On the diagonalization of the Ricci flow on Lie groups

Cynthia Will,
FaMAF and CIEM, Córdoba, Argentina

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ODE for Lie groups.

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$$\begin{aligned} M(X,Y) &= -\tfrac{1}{2}\sum \langle [X, X_i], X_j\rangle\langle [Y, X_i], X_j\rangle \\ &\quad + \tfrac{1}{4}\sum \langle [X_i, X_j], X\rangle\langle [X_i, X_j], Y\rangle. \end{aligned}$$

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Theorem

A basis of a nilpotent Lie algebra is stably Ricci-diagonal if and only if it is nice.

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The canonical basis $\{e_1, \dots, e_n\}$ *is nice for* $\mathfrak{n}$ *if and only if*

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$\rightsquigarrow$ Generalization.

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$$\mathrm{Ric} = \begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -\frac{1}{2} \end{bmatrix}.$$

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Stably Ricci diagonal basis $\Rightarrow$ Ricci diagonal.

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(Payne, Williams)

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