H-projective structures and their applications

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Based largely on:

- hamiltonian 2-forms papers with Vestislav Apostolov (UQAM), Paul Gauduchon (Ecole Polytechnique) and Christina Tønnesen-Friedman (Union College)
- joint work with Aleksandra Borowka (Bath);
- discussions with the above plus Stefan Roseman and Vladimir Matveev (Jena);
- "Hamiltonian 2-vectors in H-projective geometry", informal notes, August 2011;

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Projective geometry and conformal geometry both play an important role in riemannian geometry.

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Irony: the "H" originally stood for "holomorphic"!

The literature (name dropping)

Projective geometry: classical (Lie, Cartan,...)

H-projective geometry: large Japanese and former soviet schools (Otsuki, Tashiro, Ishihara, Tachibana, Yashimatsu, Mikes, Domashev,...).

Projective and H-projective metrics: recent works by R. Bryant, M. Dunajski, V. Matveev, S. Rosemann,...

Quaternionic geometries: S. Salamon, A. Swann,...

Parabolic geometries: A. Cap, J. Slovak, V. Soucek, T. Diemer, M. Eastwood, R. Gover, M. Hammerl,...

H-projective case: S. Armstrong, J. Hrdina,...

Projective structures

Let D be a torsion-free connection on an n-manifold M (e.g., $D = \nabla^g$ for a riemannian metric g on M).

- A curve c in M is a *geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T\}$.

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- A curve c in M is a *geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T\}$.
- Torsion-free connections D and $\tilde{D}$ have the same geodesics iff $\exists\, \gamma \in \Omega^1(M)$, a 1-form, with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^r \in C^\infty(M, \mathfrak{gl}(TM)),$$

where $$[\![X, \gamma]\!]^r(Y) := \gamma(X)Y + \gamma(Y)X.$$

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where $$[\![X, \gamma]\!]^r(Y) := \gamma(X)Y + \gamma(Y)X.$$

  Then D and $\tilde{D}$ are said to be *projectively equivalent*. We write $\tilde{D} = D + \gamma$ for short (instead of $\tilde{D} = D + [\![\cdot, \gamma]\!]^r$).
- A *projective structure* on M^n $(n > 1)$ is a projective class $\Pi^r = [D]$ of torsion-free connections.

H-projective structures

Let (M, J) be a complex manifold of real dimension $n = 2m$ and let D be a torsion-free connection on M (a smooth n-manifold) with $DJ = 0$ (e.g., $D = \nabla^g$ for a Kähler metric g on M).

- A curve c is an *H-planar geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T, JT\}$.

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- A curve c is an *H-planar geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T, JT\}$.
- Torsion-free complex connections D and $\tilde{D}$ have the same H-planar geodesics iff $\exists\, \gamma \in \Omega^1(M)$, a (real) 1-form, with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^c \in C^\infty(M, \mathfrak{gl}(TM, J)),$$
$$[\![X, \gamma]\!]^c(Y) := \tfrac{1}{2}\big(\gamma(X)Y + \gamma(Y)X - \gamma(JX)JY - \gamma(JY)JX\big).$$

  Then D and $\tilde{D}$ are said to be *H-projectively equivalent*. We write $\tilde{D} = D + \gamma$ for short.
- An *H-projective structure* on M^{2m} $(m > 1)$ is an H-projective class $\Pi^c = [D]$ of torsion-free complex connections.

Quaternionic structures

Let (M, Q) be a quaternionic manifold of real dimension $n = 4\ell$ (thus $Q \subset \mathfrak{gl}(TM)$, with fibres isomorphic to $\mathfrak{sp}(1)$, spanned by imaginary quaternions J_1, J_2, J_3) and let D be a torsion-free connection on M preserving Q (e.g., $D = \nabla^g$ for a quaternion Kähler metric g on M).

- A curve c is a *Q-planar geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T, JT : J \in Q\}$.

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- A curve c is a *Q-planar geodesic* wrt. D iff for all T tangent to c, $D_T T \in \mathrm{span}\{T, JT : J \in Q\}$.
- **Fact**. Any two torsion-free quaternionic connections D and $\tilde{D}$ have the same Q-planar geodesics: $\exists\, \gamma \in \Omega^1(M)$ with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^q \in C^\infty(M, \mathfrak{gl}(TM, Q)),$$

$$[\![X, \gamma]\!]^q(Y) := \tfrac{1}{2}\Big(\gamma(X)Y + \gamma(Y)X - \sum_i \big(\gamma(J_iX)J_iY + \gamma(J_iY)J_iX\big)\Big).$$

- The class of torsion-free quaternionic connections may be denoted analogously by $\Pi^q = [D]$.

Common framework: parabolic geometries

Projective, H-projective and quaternionic classes Π of torsion-free connections are affine spaces modelled on 1-forms. Torsion-free conformal connections ("Weyl connections") on a conformal manifold (M^n, c) also form such an affine space.

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These are all *parabolic geometries* with *abelian nilradical* which have a well developed invariant theory.

Key feature: an algebraic bracket

$$[\![,]\!] \colon TM \times T^*M \to \mathfrak{g}_0(M) \subseteq \mathfrak{gl}(TM)$$

such that $D \mapsto D + \gamma \in \Pi$ is given by $D + \gamma := D + [\![\cdot, \gamma]\!]$.

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Why "parabolic", and what is really going on?

The Cartan connection

Parabolic geometries are "Cartan geometries" modelled on a "generalized flag variety" G/P, where G is a semisimple Lie group and P a *parabolic subgroup* of G, i.e., its Lie algebra $\mathfrak{p}$ is parabolic: $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{p}^\perp$ with $\mathfrak{g}_0$ reductive and $\mathfrak{p}^\perp$ nilpotent.

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- Projective case: $G = PGL(n+1, \mathbb{R})$ (with complexification $PGL(n+1, \mathbb{C})$) acting on $\mathbb{R}P^n$.
- H-projective case: $G = PGL(m+1, \mathbb{C})$ (real, with complexification $PGL(m+1, \mathbb{C}) \times PGL(m+1, \mathbb{C})$) acting on $\mathbb{C}P^m$.
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Such a Cartan geometry on M, where $\dim M = \dim \mathfrak{g}/\mathfrak{p}$, is a principal G-bundle with a principal G-connection and a reduction to P satisfying the *Cartan condition*: the induced 1-form on M with values in the bundle associated to $\mathfrak{g}/\mathfrak{p}$ is an isomorphism on each fibre. Thus M inherits the first order geometry of G/P, and in particular, a bundle of parabolic subalgebras $\mathfrak{g}_0(M) \ltimes T^*M$.

Computing with projective connections

A function F on Π is an *invariant* if it is constant, i.e., $\forall\, D \in \Pi$, $\gamma \in \Omega^1(M)$, $\partial_\gamma F(D) := \frac{d}{dt} F(D + t\gamma)\big|_{t=0}$ is zero.

For a section s of a vector bundle E associated to the frame bundle, $\partial_\gamma D_X s = [\![X, \gamma]\!] \cdot s$ (the natural action of $\mathfrak{g}_0(M)$ on E).

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Variation of the second derivative:

$$\partial_\gamma D^2_{X,Y} s = [\![X, \gamma]\!] \cdot D_Y s + [\![Y, \gamma]\!] \cdot D_X s - D_{[\![X,\gamma]\!]\cdot Y} s + [\![Y, D_X \gamma]\!] \cdot s.$$

Hence the curvature $R^D \in \Omega^2(M, \mathfrak{g}_0(TM))$ of D, given by $D^2_{X,Y} s - D^2_{Y,X} s = R^D_{X,Y} \cdot s$, satisfies

$$\partial_\gamma R^D_{X,Y} = -[\![Id \wedge D\gamma]\!]_{X,Y} := -[\![X, D_Y \gamma]\!] + [\![Y, D_X \gamma]\!].$$

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Can write: $R^D = W + [\![Id \wedge r^D]\!]$, where W is invariant ($\partial_\gamma W = 0$), and the *normalized Ricci tensor* $r^D \in \Omega^1(M, T^*M)$ satisfies $\partial_\gamma r^D = -D\gamma$.

Projective and H-projective hessians

Consequence:

$$\partial_\gamma(D^2_{X,Y}s + [\![Y, r^D_X]\!] \cdot s) = [\![X,\gamma]\!] \cdot D_Y s + [\![Y,\gamma]\!] \cdot D_X s - D_{[\![X,\gamma]\!] \cdot Y} s$$

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On densities of weight k (sections of a certain line bundle $\mathcal{O}(k)$) this simplifies to

$$\partial_\gamma(D^2_{X,Y}s + kr^D_X(Y)s) = k\gamma(X)D_Y s + k\gamma(Y)D_X s - D_{[\![X,\gamma]\!]\cdot Y} s.$$

In (real or holomorphic) projective geometry this gives a natural hessian operator on sections of $\mathcal{O}(1)$, whose solutions yield affine coordinates. In the H-projective case, the corresponding equation describes functions with J-invariant natural hessian: in Kähler geometry, these are hamiltonians for Killing vector fields!

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A *Hessian operator* of *Hill's equation* can be used to define projective structures on 1-manifolds, and similarly H-projective structures on Riemann surfaces, also known as *Möbius structures*.

Compatible metrics

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The same is true in the H-projective case, where one can work with the corresponding J-invariant 2-vector $\phi = h(J\cdot, \cdot) \in \wedge^{1,1} TM \otimes \mathcal{O}(-1)$, which satisfies...

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If $D = \nabla^g$ for a Kähler metric g, this means that the 2-form dual to ϕ with respect to g is a *hamiltonian 2-form*!

H-projective metrics and hamiltonian 2-forms

The *mobility* of an H-projective structure is the dimension of the space of solutions of the linear equation for compatible Kähler metrics.

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- The theory of hamiltonian 2-forms provides local and global classification results for mobility ≥ 2, i.e., of H-projectively equivalent Kähler metrics which are not affinely equivalent.
- Within this classification, the mobility ≥ 3 case can be identified; such metrics are rare, and in the compact case, have constant holomorphic sectional curvature.

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The complicated geometry of these metrics can be illuminated via cone constructions, which represent Cartan connections as affine connections on a (generalized) cone manifold, but there is still much to be understood.

H-projective structures and Cartan holonomy

$\mathbb{R}P^{2m+1}$ is a circle bundle over $\mathbb{C}P^m$ (the Hopf fibration), given by a choice of complex structure on the fundamental representation $\mathbb{R}^{2m+2}$ of $GL(2m+2,\mathbb{R})$ (yielding the fundamental representation $\mathbb{C}^{m+1}$ of $GL(m+1,\mathbb{C})$).

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In general, any H-projective manifold M^{2m} has a circle bundle N^{2m+1} with a projective structure on it, and the projective Cartan connection preserves a complex structure in its fundamental representation.

Conversely, a projective structure on a $(2m+1)$-manifold whose Cartan connection has such a holonomy reduction is locally a circle bundle over an H-projective manifold.

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Conversely, a projective structure on a $(2m+1)$-manifold whose Cartan connection has such a holonomy reduction is locally a circle bundle over an H-projective manifold.

There are results about the interplay of Cartan holonomy with other structures (compatible metrics, quaternionic structures), but much remains unexplored.

Totally complex submanifolds of quaternionic manifolds

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Observation. Let (N^{4m}, Q) be a quaternionic manifold and M^{2m} a *maximal totally complex submanifold*, i.e., each tangent space of M is invariant under some $J \in Q$, but for any $I \in Q$ anticommuting with J, $I(TM)$ is complementary to TM.

Then (M, J) inherits an H-projective structure from (N, Q).

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Indeed, we just project the quaternionic connections onto TM (along the complement, which is independent of I), observing that for $X, Y \in TM$, the projection onto TM of $[\![X, \gamma]\!]^q(Y)$ is $[\![X, i^*\gamma]\!]^c(Y)$, where $i\colon M \to N$ is the inclusion.

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This prompts a further question: when does an H-projective structure arise this way?

If it does then the quaternionic manifold N is locally a neighbourhood of the zero section in $TM \otimes \mathcal{L}$ for a unitary line bundle $\mathcal{L}$ (why?).

A generalized Feix–Kaledin construction

In the early 2000's, B. Feix and D. Kaledin gave independent constructions of hyperkähler metrics on cotangent bundles of real analytic Kähler manifolds. The metrics were defined on a neighbourhood of the zero section. They placed these constructions within a more general context: hypercomplex structures on the tangent bundle of a complex manifold equipped with a real analytic torsion-free hermitian connection whose curvature has type (1,1).

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Theorem. Let (M^{2m}, J, Π^c) be a real analytic H-projective manifold whose H-projective Weyl curvature W has type (1,1). Then there is a natural quaternionic structure Q on a neighbourhood N^{4m} of the zero section in $TM \otimes \mathcal{L}$ for a certain unitary line bundle $\mathcal{L}$.

Construction via the twistor space

Idea for proof (following Feix). We construct the twistor space of (N, Q), which is a complex $2m+1$ manifold with real structure, containing real "twistor lines" (rational curves with normal bundle $\mathcal{O}(1) \otimes C^{2m}$): N is the space of such twistor lines.

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Flat model. When $M = \mathbb{C}P^m$, its complexification is $\mathbb{C}P^m \times \mathbb{C}P^m$ and the total space of $P(\mathcal{O} \oplus \mathcal{O}(1, -1))$ is birational to $\mathbb{C}P^{2m+1}$ by a partial blow down of the zero and infinity sections (inversely, write $\mathbb{C}^{2m+2} = \mathbb{C}^{m+1} \oplus \mathbb{C}^{m+1}$ and blow up two projective m-spaces in $\mathbb{C}P^{2m+1}$). This is the twistor space of $\mathbb{H}P^m$, and the fibres of $P(\mathcal{O} \oplus \mathcal{O}(1, -1))$ project to twistor lines.

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We make the same construction over the complexification M^c of M (a neighbourhood of the diagonal in $M \times \overline{M}$).

The blow-down

M^c has two complementary foliations integrating the $(1,0)$ and $(0,1)$ distributions (which restrict to $T^{1,0}M$ and $T^{0,1}M$ in $TM \otimes \mathbb{C}$ along M).

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The analogue of $P(\mathcal{O} \oplus \mathcal{O}(1,-1))$ is obtained by gluing the line bundles $\mathcal{O}(1) \otimes \overline{\mathcal{O}(-1)}$ and $\mathcal{O}(-1) \otimes \overline{\mathcal{O}(1)}$ by inversion on the complement of their zero sections. We then need to blow-down the zero sections along corresponding foliations.

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This is where the type (1,1) curvature condition on M enters: it implies that the two foliations of M^c have projectively flat leaves. Hence the hessian equation for affine sections of $\mathcal{O}(1)$ is completely integrable and we can integrate it leafwise to obtain rank $m+1$ vector bundles over the leaf spaces.