The Calabi-Yau equation for T^2-fibrations

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In collaboration with
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The Calabi-Yau equation

Yau's Theorem [Symplectic version]. *Let* (M^n, J, Ω) *be a compact Kähler manifold and let* σ *be a volume form satisfying* $\int_M \Omega^n = \int_M \sigma$. *Then there exists a unique Kähler form* $\tilde{\omega} \in [\Omega]$ *such that*

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CY equation still makes sense on an almost Kähler (AK) manifold when J is non-integrable.

CY equation on 4-manifolds

Let (M, J, Ω) be a compact AK manifold with a volume form $\sigma = \mathrm{e}^{f}\,\Omega^{n}$ satisfying $\int_M \mathrm{e}^{f}\Omega^{n} = \int_M \Omega^{n}$.

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Question: Can the Yau's Theorem be generalized to AK 4-manifolds? (At least in the special case $b^+ = 1$)

Uniqueness of solutions

Proposition [Donaldson] *In dimension* 4 *solutions to the CY equation are unique.*

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Consider $\bar{\omega} = \omega_1 + \omega_2$. $\bar{\omega}$ is a symplectic form.

$$\bar{\omega} \wedge d\alpha = 0 \implies d\alpha = 0 \,. \qquad \textit{c.v.d.}$$

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Existence of solutions

Donaldson's Conjecture. *Let* (M, Ω, J, σ) *be a compact symplectic 4-manifold with an acs* J *tamed** *by* Ω *and a volume form.*
If $\tilde{\omega} \in [\Omega]$ *is a symplectic form on* M *which is compatible with* J *and solving the CY equation*

$$\tilde{\omega}^2 = \sigma$$

then there are C^∞ *a priori bounds on* $\tilde{\omega}$ *depending only on* Ω, J *and* σ*.*

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Applications:

- Yau's theorem holds on compact 4-dimensional AK manifolds with $b^+ = 1$.
- If $b^+(M) = 1$ and there exists Ω taming J, then there exists $\tilde{\Omega}$ which is compatible with J.

* $\Omega(J\cdot, \cdot) > 0$.

The Chern connection

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Theorem [Tosatti,Weinkove,Yau] *If* $\mathcal{R} > 0$, *then the Donaldson's conjecture holds.*

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Example: An infinitesimal deformation of the F-S structure on $\mathbb{CP}^n$.

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CY equation on the Kodaira-Thurston manifold

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M has a global left-invariant coframe $\{e^1, e^2, e^3, e^4\}$

$$de^i = 0\,, \quad i = 1, 2, 3\,, \quad de^4 = e^1 \wedge e^2\,, \quad (0, 0, 0, 12)\,.$$

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$b_1(M) = 3$ and M has no Kähler structures

[K] K.Kodaira, *Amer. J. Math.*, 1964

M is a T^2-bundle over a $\mathbb{T}^2$

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Theorem[Tosatti,Weinkove] *The CY equation on (M, Ω, g) can be solved for every T^2-invariant volume form σ.*

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Theorem[Tosatti,Weinkove] *The CY equation on (M, Ω, g) can be solved for every T^2-invariant volume form σ.*

Argument of the proof:

- Writing $\sigma = e^f\,\Omega^2$, then every solution $\tilde{\omega} = \Omega + d\alpha$ of the CY equation satisfies $\boxed{\mathrm{tr_g}\,\tilde{g} \leq \mathrm{Min}_M \Delta f}$
- The continuity method gives the result.

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CY equation on the Kodaira-Thurston manifold II

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Theorem [Li] The Monge-Ampère equation on the standard torus $\mathbb{T}^n$ has always solution.

[Li] Y.Y. Li, *Comm. Pure Appl. Math.*, 1990.

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Theorem [Ue] Every orientable T^2-bundle over a $\mathbb{T}^2$ is an *infra-solvmanifold*, i.e. a finite quotient of a solvmanifold.

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Lemma *Let $M = \tilde{\Gamma}\backslash G$ be a 4-dimensional infra-solvmanifold equipped with an invariant AK structure (J, Ω). Then condition $\mathcal{R} > 0$ holds if and only if J is integrable.*

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Lemma *Let $M = \tilde{\Gamma}\backslash G$ be a 4-dimensional infra-solvmanifold equipped with an invariant AK structure (J, Ω). Then condition $\mathcal{R} > 0$ holds if and only if J is integrable.*

• In particular the Tosatti-Weinkove-Yau theorem cannot be applied to the case of a T^2-bundle over a $\mathbb{T}^2$.

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The main result

Theorem [Fino, Li, Salamon, –/ Buzano, Fino, –] *Let M be a T^2-bundle over a $\mathbb{T}^2$ equipped with an invariant AK structure (Ω, J). Then for every T^2-invariant volume form $\sigma = e^f\,\Omega^2$, $f \in C^\infty(\mathbb{T}^2)$ the associated CY equation as a unique solution.*

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- Use the classification of orientable T^2-bundles over $\mathbb{T}^2$;
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- Classify in each case *invariant Lagrangian* AK structures and *invariant Symplectic* AK structures;
- Rewrite the problem in terms of a Monge-Ampère equation;
- Show that such an equation has solution.

- *Classification of T^2-bundles over $\mathbb{T}^2$*

T^2-bundles over $\mathbb{T}^2$ were classified by Sakamoto and Fukuhara.

[SK] K. Sakamoto, S. Fukuhara, *Tokyo J. Math.*, 1983.

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T^2-bundles over $\mathbb{T}^2$ were classified by Sakamoto and Fukuhara.

- Any T^2-bundle over $\mathbb{T}^2$ can be viewed as $M = \Gamma\backslash\mathbb{R}^4$, Γ is a lattice of a group G which acts on $\mathbb{R}^4$.

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- The possible groups are
  $SO(4) \ltimes \mathbb{R}^4$, $\quad Nil^3 \times \mathbb{R}$, $\quad Sol^3 \times \mathbb{R}$, $\quad Nil^4$.

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- T^2-bundles over $\mathbb{T}^2$ are classified in 9 families

[SK] K. Sakamoto, S. Fukuhara, *Tokyo J. Math.*, 1983.

The nine families

	G	Structure equations
i, *ii*	$SO(4) \ltimes \mathbb{R}^4$	$(0,0,0,0)$
iii	$Nil^3 \times \mathbb{R}$	$(0,0,0,12)$
iv, *v*	$Sol^3 \times \mathbb{R}$	$(0,0,13,41)$
vi, *vii*, *viii*	$Nil^3 \times \mathbb{R}$	$(0,0,0,12)$
ix	Nil^4	$(0,13,0,12)$

Theorem [Geiges] *Let M be the total space of an orientable T^2-bundle over a $\mathbb{T}^2$. Then*

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- *every $a \in H^2(M,\mathbb{R})$ can be represented by a symplectic form;*
- *M has a Kähler structure if and only if $G = \mathrm{SO}(4) \ltimes \mathbb{R}^4$ and in this case all the invariant AK structures are genuine Kähler structures;*

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Theorem [Geiges] *Let M be the total space of an orientable T^2-bundle over a $\mathbb{T}^2$. Then*

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The nine families

	G	Structure equations
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- *If $G=Sol^3\times\mathbb{R}$ every AK structure is non-Lagrangian.*

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- $G = Nil^3 \times \mathbb{R} \to f^1 \in \langle e^1 \rangle, \; g(e^3, f^2) = 0, \; g(e^3, f^3)g(e^4, f^4) \geq 0.$

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In this case all the total spaces are *nilmanifolds*, all the invariant AK structures are *Lagrangian* and we can work as in the Kodaira-Thurston manifold.

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In this case the total spaces could be *infra-nilmanifolds*, invariant AK structures could be either *Lagrangian* or non-Lagrangian and the argument used in the Kodaira-Thurston case has to be modified.

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i, ii	$SO(4) \ltimes \mathbb{R}^4$	$(0, 0, 0, 0)$
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In this case the total space could be an *infra-sovmanifold*, all invariant AK structures are *non-Lagrangian* and the CY equation reduces to a Monge-Ampère equation.

Geometry type $G = Nil^4$

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i, *ii*	$SO(4) \ltimes \mathbb{R}^4$	$(0, 0, 0, 0)$
iii	$Nil^3 \times \mathbb{R}$	$(0, 0, 0, 12)$
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In this case all total spaces are *nilmanifolds*, all invariant AK structures are *Lagrangian* and the CY reduces to the same Monge-Ampère equation for *Lagrangian* AK structures in the families *vi*), *vii*), *viii*) associated to $Nil^3 \times \mathbb{R}$.

- *The Monge-Ampère equation*

The following equation covers all cases

$$\boxed{A_{11}[u]A_{22}[u] - \big(A_{12}[u]\big)^2 = E_1 + E_2\,\mathrm{e}^f}$$

where

$$\begin{aligned} A_{11}[u] &= u_{xx} + B_{11}u_y + C_{11} + Du, \\ A_{12}[u] &= u_{xy} + B_{12}u_y + C_{12}, \\ A_{22}[u] &= u_{yy} + B_{22}u_y + C_{22}, \end{aligned}$$

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In the Lagrangian case $\boxed{D = 0}$

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Goal: Show that $\boxed{A_{11}[u]A_{22}[u] - \left(A_{12}[u]\right)^2 = E_1 + E_2\,\mathrm{e}^f}$ has a solution on $\mathbb{T}^2$.

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using the a priori estimate

$$\boxed{\|u\|_{C^2} \le 2(B_{11}+1)|B_{22}|\mathrm{e}^{2C_{22}} + C_{11} + C_{22}.}$$

The CY equation on the Kodaira-Thurston manifold

(Still in progress!)

Let (M, Ω, J) be the Kodaira-Thurston manifold with the standard AK structure and let

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f can be regarded as a map $f \in C^\infty(\mathbb{R}^4, \mathbb{R})$ such that

$$f(x, y, z, t) = f(x + n, y + m, z + k + ny, t + l) , \quad (n, m, k, l) \in \mathbb{Z}^4 .$$

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The CY equation writes as

$$Jda = da \iff \begin{cases} a_{2,y} + x a_{2,z} - a_{1,x} + a_4 = -a_{4,t} + a_{3,z}, \\ a_{4,y} + x a_{4,z} - a_{1,z} = -a_{3,x} + a_{2,t} \end{cases}$$

$$(\Omega + da)^2 = \mathrm{e}^f \iff \begin{aligned} &(1 + a_{3,y} + x a_{3,z} - a_{1,t})(1 - a_{4,x} + a_{2,z}) - \\ &\qquad - (-a_{4,t} + a_{3,z})^2 - (a_{3,x} - a_{2,t})^2 = \mathrm{e}^f . \end{aligned}$$

Theorem *The CY problem is equivalent to the following Monge-Ampère type equation*

$$\Big((\partial_y + x\partial_z)^2 u + \partial_t^2 u + (\partial_y + x\partial_z) B_3 u - \partial_t B_1 u + 1\Big)\Big(\partial_x^2 u + \partial_z^2 u + 1\Big) - \Big(\partial_x \partial_t u + (\partial_y + x\partial_z)\partial_z u + \partial_z B_3 u\Big)^2 - \Big((\partial_y + x\partial_z)\partial_x u - \partial_z \partial_t u + \partial_z u + \partial_x B_3 u\Big)^2 = \mathrm{e}^F$$

where B_1 and B_3 are linear operators solving

$$\begin{cases} \partial_x(B_1 u) + \partial_z(B_3 u) = -\partial_x u \\ \partial_x(B_3 u) - \partial_z(B_1 u) = -\partial_z u. \end{cases}$$

Open related problems

- Find a (generalized) $\partial\bar{\partial}$-lemma which ensures that the CY problem reduces to a Monge-Ampère equation.
- Find a proof of the main theorem in terms of a (modified) Ricci flow.
- Find examples / classify compact AK non-Kähler manifolds with $\mathcal{R} > 0$.