Eigenvalue estimates for Dirac operators with torsion

Rauischholzhausen, Juli 2012

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We will investigate the spectrum of the **Dirac operator of a metric connection with torsion** on a manifold with **special geometric structure** through a suitable **twistor equation**.

Plan

1. The square of the Riemannian Dirac operator

2. Special geometries via connections with torsion

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3. Twistorial estimates for manifolds with reducible holonomy

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1. The square of the Riemannian Dirac operator

(M^n, g): compact Riemannian spin mnfd, Σ: spin bdle

Classical Riemannian Dirac operator D^g:

Dfn : $D^g : \Gamma(\Sigma) \longrightarrow \Gamma(\Sigma), \quad D^g\psi := \sum_{i=1}^n e_i \cdot \nabla^g_{e_i}\psi$

D^g is elliptic differential operator of first order, essentially self-adjoint on $L^2(\Sigma)$, pure point spectrum

Schrödinger(1932)-Lichnerowicz(1962)

SL formula : $(D^g)^2 = \Delta + \frac{1}{4}\mathrm{Scal}^g$

•SL formula ⇒ EV of $(D^g)^2$: $\lambda \geq \frac{1}{4}\mathrm{Scal}^g_{\mathrm{min}}$

• optimal only for spinors with $\langle \Delta\psi, \psi \rangle = \|\nabla^g \psi\|^2 = 0$, i. e. parallel spinors, and then $\mathrm{Scal}^g_{\mathrm{min}} = 0$

• no parallel spinors if $\mathrm{Scal}^g_{\mathrm{min}} > 0$

Friedrich's inequality

Thm. Optimal EV estimate: $\lambda \geq \frac{n}{4(n-1)} \mathrm{Scal}^g_{\min}$ [Friedrich, 1980]

" = " if there exists a Killing spinor (KS) ψ: $\nabla^g_X \psi = \mathrm{const} \cdot X \cdot \psi \quad \forall X$

Link to special geometries:

Thm. $\exists$ KS $\Leftrightarrow n = 5 : (M, g)$ is Sasaki-Einstein mnfd [$\in$ contact str.]

$\Leftrightarrow n = 6 : (M, g)$ nearly Kähler mnfd

$\Leftrightarrow n = 7 : (M, g)$ nearly parallel G_2 mnfd

[Friedrich, Kath, Grunewald. . .]

Friedrich's inequality has two alternative proofs:

1. by deforming the connection $\nabla^g_X \psi \rightsquigarrow \nabla^g_X \psi + cX \cdot \psi$

2. by using **twistor theory:** the twistor or Penrose operator:

$$P\psi := \sum_{k=1}^{n} e_k \otimes \left[\nabla^g_{e_k} \psi + \frac{1}{n} e_k \cdot D^g \psi \right]$$

$$\Rightarrow \|P\psi\|^2 + \frac{1}{n} \|D^g \psi\|^2 = \|\nabla^g \psi\|^2$$

together with the SL formula $\Rightarrow$ **integral formula**

$$\int_M \langle (D^g)^2 \psi, \psi \rangle dM = \frac{n}{n-1} \int_M \|P\psi\|^2 dM + \frac{n}{4(n-1)} \int_M \mathrm{Scal}^g \|\psi\|^2 dM$$

and Friedrich's inequality follows, with equality iff ψ is a **twistor spinor**,

$$P\psi = 0 \iff \nabla^g_X \psi + \frac{1}{n} X \cdot D^g \psi = 0 \quad \forall X$$

Furthermore, ψ is automatically a **Killing spinor**.

2. Special geometries via connections with torsion

Given a mnfd M^n with G-structure ($G \subset \mathrm{SO}(n)$), replace ∇^g by a *metric connection ∇ with torsion that preserves the geometric structure!*

$$\text{torsion:} \quad T(X,Y,Z) \;:=\; g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)

$$\Rightarrow \quad g(\nabla_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

1. representation theory yields

- a clear answer *which* G-structures admit such a connection; if existent, it's unique and called the *'characteristic connection'*

2. Dirac operator $\not{D}$ of the metric connection with torsion $T/3$: 'characteristic Dirac operator'
– generalizes Dolbeault operator and Kostant's cubic Dirac operator

Some characteristic connections

Ex. 1 – contact mnfd [Friedrich, Ivanov 2000]

A large class admits a char. connection ∇, and $\mathrm{Hol}_0(\nabla) \subset \mathrm{U}(n) \subset \mathrm{SO}(2n+1)$. For Sasaki manifolds, the formula is particularly simple,

$$g(\nabla^c_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2}\eta \wedge d\eta(X, Y, Z),$$

and $\nabla T = 0$ holds. [Kowalski-Wegrzynowski, 1987 for Sasaki]

Ex. 2 – almost Hermitian 6-mnfd [Friedrich, Ivanov 2000]

(M, g, J), J almost complex, compatible with g

$\exists$ a char. connection $\nabla \Leftrightarrow$ Nijenhuis tensor $g(N(X,Y),Z) \in \Lambda^3(M)$,

$$g(\nabla^c_X Y, Z) \;:=\; g(\nabla^g_X Y, Z) + \tfrac{1}{2}\left[g(N(X,Y),Z) + d\Omega(JX, JY, JZ)\right]$$

Example3 - naturally reductive homogeneous space [Agricola 2003]

$M = G/H$ reductive space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\langle , \rangle$ a scalar product on $\mathfrak{m}$.

The PFB $G \to G/H$ induces a metric connection ∇ with torsion

$$T(X, Y, Z) := -\langle [X, Y]_{\mathfrak{m}}, Z \rangle,$$

called the 'canonical connection'.

Dfn. $M = G/H$ is called *naturally reductive* if $T \in \Lambda^3(M)$; ∇ coincides then with the characteristic connection.

Naturally reductive spaces have the properties $\nabla T = \nabla \mathcal{R} = 0$

3. The square of the Dirac operator with torsion

With torsion:

(M, g): mnfd with G-structure and charact. connection ∇^c, torsion T, assume $\nabla^c T = 0$ (for convenience)

$D\!\!\!/$: Dirac operator of connection with torsion $T/3$

generalized SL formula: [Agricola-Friedrich, 2003]

$$D\!\!\!/^2 \;=\; \Delta_T + \frac{1}{4}\mathrm{Scal}^g + \frac{1}{8}||T||^2 - \frac{1}{4}T^2$$

[1/3 rescaling: Slebarski (1987), Bismut (1989), Kostant, Goette (1999), IA (2002)]

Spectrum of $\not{D}$

For eigenvalue estimates, the action of T on the spinor bundle needs to be known!

Thm. Assume $\nabla^c T = 0$ and let $\Sigma M = \oplus_\mu \Sigma_\mu$ be the splitting of the spinor bundle into eigenspaces of T. Then:

a) ∇^c preserves the splitting of Σ, i. e. $\nabla^c \Sigma_\mu \subset \Sigma_\mu \quad \forall \mu$,

b) $\not{D}^2 \circ T = T \circ \not{D}^2$, i. e. $\not{D}^2 \Sigma_\mu \subset \Sigma_\mu \quad \forall \mu$. [Agr.-Fr. 2004]

$\Rightarrow$ Estimate on every subbundle of Σ_μ

Corollary (universal estimate). The first EV λ of $\not{D}^2$ satisfies

$$\lambda \geq \frac{1}{4}\mathrm{Scal}^g_{\mathrm{min}} + \frac{1}{8}\|T\|^2 - \frac{1}{4}\max(\mu_1^2, \ldots, \mu_k^2),$$

where $\mu_1, \ldots, \mu_k$ are the eigenvalues of T.

Universal estimate:

- follows from generalized SL formula
- does not yield Friedrich's inequality for $T \to 0$
- optimal iff $\exists$ a ∇^c-parallel spinor:

  This sometimes happens on mnfds with $\mathrm{Scal}^g_{\mathrm{min}} > 0$!

deformation techniques: yield often estimates quadratic in Scal^g, require subtle case by case discussion, often restriced curvature range [Agricola, Friedrich, Kassuba [PhD], 2008]

Results

twistor techniques:

•estimate always linear in Scal^g, no curvature restriction, rather universal,

•lead to a twistor eq. with torsion and sometimes to a Killing eq. with torsion

•yield another twistorial estimate for manifolds with reducible holonomy

– submitted – [Agricola, Becker-Bender [PhD], Kim 2010-11]

Twistors with torsion

$m : TM \otimes \Sigma M \to \Sigma M$: Clifford multiplication

$p =$ projection on $\ker m$: $p(X \otimes \psi) = X \otimes \psi + \frac{1}{n} \sum_{i=1}^{n} e_i \otimes e_i \cdot X \cdot \psi$

$$\nabla^s_X Y := \nabla^g_X Y + 2sT(X, Y, -)$$

($s = 1/4$ is the standard normalisation, $\nabla^{1/4} =$ char. conn.)

twistor operator: $P^s = p \circ \nabla^s$

Fundamental relation: $\|P^s\psi\|^2 + \frac{1}{n}\|D^s\psi\|^2 = \|\nabla^s\psi\|^2$

ψ is called s-twistor spinor $\Leftrightarrow$ $\psi \in \ker P^s$ $\Leftrightarrow$ $\nabla^s_X\psi + \frac{1}{n} X D^s \psi = 0$.

Idea: to play with the parameter "s"!
different scaling in $\nabla \left[s = \frac{1}{4}\right]$ and $\not{D} \left[s = \frac{1}{4\cdot 3}\right]$

Thm (twistor integral formula). Any spinor φ satisfies

$$\int_M \langle D\!\!\!/^2\varphi,\varphi\rangle dM = \frac{n}{n-1}\int_M \|P^s\varphi\|^2 dM + \frac{n}{4(n-1)}\int_M \mathrm{Scal}^g\|\varphi\|^2 dM + \frac{n(n-5)}{8(n-3)^2}\|T\|^2\int \|\varphi\|^2 dM - \frac{n(n-4)}{4(n-3)^2}\int_M \langle T^2\varphi,\varphi\rangle dM,$$

where $s = \frac{n-1}{4(n-3)}$.

Thm (twistor estimate). The first EV λ of $D\!\!\!/^2$ satisfies $(n > 3)$

$$\lambda \geq \frac{n}{4(n-1)}\mathrm{Scal}^g_{\mathrm{min}} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 - \frac{n(n-4)}{4(n-3)^2}\max(\mu_1^2,\ldots,\mu_k^2),$$

where $\mu_1,\ldots,\mu_k$ are the eigenvalues of T, and "=" iff

- Scal^g is constant,
- ψ is a twistor spinor for $s_n = \frac{n-1}{4(n-3)}$,
- ψ lies in Σ_μ corresponding to the largest eigenvalue of T^2.

Twistor estimate:

- reduces to Friedrich's estimate for $T \to 0$
- estimate is good for $\mathrm{Scal}^g_{\min}$ dominant (compared to $\|T\|^2$)

Ex. (M^6, g) of class $\mathcal{W}_3$ ("balanced"), Stab(T) abelian

Known: $\mu = 0, \pm\sqrt{2}\|T\|$, no ∇^c-parallel spinors

twistor estimate: $$\lambda \geq \frac{3}{10}\mathrm{Scal}^g_{\min} - \frac{7}{12}\|T\|^2$$

universal estimate: $$\lambda \geq \frac{1}{4}\mathrm{Scal}^g_{\min} - \frac{3}{8}\|T\|^2$$

- better than anything obtained by deformation

On the other hand:

Ex. (M^5, g) Sasaki: deformation technique yielded better estimates.

Twistor and Killing spinors with torsion

Thm (twistor eq). ψ is an s_n-twistor spinor ($P^{s_n}\psi = 0$) iff

$$\nabla^c_X\psi + \frac{1}{n}X \cdot D\!\!\!/\psi + \frac{1}{2(n-3)}(X \wedge T)\cdot\psi \;=\; 0,$$

Dfn. ψ is a Killing spinor with torsion if $\nabla^{s_n}_X\psi = \kappa X\cdot\psi$ for $s_n = \frac{n-1}{4(n-3)}$.

$$\Leftrightarrow\;\; \nabla^c\psi - \left[\kappa + \frac{\mu}{2(n-3)}\right] X\cdot\psi + \frac{1}{2(n-3)}(X\wedge T)\psi \;=\; 0.$$

In particular:

- ψ is a twistor spinor with torsion for the same value s_n
- κ satisfies the quadratic eq.

$$n\left[\kappa + \frac{\mu}{2(n-3)}\right]^2 = \frac{1}{4(n-1)}\mathrm{Scal}^g + \frac{n-5}{8(n-3)^2}\|T\|^2 - \frac{n-4}{4(n-3)^2}\mu^2$$

- $\mathrm{Scal}^g =$ constant.

In general, this twistor equation cannot be reduced to a Killing equation.

. . . with one exception: $n = 6$

Thm. Assume ψ is a s_6-twistor spinor for some $\mu \neq 0$. Then:

- ψ is a $D\!\!\!/$ eigenspinor with eigenvalue

$$D\!\!\!/\psi = \frac{1}{3}\left[\mu - 4\frac{\|T\|^2}{\mu}\right]\psi$$

- the twistor equation for s_6 is equivalent to the Killing equation $\nabla^s\psi = \lambda X \cdot \psi$ for the same value of s.

Observation:

The Riemannian Killing / twistor eq. and their analogue with torsion behave very differently depending on the geometry!

Killing spinors on nearly Kähler manifolds

- (M^6, g, J) 6-dimensional nearly Kähler manifold

\- ∇^c its characteristic connection, torsion is parallel

\- Einstein, $\|T\|^2 = \frac{2}{15}\mathrm{Scal}^g$

\- T has EV $\mu = 0, \pm 2\|T\|$

\- $\exists$ 2 Riemannian KS $\varphi_\pm \in \Sigma_{\pm 2\|T\|}$, ∇^c-parallel

\- univ. estimate = twistor estimate, $\lambda \geq \frac{2}{15}\mathrm{Scal}^g$

Thm. The following classes of spinors coincide:

- Riemannian Killing spinors
- ∇^c-parallel spinors
- Killing spinors with torsion
- Twistor spinors with torsion

There is exactly one such spinor $\varphi_\pm$ in each of the subbundles $\Sigma_{\pm 2\|T\|}$.

A 5-dim. ex. with Killing spinors with torsion

- 5-dimensional Stiefel manifold $M = SO(4)/SO(2)$, $\mathfrak{so}(4) = \mathfrak{so}(2) \oplus \mathfrak{m}$

- Jensen metric: $\mathfrak{m} = \mathfrak{m}_4 \oplus \mathfrak{m}_1$ (irred. components of isotropy rep.),

$$\langle (X,a),(Y,b)\rangle_t = \frac{1}{2}\beta(X,Y) + 2t \cdot ab, \ \ t > 0, \ \ \beta = \text{Killing form}\Big|_{\mathfrak{m}_4}$$

- $t = 1/2$: undeformed metric: 2 parallel spinors

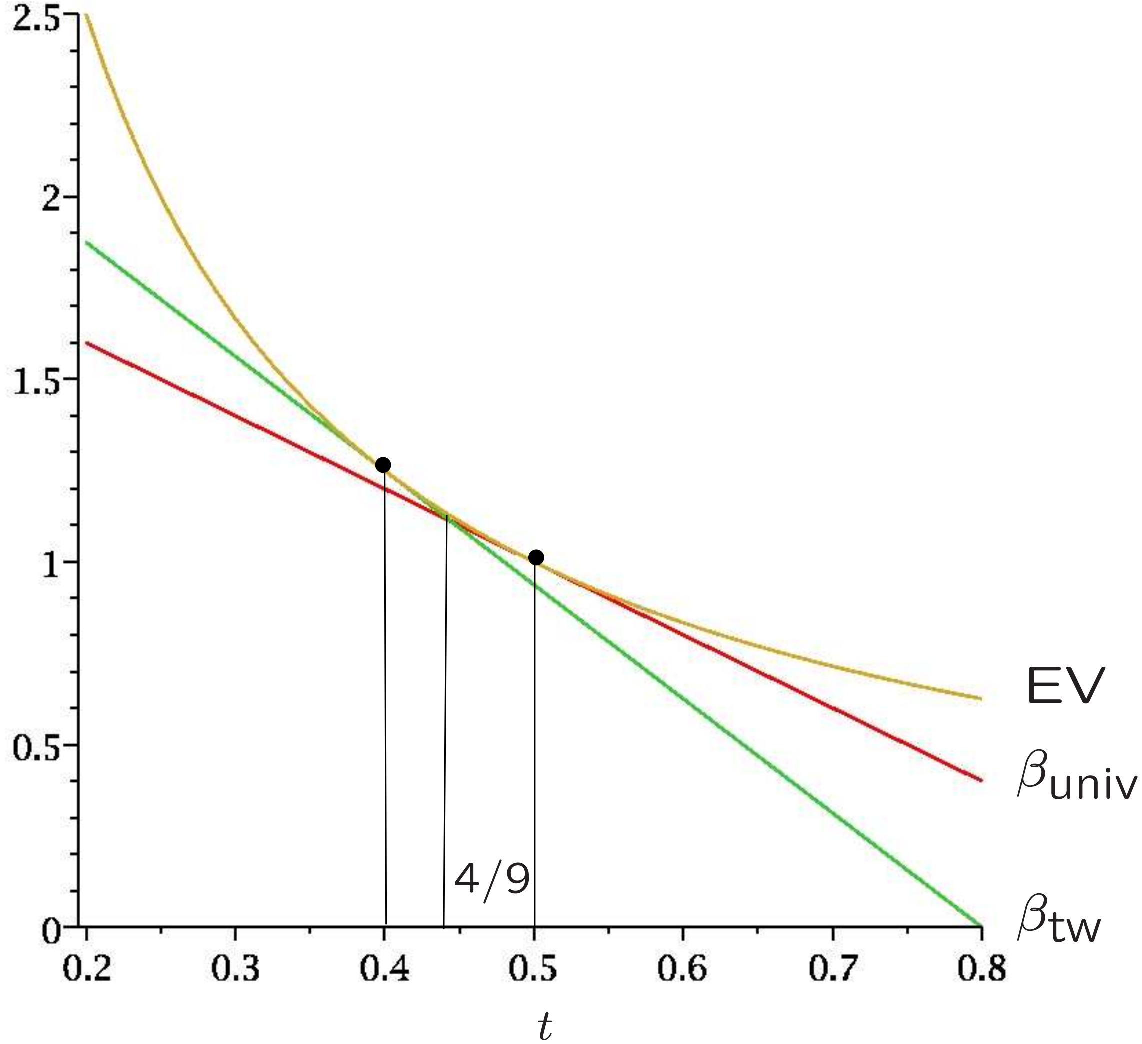
- $t = 2/3$: Einstein-Sasaki with 2 Riemannian Killing spinors

- For general t: metric contact structure in direction $\mathfrak{m}_1$ with characteristic connection ∇ satisfying $\nabla T = 0$

- $\|T\|^2 = 4t$, $\text{Scal}^g = 8 - 2t$, $\text{Ric}^g = \text{diag}(2-t, 2-t, 2-t, 2-t, 2t)$.

- Universal estimate: $\lambda \geq 2(1-t) =: \beta_{\text{univ}}$

- Twistor estimate: $\lambda \geq \frac{5}{2} - \frac{25}{8}t =: \beta_{\text{tw}}$



Result: there exist 2 twistor spinors with torsion for $t = 2/5$, and these are even Killing spinors with torsion.

4. Twistorial estimates for mfds with reducible holonomy

Parallel distributions

$\mathcal{T} \subset \mathcal{T}M^n$ is a parallel distribution $\Rightarrow$ $\nabla_X Y \in \mathcal{T}$ for $Y \in \mathcal{T}$ and $X \in \mathcal{T}M^n$

$\mathcal{T}M^n = \mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_k$ with $\mathrm{Hol}(M^n; \nabla^s) \subset \mathrm{SO}(n_1) \times \ldots \times \mathrm{SO}(n_k)$, where $\mathcal{T}_1, ..., \mathcal{T}_k$ are the parallel distributions of $\mathcal{T}M^n$.

Then **the Ricci tensor** has block structure:

$$\mathrm{Ric} = \left[\begin{array}{c|c|c} \mathrm{Ric}_1 & 0 & \\ \hline 0 & \ddots & 0 \\ \hline & 0 & \mathrm{Ric}_k \end{array}\right],$$

i. e. $\mathrm{Ric}(X, Y) \neq 0$ can only happen if $X, Y \in \mathcal{T}_i$ for some i.

And $\mathrm{Scal}_i := \mathrm{tr}\ \mathrm{Ric}_i \Rightarrow \mathrm{Scal} = \sum_{i=1}^{k} \mathrm{Scal}_i$.

1-parameter family of connections

$$\nabla^s_X Y \;=\; \nabla^g_X Y + 2s\,T(X,Y,-).$$

Dfn. [Geometry with reducible parallel torsion]
A manifold with s-parameter family of connections with torsion and a parallel distribution of TM as above has a geometry with reducible parallel torsion if

1. there exists a value s_0 with $\nabla^{s_0} T = 0$,

2. the torsion splits into a sum $T = \sum_{i=1}^k T_i$, $T_i \in \Lambda^3(\mathcal{T}_i)$.

3. the tangent bundle $\mathcal{T}M^n = \bigoplus_{i=1}^k \mathcal{T}_i$ splits into ∇^s-parallel distributions $\mathcal{T}_i$ for some parameter s and
   $\mathrm{Hol}(M^n;\nabla^s) \subset \mathrm{SO}(n_1) \times \ldots \times \mathrm{SO}(n_k)$,

(M^n, a geometry with reducible parallel torsion):

•(M^n, g) is locally a product of Riemannian manifolds [de Rham]

•$\nabla^{s_0} T_i = 0$ for each T_i [Cleyton, Moroianu 2012]

•$T^2 = \Sigma T_i^2$, $\qquad \|T\|^2 = \Sigma \|T_i\|^2$.

• $\mathcal{R}^s(X, Y, Z, V) \neq 0$ if all vectors lie in the same subspace $\mathcal{T}_i$ for some i,

• σ_T splits in $\sigma_T = \sum_{i=1}^{k} \sigma_i$ with $\sigma_i := \sigma_{T^i}$, where

$$\sigma_T \ := \ \frac{1}{2} \sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner T).$$

Partial Schrödinger-Lichnerowicz formulas

•Partial connections:

$$\nabla^{s,i}_X := \nabla^s_{p_i(X)}, \quad \text{hence } \nabla^s = \sum_{i=1}^{k} \nabla^{s,i}.$$

•Partial Dirac operators and partial spinor Laplacians:

$$D^s_i := \sum_{m=1}^{n_i} e^i_m \cdot \nabla^{s,i}_m, \quad \Delta^s_i := -\sum_{m=1}^{n_i} \nabla^{s,i}_m \nabla^{s,i}_m.$$

Then

$$D = \sum_{i=1}^{k} D^s_i, \Delta^s = \sum_{i=1}^{k} \Delta^s_i.$$

Let

$$\mathcal{D}_i^s \ := \ \sum_{m=1}^{n_i} (e_m^i \lrcorner T_i) \cdot \nabla_{e_m^i}^{s,i} \psi$$

Prop.[PSL formulas]

M with a geometry with reducible parallel torsion. Then

(i) $(D_i^s)^2 \ = \ \Delta_i^s + s(6-8s)\,\sigma_i - 4s\mathcal{D}_i^s + \frac{1}{4}\mathrm{Scal}_i^s,$

(ii) $D_i^s D_j^s + D_j^s D_i^s = 0$ for $i \neq j$,

(iii) $(D_i^{s/3})^2 \ = \ \Delta_i^s + 2s\,\sigma_i + \frac{1}{4}\mathrm{Scal}_i^g - 2s^2\|T_i\|^2.$

Adapted Twistor Operator

$$P^s\psi \;=\; \nabla^s\psi + \sum_{i=1}^{k} \frac{1}{n_i} \sum_{l=1}^{n_i} e_l^i \otimes e_l^i \cdot D_i^s \psi.$$

One checks that

$$\|P^s\psi\|^2 \;=\; \langle (\Delta^s - \sum_{i=1}^{k} \frac{1}{n_i} (D_i^s)^2)\psi, \psi \rangle.$$

Thm.[Twistorial estimate for products]

Let $n_1 \leq n_2 \leq \ldots \leq n_k$ and λ the smallest eigenvalue of $D\!\!\!/^2$. Then

$$\lambda \ \geq \ \frac{n_k}{4(n_k-1)}\mathrm{Scal}^g_{\min} + \frac{n_k(n_k-5)}{8(n_k-3)^2}\|T\|^2 + \frac{n_k(4-n_k)}{4(n_k-3)^2}\max(\mu_1^2, \ldots, \mu_k^2)$$

" = ": for $\tilde{s} = \dfrac{n_k - 1}{4(n_k-3)}$

•the Riemannian scalar curvature of (M, g) is constant,

•the eigenspinor ψ is a twistor spinor for $\tilde{s}$ on M_k,

•$i = 1, ..., k-1$:

(a)$n_i < n_k$: $\nabla^{\tilde{s}}$-parallel spinor on M_i,

(b)$n_i = n_k$: $\nabla^{\tilde{s}}$-parallel or twistor spinor for $\tilde{s}$ on M_i,

•spinors lie in $\Sigma_\mu(M_i)$ corresponding to the largest eigenvalue of T_i^2.

A generalization of

$$\lambda^g \geq \frac{n_k}{4(n_k - 1)} \mathrm{Scal}^g_{\mathrm{min}}$$

[E. C. Kim (2004), B. Alexandrov (2006)]

Ex.

Let M be a product of 5-dimensional manifolds with parallel torsion, then M is a 10-dimensional manifold with a geometry with reducible parallel torsion:

The 'twistorial eigenvalue estimate' reads

$$\lambda \geq \frac{5}{18}\mathrm{Scal}^g_{\mathrm{min}} + \frac{25}{196}\|T\|^2 - \frac{15}{49}\,\mathrm{max}(\mu^2),$$

and the 'twistorial eigenvalue estimate for products' reads

$$\lambda \geq \frac{5}{4}\mathrm{Scal}^g_{\mathrm{min}} - \frac{5}{16}\,\mathrm{max}(\mu^2).$$