WORKSHOP "GEOMETRIC STRUCTURES ON MANIFOLDS AND THEIR APPLICATIONS "- CASTLE RAUISCHHOLZHAUSEN JULY 2012

TORIC MOMENT MAPPINGS AND RIEMANNIAN STRUCTURES

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Abstract

Coadjoint orbits for the group $SO(6)$ parametrize Riemannian G-reductions in six dimensions, and we use this correspondence to interpret symplectic fibrations between these orbits, and to analyse moment polytopes associated to the standard Hamiltonian torus action on the coadjoint orbits. The theory is then applied to describe so-called intrinsic torsion varieties of Riemannian structures on the Iwasawa manifold.

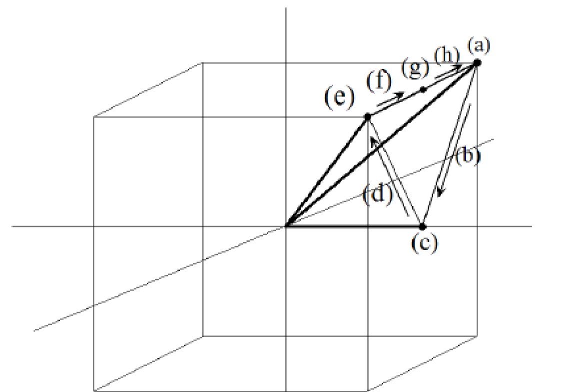
SO(6) coadjoint orbits

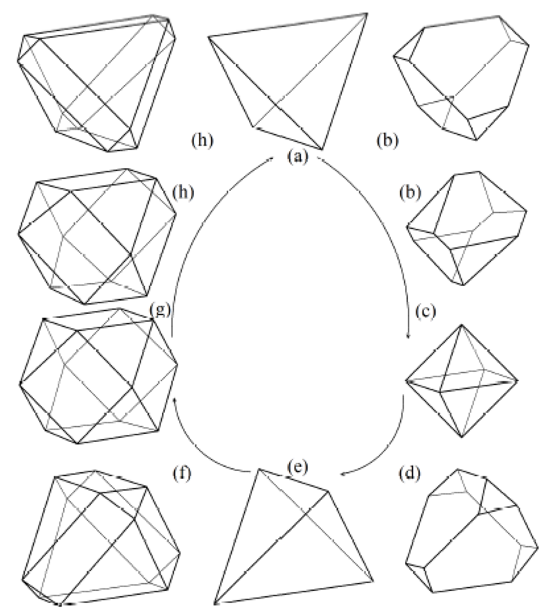
The set of adjoint orbits is parametrized by the closed fundamental Weyl chamber $\bar{B}$. The Konstant–Kirillov–Souriau is a standard symplectic structure on coadjoint orbits. Restricting the group action to the maximal torus $T \subset G$, we obtain a Hamiltonian torus action on the orbit. The Atiyah and Guillemin–Sternberg Convexity Theorem implies that the image by μ_T of an orbit passing through $\lambda \in \mathfrak{t}$ is the convex hull of the Weyl group orbit of λ:

$$\mu_T(G\cdot\lambda) = \mathrm{conv}(W\cdot\lambda).$$

$$\mathscr{O}^{SO(6)} = \frac{SO(6)}{U(1)\times U(1)\times U(1)} \cong \frac{SO(6)}{T}$$

point in $\bar{B}$	stabilizer	orbit	image by μ_T	
(α,α,α)	$U(3)$	$\mathscr{P}^+ \cong \mathbb{CP}^3$	tetrahedron $\Delta_{\mathscr{P}+}$	(a)
$(\alpha,\alpha,-\alpha)$	$\tilde{U}(3)$	$\mathscr{P}^- \cong \mathbb{CP}^3$	tetrahedron $\Delta_{\mathscr{P}-}$	(e)
$(\alpha,0,0)$	$U(1)\times SO(4)$	$\mathscr{G}\cong \mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$	octahedron $\Delta_{\mathscr{G}}$	(c)
(α,β,β)	$U(1)\times U(2)$	$\mathscr{F}^+$	truncated tetrahedron $\Delta_{\mathscr{F}+}$	(b)
$(\alpha,\beta,-\beta)$	$U(1)\times\tilde{U}(2)$	$\mathscr{F}^-$	truncated tetrahedron $\Delta_{\mathscr{F}-}$	(d)
(α,α,β)	$U(2)\times U(1)$	$\mathscr{D}^+$	skew-cuboctahedron $\Delta_{\mathscr{D}+}$	(h)
$(\alpha,\alpha,0)$	$U(2)\times SO(2)$	$\mathscr{D}^0$	cuboctahedron $\Delta_{\mathscr{D}^0}$	(g)
$(\alpha,\alpha,-\beta)$	$U(2)\times\tilde{U}(1)$	$\mathscr{D}^-$	skew-cuboctahedron $\Delta_{\mathscr{D}-}$	(f)





Differential fibrations with fibres $\mathbb{CP}^1$ and $\mathbb{CP}^2$.

$$\begin{array}{ccc} & \mathscr{O} & \\ & \downarrow{\scriptstyle \pi_0} & \\ & \mathscr{F}^+ & \\ {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} \\ \mathscr{P}^+ & & \mathscr{G} \end{array} \qquad \begin{array}{ccc} & \mathscr{O} & \\ & \downarrow{\scriptstyle \pi_0'} & \\ & \mathscr{D} & \\ {\scriptstyle\pi_1^+}\swarrow & & \searrow{\scriptstyle\pi_1^-} \\ \mathscr{P}^+ & & \mathscr{P}^- \end{array} \qquad \begin{array}{ccc} & \mathscr{O} & \\ & \downarrow{\scriptstyle \pi_0''} & \\ & \mathscr{F}^- & \\ {\scriptstyle\pi_1''}\swarrow & & \searrow{\scriptstyle\pi_2''} \\ \mathscr{G} & & \mathscr{P}^- \end{array}$$

Each fibre can be interpreted as a coadjoint orbit.

$$\mathscr{O}^{SO(6)} \cong \mathbb{G}\mathrm{r}_2(\mathbb{R}^6) \rtimes \mathscr{O}^{SO(4)} \cong \mathbb{G}\mathrm{r}_2(\mathbb{R}^6) \rtimes \mathbb{G}\mathrm{r}_2(\mathbb{R}^4) \cong \mathbb{G}\mathrm{r}_2(\mathbb{R}^6) \rtimes (S^2\times S^2)$$

$$\mathscr{O}^{SO(6)} \cong \mathscr{P} \rtimes \mathscr{O}^{SU(3)} \cong \mathbb{CP}^3 \rtimes \mathbb{CP}^2 \rtimes \mathbb{CP}^1$$

The above fibrations are *symplectic* (the fibre $\pi^{-1}(p) = F$ is a symplectic manifold,the transition mappings induce symplectomorphisms of F). Toric manifolds ($dimT = 1/2\, dimM$) can be recognized by their moment (Delzant) polytopes. In our case symplectic fibrations over (symplectic submanifolds of) coadjoint orbits are illustrated by the moment map, even though the torus actions are typically low-dimensional.

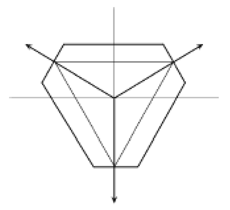


Figure 3: A generic $SU(3)$ coadjoint orbit fibres symplectically over $\mathbb{CP}^2$

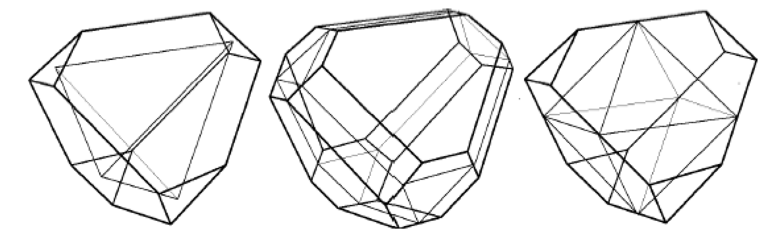


Figure 4: $\mathscr{F}^+$ fibres over $\mathscr{P}^+$; $\mathscr{O}^{SO(6)}$ fibres over $\mathscr{F}^+$; $\mathscr{F}^+$ fibres over $\mathscr{G}$.

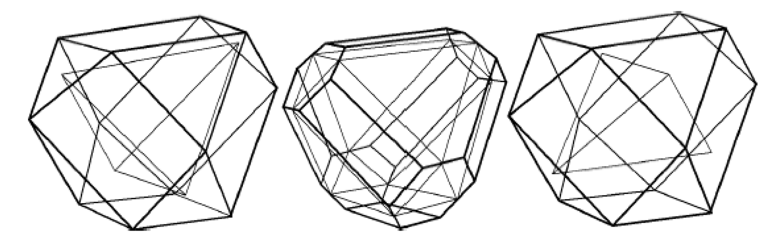


Figure 5: $\mathscr{D}$ fibres over $\mathscr{P}^+$; $\mathscr{O}^{SO(6)}$ fibres over over $\mathscr{D}$; $\mathscr{D}$ fibres over $\mathscr{P}^-$.

Riemannian geometry in six dimensions

Let (M,g) be a Riemannian manifold of dimension N. Any smooth 2-form ω determines a skew-symmetric endomorphism $\mathfrak{F}$ of each tangent space via

$$\omega(X,Y) = g(\mathfrak{F}X,Y).$$

Definition 1. Two distinguished Riemannian G-structures defined via 2-forms are called *compatible* if the associated skew-symmetric endomorphisms commute.

In six dimensions $\Lambda^2\mathbb{R}^6 \cong \mathfrak{so}(6) \cong \mathbb{R}^{15}$. Reductions of the Riemannian structure defined by a differential 2-form:

Case	2-form	SO(6) orbit
1	$a\mu_1 = a(e^{12}+e^{34}+e^{56})$	$\mathscr{P}^+$
2	$a\mu_3 = a(e^{12}+e^{34}-e^{56})$	$\mathscr{P}^-$
3	$a\mu_2 = ae^{12}$	$\mathscr{G}$
4	$a_1\mu_1+a_2\mu_2 = (a_1+a_2)e^{12}+a_1(e^{34}+e^{56})$	$\mathscr{F}^+$
5	$a_1\mu_2+a_2\mu_3 = (a_1+a_2)e^{56}+a_1(e^{34}-e^{56})$	$\mathscr{F}^-$
6	$a_1\mu_1+a_2\mu_3 = (a_1+a_2)(e^{12}+e^{23})+(a_1-a_2)e^{56}, \quad a_1>a_2$	$\mathscr{D}^+$
7	$a_1\mu_1+a_1\mu_3 = 2a_1(e^{12}+e^{34})$	$\mathscr{D}^0$
8	$a_1\mu_1+a_2\mu_3 = (a_1+a_2)(e^{12}+e^{34})+(a_1-a_2)e^{56}, \quad a_1<a_2$	$\mathscr{D}^-$
9	$a_1\mu_1+a_2\mu_2+a_3\mu_3$	$\mathscr{O}^{SO(6)}$

Case 1. Orthogonal almost complex structures (OCS) on T_pM compatible with a fixed orientation.

Case 2. OCS's inducing the opposite orientation on T_pM.

Case 3 The orbit $\mathscr{G}$ parametrizes a set orthogonal almost product structures (OPS).

A *mixed structure* (MS) is a reduction of the structure group to $U(p)\times U(q)$. This is equivalent to the assignment of an OCS J ($J^2=-I$) and an OPS P ($P^2=I$) which are compatible ($JP=PJ$).

Case 4. A $\mathscr{F}^+$ orbit parametrizes MS's determined by an OCS J in $\mathscr{P}^+$ and an OPS in $\mathscr{G}$ whose 2-plane is J-invariant and oriented consistently with J.

Case 5 $\mathscr{F}^-$ parametrizes MS's with $J\in\mathscr{P}^-$ and an OPS whose 2-plane is J-invariant and oriented consistently with J.

Cases 6 and 8. The position of the 2-form in $\bar{B}$ exhibits it as a weighted linear combination of two compatible OCS's $J_+\in\mathscr{P}^+$ and $J_-\in\mathscr{P}^-$.

Lemma 2. *If two OCS's on $\mathbb{R}^6$ are compatible then they coincide up to sign on a real 4-plane (and, therefore, on a complementary 2-plane).*

Proposition 3. *A $\mathscr{D}^\pm$ orbit parametrizes MS's determined by an OCS J in $\mathscr{P}^\pm$ and an OPS in $\mathscr{G}$ whose 2-plane is oriented consistently with $-J$.*

Case 7 Riemannian f-structure. ($f^3+f=0$, the structure group reduces to $U(p)\times SO(q)$).

Case 9 T^3-reductions of the Riemannian structure.

Moment polytopes

The reduced Riemannian G-structures are realized as smooth sections of fibre bundles with fibre $\mathscr{O}$. The mapping $SO(6)/G \to \Lambda^2T^*M$ which associates a 2-form to a specific G-reduction can be interpreted (at each point of M) as the moment map $SO(6)G \to \mathfrak{so}(6)^*$ associated to the KKS symplectic structure. The orthogonal projection $\mathfrak{so}^*(6) \to \mathfrak{t}^*$ is the moment map

$$\mu_T : \frac{SO(6)}{G} \longrightarrow \mathfrak{t}^* \cong \mathbb{R}^3.$$

Theorem 4. *The Hamiltonian action of the maximum torus T of $SO(6)$ on $\mathscr{O}$ associates a characteristic "moment polytope" to each of the Riemannian structures defined by a 2-form.*

Proposition 5. *If a 2-form is fixed by the action of some subgroup C of the maximum torus, then the corresponding skew-symmetric endomorphism $\mathfrak{F}$ commutes with the action of C on $\mathbb{R}^N$.*

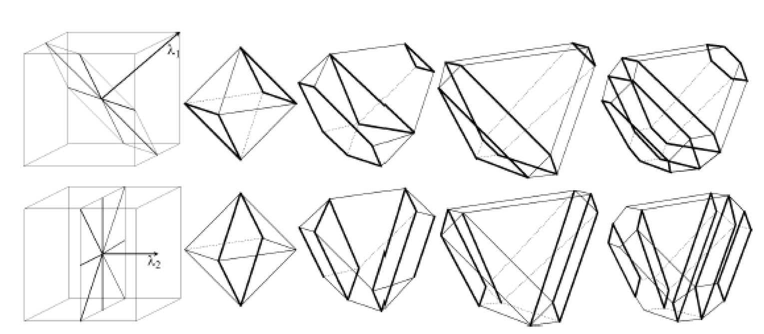


Figure 6: Roots orthogonal to λ_1 and λ_2. Projections of C_1 and C_2-invariant sets.

The sets of points fixed by C_i are F_i-coadjoint orbits.

$$\omega = -\omega_0 + 2v\wedge J_0 v,$$

$$\mu_T(\omega) = (-1+2(x_1^2+x_2^2), -1+2(x_3^2+x_4^2), -1+2(x_5^2+x_6^2)) = (x,y,z).$$

Thus the projections of this form satisfy $x+y+z=-1$,

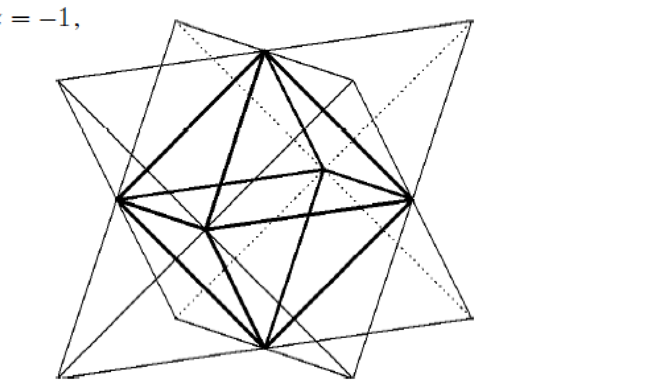


Figure 7: The moment polytope $\Delta_{\mathcal{G}}$ interpreted as intersection of J-invariant sets.

A Klein correspondence

$\mathbb{G}\mathrm{r}_2(\mathbb{R}^6) \cong \mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$ is identified with a non-degenerate quadric in $\mathbb{P}(\Lambda^2\mathbb{C}^4)$:

1. $\mathscr{G}$ parametrizes the projective lines $\mathbb{CP}^1$ in $\mathbb{CP}^3$.
2. A point $x\in\mathbb{CP}^3$ determines a plane in $\mathscr{G}$, consisting of all the lines passing through that point.
3. A point $y\in(\mathbb{CP}^3)^*$ determines a plane in $\mathscr{G}$, consisting of all the lines lying in the plane y.

A completely new interpretation:

$1'$. Given a decomposition $T_pM = \mathcal{V}\oplus\mathcal{H}$ arising from an OPS P, there is a $\mathbb{CP}^1$ worth of compatible OCS's parametrized by $\omega\in S^2\subset\Lambda^2_+\mathcal{H}^*$. This is our projective line in $\mathscr{P}^+$.

$2'$. Given an OCS J we have the J-invariant 2-planes generated by $\{v,Jv\}$ and each one determines an OPS.

$3'$. Given an OCS J we have the J-invariant oppositely-oriented 2-planes generated by $\{v,-Jv\}$.

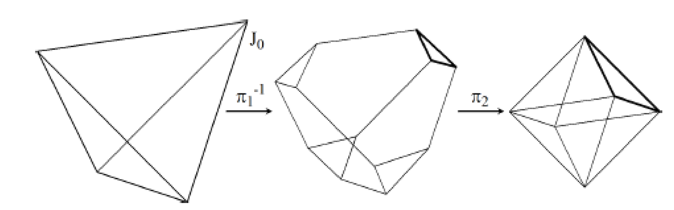
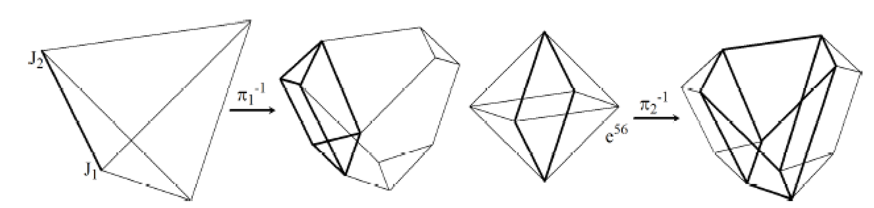


Figure 8: A polytope map induced by the Klein correspondence.

Proposition 6. *Denote by $L := \mu_T^{-1}(\mathcal{E}_{12})$. The image $\mu_T(\pi_1^{-1}(L)) \subset \Delta_{\mathscr{F}+}$ is the polytope shown on the left in Figure 9. The set $\pi_1^{-1}(L)$ is a symplectic toric manifold.*

Proposition 7. *Denote by K the set of 2-planes in the subspace $\langle e^1,e^2,e^3,e^4\rangle$. The image $\mu_T(\pi_2^{-1}(K))$ is the rectangular prismoid represented in bold on the right of Figure 9. The set $\pi_2^{-1}(K)$ is a symplectic toric manifold.*



Proposition 8. *Let F^+ and F^- denote the C_1-invariant subsets of $\mathscr{G}$ (both symplectomorphic to $\mathbb{CP}^2$) projected by μ_T on two disjoint faces of $\Delta_{\mathscr{G}}$. The images $\mu_T(\pi_2^{-1}(F^\pm))$ are respectively a hexagon and a triangular prismoid as shown in Figure 10. The set $\pi_2^{-1}(F^+)$ is a symplectic toric manifold.*

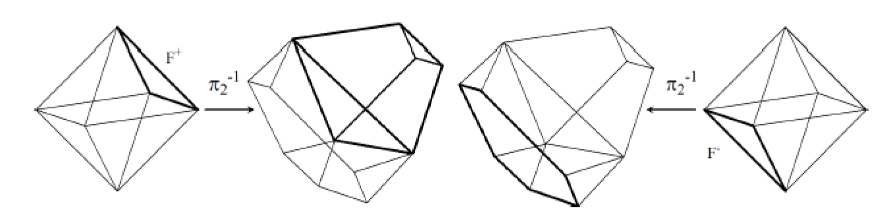


Figure 10: Projections $\mu_T(F^\pm)$ in $\Delta_{\mathscr{G}}$ and $\mu_T(\pi_2^{-1}(F^\pm))$ in $\Delta_{\mathscr{F}+}$.

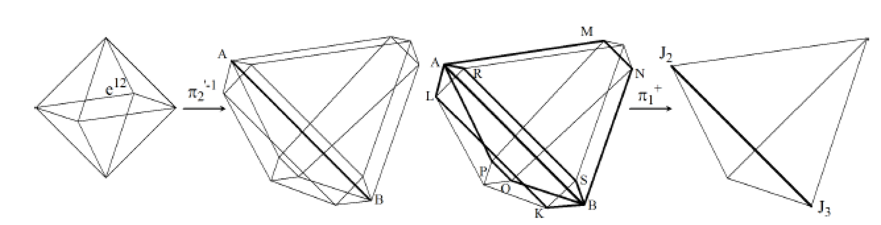


Figure 11: A polytope map induced by the Klein correspondence.

An application

The intrinsic torsion of a geometrical structure is the first order obstruction to its integrability. For this reason, a standard way of classifying Riemannian G-structures is based on criteria whereby its intrinsic torsion tensor τ reduces to a specific subset of G-irreducible components of the corresponding space of intrinsic torsion $\mathscr{W}$. The prototype case gave rise to the sixteen classes of almost Hermitian manifolds à la Gray–Hervella

$$\mathscr{W} \cong T^*M\otimes\mathfrak{u}^\perp \cong \Lambda^{1,0}\otimes[\![\Lambda^{2,0}]\!] \cong \mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_1\oplus\mathscr{W}_4,$$

$$\mathscr{W}_1\cong[\![\Lambda^{3,0}]\!],\quad \mathscr{W}_2\cong[\![R(2,1,0,...,0)]\!],\quad \mathscr{W}_3\cong[\![\Lambda_0^{2,1}]\!],\quad \mathscr{W}_4\cong[\![\Lambda^{1,0}]\!].$$

OPS's

$$\mathscr{V}\cong T^*M\otimes(\mathfrak{so}(\mathcal{V})\oplus\mathfrak{so}(\mathcal{H}))\cong(\mathcal{H}\oplus\mathcal{V})\otimes(\mathcal{H}\otimes\mathcal{V})$$

$$\mathscr{V}_1=\Lambda^2\mathcal{V}\otimes\mathcal{H},\qquad \mathscr{V}_2=S_0^2\mathcal{V}\otimes\mathcal{H},\qquad \mathscr{V}_3=1_{\mathcal{V}}\otimes\mathcal{H},$$
$$\mathscr{V}_4=\Lambda^2\mathcal{H}\otimes\mathcal{V},\qquad \mathscr{V}_5=S_0^2\mathcal{H}\otimes\mathcal{V},\qquad \mathscr{V}_6=1_{\mathcal{H}}\otimes\mathcal{V},$$

MS's

$$\mathscr{M}\cong T^*M\otimes(\mathfrak{u}(1)\oplus\mathfrak{u}(2))^\perp\cong(\mathcal{H}\oplus\mathcal{V})\otimes\Big((\mathcal{V}\otimes\mathcal{H})\oplus[\![\lambda^{2,0}]\!]\Big),$$

$$\begin{aligned}\mathscr{M} \cong\ & 3\mathcal{H}\oplus2\mathcal{V}\oplus2[\![\nu\lambda_0^{1,1}]\!]\oplus[\![\nu^2\lambda^{1,0}]\!]\oplus[\![\nu^2\lambda^{0,1}]\!]\\ &\oplus2[\![\nu\lambda^{2,0}]\!]\oplus2[\![\nu\lambda^{0,2}]\!]\oplus[\![\nu\sigma^{2,0}]\!]\oplus[\![\nu\sigma^{0,2}]\!]\oplus[\![R(2,1)]\!].\end{aligned}$$

$$\mathscr{M}=\mathscr{W}+\mathscr{V}$$

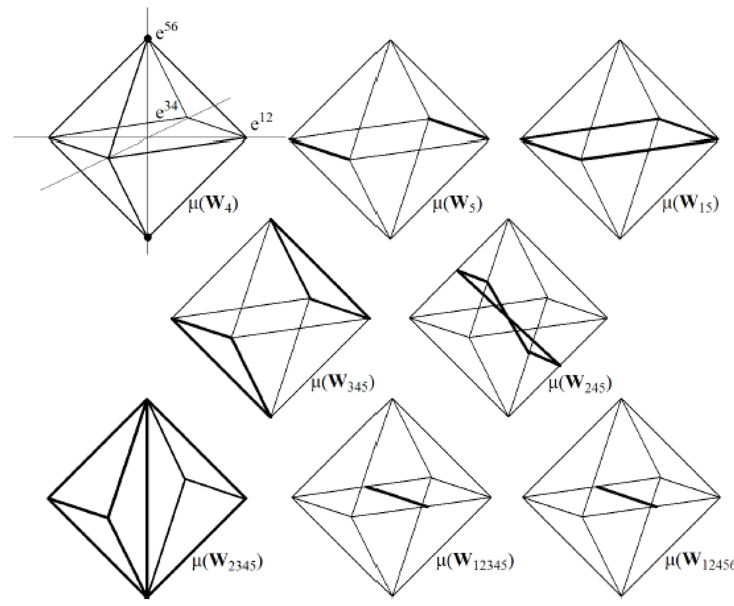
Proposition 9. *The intrinsic torsion tensor $\tau_{\mathscr{M}}$ of a MS is completely determined by the intrinsic torsion tensors $\tau_{\mathscr{W}},\tau_{\mathscr{V}}$ of the underlying OCS and OPS. Conversely, $\tau_{\mathscr{M}}$ determines the pair $(\tau_{\mathscr{W}},\tau_{\mathscr{V}})$.*

Theorem 10. *(Abbena-Garbiero-Salamon) The set I of invariant complex structures on N is given by the disjoint union of the point ω_0 and a $\mathbb{CP}^1$. This is a T-invariant subset of $\mathscr{P}^+$ and its image by μ_T is the union of a vertex and the edge $\mathcal{E}_{12}$ of $\Delta_{\mathscr{P}+}$.*

Theorem 11. *The ITV's of invariant $SO(2)\times SO(4)$ reductions of the standard Riemannian structure on the Iwasawa manifold are:*

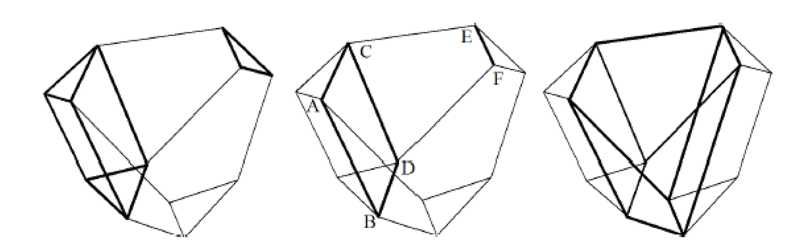
ITV	*topological description*	*geometric properties*
$\mathbf{W}_4$	*two points*	*totally geodesic horizontal foliations*
$\mathbf{W}_5 \quad (\equiv\mathscr{D})$	$\mathbb{CP}^1\sqcup\mathbb{CP}^1$	*totally geodesic horizontal foliations*
$\mathbf{W}_{15} \quad (\equiv\mathscr{H})$	$\mathbb{G}\mathrm{r}_2(\mathbb{R}^4)$	*vertical foliations*
$\mathbf{W}_{345}$	$\mathbb{CP}^2\sqcup\mathbb{CP}^2$	*horizontal foliations*
$\mathbf{W}_{245}$	$S^3\times S^1\times S^1$	*horizontal foliations*
$\mathbf{W}_{2345} \ (\equiv\mathscr{V})$	$\mathbb{R}^4$ *bundle over* $\mathbb{CP}^1\sqcup\mathbb{CP}^1$	*horizontal foliations*
$\mathbf{W}_{12345}$	$S^1\times S^1$ *bundle over* S^3	*horizontal distributions of type* D_1
$\mathbf{W}_{12456}$	$S^1\times S^1$ *bundle over* S^3	*vertical distributions of type* D_1
$\mathbf{W}_{123456}$	*generic point in* $\mathbb{G}\mathrm{r}_2(\mathbb{R}^6)$	*generic OPS*

Theorem 12. *The ITV's of invariant $SO(4)\times SO(2)$-structures on $\mathscr{I}$ exclusive of $\mathbf{W}_{12345}$ and $\mathbf{W}_{12456}$ are stable under the action of $\bar{T}$.*



Corollary 15. *The ITV inside $\mathscr{F}^+$ consisting of MS's on N of class $[\![\nu\lambda^{0,2}]\!]\oplus[\![\nu\lambda_0^{1,1}]\!]\oplus\mathcal{H}$ is a disjoint union $\mathbb{CP}^1\sqcup\mathbb{CP}^1\sqcup\mathbb{CP}^1$. The projection to $\Delta_{\mathscr{F}+}$ consists of the segments AB, CD, EF.*

Corollary 16. *The ITV inside $\mathscr{F}^+$ consisting of MS's on N of class $[\![\nu\lambda^{0,2}]\!]\oplus2[\![\nu\lambda_0^{1,1}]\!]\oplus2\mathcal{H}$ is a disjoint union $\mathbb{CP}^1\sqcup\mathbb{CP}^2\sqcup(\mathbb{CP}^1\rtimes\mathbb{CP}^1)$.*



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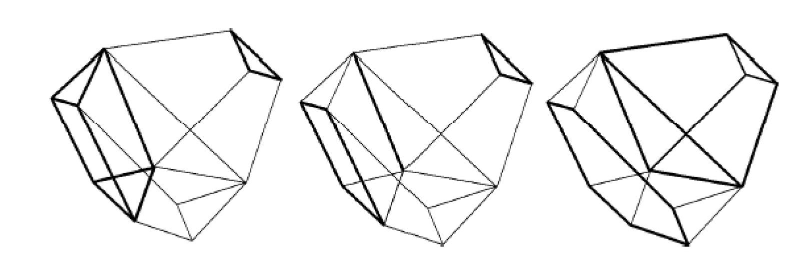

G. Mihaylov. Toric moment mappings and riemannian structures. *Geometriae Dedicata*, ISSN 0046-5755, DOI 10.1007/s10711-012-9720-6, 2012.