one needs on Z to get θ on $\tilde{Z}$.

What I just described is precisely the situation when M is a moduli space of charge 2 hyperbolic monopoles ($SU(2)$ or $SU(3)$ with minimal symmetry breaking). M^τ is then the corresponding moduli space of centred monopoles, and the resulting qK metrics are the ones due to Hitchin.

One more example: charge 2 $SU(4)$-monopoles with minimal symmetry breaking ($(\lambda, \lambda, -\lambda, -\lambda)$). Dimension 12; isometry group: $SO(3) \times SU(2) \times SU(2)$.

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Deformations of ALF gravitational instantons of type D_k

Hitchin's self-dual Einstein metrics on moduli spaces of centred hyperbolic monopoles of charge 2 are deformations of the Atiyah-Hitchin metric, which can be viewed as an ALF gravitational instanton of type D_0.

There are complete ALF gravitational instantons of type D_k, for any $k \in \mathbb{N}$, first constructed by Cherkis and Kapustin (1999) ($k = 2$ is due to Hitchin, 1983); also Cherkis & Hitchin (2005).

The next example of the construction I just described (pluricomplex $\implies$ quaternionic) produces self-dual deformations of D_k ALF instantons.

Essentially, we consider singular hyperbolic monopoles of charge 2. The location of singularities is given by k points in H^3, which corresponds to k sections q_i of $O(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\psi = \prod q_i$ - a section of $O(k,k)$, and let L^m be the line bundle $O(m,-m)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

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Set

$$Z_{m,k} = \left\{(u,v) \in L^m(k,0) \oplus L^{-m}(0,k);\ uv = \psi\right\}.$$

After resolving singularities, we obtain a twistor space of a strongly integrable pluricomplex structure in dimension 8.

If ψ is invariant w.r.t. the involution τ, we can apply the above construction and obtain a family of 4-dimensional conformal self-dual metrics.

These converge to the ALF gravitational instantons of type D_k as $m \longrightarrow \infty$.

Perhaps these are not Einstein, after all, by analogy with what happens for singular hyperbolic monopoles of charge 1.

As observed by Nash (2008, also Atiyah & LeBrun, 2012), singular monopoles of charge 1 produce a 1-dimensional family of self-dual deformations of ALF gravitational instantons of type A_k (multi-Taub-NUT). These are LeBrun's self-dual metrics on $\mathbb{P}^2 \# \mathbb{P}^2 \# \ldots \# \mathbb{P}^2$, and so not Einstein.

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