Holomorphic Poisson structures in generalized complex geometry

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Outline

Holomorphic Poisson structures

–Definition and examples

–Bondal conjecture for Fano 4-folds

Generalized complex structure

–Definition and examples

–Holomorphic Poisson is generalized complex

–B-field action on holomorphic Poisson structures

–Local Classification

$$X \text{ complex manifold}, \ \sigma \in H^0(X, \wedge^2 T)$$

$$\{f, g\} = \sigma(df, dg)$$

$$\{-,-\} \text{ is Lie} \Leftrightarrow [\sigma, \sigma] = 0 \text{ in } H^0(X, \wedge^3 T)$$

Examples:

- $X = \mathfrak{g}^*$, $\sigma \in \wedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$ such that $(\mathfrak{g}, \sigma)$ is a Lie algebra.
  Symplectic leaves: *coadjoint orbits of G*
- $X = \mathbb{C}P^2$ with $\sigma \in H^0(\mathbb{C}P^2, \wedge^2 T)$.
  Symplectic leaves:
  - *dim 0:* points on the cubic curve $C = \sigma^{-1}(0)$,
  - *dim 2:* $X \setminus C$.

Holomorphic Poisson structures

3D Fano example

$X = \mathbb{C}P^3$ and $\sigma = W(f, g)$, where

$$f, g \in H^0(\mathbb{C}P^3, \mathcal{O}(2)),$$
$$W(f, g) = fdg - gdf \in H^0(\mathbb{C}P^3, \Omega^1(\mathcal{O}(4)))$$

Note that $\Omega^1(K^{-1}) = \wedge^2 T$, and

$$[\sigma, \sigma] = W \wedge dW = 0.$$

Symplectic leaves:

- *dim 0:* points on the base locus $C = f^{-1}(0) \cap g^{-1}(0)$ (elliptic normal curve of degree 4) and singular points S of quadrics in pencil $\lambda f + \mu g$, $[\lambda : \mu] \in \mathbb{C}P^1$.
- *dim 2:* $Q \setminus (C \cup S)$, for Q a quadric in pencil.

Curious property: Expected dimension of $\sigma^{-1}(0)$ is zero.

Holomorphic Poisson structures

Theorem (Polishchuk 1997)

On any Fano 3-fold, $\sigma^{-1}(0)$ contains a curve.

Conjecture (Bondal 1993)

For Fano manifolds, the degeneracy locus $D_{2k}(\sigma)$ contains a component of dimension $\geq 2k+1$.

Theorem (M.G. and Brent Pym `arXiv:1203.4293`)

The Bondal conjecture is correct for Fano 4-folds.

Main ingredient is a detailed investigation of the geometry of Poisson modules.

$$\begin{cases} C \text{ a smooth curve of genus 1} \\ \mathcal{L} \in \mathrm{Pic}^5(C) \end{cases}$$

Then $\mathbb{P}(\mathrm{Ext}^1(\mathcal{L}, \mathcal{O}))$ has a Poisson structure, giving the Feigin-Odesskii Poisson structure on $\mathbb{C}P^4$

Symplectic leaves:

- generically symplectic,
- *dim 0*: points on elliptic normal curve C of degree 5
- *dim 2*: $S \setminus C$, for S a surface in the secant variety made from secants with fixed sum in $\mathrm{Pic}^2(C)$.

$$\mathbb{J} : T \oplus T^* \longrightarrow T \oplus T^* \,, \qquad \mathbb{J}^2 = -1$$

compatible with $O(n,n)$ structure and Courant integrable.

$$\mathbb{J} = \begin{pmatrix} A & Q \\ \sigma & -A^* \end{pmatrix} \in \mathfrak{so}(T \oplus T^*) = \wedge^2 T \oplus (T \otimes T^*) \oplus \wedge^2 T^*.$$

- Q is a real Poisson structure, Hamiltonian vector fields

  $$X_f = \pi_T(\mathbb{J} df)$$

  generate singular foliation by (smooth) symplectic leaves.
- Transverse complex structure

  $$T/Q(T^*) = (T \oplus T^*)/(\mathbb{J} T^* + T^*) \cong \mathbb{C}^k \qquad \text{type k}$$

Extreme cases:

$$\mathbb{J}_J = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}, \qquad \mathbb{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix},$$

for J an integrable complex structure and ω a real symplectic form.

Deformations of $\mathbb{J}_J$ are given by Maurer-Cartan elements

$$\{\epsilon \in \Gamma^\infty(X, \wedge^2(T_{1,0} \oplus T^*_{0,1})) \quad : \quad \overline{\partial}\epsilon + \tfrac{1}{2}[\epsilon, \epsilon] = 0\}$$

Decompose

$$\begin{array}{ccccccc} \epsilon & = & \epsilon^{2,0} & + & \epsilon^{1,1} & + & \epsilon^{0,2} \\ & & \wedge^2 T_{1,0} & & \Omega^{0,1}(T_{1,0}) & & \Omega^{0,2} \end{array}$$

Maurer-Cartan equation:

$$\begin{cases} [\epsilon^{2,0}, \epsilon^{2,0}] = 0 \\ \overline{\partial}\epsilon^{2,0} + [\epsilon^{1,1}, \epsilon^{2,0}] = 0 \\ \overline{\partial}\epsilon^{1,1} + \frac{1}{2}[\epsilon^{1,1}, \epsilon^{1,1}] + [\epsilon^{2,0}, \epsilon^{0,2}] = 0 \\ \overline{\partial}\epsilon^{0,2} + [\epsilon^{1,1}, \epsilon^{0,2}] = 0 \end{cases}$$

- If $\epsilon^{0,2} = \epsilon^{1,1} = 0$, get $\epsilon^{2,0}$ holomorphic Poisson.
- If only $\epsilon^{0,2} = 0$, get
  - $\epsilon^{1,1}$ a deformation of complex structure.
  - $\epsilon^{2,0}$ is killed by $\overline{\partial} + [\epsilon^{1,1}, -]$, hence holomorphic Poisson in new complex structure.

Holomorphic Poisson is generalized complex

If $\epsilon^{2,0} = P + iQ$ is a holomorphic Poisson structure, then we obtain a deformation

$$\begin{pmatrix} J & \\ & -J^* \end{pmatrix} \Rightarrow \begin{pmatrix} J & Q \\ & -J^* \end{pmatrix}$$

In fact we have a whole family

$$\begin{pmatrix} J & tQ \\ & -J^* \end{pmatrix}$$

B-field gauge symmetry

In addition to Diff(M), can apply $B \in \Omega^2(M, \mathbb{R})$, $dB = 0$ via

$$e^B = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \qquad \begin{cases} \text{in } O(n,n) \\ \text{preserves } [-,-] \end{cases}$$

$\mathbb{J}$ generalized complex $\Rightarrow e^B \mathbb{J} e^{-B}$ generalized complex.

For $\mathbb{J} = \begin{pmatrix} J & Q \\ & -J^* \end{pmatrix}$, $e^B \mathbb{J} e^{-B} = \begin{pmatrix} J - QB & Q \\ BJ + J^*B - BQB & BQ - J^* \end{pmatrix}$

Note that the B-transform may not be holomorphic Poisson.

Nondegenerate hol. Poisson $\cong_B$ real symplectic

$$e^B \mathbb{J} e^{-B} = \begin{pmatrix} J - QB & Q \\ BJ + J^* B - BQB & BQ - J^* \end{pmatrix}$$

If Q is nondegenerate, may take $B = Q^{-1}J$, obtain

$$e^B \mathbb{J} e^{-B} = \begin{pmatrix} & Q \\ Q^{-1} & \end{pmatrix}$$

which is a symplectic structure.
E.g.: hol. Poisson structure $(\mathbb{C}P^2, \sigma)$, is B-equivalent to a symplectic structure outside the complex locus $\sigma^{-1}(0)$.

Surgery into symplectic manifolds:

Theorem (M.G. and G. Cavalcanti, arXiv:0806.0872)

There are gen. cx. structures on $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ iff almost complex.

Theorem (Rafael Torres, arXiv:1104.3480)

Many more examples, including $m(S^2 \times S^2)$, sum with $S^1 \times S^3$...

B-equivalent but non-isomorphic hol. Poisson structures

(g, I, J, K) hyperKähler $\Rightarrow$ pair of holomorphic Poisson structures:

$$(I, \omega_J^{-1} + i\omega_K^{-1})$$
$$(J, -\omega_I^{-1} + i\omega_K^{-1})$$

While these may be non-isomorphic as hol. Poisson manifolds, the B-field transform by $B = \omega_I + \omega_J$ gives

$$e^B \begin{pmatrix} I & \omega_K^{-1} \\ & -I^* \end{pmatrix} e^{-B} = \begin{pmatrix} J & \omega_K^{-1} \\ & -J^* \end{pmatrix}$$

Local Classification

Theorem (M.G. '04)

Near a regular point of Q,

$$\mathbb{J} \cong_B \mathbb{C}^k \times (\mathbb{R}^{2n-2k}, \omega_0).$$

Theorem (Abouzaid-Boyarchenko '06)

Near any point,

$$\mathbb{J} \cong_B (\mathbb{R}^{2k}, \mathbb{J}') \times (\mathbb{R}^{2n-2k}, \omega_0),$$

$\mathbb{J}'$ of complex type at 0.

Theorem (Michael Bailey arXiv:1201.4887)

Near any point,

$$\mathbb{J} \cong_B (\mathbb{C}^k, \sigma) \times (\mathbb{R}^{2n-2k}, \omega_0),$$

where σ is a holomorphic Poisson structure.

Proof

Step 1: interpolation

For any $\mathbb{J}$ on a neighbourhood of 0 in $\mathbb{C}^n$, complex type at 0, find a smooth family $\mathbb{J}_t$ such that

$$\mathbb{J}_1 = \mathbb{J} \text{ and } \mathbb{J}_0 = \mathbb{C}^n.$$

Analogy: X vector field on vector space. Try pulling it back by rescaling $\rho_t : v \mapsto tv$:

$$(\rho_t)_*^{-1} X = \frac{1}{t} X(0) + X_{lin} \mod t$$

If $X(0) = 0$, then $X_t = (\rho_t)_*^{-1} X$ extends smoothly to $t = 0$, giving

$$X_0 = X_{lin},$$

where $X_{lin} = i_E(dX|_0)$.

Proof

Step 1: scaling problem

Same idea fails for $\mathbb{J} = \begin{pmatrix} A & Q \\ \sigma & -A^* \end{pmatrix}$ because the pullback

$$\Phi_t^* = \begin{pmatrix} (\rho_t)_*^{-1} & \\ & \rho_t^* \end{pmatrix}$$

applied to $\mathbb{J}$ blows up as $t \to 0$:

	$\wedge^2 T$	$\oplus$	$T \otimes T^*$	$\oplus$	$\wedge^2 T^*$
Φ_t^* scaling:	$t^{-2} Q$		$t^0 A$		$t^2 \sigma$

Proof

Step 1: scaling remedy

Use additional symmetry of $T \oplus T^*$: for $t \neq 0$,

$$\lambda_t = \begin{pmatrix} 1 & \\ & t \end{pmatrix}$$

is a symmetry of the Courant bracket, though not orthogonal. Scaling action is

$$\begin{array}{lccccc} & \wedge^2 T & \oplus & T \otimes T^* & \oplus & \wedge^2 T^* \\ \lambda_t \text{ scaling:} & t^{-1} Q & & t^0 A & & t^1 \sigma \end{array}$$

Compatibility with B-field action:

$$\lambda_t e^B \lambda_t^{-1} = e^{tB}.$$

Proof

Step 1: interpolation

Apply both pullback and scaling

$$\mathbb{J}_t = \lambda_{t^{-2}} \Phi_t^* \mathbb{J}$$

	$\wedge^2 T$	$\oplus$	$T \otimes T^*$	$\oplus$	$\wedge^2 T^*$
$\lambda_{t^2}\Phi_t^*$ scaling:	$t^0 Q$		$t^0 A$		$t^0 \sigma$

First order parts left alone, higher order components killed.

$\mathbb{J}_t$ is **smooth in** t, **integrable** $\forall t$, and

$$\mathbb{J}_0 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$$

Proof

Step 2: Implicit function theorem

View $\mathbb{J}_t$ as deformation ϵ_t of $\mathbb{J}_0 = \mathbb{C}^n$.

For sufficiently small $t > 0$ find $B = dA$ on a small ball such that

$$(e^B \epsilon_t)^{0,2} = 0.$$

$$\begin{cases} (e^B\epsilon)^{0,2} = \epsilon^{0,2} + B^{0,2} + B^{1,1}\epsilon^{1,1} - \epsilon^{1,1}B^{1,1} - B^{1,1}\epsilon^{2,0}B^{1,1} - \epsilon^{1,1}B^{2,0}\epsilon^{1,1} + \cdots \\ \overline{\partial}\epsilon^{0,2} + [\epsilon^{1,1}, \epsilon^{0,2}] = 0 \end{cases}$$

Linearized equations about a Poisson structure are

$$\begin{cases} \epsilon^{0,2} + B^{0,2} = 0 \\ \overline{\partial}\epsilon^{0,2} = 0 \end{cases}$$

Solvable by Dolbeault lemma. Appropriate implcit function gives $\exists B$ for the nonlinear equation.

Details follow papers of J. Conn, Ann. of Math. '84, '85, based on Nash-Moser implicit function theorem interpreted by R. Hamilton.

Theorem (Conn, 1984-5)

If P is Poisson on $\mathbb{R}^n$, with $P(0) = 0$, and P_{lin} is semisimple and compact, then $\exists$ neighbourhood with $P \cong P_{lin}$.

Recent work of Miranda-Monnier-Zung packages J. Conn's use of Nash-Moser techniques in a convenient way for finding normal forms for various geometric problems.

Final local classification problem

In the output of Bailey's theorem,

$$\mathbb{J} \cong_B (\mathbb{C}^k, \sigma) \times (\mathbb{R}^{2n-2k}, \omega_0),$$

is σ uniquely defined?

Theorem (M. Bailey, M.G.)

The holomorphic Poisson local model of a gen. cx. structure is unique up to icthyomorphism.

Proof

Part 1: interpolation in families

If $\mathbb{J}$ is locally B-equivalent to two hol. Poisson structures (I_0, σ_0), (I_1, σ_1), apply a version of Bailey's theorem in families to obtain a path of Poisson structures (I_t, σ_t) which are all B-equivalent.

Proof

Part 2: exchange B-transform with Diffeomorphism

Recall that B acts on holomorphic Poisson via

$$e^B \begin{pmatrix} J & Q \\ & -J^* \end{pmatrix} e^{-B} = \begin{pmatrix} J - QB & Q \\ BJ + J^*B - BQB & BQ - J^* \end{pmatrix}$$

The infinitesimal action by $\dot{B}$ is

$$\begin{pmatrix} -Q\dot{B} & 0 \\ \dot{B}J + J^*\dot{B} & \dot{B}Q \end{pmatrix}$$

this remains hol. Poisson iff $\dot{B}$ is of type $(1,1)$. This implies $\dot{B}(t) = dd^c f_t$ for

$$f_t \in C^\infty(M, \mathbb{R}).$$

The complex structure J_t changes via $\dot{J}_t = -Q\dot{B}_t$, but we have

$$Q(dd^c f_t) = \mathcal{L}_{Qdf_t} J,$$

proving that the time-1 flow of the Hamiltonian vector field of f_t takes (I_0, σ_0) to (I_1, σ_1).

Conclusion

- Local structure of a generalized complex manifold is governed by a *canonical holomorphic Poisson structure*
- Complexity is hidden in the holomorphic Poisson structure itself, as well as in the gluing by B–field transforms.
- Quantization? Branes? Groupoids?