Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems

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1 Nilmanifolds

G: a simply connected "nilpotent" Lie group
($\mathfrak{g}$: Lie algebra)
Γ a lattice(cocompact discrete subgroup of G)
Consider nilmanifold G/Γ
Invariant differental forms

$$\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$$

Theorem 1 (Nomizu) *The inclusion*

$$\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$$

induces an isomorphism

$$H^*(\mathfrak{g}) \cong H^*(G/\Gamma)$$

2 Formality and Lefschetz property of nilmanifolds

Definition 1 (D. Sullivan) *A DGA A is formal*
$\Longleftrightarrow$ *There exists a sequence of DGA(differential graded algebra) homomorphisms*

$$A \to C_1 \leftarrow C_2 \cdots \leftarrow (H^*(A), d = 0)$$

such that all homomorphisms induce cohomology isomorphisms. A manifold M is formal if the de Rham DGA $A^*(M)$ is fromal.

Theorem 2 (Hasegawa) *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then the DGA $\bigwedge \mathfrak{g}^*$ is formal if and only if $\mathfrak{g}$ is abelian. Hence a nilmanifold G/Γ is formal if and only if G/Γ is a torus.*

Definition 2 *A $2n$-symplectic manifold (M, ω) satisfies Hard Lefschetz property*
$\Longleftrightarrow$ *The linear map*

$$[\omega]^{n-i}\wedge : H^i(M) \to H^{2n-i}(M)$$

is an isomorphism for any $1 \leq i \leq n$.

Theorem 3 (Beson-Gordon) *Let $\mathfrak{g}$ be a $2n$-dimensional nilpotent Lie algebra. Suppose $\mathfrak{g}$ has a symplectic form ω (i.e. $\omega \in \bigwedge \mathfrak{g}^*$ such that $d\omega = 0$ and $\omega^n \neq 0$). Then the linear map $[\omega]\wedge : H^1(M) \to H^{2n-1}(M)$ is an isomorphism if and only if g is abelian. Hence a symplectic nilmanifold $(G/\Gamma, \omega)$ satisfies hard Lefschetz property if and only if G/Γ is a torus.*

3 Twiste de Rham DGA

M: a manifold
$\rho : \pi_1(M, x) \to (\mathbb{C}^*)^n$ a diagonal representation.
T: the Zariski-closure of $\rho(\pi_1(M, x))$.
$\{V_\alpha\}$: the set of one-dimensional representations of T.
(E_α, D_α): a rank one flat bundle with the monodromy $\alpha \circ \rho$
$A^*(M, E_\alpha)$ the space of E_ρ-valued C^∞-differential forms.

$$A^*(M, \mathcal{O}_\rho) = \bigoplus_\alpha A^*(M, E_\alpha)$$

with the derivation $\bigoplus_\alpha D_\alpha$.

4 solvmanifolds

G: a simply connected "Solvable" Lie group
($\mathfrak{g}$: Lie algebra)
Γ a lattice(cocompact discrete subgroup of G)
Consider solvmanifold G/Γ
· In general the isomorphism $H^*(\mathfrak{g}) \cong H^*(G/\Gamma)$ does not hold.
· There exist many non-toral solvmanifolds which are formal and hard Lefschetz.

Purpose I will generalize Nomizu's, Hasegawa's and Benson-gordon's theorem by using local systems (differential forms on flat bundles).

5 Algebraic hull

G: a simply connected solvable Lie group
Then we have a unique $\mathbb{R}$-algebraic group $\mathbf{H}_G$ with an injective homomorphism $\psi : G \to \mathbf{H}_G(\mathbb{R})$ so that:
(1) $\psi(G)$ is Zariski-dense in $\mathbf{H}_G$.
(2) The centralizer $Z_{\mathbf{H}_G}(\mathbf{U}(\mathbf{H}_G))$ of $\mathbf{U}(\mathbf{H}_G)$ is contained in $\mathbf{U}(\mathbf{H}_G)$.
(3) $\dim \mathbf{U}(\mathbf{H}_G) = \dim G$.
Such $\mathbf{H}_G$ is called the algebraic hull of G.
We call the unipotent radical $\mathbf{U}(\mathbf{H}_G)$ of $\mathbf{H}_G$ the unipotent hull of G.

6 Main theorem

onsider $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$.
By Lie's Theorem, Ad is trigonalizable.
$\rho : G \to (\mathbb{C}^*)^{\dim G}$: the diagonal part of Ad. Consider the DGA

$$A^*(G/\Gamma, \mathcal{O}_{\rho|\Gamma}).$$

Theorem 4 (K.) *Let $\mathfrak{u}$ be the Lie algebra of $\mathbf{U}(\mathbf{H}_G)$.*
We have a injection

$$\bigwedge \mathfrak{u}^* \to A^*(G/\Gamma, \mathcal{O}_{\rho|\Gamma}).$$

inducing a cohomology isomorphism. Thus $\bigwedge \mathfrak{u}^*$ is Sullivan's minimal model of $A^*(G/\Gamma, \mathcal{O}_{\rho|\Gamma})$ which induces a cohomology isomorphism.

Theorem 5 (K.) *Let G be a simply connected solvable Lie group with a lattice Γ. Then the following conditions are equivalent:*
(A) $\mathbf{U}(\mathbf{H}_G)$ *is abelian.*
(B) $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *such that the action* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple.*
(C) $A^*(G/\Gamma, \mathcal{O}_{\rho|\Gamma})$ *is formal.*

Theorem 6 (K.) *Let G be a simply connected solvable Lie group with a lattice Γ. Suppose solvmanifold G/Γ admits a symplectic form ω. Then the following conditions are equivalent:*
(A) $\mathbf{U}(\mathbf{H}_G)$ *is abelian.*
(B) $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *such that the action* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple.*
(C)

$$[\omega]^{n-i}\wedge : H^i(A^*(G/\Gamma, \mathcal{O}_{\rho|\Gamma})) \to H^{2n-i}(A^*(G/\Gamma, \mathcal{O}_{\rho|\Gamma}))$$

is an isomorphism for any $i \leq n$.