Riemannian Geometries with Parallel Torsion

Geometric structures on manifolds and their applications
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Outline of talk

- Problem
- Definitions
- Main results
- Special cases and examples
- To do

Problem:

Characterize the geometry of a Riemannian manifold admitting a metric connection with parallel torsion

All statements are local

Definitions

- (M^n, g) Riemannian manifold
- τ one-form with values in $\mathfrak{so}(n)$: torsion
- $\nabla^\tau := \nabla^g - \tau$: torsion connection
- $\nabla^\tau \tau = 0$ parallel torsion

(M, g, τ) a torsion geometry:

- Skew torsion geometry:
  Same as above but $\tau_X Y = -\tau_Y X$

Associated structures

- $G = \text{Stab}(\tau) \subset O(n)$
- Bundle of frames adapted to τ

$$\begin{array}{c} Q \\ \downarrow {\scriptstyle G} \\ M \end{array}$$

- Holonomy of $\nabla^\tau \subset G$

Decomposition Lemma

The torsion τ is a section of $T^*M \otimes \mathfrak{so}(TM)$.

Suppose $TM = V_1 \oplus V_2$ w.r.t G such that

$$\tau \in (V_1^* \otimes \mathfrak{so}(V_1)) \oplus (V_2^* \otimes \mathfrak{so}(V_2))$$

Then M is the Riemannian product of two torsion geometries.

Cf. Nagy 2007 (nearly Kähler case) Kowalski & Vanhecke 1983 (naturally reductive spaces)

Structure of skew-torsion geometries

Here on: skew torsion only

Suppose (M^n, g, τ) a torsion geometry

Then

$$K/G \longrightarrow M$$

Locally homogeneous space

$$M \xrightarrow{\ \pi\ } B$$

π totally geodesic Riemannian submersion

Sketch of proof, I

The 'totally geodesic part'

- $TM = V \oplus H$, V maximal G-bundle s.t. $\mathfrak{g} \cap \mathfrak{so}(V)$ is trivial.
- Key observation
  $$\tau = \tau_V + \tau_{mix} + \tau_H, \tau_{mix} \in V^* \wedge \mathfrak{so}(H)$$
- V involutive
- Calculate O'Neill tensors

Sketch of proof, II

The ‘locally homogeneous’ fibre part:

- Restriction of R^τ to V is parallel w.r.t. ∇^τ ,
  using Berger algebra of $\mathfrak{g}$
- Infinitesimal model $\mathfrak{k} = \mathfrak{g} \bigoplus V$
- Fiber is isometric to quotient K/G
  Warning: Locally homogeneous, non-global quotient,
  cf.: papers by Kowalski, Tricerri, Vanhecke 1990’s

Notes

Decomposition lemma not used in proof of structural theorem

Obs: (M, g, τ) irreducible & non-trivial V
$\Longrightarrow \tau_{mix}$ is 'non-degenerate' i.e. $\mathrm{V} \hookrightarrow^{\tau} \Lambda^2(H)$

Special cases & Examples

- V trivial and (M, g, τ) irreducible
  $\Rightarrow H$ irreducible w.r.t $\mathfrak{g}$

  Classification (Swann,-,2004)
  - 7-dim weak holonomy
  - 6-dim nearly Kaehler
  - Isotropy irreducible homogeneous spaces

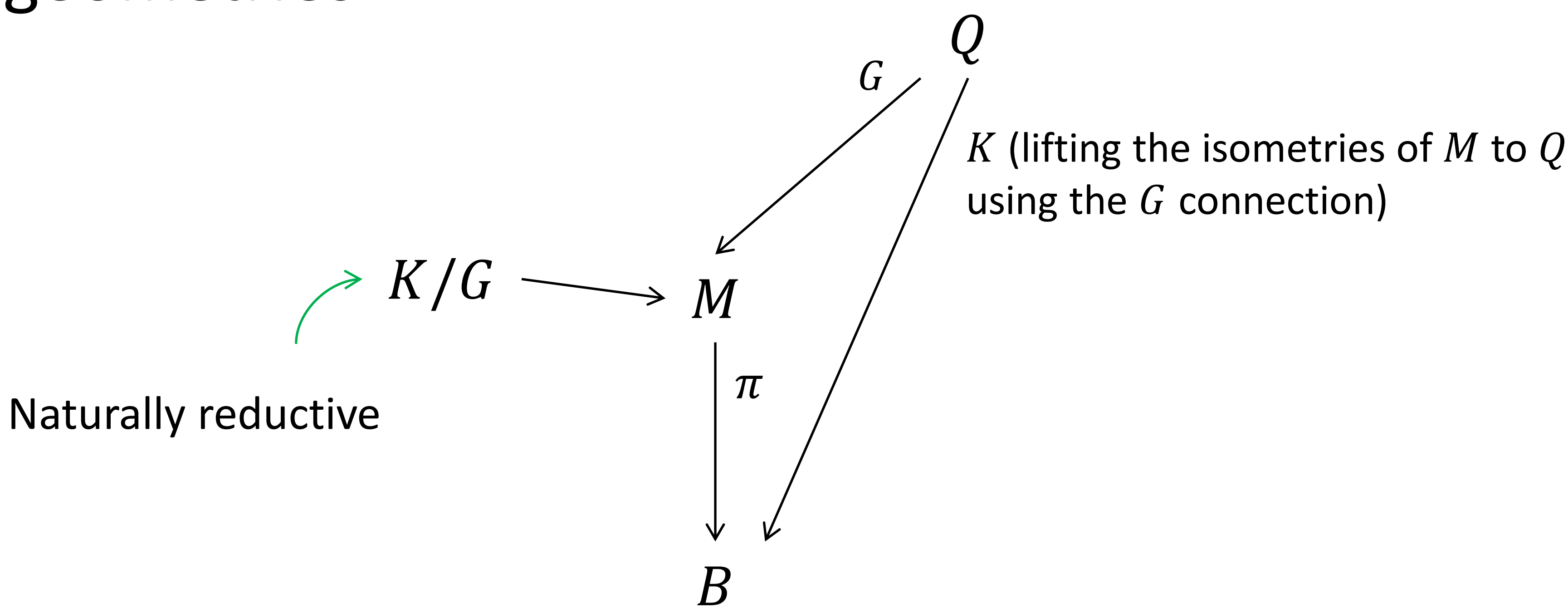
- M homogeneous:
  Results by Nagy 2008, Olmos & Regianni 2008 on full isometry group of nat. reductive spaces

Special cases & Examples

- Sasakian manifolds: $\tau = \xi \wedge \phi$, B Kähler
- 3-Sasakian manifolds: $\tau = \xi_1 \wedge \phi_1 + \xi_2 \wedge \phi_2$ $+\xi_3 \wedge \phi_3 - c\, \xi_1 \wedge \xi_2 \wedge \xi_3$, B quaternionic Kähler
- Classifications in low-dimensions [Friedrich, Puhle, Schoeman,…]
- Clifford structures(?) [Moroianu & Semmelmann, 2011]

Structure of skew-torsion geom's, II

Speculative: global structure of skew torsion geometries



More problems

- New problem:
  What conditions on (B, K, G, Q) give
  a torsion geometry on $M = Q/G$?

- Classification of irreducible skew torsion geometries?
  (with non-trivial Berger algebra)

Geometric structures on manifolds and their applications

- Einstein metrics ✓
- Sasakian geometry ✓
- Almost Hermitian geometry ✓
- G_2 structures ✓
- Holonomy theory ✓
- Dirac operators ÷
- Spin geometry (✓)
- generalized Kähler geometry - calibrated geometry - applications to super strings & mathematical physics - etc. ÷