G-structures on 8-manifolds

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If an oriented (M^n, g) *admits a global 3-form in an open orbit, then* $n = 6, 7, 8$ *and* (M^n, g) *admits a* G*-structure with* G *one of the compact groups*

$$\mathrm{SU}(3), \quad G_2, \quad \mathrm{PSU}(3),$$

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Fact (Witt 2005/2008)

The $\mathrm{GL}(8)$*-orbit of* $f \in \Lambda^3(\mathbb{R}^8)$*, defining the structure constants,*

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(Almost) every physicist knows a 3-form in an open $\mathrm{GL}(8)$*-orbit.*

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Proposition (Cartan 1925)

Any metric connection ∇ is uniquely determined by its torsion tensor T,

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + A(X,Y,Z),$$

$$A(X,Y,Z) := \frac{1}{2}\big(T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y)\big).$$

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Proposition (Cartan 1925)

If $n \geq 3$*, the space* $\mathcal{T} := \Lambda^2(TM^n) \otimes TM^n$ *of possible torsion tensors splits into* 3 *irreducible* $\mathrm{O}(n)$*-modules,* $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$*,*

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Lemma

A metric connection is compatible with the G-structure iff the corresponding torsion tensor satisfies

$$\Gamma(X) = -\mathrm{pr}_{\mathfrak{g}^\perp}(A(X,\cdot,\cdot)).$$

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Alternative definition and Fernández classes

Definition

A Spin(7)-structure is a Riemannian manifold (M^8, g) equipped with a 4-form Φ s.t. there exists an oriented ONF $(e_1, \ldots, e_8)$ realizing

$$\Phi = e_{1278} + e_{3478} + e_{5678} + e_{2468} - e_{2358} - e_{1458} - e_{1368} + e_{3456} + e_{1256} + e_{1234} + e_{1357} - e_{1467} - e_{2367} - e_{2457}.$$

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Theorem (Ivanov 2004)

Any Spin(7)*-structure* (M^8, g, Φ) *admits a unique compatible connection* ∇^c *with totally skew-symmetric torsion*

$$T^c = -\delta\Phi - \frac{7}{6} * (\theta \wedge \Phi), \quad \theta = \frac{1}{7} * (\delta\Phi \wedge \Phi).$$

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Proposition (Cabrera 1995, P' 2009)

Let (M^8, g, Φ) *be a* Spin(7)*-structure. Then*

(M^8, g, Φ) *is*	*if and only if*	*or equivalently if*
parallel	$T^c_8 = 0,\ T^c_{48} = 0$	$d\Phi = 0,\ \theta = 0$
locally conformal parallel	$T^c_{48} = 0$	$d\Phi = \theta \wedge \Phi$
balanced	$T^c_8 = 0$	$\theta = 0$

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If $\nabla^c T^c = 0$*, any* ∇^c*-parallel spinor field* Ψ *on* (M^8, g, Φ) *satisfies*

$$(T^c)^2 \cdot \Psi = 7\|T^c_8\|^2 \cdot \Psi, \quad -4\,\mathrm{Ric}^c(X) \cdot \Psi = \left((T^c)^2 - 7\|T^c_8\|^2\right) \cdot X \cdot \Psi. \quad (*)$$

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There exists at least one ∇^c-parallel spinor field Ψ_0 on (M^8, g, Φ),

$$\Phi \cdot \Psi_0 = -14 \Psi_0.$$

Algebraic classification

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Classification recipe:

(1) Fix $\mathfrak{h} = \mathfrak{hol}(\nabla^c) \subseteq \mathfrak{iso}(T^c) = \mathfrak{g}$ with $\mathfrak{g} \subsetneq \mathfrak{spin}(7)$ non-Abelian.

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(2) Determine the spaces of $\mathfrak{h}$-invariant spinors and $\mathfrak{g}$-invariant 3-forms.

(3) Solve equations $(*)$ on these spaces.

Results

Proposition

Only the following isotropy algebrae $\mathfrak{iso}(T^c)$ *allow to carry out (1) to (3) consistently:*

$\mathfrak{g}_2$, $\mathfrak{su}(3)$, $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$, $\mathfrak{u}(2)$, $\mathbb{R} \oplus \mathfrak{su}(2)$	$\mathbb{R} \oplus \mathfrak{su}_c(2)$, $\mathfrak{so}(3)$

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Proposition

There exist at least two non-zero ∇^c*-parallel spinor fields on non-parallel* $\mathrm{Spin}(7)$*-manifolds with* $\nabla^c T^c = 0$ *and* $\mathfrak{iso}(T^c)$ *non-Abelian.*

Theorem (P' 2009)

Let (M^8, g, Φ) *be a complete, simply connected* $\mathrm{Spin}(7)$*-manifold with* $\nabla^c T^c = 0$ *and* $\mathfrak{iso}(T^c)$ *equal to*

a) $\mathfrak{g}_2$ *or* $\mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$.

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Theorem (continuation)

Moreover, any $\mathrm{Spin}(7)$*-manifold with* $\nabla^c T^c = 0$ *and* $\mathfrak{iso}(T^c) = \mathbb{R} \oplus \mathfrak{su}_c(2)$ *is locally isometric to a naturally reductive homogeneous space.*

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Let (M^n, g, T) *be a complete, simply connected Riemannian manifold with 3-form* T*. Suppose that the tangent bundle*

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splits under the action of the holonomy group of $\nabla_X Y = \nabla^g_X Y + \frac{1}{2} \cdot T(X, Y, \cdot)$ *so that*

$$T(X_+, X_-, \cdot) = 0, \qquad T(X_+, Y_+, \cdot) \in TM_+, \qquad T(X_-, Y_-, \cdot) \in TM_-.$$

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Let $T = T_+ + T_-$ *denote the corresponding decomposition of* T*. Then* (M, g, T) *is isometric to a Riemannian product* $(M_+, g_+, T_+) \times (M_-, g_-, T_-)$*.*

Outline

Definitions

Definition

A PSU(3)-structure is a Riemannian manifold (M^8, g) equipped with a 3-form ρ s.t. there exists an oriented ONF $(e_1, \dots, e_8)$ realizing

$$\rho = e_{246} - e_{235} - e_{145} - e_{136} + (e_{12} + e_{34} - 2\,e_{56}) \wedge e_7 + \sqrt{3}(e_{12} - e_{34}) \wedge e_8$$

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$$TM^8 \otimes \mathfrak{psu}(3)^{\perp} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6.$$

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A $\mathrm{PSU}(3)$-structure (M^8, g, ρ) is said to be of class $\mathcal{W}_{i_1} \oplus \ldots \oplus \mathcal{W}_{i_k}$ if

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Classes vs. differential equations

Proposition

Let (M^8, g, ρ) *be a* PSU(3)*-structure. Then*

(M^8, g, ρ) *is of class*	*if and only if* ρ *satisfies*
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$	$\rho \lrcorner * d\rho = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$	$6\,\delta\rho = (\delta\rho \lrcorner \rho) \lrcorner \rho$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$	$\delta\rho \lrcorner \rho = 0$ *or* $\rho \lrcorner d\rho = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$	$\delta\rho = 0$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_6$	$10 * d\rho = \rho \wedge (\rho \lrcorner * d\rho)$
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We now restrict to non-integrable (i.e. $\Gamma \neq 0$) PSU(3)-structures of class $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_5$.

Compatible connections

Via PSU(3)-equivariant isomorphisms we identify

$$\Gamma_1 + \Gamma_2 + \Gamma_3 \quad \text{with} \quad T_\Gamma \in \Lambda^3, \qquad \Gamma_4 + \Gamma_5 \quad \text{with} \quad F_\Gamma \in \Lambda^4.$$

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Any PSU(3)-structures of class $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_5$ admits a (unique) compatible connection ∇^c with torsion tensor

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Link to Spin(7)-structures and Bianchi identity

Proposition

Any Spin(7)*-manifold* $(M^8, g, \Phi, \overline{\nabla^c}, \overline{T^c})$ *with* $\overline{\nabla^c}\,\overline{T^c} = 0$ *and* $\mathfrak{iso}(\overline{T^c}) \subseteq \mathbb{R} \oplus \mathfrak{su}_c(2)$ *admits a* PSU(3)*-structure* $(M^8, g, \rho, \nabla^c, T^c)$ *of class* $\mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_3$ *with*

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$$\mathfrak{S}_{X,Y,Z}\mathrm{R}^c(X,Y,Z,V) = \sum_i \big((e_i \lrcorner T_\Gamma) - ((e_i \lrcorner \rho) \lrcorner F_\Gamma)\big) \wedge \\ \big((e_i \lrcorner V \lrcorner T_\Gamma) - (e_i \lrcorner ((V \lrcorner \rho) \lrcorner F_\Gamma))\big)(X,Y,Z).$$

Main results

Theorem (P' 2012)

Let (M^8, g, ρ) *be a* $\mathrm{PSU}(3)$*-structure of class* $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_5$ *with* $\nabla^c T^c = 0$ *and* $\mathfrak{hol}(\nabla^c) = \mathbb{R} \oplus \mathfrak{su}_c(2), \mathfrak{su}_c(2), \mathfrak{t}^2$.

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a) *is of class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ and*

b) *admits a $\mathrm{Spin}(7)$-structure preserved by ∇^c.*

Moreover, if (M^8, g, ρ) is regular, it is a principal S^1-bundle and a Riemannian submersion over a co-calibrated G_2-manifold $(\overline{N}, \overline{g}, \overline{\varphi}, \overline{\nabla^c}, \overline{T^c})$ with $\overline{\nabla^c}\, \overline{T^c} = 0$ and $\mathfrak{hol}(\overline{\nabla^c}) \subseteq \mathbb{R} \oplus \mathfrak{su}_c(2)$.

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Proposition (Friedrich 2009)

There exists a unique simply connected, complete co-calibrated G_2*-manifold* $\overline{N}$ *with* $\overline{\nabla^c}\,\overline{T^c} = 0$ *and* $\mathfrak{hol}(\overline{\nabla^c}) = \mathfrak{su}_c(2)$.

Main results

Theorem (P' 2012)

Let (M^8, g, ρ) *be a* PSU(3)*-structure of class* $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_5$ *with* $\nabla^c T^c = 0$ *and* $\mathfrak{hol}(\nabla^c) = \mathbb{R} \oplus \mathfrak{su}_c(2), \mathfrak{su}_c(2), \mathfrak{t}^2$. *Then* (M^8, g, ρ)

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Remark

In all considered cases, T^c *is either totally skew-symmetric or traceless cyclic.*

Proof (for the case $\mathfrak{hol}(\nabla^c) = \mathbb{R} \oplus \mathfrak{su}_c(2)$)

i) *The following are globally well defined and ∇^c-parallel*

$$e_8, \quad \omega := e_8 \lrcorner \rho, \quad \varphi_1 := e_{246} - e_{235} - e_{145} - e_{136} + e_{127} + e_{347}, \quad \varphi_2 := e_{567}$$

and $\Phi = (\varphi_1 + \varphi_2) \wedge e_8 + *((\varphi_1 + \varphi_2) \wedge e_8)$.

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$$T_\Gamma = a_1(\varphi_1 + 3\varphi_2) + a_2(\omega \wedge e_8 + 3\varphi_2), \quad F_\Gamma = 0, \quad (a_1, a_2) \in A \subsetneq \mathbb{R}^2.$$

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$$\mathscr{L}_{e_8}\omega, \quad \mathscr{L}_{e_8}\varphi_i, \quad \mathscr{L}_{e_8}(e_8 \lrcorner * \varphi_i) = 0, \quad i = 1, 2.$$

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ix) *Consequently,* $\overline{N}$ *is a co-calibrated* G_2*-manifold with fund. form* $\overline{\varphi}$. □

Construction

i) Start from a co-calibrated G_2-manifold $(\overline{N}, \overline{g}, \overline{\varphi}, \overline{\nabla^c}, \overline{T^c})$ with $\overline{\nabla^c}\,\overline{T^c} = 0$ and $\mathfrak{hol}(\overline{\nabla^c}) = \mathbb{R} \oplus \mathfrak{su}_c(2), \mathfrak{su}_c(2)$.

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iii) Suppose the equation $de_8 = a_2\omega$ defines a principal S^1-bundle $\pi : M^8 \to \overline{N}$.

iv) Then, M^8 admits a $\mathrm{PSU}(3)$-structure

$$\rho = \pi^*(\overline{\varphi_1}) - 2\pi^*(\overline{\varphi_2}) + \pi^*(\overline{\omega}) \wedge e_8$$

with parallel torsion

$$T^c = a_1(\pi^*(\overline{\varphi_1}) + 3\pi^*(\overline{\varphi_2})) + a_2(\pi^*(\overline{\omega}) \wedge e_8 + 3\pi^*(\overline{\varphi_2}))$$

and $\mathfrak{hol}(\nabla^c) \subseteq \mathbb{R} \oplus \mathfrak{su}_c(2)$.