Conformal Properties of the Generalized Dirac Operator

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HyperKähler Manifolds

Definition. A HyperKähler manifold is a Riemann manifold endowed with automorphisms I,J,K of the tangent bundle and satisfying:

1. $\nabla I = \nabla J = \nabla K = 0$, where ∇ is the Levi-Civita connection
2. $I^2 = J^2 = K^2 = IJK = -1$

Example 1. The quaternionic vector space $\mathbb{H}^n$.

Example 2. 4-dimensional Calabi-Yau manifolds.

Example 3. Swann bundles [1] Swann details a construction of hyperKähler manifolds (Swann bundles), over quaternionic Kähler manifolds with positive curvature, having a hyperKähler potential.

Generalized Dirac Operator

$\mathrm{C\,Spin}^{\epsilon}_{G}$-structure

Additionally, to introduce generalized Seiberg-Witten equations we need a hyperKähler action of a compact Lie group G.
Let $\epsilon \in Z(G)$. The element $(-1,\epsilon) \in \mathrm{Spin}(4) \times G$ generates a normal subgroup of order 2, which we denote by ± 1.

$$\hat{G} := \mathrm{Spin}^{\epsilon}_{G}(4) := \mathrm{Spin}(4) \times_{\pm 1} G = \mathrm{Spin}(4) \times_{\pm 1} G.$$

We have the exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}^{\epsilon}_{G}(4) \xrightarrow{\lambda^G} \mathrm{SO}(4) \times G/\epsilon \to 1$$

Principal $\mathrm{Spin}^{\epsilon}_{G}(4)$- bundle is λ^G-reduction
$\pi : Q \longrightarrow P_{SO(4)} \times_X P_{G/\epsilon}$.

The idea of generalization: Replace the fibre $\mathbb{H}$ by a hyperKähler manifold M with permuting action $\mathrm{Sp}(1) \curvearrowright M$. An example of the requisite hyperKähler manifolds $\Longrightarrow$ Swann bundles. We define the set $C^{\infty}(Q,M)^{\hat{G}}$ to be the set of generalized spinors. The covariant derivative of a generalized spinor $u \in C^{\infty}(Q,M)^{\hat{G}}$ is given by:

$$D_A u = Tu + \mathcal{K}^M_A|_u \in C^{\infty}(Q, (\mathbb{R}^4)^* \otimes u^* E^+)^{\hat{G}}$$

Composed with the Clifford multiplication $\mathfrak{c}$, we get the (**nonlinear!**) generalized Dirac operator:

$$\mathfrak{D}_A u \in C^{\infty}(Q, u^* E^-)^{\hat{G}}.$$

Generalized Seiberg-Witten equations

Let μ_G denote the hyperKähler moment map for the G action on the hyperKähler manifold. The generalized Seiberg-Witten equations are now given by

$$\begin{cases} \mathfrak{D}_A u = 0 \\ F_A - \mu_G \circ u = 0 \end{cases}$$

We are interested in the case when M is a Swann bundle and the group $G = \mathrm{SU(n)}$.

Conformal Properties

Restricting to the case of Swann bundles:

Swann bundles + HyperKähler potential → Metric cone over 3-Sasakian → Action of $\mathbb{R}^+ \times Sp(1)$

In the usual case, under the conformal change of metric $g \rightsquigarrow g' = e^{2f} g$ on the base manifold, the Dirac operator transforms as

$$\mathfrak{D}_{A'}(\bar{u}) = \overline{e^{-5/2f}\mathfrak{D}_A(e^{3/2f}u)},$$

where the bar denotes the isomorphism between the corresponding frame bundles. The generalized Dirac operator exhibits a similar behaviour under the conformal change of metric i.e:

$$\mathfrak{D}_{A'}(\bar{u}) = \overline{Te^{-5/2f}\mathfrak{D}_A(e^{3/2f}u)}$$

L^{∞} estimates

For the original S^1- Seiberg-Witten equations:

L^{∞} estimate → Coulomb Gauge → Elliptic estimates → Arzela-Ascoli

In the case of generalized Seiberg-Witten equations, we restrict ourselves to the following case.

$$\begin{array}{ccc} \mathbb{H}^{n+1}\setminus\{0\} & \xrightarrow{U(1)} & \mathcal{O}_{n+1} \\ \downarrow \mathbb{H}^* & & \downarrow \mathbb{H}^*\ \mathbb{Z}/2 \\ \mathbb{HP}^n & \xrightarrow{U(1)} & X(n-1) \end{array}$$

The Seiberg-Witten equations with the target manifold $\mathcal{O}_{n+1}$ can be equivalently written as Seiberg-Witten equations with the target manifold as $\mathbb{H}^{n+1}\setminus\{0\}$ with an additional condition $\mu_{U(1)} \circ u = 0$. For a generalized spinor $u \in C^{\infty}(Q_{\hat{G}}, \mathbb{H}^{n+1}\setminus\{0\})$ with $G = SU(n)$ we have the following result. Let $u \in C^{\infty}(Q_{\hat{G}}, \mathbb{H}^{n+1}\setminus\{0\})$ be a solution to the generalized Seiber-Witten equations. Then we have the a-priori estimate

$$\|u\|_{\infty} \le \max\{0, \tfrac{-1}{16}\min_{x\in X} s_X(x)\}$$

A look ahead: Exploiting this behaviour, can we have Uhlenbeck-like compactness for the generalized Seiberg-Witten equations in the case where $G = U(1)$? Motivating examples for this are the $PU(2)$-monopole equations and [3].

References

1. 1 Andrew Swann, HyperKähler and Quaternionic Kähler Geometry, *Mathematische Annalen* **289** (1991), no.3, 421-450.
2. V. Ya. Pidstrygach, HyperKähler manifolds and Seiberg-Witten Equations,*Proceedings of Steklov Institute of Mathematics* **246** (2004), 249-262.
3. Clifford H. Taubes, PSl(2;ℂ) connections on 3-manifolds with L^2 bounds on curvature, *arXiv:1205.0514v1 [math.DG]*

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