Geometric formality of some homogeneous spaces

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based on joint works with DIETER KOTSCHICK

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Content of the talk

- Notion of geometric formality
- Examples of non-geometrically formal homogeneous spaces
- Examples of geometrically formal homogeneous spaces

Notion of geometric formality

(M, g) - Riemannian manifold, $\Omega^*(M)$ -de Rham algebra of differential forms;

$\omega \in \Omega^k(M)$ is **harmonic** if

$$\Delta\omega = d\delta\omega + \delta d\omega = (d+\delta)^2\omega = 0$$

d – exterior derivative; δ – coderivative; Δ - Laplace-de Rham operator

Notion of geometric formality

In more detail:

$\langle , \rangle : \Omega^k(M) \to R$ - scalar product:

$$\langle \alpha, \beta \rangle = \int_M g(\alpha_x, \beta_x) dvol_g \ .$$

The *Hodge star operator*

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

is defined with

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dvol_g .$$

Notion of geometric formality

If $\alpha \in \Omega^{k-1}$ and $\beta \in \Omega^{k}$ then

$$\langle d\alpha, \beta \rangle = (-1)^{k} \langle \alpha, *^{-1} d * \beta \rangle.$$

$\delta = (-1)^{k} *^{-1} d*$ is conjugate to d in the space of k - forms.

$*^{-1} = (-1)^{(n-k+1)(k-1)} * \Rightarrow \delta = (-1)^{nk+n+1} * d*$.

Notion of geometric formality

$\Upsilon(M, g) \subseteq \Omega^*(M)$ – graded linear subspace of harmonic forms;

Well known properties of harmonic forms:

- $\alpha \in \Upsilon(M, g) \implies d\alpha = 0$ (α is closed);
- No harmonic form is a exact: $d\omega \neq \alpha$ for any $\omega \in \Omega$.
- **Hodge theorem**: Any cohomology class $[\omega] \in H^*(M, R)$ has unique harmonic representative:

$$\Upsilon(M, g) \to H^*(M, R) \quad \text{is bijection.}$$

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<u>Question:</u> Is $(\Upsilon(M, g), \wedge)$ ($\wedge$ - exterior product) is an algebra?

Affirmative answer to question $\Longrightarrow (\Upsilon(M, g), \wedge) \cong (H^*(M, R), \wedge)$

Definition

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Example

- Real cohomology spheres are geometrically formal;
- Symmetric spaces G/H are geometrically formal related to an invariant metric g;

Proof: $\omega \in \Omega^G(G/H) \Rightarrow d\omega = 0$ and $\omega \neq d\alpha$ for $\alpha \in \Omega^*(G/H)$

$$\Omega^G(G/H) \equiv \Upsilon(G/H, g)$$

.

Relation to rational formality

Definition

A manifold M is formal in the sense of rational homotopy theory if $\Omega^*(M)$ is weakly equivalent to $H^*(M,R)$:

$$(\Omega^*(M), d) \leftarrow (C, d) \rightarrow (H^*(M), d = 0),$$

where the both homomorphisms induce isomorphisms in cohomology.

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Well known examples of formal spaces:

- Any manifold having free cohomology algebra is formal.
- Any Kaehler manifold is formal.
- Compact symmetric spaces are formal.
  The first proof: Take $(C, d) = (\Upsilon(G/H), 0)$ — (geometrical proof).

Relation to rational formality

Formality createrion for compact homogeneous spaces:

Theorem

G/H is formal if and only if it is of Cartan type:

$$H^*(G/H, R) \cong R^{W_H}[s]/\langle P_1, \ldots, P_k\rangle \otimes \wedge(z_{k+1}, \ldots, z_n),$$

where $P_1, \ldots, P_k$ are functionally independent and $rkH = k$, $rkG = n$.

- s - Cartan algebra for H
- $H^*(BG, R) \cong R[t]^{W_H} \cong \langle P_1, \ldots, P_n\rangle$, $\deg P_i = 2k_i - 1$

Relation to rational formality

It is then proved:

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Theorem

All generalised symmetric spaces are formal.

- (G, H, θ) $\theta \in Aut(G)$ $\theta^k = Id$, $k \geq 2$ $G_o^\theta \subseteq H \subseteq G$
- G -semisimple and simply connected $\Rightarrow (G, H, \theta) = (G, G^\theta, \theta)$
- In this case there is bijection with generalised symmetric algebras (g, g^θ, θ)

— Purely topological proof of formality!

Relation to rational formality

- Geometrically formal manifold M is formal:

$$(\Omega(M), d) \leftarrow (\Upsilon(M), d) \rightarrow (H^*(M), d = 0).$$

- Vice versa is not true.

Geometric formality

Results of D. Kotschick:

Theorem

Let closed orineted M^n be geometrically formal:

1. $b_k(M^n) \leq b_k(T^n)$;
2. $n = 4m \Rightarrow b^{\pm}_{2m}(M^n) \leq b^{\pm}_{2m}(T^n)$;
3. $b_1(M^n) \neq n - 1$.

Theorem

If M^n is closed orineted which fibers smoothly over S^1 aand $b_1(M^n) = 1$, $b_k(M^n) = 0,\ 1 < k < n - 1$ then M is geometrically formal.

Theorem

All closed oriented geometrically formal manifolds of dimension ≤ 4 have real cohomology of compact symmetric spaces.

Corollary

A closed oriented three (four)-manifold with $b_1(M) = 1$ (and $b_2(M) = 0$) is geometrically formal iff it fibers over S^1.

$$H^*(M) \cong H^*(S^1) \otimes H^*(S^2)(H^*(S^3))$$

This gives many non-symmetric examples of geometrically formal three and four manifolds.

Joint results with D. Kotschick

Seach for geometrically formal manifolds in the class of:

- k-symmetric spaces;
- isotropy irreducible spaces;
- fibrations with geometrically formal base and the fiber;
- homogeneous spaces or biquotients with the "nice" metrics.

Non-geometrically formal exmaples

M is geometrically formal $\Rightarrow$ all cohomology relations hold for the harmonic forms.

Theorem

There are no formal metrics on complete flag manifolds $U(n)/T^n$, $Sp(n)/T^n$, $SO(2n)/T^n$, $SO(2n+1)/T^n$.

Theorem

There are no formal metrics on $SU(2n+1)/T^n$, $SU(2n)/T^n$, $Spin(2n+2)/T^n$.

Remark:

- The cohomology ring structure is obstruction.
- They are all k-symmetric spaces.
- $CP^1 \to U(3)/T^3 \to CP^2$

Partial flag manifolds

- $F_n = SU(n+1)/S(U(n) \times U(1) \times U(1))$; 3 - symmetric space;
- $F_n = \{(L, P) | P$ a 2-plane in C^{n+2}, L a line in $P\}$;

$F_n \to L$ gives fibration $p : F_n \to CP^{n+1} \Rightarrow F_n$ is projectivized tangent bundle of CP^{n+1};
$H^*(F_n, R)$ is generated by two elements x and y of degree 2 with:

$$x^{n+2} = 0, \quad \frac{(x+y)^{n+2} - x^{n+2}}{y} = 0.$$

Theorem

Any closed oriented manifold M with cohomology ring of F_n is not geometrically formal.

Isotropy irreducible spaces

Isotropy irreducible homogeneous spaces need not to be formal.

Example

- $SU(pq)/(SU(p) \times SU(q))$ for $p, q \geq 3$
- $SO(78)/E_6$
- $SO(n^2 - 1)/SU(n)$ for $n \geq 3$

are strongly isotropy irreducible, but are not formal (not Cartan type) $\rightarrow$ not geometrically formal.

S^2-bundle over CP^2

Theorem

Let M^6 be the total space of an S^2-bundle over CP^2. It is geometrically formal if and only if it is the trivial bundle $S^2 \times CP^2$.

- The orientation-preserving diffeomorphism group of S^2 is homotopy equivalent to $SO(3) \Rightarrow$ the bundle structure group is $SO(3)$.
- $SO(3) = PU(2) \Rightarrow$ every S^2-bundle is the projectivisation of a complex rank 2 vector bundle E.
- $$H^*(M^6, R) = R[x, y], \ \ degx = degy = 2,$$

  x is pulled back from CP^2 and y restricts as a generator to every fiber. We choose y so that

  $$y^2 + c_1(E)xy + c_2(E)x^2 = 0,$$

  where $c_i(E) = \langle c_i(E), [CP^i] \rangle$ are the Chern numbers . We also have $x^3 = 0 \Rightarrow Kerx \neq 0$.

S^2-bundle over CP^2

- We do the base change to obtain $y^2 + cx^2 = 0$, where c vanishes iff M is trivial bundle.
- $x^3 = 0 \Rightarrow xy^2 = 0 \Rightarrow y^3 \neq 0$ for $c \neq 0$.
- If $c \neq 0$ and M is geometrically formal then y is nondegenerate.
- But $i_v(y^2) = 2i_v y \wedge y = 0$ for $v \in Ker x$- contradiction!

Eschenburg's and Totaro's biquotients

Source of examples having positive sectional curvature.

Example

$G = SU(3)$,
$U = T^1 \times T^1 = \{D(a^k, a^l, a^{-k-l}), D(b^m, b^n, b^{-n-m}),\ a, b \in S^1\}$

$$H^*(G/U) = R[x, y]/\left\langle x^2 = y^2, x^3 = y^3 \right\rangle$$

— M geometrically formal $\Rightarrow x + y$ is a symplectic form.
— Since $(x + y)(x - y) = 0 \Rightarrow x - y$ vanishes - contradiction.

Example

Totaro's biquotients $M = (S^3)^3/(S^1)^3$ are with infinitely many rational cohomolog rings.
$H^*(M)$ has three generators x_1, x_2, x_3 in degree 2 with relations:

$$x_1^2 = 0,\ x_2(ax_1 + x_3 + x_2) = 0,\ x_3(bx_1 + 2x_2 + x_3) = 0.\ a, b \in Z.$$

None of them is geometrically formal (each cohomology ring gives obstruction).

Aloff-Wallach spaces $N_{k,l} = SU(3)/T^1$

- $T^1 \subset SU(3)$ as $D(z^k, z^l, z^{-k-l})$, k, l-coprime integers with $kl(k+l) \neq 0$;
- Normal homogeneous metric on $N_{k,l}$: submersion metric for a biinvariant metric on $SU(3)$ from principal circle fibration $T^1 \to SU(3) \to N_{k,l}$;
- It is with positive sectional curvature (proved by Aloff and Wallach);

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- It is with positive sectional curvature (proved by Aloff and Wallach);
- $N_{k,l}$ has real cohomology of $S^2 \times S^5 \Longrightarrow$ no cohomology obstructions to geometric formality;

Theorem

The normal homogeneous metrics on Aloff-Wallach spaces are not formal.

- Assume $N_{k,l}$ — geometrically formal $\Rightarrow$ for harmonic ω_2, ω_5 we have $\omega_2^2 = 0$ and $\omega_2 \wedge \omega_5$ is a volume form.
- $e = \lambda[\omega_2]$, $\lambda \neq 0$ - Euler class of this principal bundle, normalize to get $\lambda = 1$;
- $\eta_2 = \pi^*(\omega_2) = d\alpha$, for connection form α;
- $\eta_5 = \pi^*(\omega_5) \Rightarrow \eta_2^2 = 0$, $*\eta_5 = \alpha \wedge \eta_2$, $d(*\eta_5) = 0$;
- $\eta_3 = *\eta_5$ - harmonic on $SU(3) \Rightarrow \eta_3(X, Y, Z) = \langle X, [Y, Z]\rangle$;
- There exists $K \subseteq T_e(SU(3)/T^1)$, $\dim K = 5$ such that $i_v(\eta_3) = 0$ for $v \in K$;
- Let $H_1, H_2, E_1, E_2, F_1, F_2$ - canonical (Chevalley) generators for $SU(3)$;
- Let L is spanned by $H_1, H_2, E_1, F_1 \Rightarrow i_x(\eta_3) \neq 0$ for $x \in L$;
- $K \cap L = 0$ - impossible for dimension reasons.

Examples of geom. formal homogeneous spaces

Harmonic forms of an invariant metric on G/H are invariant:

h - harmonic $\Rightarrow h = h_i + d\alpha$, $*h = (*h)_i + d\beta$ for
$h_i = \int_G g^* h$ and $(*h)_i = \int_G g^*(*h)$; h_i and $(*h)_i$ are invariant $\Rightarrow$

$$*h_i = *\int_G g^* h = \int_G *(g^* h) = \int_G g^*(*h) = (*h)_i$$

$\Rightarrow h_i$ is harmonic $\Rightarrow h = h_i$.

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Lemma

Let $H^*(G/H, R) = \wedge(x, y)$ where x and y are of odd degrees. Then any homogeneous metric on G/H is geometrically formal.

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Lemma

Let $H^*(G/H, R) = \wedge(x, y)$ where x and y are of odd degrees. Then any homogeneous metric on G/H is geometrically formal.

Lemma

$SU(4)$ acts transitively on $S^5 \times S^7$ with isotropy subgroup $SU(2)$. All $SU(4)$-homogeneous metrics for this action are formal. Furthermore, the normal homogeneous metrics are not symmetric.

Stiefel manifolds

Theorem

All homogeneous metrics on the following homogeneous spaces are geometrically formal:

1. *the real Stiefel manifolds* $V_4(R^{2n+1}) = SO(2n+1)/SO(2n-3)$ *for* $n \geq 3$,
2. *the real Stiefel manifolds* $V_3(R^{2n}) = SO(2n)/SO(2n-3)$ *for* $n \geq 3$,
3. *the complex Stiefel manifolds* $V_2(C^n) = SU(n)/SU(n-2)$, *for* $n \geq 5$,
4. *quaternionic Stiefel manifolds* $V_2(H^n) = Sp(n)/Sp(n-2)$, *for* $n \geq 3$,
5. *the octonian Stiefel manifold* $V_2(O^2) = Spin(9)/G_2$, *and*
6. *the space* $Spin(10)/Spin(7)$.

Moreover, none of these spaces is homotopy equivalent to a symmetric space. They are not homotopy equivalent to products of real cohomology spheres, except possibly for $V_3(R^{2n})$ *with* n *even.*

- By James-Whitehead: if $V_2(C^n)$ is homotopy equivalent to $S^{2n-1} \times S^{2n-3}$ then $\pi_{4n-1}(S^{2n})$ contains an element of Hopf invariant one;
- By Adams's result it follows $n \in \{1, 2, 4\}$;
- If $V_2(C^n) \approx X_1 \times X_2$, where X_i are real homology spheres $\Rightarrow X_i$ are homotopy spheres $\Rightarrow n = 4$. But, $V_2(C^4)$ is an S^5-bundle over S^7 with structure group $SU(4)$. The clutching element is from $\pi_6(SU(4)) = 0 \Rightarrow V_2(C^4)$ is diffeomorphic to $S^5 \times S^7$.
- Non homotopy equivalence to symmetric space follows using classification of homogeneous spaces having cohomology of odd-dimensional spheres (given by Onischick or Kramer).