Gradient Ricci Solitons of Cohomogeneity One

joint work with A. Dancer, Oxford
and Stuart Hall, Buckingham

Rauischholzhausen Workshop, July 4, 2012

1. Basic Definitions and Facts

Ricci soliton: special solution of

Ricci flow equation $\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g)$

of form $g(t) = \lambda(t)\phi_t^*(g_0)$ where

ϕ_t is a 1-parameter family of diffeomorphisms with $\phi(0) = \mathrm{id}_M$

$\lambda(t)$ smooth function with $\lambda(0) = 1$ (scale change)

"Static" *Ricci soliton equation* for pair (g, X) on manifold M:

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g + \frac{\epsilon}{2} g = 0$$

where g is a *complete metric*,

X is a vector field

(necessarily complete, Z-H Zhang 2009)

$\epsilon = -\frac{\Lambda}{2}$ is a real constant

$\epsilon > 0$ *expanding soliton* $(\Lambda < 0)$

$\epsilon = 0$ *steady soliton* $(\Lambda = 0)$

$\epsilon < 0$ *shrinking soliton* $(\Lambda > 0)$

X Killing $\Longrightarrow g$ Einstein ("*trivial*" solitons)

In Einstein case, $\Lambda = -\frac{\epsilon}{2} \approx$ Einstein constant.

gradient Ricci soliton: special solution where

$X^\flat = du$

$u : M \longrightarrow \mathbb{R}$ (*soliton potential*)

static equation becomes

$$\mathrm{Ric}(g) + \mathrm{Hess}_g\,(u) + \frac{\epsilon}{2}\,g = 0$$

[Petersen-Wylie] g Einstein $\Longrightarrow$

Gaussian or du parallel

3. Cohomogeneity One GRS Equations

Assume compact Lie group G acts isometrically

on manifold M^{n+1} with

- orbit space an interval I (closed or half-open)
- generic (principal) orbit type G/K
- singular orbits G/H_i with $H_i/K \approx S^{k_i}$

Write metric as $\overline{g} = dt^2 + g_t$ where

g_t : a curve of G-invariant metrics on $P := G/K$

GRS equations become the system:

$$-(\delta^{\nabla^t} L_t)^\flat - d(\mathrm{tr} L_t) = 0 \qquad (1)$$

$$-\mathrm{tr}(\dot{L}_t) - \mathrm{tr}(L_t^2) + \ddot{u} + \frac{\epsilon}{2} = 0 \qquad (2)$$

$$\mathrm{ric}_t - \mathrm{tr}(L_t)L_t - \dot{L}_t + \dot{u}\, L_t + \frac{\epsilon}{2}\, \mathbb{I} = 0 \qquad (3)$$

where

- L_t is the *shape operator* of hypersurface P_t

- $\delta^{\nabla^t} : T^*(P) \otimes TP \to TP$ codifferential,

- ric_t is the *Ricci operator* of P_t defined by

$$\text{Ric}(g_t)(X, Y) = g_t(\text{ric}_t(X), Y)$$

Plus appropriate *boundary conditions* at endpoints of I to guarantee smoothness and completeness

Conservation Law: two formulations

$$\ddot{u} + (-\dot{u} + \mathrm{tr}L)\,\dot{u} = C + \epsilon u \text{ (R. Hamilton)} \quad (4)$$

$$\Leftrightarrow S_t + \mathrm{tr}(L^2) - (-\dot{u} + \mathrm{tr}(L))^2 + (n-1)\,\frac{\epsilon}{2} = C + \epsilon u$$

Useful Fact: (A. Back for Einstein case)

smoothness (e.g. C^3) of $\bar{g}, u$ + Eq. (3) +

codim$(G/H) \geq 2 \Longrightarrow$ Eq. (1)

above + conservation law $\Longrightarrow$ Eq. (2)

Hamiltonian Formulation:

$$\mathcal{C} = S^2_+(\mathfrak{p})^K \times \mathbb{R}$$

On $T^*\mathcal{C}$ (with canonical symplectic structure)

take Hamiltonian function

$$\mathcal{H} = v(q)e^{-u}\Big(\Big(2\langle L, L\rangle + \dot{u}^2 - 2\dot{u}\,\mathrm{tr}L\Big)$$

$$+ E - \epsilon(n+1-u) - S(q))$$

(from Perelman's $\mathcal{W}$-functional)

$v(q)$ relative volume, E Lagrangian multiplier

KE has Lorentz signature

Then integral curves in $\{\mathcal{H} = 0\}$ are equivalent to solutions of Eq. (2) and (3).

Initial Value Problem at Singular Orbit:

existence of a local solution (arbitrary ϵ) in a G-invariant nbd of singular orbit G/H with *prescribed metric and shape operator* on G/H

M. Buzano (JGP 2011)

under *assumption*: at special orbit G/H, the slice rep. and the isotropy rep. as K-reps, have no common irred. summands

4. Non-existence Result ($[DHW]$, after Böhm)

Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (Ad_K-invariant decomposition)

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r \qquad (5)$$

where $\mathfrak{p}_i$ is the sum of all equivalent Ad_K-irreducible summands of a fixed type. This decomposition is unique up to permutation of summands.

Special orbits G/H_i: $\mathfrak{h}_i = \mathfrak{s}_i \oplus \mathfrak{k}$, $\mathfrak{p} = \mathfrak{s}_i \oplus \mathfrak{q}_i$

Theorem 1. *Let M be a closed cohomogeneity one G-manifold as described above. Assume that some summand $\mathfrak{p}_{i_0}$ in (5) is actually Ad_K-irreducible and that for any G-invariant metric on G/K, the restriction to $\mathfrak{p}_{i_0}$ of its traceless Ricci tensor is always negative definite. Assume further that $\mathfrak{p}_{i_0} \cap \mathfrak{s}_j = \{0\}$ for $j = 1, 2$.*

Then there cannot be any G-invariant gradient Ricci soliton structure on $\overline{M}$.

Sketch of Proof:

Consider $\tilde{g} = v^{-\frac{2}{n}} g$, where $v := \sqrt{\det g_t}$

Set $F_i := \frac{1}{2}\ \mathrm{tr}_i(\tilde{g}^2)$. Then one computes that

$\ddot{F}_i + \xi \dot{F}_i = \mathrm{tr}_i(\dot{\tilde{g}}^2) + \mathrm{tr}_i(\dot{\tilde{g}} \tilde{g}^{-1} \dot{\tilde{g}} \tilde{g}) + 2\ \mathrm{tr}_i(\tilde{g}^2 r^{(0)})$

Pick i_0.

At the singular orbits, F_{i_0} tend to $+\infty$. So F_{i_0} has an interior minimum.

There, $\dot{F}_{i_0} = 0$, while $\ddot{F}_{i_0} \geq 0$. $\square$

Explicit example (C. Böhm):

$$S^{k+1} \times (G'/K') \times M_3 \times \cdots \times M_r$$

with M_i compact isotropy irreducible and

$$G'/K' = \mathrm{SU}(\ell + m)/(\mathrm{SO}(\ell) \cdot \mathrm{U}(1) \cdot \mathrm{U}(m)),$$

$$\ell \geq 32, \quad m = 1, 2, \quad k = 1, 2, \cdots, [\ell/3]$$

Complete, non-compact, non-trivial GRS

I. Steady Case ($\epsilon = 0$)

can apply and/or sharpen results of B. L. Chen ($\bar{R} > 0$), Munteanu-Sesum, Peng Wu, ...

Proposition 2. *For a complete, non-compact, non-trivial steady GRS of cohomogeneity one:*

(a) u *is strictly decreasing and concave (as function of* t*) with* $\ddot{u}(0) = C/(k+1) < 0$

(*note: no curvature assumptions*)

(b) $\mathrm{tr}L$ *is strictly decreasing;* $0 < \mathrm{tr}L \leq \frac{n}{t}$.

(c) *generalized mean curvature* $\xi := -\dot{u} + trL$ *is strictly decreasing with asymptotic limit* $\sqrt{-C}$*. Hence* $C\,\xi^{-2}$ *is a general Lyapunov function.*

(d) *ambient scalar curvature is strictly decreasing with asymptotic limit* 0 *(since* $\bar{R} + \dot{u}^2 = -C$ *).*

(e) *quantity* $\mathcal{F} := v^{\frac{2}{n}}(S + \mathrm{tr}(L_0)^2)$ *is non-increasing on any trajectory corresponding to a non-trivial soliton* (*Lyapunov function*)

Example 1. $[DW2009]$ $M = \mathbb{R}^{d_1+1} \times M_2 \times \cdots \times M_r$

$M_1 = S^{d_1}, d_1 > 1$, equipped with the constant curvature 1 metric h_1

(M_i, h_i), $2 \leq i \leq r$ Einstein with Einstein constants $\lambda_i > 0$ and dimension d_i.

$\exists\ r-1$ parameter family of non-trivial steady GRS structures with $\bar{g} = dt^2 + g_1(t)^2 h_1 + \cdots + g_r(t)^2 h_r$, $\mathrm{Ric}(\bar{g}) \geq 0$ (positive off the zero section)

Remarks:

(i) generally non-Kähler; generally not locally conformally flat if $r \geq 2$

(ii) $r = 1$: Bryant solitons on $\mathbb{R}^n$, $n \geq 3$

($n = 2$ is Hamilton's cigar, which is Kähler)

These have positive sectional curvature.

(iii) $r = 2$ Ivey's generalization of Bryant solitons, PAMS (1994)

(iv) asymptotics: $g_i \sim \sqrt{t}$, tr$L \sim \frac{n}{2t} + O(t^{-2})$ and $u(t) \sim -\sqrt{-C}t + \frac{n}{4} \log t$.

(iv) C. Böhm (1999): $r - 2$ parameter family of complete Ricci-flat metrics (C=0); asymptotically Euclidean

Example 2. [DZ Chen 2010]

$M = S^1 \times L_q$ where L_q is the complex line bundle over a Fano KE manifold with $|q|$ the first Chern number.

$\exists$ a 3-parameter family of "explicit" steady soliton solutions (modulo homothety)

Hypersurfaces are T^2 bundles over Fano with connection metric

Metric on $T^2 = S^1 \times S^1$ is not "diagonal"

Asymptotically, metric components $\sim t$ (paraboloidal)

II. Expanding Case: ($\epsilon > 0$) Set

$$\xi := -\dot{u} + \mathrm{tr}L \text{ (generalized mean curvature)}$$

and $\mathcal{E} = C + \epsilon u$.

Conservation law becomes: $\ddot{\mathcal{E}} + \xi\dot{\mathcal{E}} - \epsilon\mathcal{E} = 0$.

Its derivative yields for $y = \dot{u}$:

$$\ddot{y} + \xi\dot{y} - (\tfrac{\epsilon}{2} + \mathrm{tr}(L^2))y = 0$$

can apply and/or sharpen results of B.L. Chen ($\bar{R} + \frac{\epsilon}{2}(n+1) > 0$), Shijin Zhang, Zhuhong Zhang, Carillo-Ni, Munteanu-Sesum, Pigola-Rimoldi-Setti,....

Proposition 3. *For a non-trivial complete expanding GRS with $u(0) = 0$*

(a) u is strictly decreasing and strictly concave; $\ddot{u}(0) = C/(k+1) < 0$

(b) volume grows at least logarithmically

(c) $\exists\, t_1 > 0$ such that $-\sqrt{\frac{\epsilon}{2}n} < \mathrm{tr}L < \sqrt{\frac{\epsilon}{2}n}$ for $t \geq t_1$

Proposition 4. (*gradient bound*) $\exists\, t_1 > 0$ *and* $a > 0$ *such that for* $t \geq t_1$

(*a*) $\frac{9}{10}\left(\frac{-\dot{u}(t_1)}{\frac{\epsilon}{2}t+a}\right)\left(\frac{\epsilon}{2}t + a\right) < |\overline{\nabla} u| < \frac{\epsilon}{2}t + \sqrt{-C}$

Hence u *is asymptotically bounded above and below by quadratics.*

(*b*) $\lim_{t\to+\infty} \xi = +\infty$

(*c*) *For* t *large, the quantity* $\mathcal{F} := v^{\frac{2}{n}}(S + \mathrm{tr}(L_0)^2)$ *is strictly decreasing on any trajectory in velocity phase space except when* L_0 *vanishes*

(*d*) $\ddot{u}+\frac{\epsilon}{2} = -\mathrm{Ric}_{\bar{g}}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \leq \frac{\epsilon}{2}\left(1 + \frac{9}{10}\left(\frac{-\dot{u}(t_1)}{\frac{\epsilon}{2}t+a}\right)\left(\frac{\epsilon}{2}t + a\right)\right)$ *provided* $t \geq t_1$.

(*e*) *If* $\ddot{u} + \epsilon/2 \leq 0$ *for* $t \geq t_0$*, then* $\mathrm{tr}L$ *is strictly decreasing,* $0 < \mathrm{tr}L < n/t$ *and* $\bar{R} > -\frac{\epsilon}{2}n$.

Example 3 $[DW2009]$

On the same manifolds as in Example 1, there is an r-parameter family of non-trivial (generally non-Kähler, non-locally conformally flat) expanding GRS structures

Remarks:

(i) $r = 1$ Ivey in [Chow et al] ;

(ii) $r = 2$ Gastel and Kronz (2004)

(iii) Böhm (1999): $r - 1$ parameter family of complete negative Einstein metrics on these manifolds

Einstein metric components grow exponentially with t, mean curvature asymptotically constant $\sim \sqrt{n\epsilon/2}$.

(iv) solitons are asymptotically conical and satisfy $\ddot{u} + \epsilon/2 \leq 0$ for $t \geq t_0$; also, ambient scalar curvature tends to 0 and $\xi \sim \frac{\epsilon}{2} t$.

4. A General Winding Number for Shrinkers

Recall $\xi = -\dot{u} + \mathrm{tr} L$ (generalized mean curvature for measure $e^{-u} d\mu_{\bar{g}}$)

Eq. (2) $\Longrightarrow$ ξ is strictly decreasing from

$+\infty$ to $-\infty$ in all cases (unique zero)

Let $\mathcal{E} := C + \epsilon u$ and $\mathcal{F} := \dot{u}$.

Recall Conservation law (4) in the form

$$\ddot{\mathcal{E}} + \xi \dot{\mathcal{E}} - \epsilon \mathcal{E} = 0$$

Let $ds := \xi \, dt$ and $'$ denote differentiation wrt s.

Note: insert -1 for change of variables *after* unique zero of ξ.

We now have (with $W := \xi^{-1}$)

$$\mathcal{E}' = \epsilon W \mathcal{F}$$

$$\mathcal{F}' = W \mathcal{E} - \mathcal{F}$$

associated vector field

Theorem 5. (*[DHW 2011]*) *For trajectories of the flow of* $(\mathcal{F}, \mathcal{E})$ *starting from either the positive or negative* $\mathcal{E}$ *axis, the winding number about the origin up to the (unique) turning point is finite, non-positive and bounded from below by* $-(6+\frac{\pi}{4})$.

Remark: The origin corresponds to Einstein trajectories.

Some General Facts about Shrinking GRS:

(a) $\bar{R} \geq 0$. It is positive unless the soliton metric is flat. (B. L. Chen without sectional curvature bounds, Pigoli-Rimoldi-Setti for rigidity)

(b) Quadratic bound for soliton potential in complete, non-compact case

[H. D. Cao-D. Zhou 2010]

$$-\frac{\epsilon}{2}(n+1) + \frac{\epsilon}{4}(t\sqrt{-\epsilon} + c_2)^2 \leq \mathcal{E}(t) = C + \epsilon u(t)$$

$$\leq -\frac{\epsilon}{2}(n+1) + \frac{\epsilon}{4}(t\sqrt{-\epsilon} - c_1)^2$$

Note: c_i depend only on $n+1 = \dim M$. (Haslhofer-Müller 2011)

Ambient scalar curvature

$$\overline{R} = -2\,\mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2) - (\mathrm{tr}L)^2 + S$$

$$= -\mathcal{E} - \frac{\dot{\mathcal{E}}^2}{\epsilon^2} - \frac{\epsilon}{2}(n+1)$$

So General Fact (a) implies in non-flat cases

$$\mathcal{E} < -\tfrac{\epsilon}{2}(n+1), \text{ and}$$

$$\ddot{u}(0) < -\frac{\epsilon}{2}\left(\frac{n+1}{k+1}\right)$$

Theorem 6. $[DHW2011]$

Let $(M, \bar{g}, u)$ be a non-trivial complete shrinking GRS of cohomogeneity one with invariant soliton potential and orbit space I. Then, regarding $\mathcal{E}$ as a function of t:

(*i*) *$\mathcal{E} = C + \epsilon u$ must change sign and is a Morse-Bott function on M.*

(*ii*) *If $\bar{g}$ is nonflat, then $\mathcal{E} < -\frac{\epsilon}{2}(n+1)$.*

(*iii*) *If M is compact, $\mathcal{E}$ has at most 4 critical points in* int I*. As a function of t, $\mathcal{E}$ is either a local max (where $\mathcal{E} > 0$) or a local min (where $\mathcal{E} < 0$).*

(*iv*) *If M is complete, noncompact, $\mathcal{E}$ has at most 5 critical points in* int I*.*

Remark: In known examples, $\mathcal{E}$ is monotone decreasing. But these are all Kähler.

Theorem rules out

Example 1 smooth Gaussian (rigid in Petersen-Wylie sense)

$M = \mathbb{R}^{d_1+1} \times M_2 \times \cdots \times M_r$

$\mathbb{R}^{d_1+1}$ Euclidean, M_i positive Einstein $i > 1$

$u(t) = -\frac{\epsilon}{4}t^2$, $\mathrm{tr}L = \frac{d_1}{t}$, $\overline{R} = -\frac{\epsilon}{2}(n - d_1)$

Example 2 [FIK 2003], [DW 2008]

$(V_i, J_i, h_i), 1 \leq i \leq r,\ r \geq 2$, Fano KE manifolds with complex dimension n_i and $c_1(V_i) = p_i a_i$ where $p_i > 0$ and a_i are indivisible classes in $H^2(V_i, \mathbb{Z})$

$V_1 = \mathbb{CP}^{n_1}, n_1 \geq 0$, with normalised Fubini-Study metric

P_q : principal S^1 bundle over $V_1 \times \cdots \times V_r$ with Euler class $-\pi_1^*(a_1) + \sum_{i=2}^r q_i \pi_i^*(a_i)$, i.e., $q_1 = -1$.

Assume $0 < -(n_1 + 1)\, q_i < p_i$ for all $2 \leq i \leq r$.

Then there is a complete shrinking GKRS structure on the space $\overline{M}$ obtained from the line bundle $P_q \times_{S^1} \mathbb{C}$ by blowing the zero section down to $V_2 \times \cdots \times V_r$.

soliton metric has an asymptotically conical end

Remarks: (a) Feldman-Illmanen-Knopf considered case with $r = 2, n_1 = 0$ and V_2 to be a complex projective space.

(b) The case $r = 2, n_2 = 0$ corresponds to flat $\mathbb{C}^{n_1+1}$ as a shrinking soliton.

(c) Also: Bo Yang (2008), A.Futaki-M.T.Wang (2010), Chi Li (2010)

(d) There is a version of theorem where the base is a coadjoint orbit and the principal orbits are suitable circle bundles over it.

(e) The condition $\ddot{u} + \frac{\epsilon}{2} > 0$ holds except in flat case.

Proposition 7. *Assume $\ddot{u} \leq -\frac{\epsilon}{2}$ on some $[a, +\infty)$, $a > 0$.*

- *from some $t_0 \geq a$ on,* $\mathrm{tr}L$ *is decreasing and* $0 < \mathrm{tr}L < (\frac{t}{n} + c(t_0))^{-1}$ *and*

- *ambient scalar curvature $< -\frac{\epsilon}{2} n$.*

Numerical Search: $[DHW]$

negative search results in compact cases

(i) S^5 with SO(3) $\times$ SO(3) action

(ii) $S^2 \times S^3$ with SO(3) $\times$ SO(3) action

(iii) S^{11} with SO(6) $\times$ SO(6) action

(iv) $\mathbb{HP}^{n+1} \sharp \mathbb{HP}^{n+1}$ with Sp(1) $\times$ Sp($n+1$) action;

connected sum of Cayley projective planes

(v) $\mathbb{R}^3$ bundle over $\mathbb{HP}^n$ with $G =$ Sp($n+1$); principal orbit is twistor fibration over $\mathbb{HP}^n$

(vi) non-trivial sphere bundles over S^2 (Hashimoto-Sakaguchi-Yasui): principal orbit $S^3 \times S^{d-2}$