Extremal Sasakian metrics on S^3 bundles over compact Riemann Surfaces

Castle Rauischholzhausen
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Joint work in progress with

- Charles Boyer

Also relies heavily on work

- with and by The ACG Team; Vestislav Apostolov, David Calderbank and Paul Gauduchon

- by Charles Boyer et all.

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Geometrically Ruled Surfaces:

$$(M, J) = P(E) \to \Sigma_g$$

- $E \to \Sigma_g$: holomorphic rank 2 vector bundle.

- Σ_g compact connected Riemann surface of genus g with a fixed complex structure

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By Narasimhan-Seshadri this is equivalent to E being projectively flat Hermitian.

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CASE 3: $E \to \Sigma_g$ is indecomposable and not (poly)stable.

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In this case we call g, corresponding to ω, an **extremal Kähler metric**. A Kähler metric with constant scalar curvature (CSC) is in particular extremal.

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be the symplectic 2-form on $\Sigma_g \times S^2$, where ω_g and ω_0 are the standard area measures on Σ_g and S^2, respectively. Let $\alpha_{k_1,k_2} \in H^2(M, \mathbb{R})$ denote the cohomology class of ω_{k_1,k_2}.

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Lemma: For any $(M, J) = P(\mathcal{O} \oplus \mathcal{L}_{2m}) \to \Sigma_g$, $deg(\mathcal{L}_{2m}) = 2m$ (Case 2) α_{k_1,k_2} is a Kähler class if and only if $k_2 > 0$ and $\frac{k_1}{k_2} > m$.

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(This is essentially due to Fujiki)

So with each choice of α_{k_1,k_2} on $M = \Sigma_g \times S^2$ comes from Case 2 a family of complex structures J_{2m}, $m = 1, ..., \lceil \frac{k_1}{k_2} \rceil - 1$. ($J_{2m}$ might not be unique, even up to biholomorphism, unless $g = 0, 1$.)

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Moreover each J_{2m} defines a natural Hamiltonian S^1 action on (M, ω_{k_1,k_2}), generated by K_{2m}.

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Ceiling function: $\lceil x \rceil$ = smallest integer $\geq x$.

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and ξ is a Killing vector field of g which generates a one dimensional foliation $\mathcal{F}_\xi$ of M whose transverse structure is Kähler.

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If ξ is **regular**, the transverse Kähler structure lives on a smooth manifold.

If ξ is **quasi-regular**, the transverse Kähler structure has orbifold singularities.

Assume now that $k_2 = 1$, $k_1 = l \in \mathbb{Z}^+$ and J is some complex structure on $\Sigma_g \times S^2$ which makes $\alpha_{l,1}$ a Kähler class (and is compatible with $\omega_{l,1}$). Then by the Boothby-Wang construction the total space X of the principal circle bundle over $\Sigma_g \times S^2$ corresponding to the cohomology class $\alpha_{l,1} \in H^2(\Sigma_g \times S^2, \mathbb{Z})$ has a natural Sasakian structure

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This is seen by viewing $(X, \xi, \eta, \Phi_J, g)$ as arising from a Wang–Ziller join construction and using topological arguments and a recent result by Kreck and Lück.

When the Kähler class $\alpha_{l,1}$ on (M, J) admits extremal Kähler metrics, the Sasaki Structure (ξ, η, Φ_J, g) is extremal up to a suitable deformation of the Sasakian structure ($\eta \mapsto \eta + t\chi$, χ a basic 1-form of the foliation defined by ξ, new contact structure isotopic to old). It is convention still to call (ξ, η, Φ_J, g) extremal.

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For J from Case 1 (and there are many of those!), there is a constant scalar curvature (CSC) Kähler metric in each Kähler class of the Kähler cone on $\Sigma_g \times \mathbb{CP}^1$. It is simply a local product of the constant curvature Kähler metrics on Σ_g and $\mathbb{C}P^1$ respectively.

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For J_{2m}, $m = 1, ..., k-1$ from Case 2, IF there is an extremal Kähler metric (non-CSC) in the Kähler class $\alpha_{l,1}$ THEN it must arise from a Calabi type construction (joint work with ACG).

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Writing $F(\mathfrak{z}) = \Theta(\mathfrak{z})(1 + r\mathfrak{z})$, the corresponding metric is extremal exactly when $F(\mathfrak{z})$ is a polynomial of degree at most 4 and $F''(-1/r) = 2r(\frac{1-g}{m})$. This, as well as the endpoint conditions of (1), is satisfied precisely when $F(\mathfrak{z})$ is given by

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CSC solutions would correspond to $h(\mathfrak{z})$ being a linear function and this is clearly never possible.

Conversely, $\Theta(\mathfrak{z}) = F(\mathfrak{z})/(1 + r\mathfrak{z})$ with $F(\mathfrak{z})$ defined by (2)satisfies conditions (ii) and (iii) in (1) and thus we have an extremal Kähler metric precisely when $\Theta(\mathfrak{z})$ also satisfies (i).

It is now a calculus exercise to check that

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1. For any choice of genus $g = 1, 2, ..., 19$, any choice of $l = 2, 3, ...$, and any choice of complex structures J_{2m} with $m = 1, ..., l-1$ $\Theta(\mathfrak{z}) = F(\mathfrak{z})/(1 + r\mathfrak{z})$ with $F(\mathfrak{z})$ defined by (2) satisfies (i).

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2. For any choice of genus $g = 20, 21, ...$ there exists a $l_g \in \{2, 3, 4...\}$ such that for any choice of $l = l_g, l_g + 1, ...$, and any choice of complex structure J_{2m} with $m = 1, ..., l-1$ $\Theta(\mathfrak{z}) = F(\mathfrak{z})/(1 + r\mathfrak{z})$ with $F(\mathfrak{z})$ defined by (2)satisfies (i).

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3. For any choice of genus $g = 20, 21, ...$ there exist at least one pair (l, m) with $1 \leq m \leq l-1$ such that $\Theta(\mathfrak{z}) = F(\mathfrak{z})/(1 + r\mathfrak{z})$ with $F(\mathfrak{z})$ defined by (2) does not satisfy (i).

Corollary

For any genus $g \geq 1$, $\Sigma_g \times S^3$ has regular Sasakian Structures with CSC as well as regular extremal non-CSC Sasakian Structures.

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What about the non-trivial S^3-bundle over Σ_g?

Let $\omega_{(p,q)}$ be the standard area measure on $\mathbb{CP}_{(p,q)}$.

By the Boothby-Wang construction the total space X of the S^1 orbibundle over $\Sigma_g \times \mathbb{CP}_{(p,q)}$ corresponding to the cohomology class $\alpha_{l,1} = l\omega_g + \omega_{(p,q)}$ (is smooth and) has natural Sasakian structures

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Let $\omega_{(p,q)}$ be the standard area measure on $\mathbb{CP}_{(p,q)}$.

By the Boothby-Wang construction the total space X of the S^1 orbibundle over $\Sigma_g \times \mathbb{CP}_{(p,q)}$ corresponding to the cohomology class $\alpha_{l,1} = l\omega_g + \omega_{(p,q)}$ (is smooth and) has natural Sasakian structures

$$(\xi, \eta, \Phi_J, g)$$

whose contact form η satisfies $d\eta = \pi^* \omega_{l,1}$ where π is the natural bundle projection.

Diffeomorphically the 5 dimensional manifold is $\Sigma_g \times S^3$ if $l(p+q)$ is even and the non-trivial S^3-bundle over Σ_g if $l(p+q)$ is odd.

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Remark: The Kähler class $[\omega]$ as well as n should be completely determined by (g, l, p, q) (work in progress)

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"For any genus $g \geq 1$, the non-trivial S^3-bundle over Σ_g has a regular extremal non-CSC Sasakian Structure."

What about the quasi-regular CSC Sasakian Structures?

Changing the rules in the Calabi Construction

If we allow for orbifold singularities along the zero and infinity sections of $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma_g$ we may tinker a bit with (1):

$$\begin{array}{l} (i)\ \Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \\ (ii)\ \Theta(\pm 1) = 0, \\ (iii)\ \Theta'(-1) = 2/p \quad \text{and} \quad \Theta'(1) = -2/q, \end{array} \tag{3}$$

where p and q are positive co-prime integers. Then the extremal solution $\Theta(\mathfrak{z})$ is

$$\frac{(1-\mathfrak{z}^2)h(\mathfrak{z})}{(1+r\mathfrak{z})(4pq(3-r^2))}$$

where

$$\begin{aligned} h(\mathfrak{z}) &= q(6-3r-4r^2+r^3)+p(6+3r-4r^2-r^3) \\ &+ 2(3-r^2)(q(r-1)+p(1+r))\mathfrak{z} \\ &+ r(p(3+2r-r^2)-q(3-2r-r^2))\mathfrak{z}^2 \\ &+ 2pqr^3(\tfrac{1-g}{m})(1-\mathfrak{z}^2), \end{aligned}$$

Conversely, such $\Theta(\mathfrak{z})$ satisfies conditions (ii) and (iii) in (3) and thus we have an extremal Kähler metric precisely when $\Theta(\mathfrak{z})$ also satisfies (i). Further, a CSC metric arise whenever $\deg h \leq 1$.

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For example when

- $g = 1$, $m/l = 1/2$, $p = 7$, and $q = 15$

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- $g \geq 2$ and even, $m = g - 1$, $l = 3(g - 1)$, $p = 5$ and $q = 16$.

- $g \geq 3$ and odd, $m = gp(g - 2)$, $l = pg^2$, $p = 3g - 2$, and $q = 2g(g - 1)$.

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DANKESCHÖN!!!

THANKS FOR A GREAT WORKSHOP!!!