Positively curved GKM manifolds

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Let \((M, g)\) be a Riemannian manifold, with its Levi-Civita connection \(\nabla\) and associated curvature tensor \(R\).

For a two-dimensional subspace \(\sigma \subset T_pM\), spanned by orthonormal vectors \(X\) and \(Y\), the sectional curvature of \(\sigma\) is defined by

\[
K(\sigma) = \langle R(X, Y)Y, X \rangle.
\]

Unsolved problem: classify Riemannian manifolds with positive sectional curvature, i.e., \(K(\sigma) > 0\) for all \(\sigma\).
In this talk we consider even-dimensional compact connected orientable Riemannian manifolds with positive sectional curvature.

The only known examples are:

1. Spheres $S^{2n}$
2. The projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$
3. The Wallach spaces $SU(3)/T^2$, $Sp(3)/Sp(1)^3$, $F_4/Spin(8)$
4. Eschenburg’s twisted flag manifold $SU(3)//T^2$.

Note: these spaces are all highly symmetric!
Examples of structural results:

- (Grove–Searle) If a torus $T^k$ acts effectively and isometrically on a positively curved simply-connected compact Riemannian manifold of dimension $n$, then $k \leq \lfloor \frac{n+1}{2} \rfloor$, and equality can occur if and only if $M$ is diffeomorphic to $S^n$ or $\mathbb{C}P^{n/2}$.

- (Wilking) If $\dim \text{Iso}(M) \geq 2n - 5$ (same assumptions on $M$), then $M$ is homotopy equivalent to a compact rank-one symmetric space, or isometric to a homogeneous space of positive curvature.

- (Amann–Kennard) If $n$ is even, and a torus $T$ of dimension at least $\log_{4/3}(n)$ acts effectively and isometrically on $M$, then

$$
\chi(M) \leq \sum b_{2i}(M^T) \leq \left( \frac{2}{n} + 1 \right)^{1+\log_{4/3}(\frac{n}{2}+1)}
$$
Torus actions of GKM type

An action of a torus $T$ on an orientable differentiable manifold $M$ satisfying $H^{\text{odd}}(M, \mathbb{R}) = 0$ is called GKM$_k$ (named after a paper by Goresky–Kottwitz–MacPherson) if

1. The action has only finitely many fixed points
2. For each fixed point $p \in M^T$, any $k$ weights of the isotropy representation are linearly independent.

If $k = 2$ then we simply say that the action is GKM.
Torus actions of GKM type

Geometric interpretation of the second condition: Let $p \in M^T$, and decompose

$$T_p M = \bigoplus_{\alpha} V_{\alpha}$$

into weight spaces, $\dim V_{\alpha} = 2$. Then for a subtorus $T' \subset T$ we have

$$T_p M^{T'} = \bigoplus_{\alpha: \alpha|_{T'}=0} V_{\alpha}.$$

Condition 2: If $\dim T' = \dim T - 1$, then $\dim M^{T'} \leq 2$. 
An example

Consider the $T^n$-action on $\mathbb{C}P^n$ by

$$(e^{i\varphi_1}, \ldots, e^{i\varphi_n}) \cdot [z_0 : \ldots : z_n] = [z_0 : e^{i\varphi_1}z_1 : \ldots e^{i\varphi_n}z_n].$$

Fixed points: $[1 : 0 : \ldots : 0], \ldots, [0 : \ldots : 0 : 1]$. Components of $M^{T'}$, where $\dim T' = \dim T - 1$: either a fixed point or of the form

$$\{[0 : \ldots : 0 : u : 0 : \ldots : 0 : v : 0 : \ldots : 0]\} = \mathbb{C}P^1 \cong S^2.$$

Each of these $S^2$ is $T$-invariant and contains two fixed points!
The GKM graph

In general: any two-dimensional component of a submanifold $M^{T'}$ as above is a two-sphere, and contains exactly two fixed points.

Thus: if $\text{dim } M = 2n$, then for any fixed point $p$ there are $n$ invariant two-spheres containing $p$.

To any GKM action we can hence assign the GKM graph:

- **Vertices**: the fixed points.
- **Edges**: an edge connecting two fixed points for each $T$-invariant $S^2$ as above containing them.
- **Labeling**: An edge is labeled with the corresponding weight of the isotropy representation.
More generally: any toric manifold satisfies the GKM condition, and the GKM graph is the one-skeleton of the momentum image.
All the known examples of positively curved even-dimensional orientable manifolds admit an action of GKM type:

Guillemin–Holm–Zara: Let $G/H$ be a homogeneous space of compact Lie groups with $\text{rk } G = \text{rk } H$, and let $T \subset H$ be a maximal torus. Then the $T$-action on $G/H$ is GKM.

E.g.: weights of isotropy representation at $eH$: roots of $G$ which are not roots of $H$. In particular: pairwise linearly independent.

Also Eschenburg’s twisted flag admits a GKM action.
GKM actions on the positively curved examples

- $S^{2n}$
- $\mathbb{C}P^n$
- $\mathbb{H}P^n$
- $\mathbb{O}P^2$
- $SU(3)/T^2$ and $SU(3)\sslash T^2$
- $Sp(3)/Sp(1)^3$
- $F_4/Spin(8)$
Fact: The GKM graph of a GKM action of a torus $T$ on $M$ determines the real cohomology ring $H^*(M)$.

Sketch: Consider equivariant cohomology $H^*_T(M)$. The condition $H^\text{odd}(M) = 0$ implies that $H^*_T(M)$ is a free module over $H^*(BT)$.

Chang-Skjelbred-Lemma: Denote by $M_1 = \{ p \in M \mid \dim Tp \leq 1 \}$ the one-skeleton of the action. Then freeness of $H^*_T(M)$ implies that there is a short exact sequence

$$0 \longrightarrow H^*_T(M) \longrightarrow H^*_T(M^T) \longrightarrow H^*_T(M_1, M^T).$$

Thus the GKM graph determines $H^*_T(M)$. Freeness of $H^*_T(M)$ implies also $H^*(M) = H^*_T(M) \otimes_{H^*(BT)} \mathbb{R}$, hence $H^*_T(M)$ determines $H^*(M)$. 
The main result

Theorem (—, Wiemeler)

Let $M$ be a compact connected positively curved orientable Riemannian manifold.

1. If $M$ admits an isometric torus action of type GKM$_4$, then $M$ has the real cohomology ring of $S^{2n}$ or $\mathbb{C}P^n$.

2. If $M$ admits an isometric torus action of type GKM$_3$, then $M$ has the real cohomology ring of a compact rank one symmetric space.

Idea of proof: determine all possible GKM graphs of a GKM$_3$-action on $M$ and show that they are one of those described above. Easy in case 1, rather technical in case 2.
Main ingredient

Given a GKM$_3$-action on $M$, then any component of $M^{T'}$, where $\dim T' = \dim T - 2$, is at most 4-dimensional.

It thus makes sense to speak about two-dimensional faces of the GKM graph: any two edges emanating from the same vertex determines a two-dimensional face, i.e., a subgraph corresponding to a four-dimensional submanifold.

These submanifolds are totally geodesic and admit an effective isometric $T^2$-action, hence the result of Grove and Searle applies: they are either $S^4$ or $\mathbb{C}P^2$. In particular: the two-dimensional faces have either 2 or 3 vertices!
Note that this condition is violated for the GKM graphs of the Wallach spaces and the twisted flag:
Proof of case 1

Consider the GKM$_4$-case. If there are only 2 vertices, then the GKM graph is necessarily that of an action on a sphere.

Let $v_1, v_2, v_3$ be three vertices, and denote by $K_{ij}$ the set of edges between $v_i$ and $v_j$. Define a map

$$\phi : K_{12} \times K_{13} \longrightarrow K_{23}$$

sending two edges $(e_1, e_2)$ to the third edge of the two-dimensional face determined by $e_1$ and $e_2$. If $\alpha_1$ and $\alpha_2$ are the weights of $e_1$ and $e_2$, then the weight of $\phi(e_1, e_2)$ is of the form $a\alpha_1 + b\alpha_2$. If $\phi(e'_1, e'_2) = \phi(e_1, e_2)$, then $a'\alpha'_1 + b'\alpha'_2 = a\alpha_1 + b\alpha_2$, a contradiction to the 4-independence of the weights.
Hence

\[ \phi : K_{12} \times K_{13} \rightarrow K_{23} \]

is injective, i.e., \(|K_{12}| \cdot |K_{13}| \leq |K_{23}| \).

This implies: if \(|K_{ij}| > 1\) for some \(i, j\), then the other two \(K_{ij}\) must be empty. Because the graph is connected, this implies that the graph is necessarily a complete graph, i.e., the graph of a GKM action on \(\mathbb{C}P^n\).
We can prove two generalizations of the main result:

- **Integer coefficients:** If $M$ satisfies $H^{\text{odd}}(M, \mathbb{Z}) = 0$ and admits an isometric GKM$_3$-action such that any two weights are coprime, then $M$ has the integer cohomology of a compact rank one symmetric space.

- **Non-orientable manifolds:** If a non-orientable $M$ admits an isometric GKM$_3$-action, then $M$ has the real cohomology of a real projective space, i.e., $H^*(M) = H^0(M) = \mathbb{R}$. 