Flat manifolds with holonomy group $\mathbb{Z}_2^k$ of diagonal type

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1. Summary

We consider relations between two families of flat manifolds with holonomy group $(\mathbb{Z}_2)^k$ of diagonal type. The family $\mathcal{RBM}$ of real Bott manifolds and the family $\mathcal{GHW}$ of generalized Hantzsche-Wendt manifolds. In particular, we prove that the intersection $\mathcal{GHW} \cap \mathcal{RBM}$ is not empty. We also consider some class of real Bott manifolds without Spin and $\mathrm{Spin}^{\mathbb{C}}$ structure. There are given conditions for the (non)existence of such structures.

2. Introduction

Let M^n be a flat manifold of dimension n. By definition, this is a compact connected, Riemannian manifold without boundary with sectional curvature equal to zero. From the theorems of Bieberbach ([2]) the fundamental group $\pi_1(M^n) = \Gamma$ determines a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma \overset{p}{\to} G \to 0, \tag{1}$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and G is a finite group which is isomorphic to the holonomy group of M^n. The universal covering of M^n is the Euclidean space $\mathbb{R}^n$ and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group $Isom(\mathbb{R}^n) = O(n) \times \mathbb{R}^n = E(n)$. Conversely, given a short exact sequence of the form (1), it is known that the group Γ is (isomorphic to) the fundamental group of a flat manifold if and only if Γ is torsion free. In this case Γ is called a Bieberbach group. We can define a holonomy representation $\phi : G \to GL(n, \mathbb{Z})$ by the formula:

$$\forall g \in G, \phi(g)(e_i) = \tilde{g} e_i (\tilde{g})^{-1}, \tag{2}$$

where $e_i \in \Gamma$ are generators of $\mathbb{Z}^n$ for $i = 1, 2, ..., n$, and $\tilde{g} \in \Gamma$ such that $p(\tilde{g}) = g$. In this article we shall consider only the case

$$G = \mathbb{Z}_2^k, 1 \leq k \leq n-1, \text{ with } \phi(\mathbb{Z}_2^k) \subset D \subset GL(n, \mathbb{Z}), \tag{3}$$

where D is the group of all diagonal matrices.

3. Generalized Hantzsche-Wendt manifolds

Definition 1 *A generalized Hantzsche-Wendt manifold ($\mathcal{GHW}$-manifold) is a flat manifold of dimension n with holonomy group $(\mathbb{Z}_2)^{n-1}$.*

Let $M^n \in \mathcal{GHW}$. In [16, Theorem 3.1] it is proved that the holonomy representation (2) of $\pi_1(M^n)$ satisfies (3).

The (co)homology groups and cohomology rings with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$, of $\mathcal{GHW}$ manifolds are still not known, see [4] and [5]. We finish this overview with an example of $\mathcal{GHW}$ manifolds which have been known already in 1974.

Example 1 *Let $M^n = \mathbb{R}^n/\Gamma_n, n \geq 2$ be manifolds defined in* [11] (*see also* [16, page 1059]), *where $\Gamma_n \subset E(n)$ is generated by $\gamma_0 = (I = id, (1, 0, ..., 0))$ and*

$$\gamma_i = \left(\begin{bmatrix} 1 & 0 & 0 & . & . & \ldots & 0 \\ 0 & 1 & 0 & . & . & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & . & \ldots & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 0 \\ \frac{1}{2} \\ \ldots \\ 0 \end{pmatrix} \right) \in E(n), \tag{4}$$

where the -1 is placed in the (i, i) entry and the $\frac{1}{2}$ as an $(i+1)$ entry, $i = 1, 2, ..., n-1$. Γ_2 is the fundamental group of the Klein bottle.

4. Real Bott manifolds

We follow [3], [10] and [14]. To define the second family let us introduce a sequence of $\mathbb{R}P^1$-bundles

$$M_n \overset{\mathbb{R}P^1}{\to} M_{n-1} \overset{\mathbb{R}P^1}{\to} \ldots \overset{\mathbb{R}P^1}{\to} M_1 \overset{\mathbb{R}P^1}{\to} M_0 = \{\text{a point}\} \tag{5}$$

such that $M_i \to M_{i-1}$ for $i = 1, 2, ..., n$ is the projective bundle of a Whitney sum of a real line bundle L_{i-1} and the trivial line bundle over M_{i-1}. We call the sequence (5) a *real Bott tower* of height n, [3].

Definition 2 ([10]) *The top manifold M_n of a real Bott tower* (5) *is called a real Bott manifold ($\mathcal{RBM}$).*

Let γ_i be the canonical line bundle over M_i and set $x_i = w_1(\gamma_j)$. Since $H^1(M_{i-1}, \mathbb{Z}_2)$ is additively generated by $x_1, x_2, .., x_{i-1}$ and L_{i-1} is a line bundle over M_{i-1}, one can uniquely write

$$w_1(L_{i-1}) = \Sigma_{k=1}^{i-1} a_{k,i} x_k \tag{6}$$

with $a_{k,i} \in \mathbb{Z}_2 = \{0, 1\}$ and $i = 2, 3, ..., n$. From above $A = [a_{ki}]$ is an upper triangular matrix, $a_{k,i} = 0$ unless $k < i$, of size n whose diagonal entries are 0 and other entries are either 0 or 1. Summing up, we can say that the tower (5) (1) is completely determined by the matrix A.

From [10, Lemma 3.1] we can consider any $\mathcal{RBM}$ $M(A)$ in the following way. Let $M(A) = \mathbb{R}^n/\Gamma(A)$, where $\Gamma(A) \subset E(n)$ is generated by elements

$$s_i = \left(\begin{bmatrix} 1 & 0 & 0 & . & . & \ldots & 0 \\ 0 & 1 & 0 & . & . & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & (-1)^{a_{i,i+1}} & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & (-1)^{a_{i,n}} \end{bmatrix}, \begin{pmatrix} 0 \\ . \\ 0 \\ \frac{1}{2} \\ 0 \\ . \\ 0 \\ 0 \end{pmatrix} \right) \in E(n), \tag{7}$$

where $(-1)^{a_{i,i+1}}$ is placed in $(i+1, i+1)$ entry and $\frac{1}{2}$ as an (i) entry, $i = 1, 2, ..., n-1$. $s_n = (I, (0, 0, ..., 0, 1)) \in E(n)$. From [10, Lemma 3.2,3.3] $s_1^2, s_2^2, ..., s_n^2$ commute with each other and generate a free abelian subgroup $\mathbb{Z}^n$. It is easy to see that it is not always a maximal abelian subgroup of $\Gamma(A)$. Moreover, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & \Gamma(A) & \longrightarrow & \mathbb{Z}_2^k & \longrightarrow & 0 \\ & & \big| i & & \big\| & & \big| p & & \\ 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma(A) & \longrightarrow & \mathbb{Z}_2^n & \longrightarrow & 0 \end{array}$$

where $k = rk_{\mathbb{Z}_2}(A)$, N is the maximal abelian subgroup of $\Gamma(A)$, and $p : \Gamma(A)/\mathbb{Z}^n \to \Gamma(A)/N$ is a surjection induced by the inclusion $i : \mathbb{Z}^n \to N$. From the first Bieberbach theorem, see [2], N is a subgroup of all translations of $\Gamma(A)$ i.e. $N = \Gamma(A) \cap \mathbb{R}^n = \Gamma(A) \cap \{(I, a) \in E(n) \mid a \in \mathbb{R}^n\}$.

Definition 3 ([3]) *A binary square matrix A is a Bott matrix if $A = PBP^{-1}$ for a permutation matrix P and a strictly upper triangular binary matrix B.*

Let $\mathcal{B}(n)$ be the set of Bott matrices of size n. Since two different upper triangular matrices A and B may produce (affinely) diffeomorphic ($\sim$) real Bott manifolds $M(A), M(B)$, see [3] and [10], there are three operations on $\mathcal{B}(n)$, denoted by (Op1), (Op2) and (Op3), such that $M(A) \sim M(B)$ if and only if the matrix A can be transformed into B through a sequence of the above operations, see [3, part 3]. The operation (Op1) is a conjugation by a permutation matrix,

(Op2) is a bijection $\Phi_k : \mathcal{B}(n) \to \mathcal{B}(n)$

$$\Phi_k(A)_{*,j} := A_{*,j} + a_{kj} A_{*,k}, \tag{8}$$

for $k, j \in \{1, 2, ..., n\}$ such that $\Phi_k \circ \Phi_k = 1_{\mathcal{B}(n)}$.

Finally (Op3) is, for distinct $l, m \in \{1, 2, .., n\}$ and the matrix A with $A_{*,l} = A_{*,m}$

$$\Phi^{l,m}(A)_{i,*} := \begin{cases} A_{l,*} + A_{m,*} & \text{if } i = m \\ A_{i,*} & \text{otherwise} \end{cases} \tag{9}$$

Here $A_{*,j}$ denotes j-th column and $A_{i,*}$ denotes i-th row of the matrix A.

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