Solvable Lie groups and Hermitian geometry

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Overview

$M = G/\Gamma$: nilmanifold/solvmanifold

G: simply connected real Lie group nipotent/solvable

Γ: lattice (cocompact discrete subgroup)

Definition

• A complex structure J on $M^{2n} = G/\Gamma$ is invariant if it is induced by a complex structure on $\mathfrak{g}$.
• (M^{2n}, J) has holomorphically trivial canonical bundle if it has a non-zero holomorphic $(n, 0)$-form.

Problem

Study invariant cpx structures J and Hermitian geometry on (M^6, J) with holomorphically trivial canonical bundle.

Remark

Any compact complex surface (M^4, J) with holomorphically trivial canonical bundle is isomorphic to a $K3$ surface, a torus T^4, or a Kodaira surface KT.

The first two are Kähler, and the latter KT is a nilmanifold!

Nilpotent and solvable Lie groups

Definition

- G is *k-step nilpotent* $\iff$ the chain
$G_0 = G \supset G_1 = [G, G] \supset \cdots \supset G_{i+1} = [G_i, G] \supset \ldots$.
degenerates, i.e. $G_i = \{e\}$, $\forall i \geq k$, (e neutral element).
- G is *k-step solvable* $\iff$ the series of normal subgroups
$G_{(0)} = G \supset G_{(1)} = [G, G] \supset \cdots \supset G_{(i+1)} = [G_{(i)}, G_{(i)}] \supset \ldots$.
degenerates.

A solvable G is *completely solvable* if every eigenvalue of Ad_g is real, for every $g \in G$.

A nilpotent Lie group is then completely solvable!

Existence of lattices for nilpotent Lie groups

G: simply connected nilpotent Lie group

- $\exp : \mathfrak{g} \to G$ is a diffeomorphism

- $\exists$ a simple criteria for the existence of lattices:

Theorem (Malčev)

$\exists$ a lattice Γ of $G \iff \mathfrak{g}$ has a basis for which the structure constants are rational $\iff \exists\, \mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$

If $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$, one says that $\mathfrak{g}$ has a rational structure.

Existence of lattices for solvable Lie groups

G: connected and simply connected Lie group

- $G \overset{\text{diffeo}}{\simeq} \mathbb{R}^n$
- $\exp : \mathfrak{g} \to G$ is not necessarily injective or surjective!

Remark

There is no a simple criteria for the existence of lattices for G.

A necessary criteria:

Proposition (Milnor)

If G has a lattice, then G is unimodular, i.e. $\operatorname{tr}\operatorname{ad}_X = 0$, $\forall X \in \mathfrak{g}$.

Kähler structures

Theorem (Benson, Gordon; Hasegawa)

A nilmanifold G/Γ has a Kähler structure if and only if it is a complex torus.

Conjecture (Benson, Gordon)

If a solvmanifold has a Kähler structure, then it is a complex torus.

For the completely solvable case it has been proved by Baues and Cortes.

Theorem (Hasegawa)

A solvmanifold has a Kähler structure if and only if it is a finite quotient of a complex torus, which is the total space of a complex torus over a complex torus.

Invariant complex structures on nilmanifolds

Theorem (Salamon)

If a nilpotent $\mathfrak{g}$ admits a complex structure, then $\exists$ a basis of $(1,0)$ forms $\{\omega^1, \dots, \omega^n\}$ such that $d\omega^1 = 0$ and

$$d\omega^i \in \mathcal{I} < \omega^1, \dots, \omega^{i-1} >, \quad i > 1.$$

$\Rightarrow$ $\exists$ a closed non-zero invariant $(n,0)$-form and the canonical bundle of $(\Gamma\backslash G, J)$ is holomorphically trivial.

• $\exists$ a classification of real 6-dimensional nilpotent $\mathfrak{g}$ in 34 classes (Magnin, 1986 and Goze-Khakimdjanov, 1996).

• 18 classes admit a complex structure (Salamon, 2001).

Invariant complex structures on solvmanifolds

For solvable Lie algebras admitting a complex structure there exists a general classification only in real dimension four [Ovando].

In higher dimensions there are the following classifications:

• for solvable Lie algebras $\mathfrak{g}$ admitting a bi-invariant complex structure J (i.e. $J \circ ad_X = ad_X \circ J, \forall X \in \mathfrak{g}$) in dimension 6 and 8 [Nakamura].

• for 6-dimensional solvable Lie algebras $\mathfrak{g}$ admitting an abelian complex structure J (i.e. $[JX, JY] = [X, Y], \forall X, Y \in \mathfrak{g}$) [Andrada, Barberis, Dotti].

Existence of an Invariant holomorphic $(n, 0)$-form

Proposition (-, Otal, Ugarte)

Let $(M^{2n} = \Gamma \backslash G, J)$ be a solvmanifold with an invariant cpx structure J. If Ω is a nowhere vanishing holomorphic $(n, 0)$-form, then Ω is necessarily invariant.

If (M^{2n}, J) has holomorphically trivial canonical bundle $\Rightarrow \mathfrak{g}$ has to be an unimodular solvable Lie algebra admitting a cpx structure with non-zero closed $(n, 0)$-form.

Stable forms

(V, ν): oriented 6-dimensional vector space

Definition

A 3-form ρ is stable if its orbit under the action of $\mathrm{GL}(V)$ is open.

Consider $\kappa \colon \Lambda^5 V^* \xrightarrow{\cong} V, \eta \mapsto X$, where X is such that $\iota_X \nu = \eta$, and the endomorphism $K_\rho : V \longrightarrow V, X \mapsto \kappa(\iota_X \rho \wedge \rho)$.

Proposition (Reichel; Hitchin)

ρ is stable $\Leftrightarrow \lambda(\rho) = \frac{1}{6} trace(K_\rho^2) \neq 0$.

When $\lambda(\rho) < 0$, $J_\rho := \frac{1}{\sqrt{|\lambda(\rho)|}} K_\rho$ gives rise to an almost cpx structure on V.

The action of the dual J_ρ^* on V^* is given by

$$((J_\rho^*\alpha)(X))\,\phi(\rho) = \alpha \wedge \iota_X \rho \wedge \rho, \quad \forall \alpha \in V^*, \forall X \in V,$$

where $\phi(\rho) := \sqrt{|\lambda(\rho)|}\nu \in \Lambda^6 V^*$.

There is a natural mapping

$$\begin{array}{rcl} \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) < 0\} & \to & \{J\colon V \to V \mid J^2 = -Id_V\} \\ \rho & \mapsto & J = J_\rho \end{array}$$

Remark

This map is not injective but it is onto and therefore it covers the space of almost complex structures on V.

Let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3\mathfrak{g}^* \mid d\rho = 0\}$.

Lemma (-, Otal, Ugarte)

Let ν a volume form on $\mathfrak{g}$. Then, $\mathfrak{g}$ admits an almost cpx structure with a non-zero closed $(3,0)$-form if and only if there exists $\rho \in Z^3(\mathfrak{g})$ such that the endomorphism $\tilde{J}^*_\rho \colon \mathfrak{g}^* \to \mathfrak{g}^*$ defined by

$$\left((\tilde{J}^*_\rho \alpha)(X)\right)\nu = \alpha \wedge \iota_X\rho \wedge \rho,$$

for any $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$, satisfies that $\tilde{J}^*_\rho \rho$ is closed and $\mathrm{tr}(\tilde{J}^{*2}_\rho) < 0$.

An obstruction

Remark

If $\dim \mathfrak{g} = 6$, the unimodularity of $\mathfrak{g}$ is equivalent to $b_6(\mathfrak{g}) = 1$.

Lemma

If $\mathfrak{g}$ is an unimodular Lie algebra admitting a complex structure with a non-zero closed $(3,0)$-form Ψ, then $b_3(\mathfrak{g}) \geq 2$.

So in particular, if $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}$, we have

$$b_3(\mathfrak{b})b_0(\mathfrak{c}) + b_2(\mathfrak{b})b_1(\mathfrak{c}) + b_1(\mathfrak{b})b_2(\mathfrak{c}) + b_0(\mathfrak{b})b_3(\mathfrak{c}) = b_3(\mathfrak{g}) \geq 2.$$

Classification of Lie algebras

Theorem (-,Otal,Ugarte)

A 6-dim unimodular (non-nilpotent) solvable $\mathfrak{g}$ admits a complex structure with a non-zero closed $(3,0)$-form if and only if it is isomorphic to one of the following:

$$\mathfrak{g}_1 = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0),$$
$$\mathfrak{g}_2^{\alpha} = (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0), \ \alpha \geq 0,$$
$$\mathfrak{g}_3 = (0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45}),$$
$$\mathfrak{g}_4 = (e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$$
$$\mathfrak{g}_5 = (e^{24} + e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0),$$
$$\mathfrak{g}_6 = (e^{24} + e^{35}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$$
$$\mathfrak{g}_7 = (e^{24} + e^{35}, e^{46}, e^{56}, -e^{26}, -e^{36}, 0),$$
$$\mathfrak{g}_8 = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} + e^{45}, -e^{35} - e^{46}, 0, 0),$$
$$\mathfrak{g}_9 = (e^{45}, e^{15} + e^{36}, e^{14} - e^{26} + e^{56}, -e^{56}, e^{46}, 0).$$

Existence of lattices

The simply-connected solvable Lie groups G_k with Lie algebra $\mathfrak{g}_k$ admit co-compact lattices Γ_k

$\hookrightarrow$ compact $\Gamma_k \backslash G_k$ with holomorphically trivial canonical bundle

Definition

A connected and simply-connected Lie group G with nilradical H is called almost nilpotent (resp. almost abelian) if $G = \mathbb{R} \ltimes_\mu H$ (resp. $G = \mathbb{R} \ltimes_\mu \mathbb{R}^m$).

$d_e(\mu(t)) = \exp^{\mathrm{GL}(m,\mathbb{R})}(t\varphi)$, where e the identity element of H and φ a derivation of the Lie algebra $\mathfrak{h}$ of H.

Lemma (Bock)

Let $G = \mathbb{R} \ltimes_\mu H$ almost nilpotent with nilradical H. If there exists $t_1 \in \mathbb{R} - \{0\}$ and a rational basis $\{X_1, \ldots, X_m\}$ of $\mathfrak{h}$ such that the matrix of $d_e(\mu(t_1))$ in such basis is integer, then $\Gamma = t_1\mathbb{Z} \ltimes_\mu \exp^H(\mathbb{Z}\langle X_1, \ldots, X_m\rangle)$ is a lattice in G.

Proposition (-, Otal, Ugarte)

For any $k \neq 2$, the connected and simply-connected Lie group G_k admits a lattice.
For $k = 2$, there exists a countable number of distinct α's, including $\alpha = 0$, for which G_2^α admits a lattice.

Moduli of complex structures

We classify, up to equivalence, the complex structures having closed (3,0)-form on the Lie algebras $\mathfrak{g}_i$, $I = 1, \ldots, 9$.

Definition

Two complex structures J and J' on $\mathfrak{g}$ are said to be equivalent if there exists an automorphism F of $\mathfrak{g}$ such that $F \circ J = J' \circ F$.

Complex structures on 6-dimensional nilpotent Lie algebras have been classified up to equivalence by Ceballos, Otal, Ugarte and Villacampa.

Decomposable Lie algebras

Proposition (-, Otal, Ugarte)

Up to isomorphism, we have the following cpx structures with closed $(3,0)$-form:

$$(\mathfrak{g}_1, J') : \ d\omega^1 = \omega^{13} + \omega^{1\bar{3}}, \ d\omega^2 = -\omega^{23} - \omega^{2\bar{3}}, \ d\omega^3 = 0,$$

$$(\mathfrak{g}_2^0, J') : \ d\omega^1 = i\,(\omega^{13} + \omega^{1\bar{3}}), \ d\omega^2 = -i\,(\omega^{23} + \omega^{2\bar{3}}), \ d\omega^3 = 0$$

$$(\mathfrak{g}_2^\alpha, J_\pm) : \begin{cases} d\omega^1 = (\pm\cos\theta + i\sin\theta)\,(\omega^{13} + \omega^{1\bar{3}}), \\ d\omega^2 = -(\pm\cos\theta + i\sin\theta)\,(\omega^{23} + \omega^{2\bar{3}}), \\ d\omega^3 = 0, \quad \alpha = \frac{\cos\theta}{\sin\theta}, \ \theta \in (0, \pi/2) \end{cases}$$

$$(\mathfrak{g}_3, J_x) : \begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\frac{1}{2}\omega^{13} - (\frac{1}{2} + xi)\omega^{1\bar{3}} + xi\,\omega^{3\bar{1}}, \\ d\omega^3 = \frac{1}{2}\omega^{12} + (\frac{1}{2} - \frac{i}{4x})\omega^{1\bar{2}} + \frac{i}{4x}\,\omega^{2\bar{1}}, \quad x \in \mathbb{R}^+ \end{cases}$$

Indecomposable Lie algebras

Proposition (-, Otal, Ugarte)

Up to isomorphism we have the following cpx structures with closed $(3,0)$-form

$$(\mathfrak{g}_4, J_\pm): \begin{cases} d\omega^1 = i(\omega^{13}+\omega^{1\bar 3}), \\ d\omega^2 = -i(\omega^{23}+\omega^{2\bar 3}), \\ d\omega^3 = \pm\,\omega^{1\bar 1}; \end{cases} \quad (\mathfrak{g}_5, J'): \begin{cases} d\omega^1 = \omega^{13}+\omega^{1\bar 3} \\ d\omega^2 = -\omega^{23}-\omega^{2\bar 3} \\ d\omega^3 = \omega^{1\bar 2}+\omega^{2\bar 1} \end{cases}$$

$$(\mathfrak{g}_6, J'): \begin{cases} d\omega^1 = i\,(\omega^{13}+\omega^{1\bar 3}) \\ d\omega^2 = -i\,(\omega^{23}+\omega^{2\bar 3}) \\ d\omega^3 = \omega^{1\bar 1}+\omega^{2\bar 2} \end{cases} \quad (\mathfrak{g}_7, J_\pm): \begin{cases} d\omega^1 = i\,(\omega^{13}+\omega^{1\bar 3}), \\ d\omega^2 = -i\,(\omega^{23}+\omega^{2\bar 3}), \\ d\omega^3 = \pm(\omega^{1\bar 1}-\omega^{2\bar 2}) \end{cases}$$

$$(\mathfrak{g}_9, J'): \begin{cases} d\omega^1 = -\omega^{3\bar 3}, \\ d\omega^2 = \frac{i}{2}\omega^{12}+\frac{1}{2}\omega^{1\bar 3}-\frac{i}{2}\omega^{2\bar 1}, \\ d\omega^3 = -\frac{i}{2}\omega^{13}+\frac{i}{2}\omega^{3\bar 1}. \end{cases}$$

There are infinitely many non-isomorphic complex structures on $\mathfrak{g}_8$.

Proposition (-, Otal, Ugarte)

Up to isomorphism we have the following cpx structures on $\mathfrak{g}_8$ with closed $(3,0)$-form:

$$(\mathfrak{g}_8, J')\colon\ d\omega^1 = 2i\,\omega^{13} + \omega^{3\bar{3}},\ \ d\omega^2 = -2i\,\omega^{23},\ \ d\omega^3 = 0;$$

$$(\mathfrak{g}_8, J'')\colon\ d\omega^1 = 2i\,\omega^{13} + \omega^{3\bar{3}},\ \ d\omega^2 = -2i\,\omega^{23} + \omega^{3\bar{3}},\ \ d\omega^3 = 0;$$

$$(\mathfrak{g}_8, J_A)\colon \begin{cases} d\omega^1 = -(A-i)\omega^{13} - (A+i)\omega^{1\bar{3}}, \\ d\omega^2 = (A-i)\omega^{23} + (A+i)\omega^{2\bar{3}}, \\ d\omega^3 = 0, \qquad A \in \mathbb{C} - \{0\}. \end{cases}$$

Moreover, J', J'' and J_A are non-isomorphic.

SKT and strongly Gauduchon metrics

Let (M^{2n}, J, g) be a Hermitian manifold $2n$ with fundamental 2-form $F(\cdot,\cdot) = g(J\cdot,\cdot)$.

Definition

The Hermitian metric g is called

- SKT (strong Kähler with torsion) if $\partial\overline{\partial}F = 0$;
- balanced if $dF^{n-1} = 0$;
- strongly Gauduchon if the $(n, n-1)$-form ∂F^{n-1} is $\overline{\partial}$-exact.

balanced $\Rightarrow$ strongly Gauduchon

SKT metrics on nilmanifolds

Theorem (Enrietti, -, Vezzoni)

$M^{2n} = \Gamma \backslash G$ nilmanifold (not a torus), J left-invariant. If (M^{2n}, J) has a SKT metric, then G has to be 2-step and (M^{2n}, J) is a principal holomorhic torus bundle over a torus.

To prove the theorem we show that J has to preserve the center ξ of $\mathfrak{g}$ and that a SKT metric on $\mathfrak{g}$ induces a SKT metric on $\mathfrak{g}/\xi$.

Symmetrization

If a simply-connected Lie group G admits a uniform discrete subgroup Γ, then G is unimodular and has a bi-invariant volume form $d\mu$.

Proposition (-, Grantcharov)

$(M = \Gamma \backslash G, d\mu)$

• if g is a Riemannian metric, then $\tilde{g}(A, B) = \int_M g_m(A_m, B_m) d\mu$, is left-invariant.

• if ω is a k-form, then $\tilde{\omega}(B_1, \ldots, B_k) = \int_{m \in M} \omega_m(B_1|_m, \ldots, B_k|_m) d\mu$ is left-invariant and

$$d\tilde{\omega}(B_1, \ldots, B_{k+1}) = \int_{m \in M} d\omega_m(B_1|_m, \ldots B_{k+1}|_m) d\mu.$$

Theorem (-, Grantcharov)

If $M = \Gamma\backslash G$ admits a left-invariant J and F is a Kähler form of a non-invariant Hermitian metric $g \Rightarrow$

$$\alpha(A_1, ..., A_{2n-2}) = \int_M F^{n-1}|_m(A_1|_m, ...A_{2n-2}|_m)d\mu,$$

is equal to $\tilde{F}^{n-1}$ for some Kähler form $\tilde{F}$ of a left-invariant Hermitian metric $\tilde{g}$. If $dF^{n-1} = 0 \Rightarrow d\tilde{F}^{n-1} = 0$.

If g is SKT (resp. strongly Gauduchon), then $\tilde{g}$ is SKT (resp. strongly Gauduchon) [Ugarte].

SKT metrics on 6-solvmanifolds

The symmetrization process can be applied to conclude that the existence of SKT, balanced or strongly Gauduchon metrics on $(M^6 = \Gamma\backslash G, J)$ reduces to the level of the Lie algebra $\mathfrak{g}$ of G.

Remark

Given a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ for J, a generic Hermitian structure F on $\mathfrak{g}$ is expressed as

$$\begin{aligned} 2\,F \;&=\; i\,(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}}) + u\omega^{1\bar{2}} - \bar{u}\omega^{2\bar{1}} \\ &\quad + v\omega^{2\bar{3}} - \bar{v}\omega^{3\bar{2}} + z\omega^{1\bar{3}} - \bar{z}\omega^{3\bar{1}}, \end{aligned}$$

where r^2, s^2, t^2 are non-zero real numbers and u, v, $z \in \mathbb{C}$ satisfy

$$\begin{aligned} &r^2s^2 > |u|^2, s^2t^2 > |v|^2, r^2t^2 > |z|^2, \\ &r^2s^2t^2 + 2\mathrm{Re}(i\bar{u}\bar{v}z) > t^2|u|^2 + r^2|v|^2 + s^2|z|^2. \end{aligned}$$

Theorem (-, Otal, Ugarte)

$M^6 = \Gamma\backslash G$ solvmanifold, J invariant with holomorphically trivial canonical bundle. Then, (M^6, J) has an SKT metric if and only if $\mathfrak{g}$ is isomorphic to

$$\mathfrak{g}_2^0 = (e^{25}, -e^{15}, e^{45}, -e^{35}, 0, 0) \quad \textit{or}$$
$$\mathfrak{g}_4 = (e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0).$$

Remark

- Any cpx structure with non-zero closed (3,0)-form on $\mathfrak{g}_2^0$ or $\mathfrak{g}_4$ admits SKT metrics.
- $\mathfrak{g}_2^0$ admits Kähler metrics. A solvmanifold $\Gamma\backslash G$ with $\mathfrak{g} \cong \mathfrak{g}_4$ provides a new example of 6-dim compact SKT manifold.

Balanced metrics

Theorem (-, Otal, Ugarte)

$M^6 = \Gamma\backslash G$ solvmanifold, J invariant with holomorphically trivial canonical bundle. If (M^6, J) has a balanced metric then $\mathfrak{g} \cong \mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$ or $\mathfrak{g}_8$.
Moreover, in such cases, any J admits balanced metrics except for the complex structures which are isomorphic to J' or J'' on $\mathfrak{g}_8$.

The necessary condition for the existence of balanced metrics is necessary and sufficient for the existence of strongly Gauduchon metrics!

Strongly Gauduchon metrics

Theorem (-, Otal, Ugarte)

Let $(M^6 = \Gamma \backslash G, J)$ be a solvmanifold endowed with an invariant cpx structure J with holomorphically trivial canonical bundle. Then, (M^6, J) has a strongly Gauduchon metric if and only if $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^{\alpha}, \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7$ or $\mathfrak{g}_8$.
Moreover, if $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^{\alpha}, \mathfrak{g}_3$ or $\mathfrak{g}_8$, then any invariant Hermitian metric on (M^6, J) is strongly Gauduchon.

Holomorphic deformations

Problem

Study existence of balanced metrics under deformation of the complex structure.

Consider a holomorphic family of compact complex manifolds $(M, J_a)_{a \in \Delta}$, where Δ is an open disc around the origin in $\mathbb{C}$.

Definition

A property is said open under holomorphic deformations if when it holds for a given compact (M, J_0), then (M, J_a) also has that property for all $a \in \Delta$ sufficiently close to 0.

Theorem (Alessandrini, Bassanelli)

The property of existence of balanced Hermitian metrics is not open under holomorphic deformations.

An example is provide by a nimanifold.

In contrast to the balanced case

Theorem (Popovici)

The property of existence of strongly Gauduchon metrics is open under holomorphic deformations.

Definition

A property is said to be closed under holomorphic deformations if whenever (M, J_a) has that property for all $a \in \Delta \setminus \{0\}$ then the property also holds for the central limit (M, J_0).

Theorem (Ceballos, Otal, Ugarte, Villacampa)

The strongly Gauduchon property and the balanced property of compact complex manifolds are not closed under holomorphic deformations.

More concretely, $\exists$ a holomorphic family of compact $(M, J_a)_{a\in\Delta}$ such that (M, J_a) has balanced metric for any $a \neq 0$ but the central limit (M, J_0) does not admit any strongly Gauduchon metric.

$\partial\overline{\partial}$ Lemma

Definition (Deligne, Grifftihs, Morgan, Sullivan)

A compact complex manifold M satisfies the $\partial\overline{\partial}$-lemma if for any d-closed form α of pure type on M, the following exactness properties are equivalent:
α is d-exact $\Longleftrightarrow$ α is ∂-exact $\Longleftrightarrow$ α is $\overline{\partial}$-exact $\Longleftrightarrow$ α is $\partial\overline{\partial}$-exact.

Under this strong condition, the existence of strongly Gauduchon metric in the central limit is guaranteed:

Proposition (Popovici)

If the $\partial\overline{\partial}$-lemma holds on (M, J_a) for every $a \in \Delta \setminus \{0\}$, then (M, J_0) has a strongly Gauduchon metric.

Problem

Does the central limit admit a Hermitian metric, stronger than strongly Gauduchon, under the $\partial\overline{\partial}$-lemma condition?

The $\partial\overline{\partial}$-lemma property is open, but it is not closed [Angella, Kasuya].

We provide a negative answer to the problem.

Theorem (-, Otal, Ugarte)

There exists a solvmanifold M with a holomorphic family of cpx structures J_a, $a \in \Delta$, such that (M, J_a) satisfies the $\partial\overline{\partial}$-lemma and admits balanced metric for any $a \neq 0$, but the central limit (M, J_0) neither satisfies the $\partial\overline{\partial}$-lemma nor admits balanced metrics.

The solvmanifold is the Nakamura manifold G/Γ.

Description of the Nakamura manifold

G is the simply connected complex Lie group:

$$\left\{ \begin{pmatrix} e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, w_1, w_2, z \in \mathbb{C} \right\} \cong \mathbb{C} \ltimes_\varphi \mathbb{C}^2,$$

with $\varphi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}$ and the lattice Γ is given by $\Gamma = L_{1,2\pi} \ltimes_\varphi L_2$, with

$$L_{1,2\pi} = \mathbb{Z}[t_0, 2\pi i] = \{t_0 k + 2\pi h i, h, k \in \mathbb{Z}\}, \; L_2 = \left\{ P \begin{pmatrix} \mu \\ \alpha \end{pmatrix}, \mu, \alpha \in \mathbb{Z}[i] \right\},$$

where $P \in GL(2, \mathbb{R})$ such that $PBP^{-1} = diag(e^{t_0}, e^{-t_0})$ and $B \in SL(2, \mathbb{Z})$.

Proof

Let J_0 be the complex structure on $\mathfrak{g}_8$ defined by
$d\omega^1 = 2i\,\omega^{13} + \omega^{3\bar{3}},\ \ d\omega^2 = -2i\,\omega^{23},\ \ d\omega^3 = 0.$

For each $a \in \mathbb{C}$ such that $|a| < 1$, we consider the cpx structure J_a on M defined by $\Phi^1 = \omega^1,\ \ \Phi^2 = \omega^2,\ \ \Phi^3 = \omega^3 + a\,\omega^{\bar{3}}.$

$$\Longrightarrow \begin{cases} d\Phi^1 = \frac{2i}{1-|a|^2}\Phi^{13} - \frac{2ia}{1-|a|^2}\Phi^{1\bar{3}} + \frac{1}{1-|a|^2}\Phi^{3\bar{3}}, \\ d\Phi^2 = -\frac{2i}{1-|a|^2}\Phi^{23} + \frac{2ia}{1-|a|^2}\Phi^{2\bar{3}}, \\ d\Phi^3 = 0. \end{cases}$$

• For any J_a, $a \in \Delta - \{0\}$, the structures

$$2\,F = i\,(r^2\Phi^{1\bar{1}} + s^2\Phi^{2\bar{2}} + t^2\Phi^{3\bar{3}}) + \frac{(1-|a|^2)r^2}{2\bar{a}}\Phi^{1\bar{3}} - \frac{(1-|a|^2)r^2}{2a}\Phi^{3\bar{1}},$$

with $r, s \neq 0$ and $t^2 > \frac{(1-|a|^2)^2 r^2}{4|a|^2}$, are balanced.

• By a result by Angella and Kasuya we have that for any $a \neq 0$ the compact $(\Gamma\backslash G, J_a)$ satisfies the $\partial\bar{\partial}$-lemma.

• The central limit J_0 does not satisfy the $\partial\bar{\partial}$-lemma. By the symmetrization process, it suffices to prove that it is not satisfied at the Lie algebra level $(\mathfrak{g}_8, J_0)$. But the form ω^{23} is ∂-closed, $\bar{\partial}$-closed and d-exact, however it is not $\partial\bar{\partial}$-exact!

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THANK YOU VERY MUCH FOR YOUR ATTENTION!