

HyperKähler Manifolds

Definition. A **HyperKähler manifold** is a Riemann manifold endowed with automorphisms I, J, K of the tangent bundle and satisfying:

- $\nabla I = \nabla J = \nabla K = 0$, where ∇ is the Levi-Civita connection
- $I^2 = J^2 = K^2 = IJK = -1$

Example 1. The quaternionic vector space \mathbb{H}^n .

Example 2. 4-dimensional Calabi-Yau manifolds.

Example 3. **Swann bundles** [1] Swann details a construction of hyperKähler manifolds (Swann bundles), over quaternionic Kähler manifolds with positive curvature, having a hyperKähler potential.

Generalized Dirac Operator

$C\text{Spin}_G^\epsilon$ -structure

Additionally, to introduce generalized Seiberg-Witten equations we need a **hyperKähler** action of a compact Lie group G .

Let $\epsilon \in Z(G)$. The element $(-1, \epsilon) \in \text{Spin}(4) \times G$ generates a normal subgroup of order 2, which we denote by ± 1 .

$$\hat{G} := \text{Spin}_G^\epsilon(4) := \text{Spin}(4) \times_{\pm 1} G = \text{Spin}(4) \times_{\pm 1} G.$$

We have the exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Spin}_G^\epsilon(4) \xrightarrow{\lambda^G} \text{SO}(4) \times G/\epsilon \rightarrow 1$$

Principal $\text{Spin}_G^\epsilon(4)$ - bundle is λ^G -reduction
 $\pi : Q \rightarrow P_{\text{SO}(4)} \times_X P_{G/\epsilon}$.

The idea of generalization: Replace the fibre \mathbb{H} by a hyperKähler manifold M with **permuting action** $\text{Sp}(1) \curvearrowright M$. An example of the requisite hyperKähler manifolds \implies **Swann bundles**. We define the set $C^\infty(Q, M)^{\hat{G}}$ to be the set of **generalized spinors**. The covariant derivative of a generalized spinor $u \in C^\infty(Q, M)^{\hat{G}}$ is given by:

$$D_A u = Tu + \mathcal{K}_A^M|_u \in C^\infty(Q, (\mathbb{R}^4)^* \otimes u^* E^+)^{\hat{G}}$$

Composed with the Clifford multiplication \mathfrak{c} , we get the **(nonlinear!) generalized Dirac operator**:

$$\mathcal{D}_A u \in C^\infty(Q, u^* E^-)^{\hat{G}}.$$

Generalized Seiberg-Witten equations

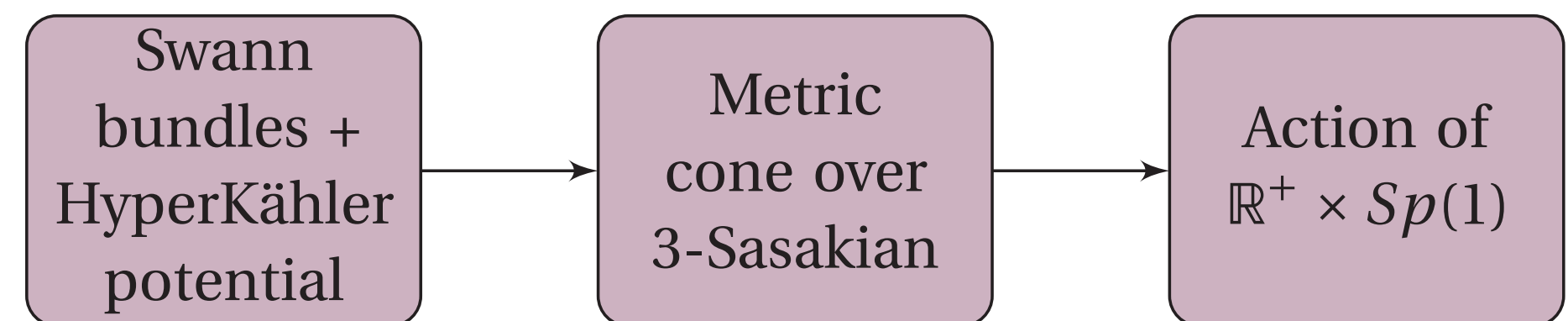
Let μ_G denote the **hyperKähler moment map** for the G action on the hyperKähler manifold. The generalized Seiberg-Witten equations are now given by

$$\begin{cases} \mathcal{D}_A u = 0 \\ F_A - \mu_G \circ u = 0 \end{cases}$$

We are interested in the case when M is a Swann bundle and the group $G = \text{SU}(n)$.

Conformal Properties

Restricting to the case of **Swann bundles**:



In the usual case, under the conformal change of metric $g \rightsquigarrow g' = e^{2f}g$ on the base manifold, the Dirac operator transforms as

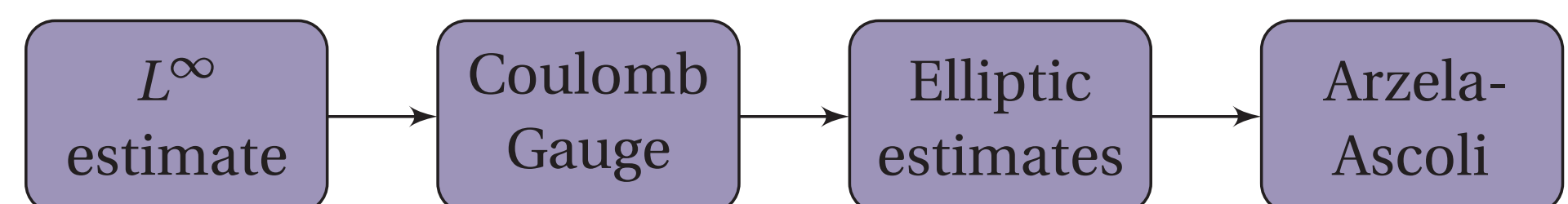
$$\mathcal{D}_{A'}(\bar{u}) = \overline{e^{-5/2f} \mathcal{D}_A(e^{3/2f} u)},$$

where the bar denotes the isomorphism between the corresponding frame bundles. The generalized Dirac operator exhibits a similar behaviour under the conformal change of metric i.e:

$$\mathcal{D}_{A'}(\bar{u}) = \overline{T e^{-5/2f} \mathcal{D}_A(e^{3/2f} u)}$$

L^∞ estimates

For the original S^1 - Seiberg-Witten equations:



In the case of generalized Seiberg-Witten equations, we restrict ourselves to the following case.

$$\begin{array}{ccc} \mathbb{H}^{n+1} \setminus \{0\} & \xrightarrow{U(1)} & \mathcal{O}_{n+1} \\ \downarrow \mathbb{H}^* & & \downarrow \mathbb{H}^* \mathbb{Z}/2 \\ \mathbb{H}\mathbb{P}^n & \xrightarrow{U(1)} & X(n-1) \end{array}$$

The Seiberg-Witten equations with the target manifold \mathcal{O}_{n+1} can be equivalently written as Seiberg-Witten equations with the target manifold as $\mathbb{H}^{n+1} \setminus \{0\}$ with an additional condition $\mu_{U(1)} \circ u = 0$. For a generalized spinor $u \in C^\infty(Q_{\hat{G}}, \mathbb{H}^{n+1} \setminus \{0\})$ with

$G = \text{SU}(n)$ we have the following result. Let $u \in C^\infty(Q_{\hat{G}}, \mathbb{H}^{n+1} \setminus \{0\})$ be a solution to the generalized Seiberg-Witten equations. Then we have the a-priori estimate

$$\|u\|_\infty \leq \max\left\{0, \frac{-1}{16} \min_{x \in X} s_X(x)\right\}$$

A look ahead: Exploiting this behaviour, can we have Uhlenbeck-like compactness for the generalized Seiberg-Witten equations in the case where $G = U(1)$? Motivating examples for this are the $PU(2)$ -monopole equations and [3].

References

- 1 Andrew Swann, HyperKähler and Quaternionic Kähler Geometry, *Mathematische Annalen* **289** (1991), no.3, 421-450.
2. V. Ya. Pidstrygach, HyperKähler manifolds and Seiberg-Witten Equations, *Proceedings of Steklov Institute of Mathematics* **246** (2004), 249-262.
3. Clifford H. Taubes, $\text{PSU}(2; \mathbb{C})$ connections on 3-manifolds with L^2 bounds on curvature, *arXiv:1205.0514v1 [math.DG]*