Hilbert Schemes, Symmetric Quotient Stacks, and Categorical Heisenberg Actions

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Plan of the Talk

Theorem (Göttsche / Nakajima / Grojnowski)

The cohomology of the Hilbert schemes (Douady spaces) of points on a smooth quasi-projective surface carry the structure of the Fock space representation of a Heisenberg algebra.

In this Talk

Discuss three approaches to a categorification of this Heisenberg action:

1. Lift the **Nakajima operators** to the derived categories.
2. Lift other generators (half of the **vertex operators**).
3. Give the derived categories of the Hilbert schemes the structure of a **Hopf category**.

Outline

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Symmetric Quotient Varieties

X: Smooth quasi-projective variety over $\mathbb{C}$.
$\mathfrak{S}_n$: Symmetric group. $\mathfrak{S}_n \curvearrowright X^n$ by permutation of factors:

$$(x_1, \dots, x_n) \overset{\sigma}{\mapsto} (x_{\sigma(1)}, \dots, x_{\sigma(n)}) .$$

Definition

Quotient $X^{(n)} := X^n/\mathfrak{S}_n$ is called **symmetric quotient variety.**

Examples (Curve Case)

$\mathbb{C}^{(n)} \cong \mathbb{C}^n$ (Theorem on symmetric functions), $\mathbb{P}^{1(n)} \cong \mathbb{P}^n$.

Non-Smoothness

For $\dim(X) \geq 2$, the symmetric quotient variety is not smooth. The singular locus consists of the partial diagonals.

Hilbert Schemes as Resolutions of Singularities

In the case that X is a **surface**, there is a **resolution of singularities** $\mu\colon X^{[n]} \to X^{(n)}$ with very good properties: The **Hilbert scheme of points on** X.

Example (n=2)

$\mu\colon X^{[2]} \to X^{(2)}$ is the blow-up along the diagonal.

Definition (General n)

The Hilbert scheme (Douady space) $X^{[n]}$ is the fine moduli space of n-**Clusters** on X. The **Hilbert–Chow morphism** $\mu\colon X^{[n]} \to X^{(n)}$ sends an n-Cluster to its weighted support.

Zero-Dimensional Subschemes (Clusters)

Definition

An *n*-**Cluster** on X is a zero-dimensional closed subscheme $Z \subset X$ of length $\ell(Z) := \dim_{\mathbb{C}}(\mathcal{O}(Z)) = n$.

Examples of Clusters

- **Collections of *n* distinct points:** $Z = \{x_1, \ldots, x_n\} \subset X$, $\mu(Z) = x_1 + \cdots + x_n \in X^{(n)}$.
- **Fat points:** Non-reduced schemes concentrated in one point.
  - $n = 2$: Fat points are points with infinitesimal tangent direction. $Z \cong \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$, $\mu(Z) = 2x$.

Recall: $\mu\colon X^{[2]} \to X^{(2)}$ is blow-up along diagonal.

Outline

Betti Numbers of Hilbert schemes

Fix smooth projective surface X
$\rightsquigarrow \quad \mathbb{H} := \bigoplus_{n\geq 0} \mathrm{H}^*(X^{[n]}, \mathbb{C})$ is double graded vector space.
The Betti numbers are the dimensions of graded pieces
$b_i(X^{[n]}) := \dim_{\mathbb{C}} \mathrm{H}^i(X^{[n]}, \mathbb{C})$.

Theorem (Göttsche, 1992)

$$\sum_{n=0}^{\infty} \sum_{i=0}^{4n} b_i(X^{[n]}) s^{i-2n} t^n = \prod_{m=1}^{\infty} \prod_{j=0}^{4} (1 - (-1)^j s^{j-2} t^m)^{(-1)^{j+1} b_j(X)}$$

$\mathbb{F}$: **Fock space** representation of the **Heisenberg Lie algebra** $\mathfrak{h}_V$ associated to $V := \mathrm{H}^*(X, \mathbb{C})$.

Corollary

$\mathbb{H} \cong \mathbb{F}$ as double graded vector spaces.

Heisenberg Lie Algebra (Basic Case)

Definition of the (infinite dimensional) Heisenberg Lie algebra $\mathfrak{h}$:

- As a **vector space:**

$$\mathfrak{h} := \mathbb{C} \cdot c \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathbb{C} \cdot a(n) \,.$$

- **Lie bracket** of $\mathfrak{h}$: c is central, i.e. $[c, v] = 0 \, \forall v \in \mathfrak{h}$, and

$$[a(m), a(n)] = \delta_{m,-n} \cdot n \cdot c = \begin{cases} n \cdot c & \text{if } m = -n, \\ 0 & \text{else.} \end{cases}$$

Alternative notation $t^m := a(m)$: For $f, g \in \mathbb{C}[t, t^{-1}]$ have $[f(t), g(t)] = \operatorname{res}\left(f(t) \cdot \frac{\partial g}{\partial t}(t)\right) \cdot c$.

Heisenberg Lie Algebra (General Case)

- **Given data:** Finite dimensional $\mathbb{C}$ vector space V together with bilinear form $\langle \_, \_ \rangle$.
- As a **vector space:** $\mathfrak{h}_V := \mathbb{C} \cdot c \oplus V^{\oplus \mathbb{Z} \setminus \{0\}}$.
- For $\beta \in V$ and $n \in \mathbb{Z}$, denote by $a_\beta(n) \in \mathfrak{h}_V$ the image of β under the inclusion of the n-th summand.
- **Lie bracket:** c is central and

$$[a_\alpha(m), a_\beta(n)] = \delta_{m,-n} \cdot n \cdot \langle \alpha, \beta \rangle \cdot c\,.$$

Question (Geometric Interpretation of Göttsche's Theorem)

Let $V = \mathrm{H}^*(X, \mathbb{C})$ together with **intersection form** $\langle \alpha, \beta \rangle = \int_X \alpha \cup \beta$. How to construct an action of $\mathfrak{h}_V$ on $\mathbb{H} = \bigoplus_{n \geq 0} \mathrm{H}^*(X^{[n]}, \mathbb{C})$ in a natural (geometric) way?

$\rightsquigarrow$ Constructions of Nakajima/Grojnowski (1996)

Nakajima Operators

- **Nakajima correspondences:** For $\ell, n \in \mathbb{N}$ consider closed subscheme $Z^{n,\ell} \subset X \times X^{[\ell]} \times X^{[\ell+n]}$

$$Z^{\ell,n} := \left\{(x, Z, Z') \mid Z \subset Z',\ Z \text{ and } Z' \text{ only differ in } x\right\}.$$

$\rightsquigarrow$ Induced operators $a(\ell, n)\colon\ \mathrm{H}^*(X \times X^{[\ell]}) \to \mathrm{H}^*(X^{[\ell+n]})$.

- **Fix** $\beta \in \mathrm{H}^*(X)$. Define **Nakajima operator** $a_\beta(\ell, n) := a(\ell, n)(\beta \otimes \_)$:

$$\begin{array}{ccccc}
\mathrm{H}^*(X) \otimes \mathrm{H}^*(X^{[\ell]}) & \xrightarrow[\cong]{\text{Künneth}} & \mathrm{H}^*(X \times X^{[\ell]}) & \xrightarrow{a(\ell,n)} & \mathrm{H}^*(X^{[\ell+n]}) \\
\uparrow{\scriptstyle \beta\otimes_} & & & \nearrow{\scriptstyle a_\beta(\ell,n)} & \\
\mathrm{H}^*(X^{[\ell]}) & & & &
\end{array}$$

- Set $a_\beta(n) := \bigoplus_{\ell \geq 0} a_\beta(\ell, n) \colon \mathbb{H} = \bigoplus_{\ell \geq 0} \mathrm{H}^*(X^{[\ell]}) \to \mathbb{H}[n]$.
- **For $n < 0$,** define $a_\beta(n) \colon \mathbb{H}[n] \to \mathbb{H}$ as the **adjoint** of $a_\beta(-n)$ with respect to the intersection pairing on the cohomology of the Hilbert schemes.

Theorem (Nakajima)

The commutator relations between these operators satisfy

$$[a_\alpha(m), a_\beta(n)] = \delta_{m,-n} \cdot n \cdot \langle \alpha, \beta \rangle \cdot \mathrm{id}_{\mathbb{H}} \ .$$

Corollary

The above operators equip $\mathbb{H}$ with the structure of a module over the Heisenberg Lie algebra $\mathfrak{h}_V$ associated to $V = \mathrm{H}^*(X)$ where c acts as the identity.

Outline

Lifting/Categorification Problem

Goal

Would like to lift the Heisenberg action from the cohomology of the Hilbert schemes to the level of **Grothendieck groups** or, even better, **derived categories** of coherent sheaves **('categorification')**.

$$\begin{array}{ccc}
\mathbb{D} := \bigoplus_{\ell \geq 0} \mathrm{D}(X^{[\ell]}) & \xrightarrow{?} & \mathbb{D} \\
\downarrow [_] & & \downarrow [_] \\
\mathbb{K} := \bigoplus_{\ell \geq 0} \mathrm{K}(X^{[\ell]}) & \xrightarrow{?} & \mathbb{K} \\
\downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\
\mathbb{H} := \bigoplus_{\ell \geq 0} \mathrm{H}^*(X^{[\ell]}) & \xrightarrow{a_\beta(n)} & \mathbb{H}
\end{array}$$

Coherent Sheaves and Complexes

- Y: smooth projective variety.
- $\mathrm{Coh}(Y)$: abelian category of coherent sheaves.
- $\mathrm{Kom}(Y) := \mathrm{Kom}(\mathrm{Coh}(Y))$ category of complexes

$$\begin{array}{lccccc} \text{objects:} & \cdots A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1}\cdots \\ \text{morphisms:} & \downarrow \varphi^{i-1} & & \downarrow \varphi^i & & \downarrow \varphi^{i+1} \\ & \cdots B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1}\cdots \end{array}$$

- $\varphi^\bullet : A^\bullet \to B^\bullet \quad \rightsquigarrow \quad \mathcal{H}^i(\varphi^\bullet) : \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$.
- $\varphi^\bullet$ is a **quasi-isomorphism (qis)**
  $:\iff \quad \mathcal{H}^i(\varphi^\bullet)$ is an isomorphism $\forall i \in \mathbb{Z}$.

Derived Category

Definition (Derived Category)

$\mathrm{D}(Y) := \mathrm{D}(\mathrm{Coh}(Y)) := \mathrm{Kom}(Y)[\mathrm{qis}^{-1}]$

- **Objects:** (Bounded) Complexes of coherent sheaves.
- **Morphisms:** Morphisms of complexes
  $+$ Formal inverses of quasi-isomorphisms.

Features of the Derived Category

- **Shift** autoequivalence $[1]\colon \mathrm{D}(Y) \to \mathrm{D}(Y)$.
- Fully faithful **embedding** $\mathrm{Coh}(Y) \hookrightarrow \mathrm{D}(Y)$, $E \mapsto E[0]$.
- **Graded Hom-spaces**
  $\mathrm{Hom}^*(A^\bullet, B^\bullet) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}(Y)}(A^\bullet, B^\bullet[i])[-i]$.
- For $E, F \in \mathrm{Coh}(Y)$: $\quad \mathrm{Hom}^*(E, F) \cong \mathrm{Ext}^*(E, F)$.

Grothendieck Groups and Euler Characteristic

Definition (Grothendieck Group and its Natural Bilinear Form)

$\mathrm{K}(Y) := \mathrm{K}(\mathrm{Coh}(Y)) := \mathbb{Z} \cdot \mathrm{Coh}(Y)/\langle \text{short exact seq.} \rangle$
is equipped with a bilinear form, the **Euler bicharacteristic**
$\langle [E], [F] \rangle := \chi(E, F) := \chi(\mathrm{Ext}^*(E, F)) := \sum (-1)^i \,\mathrm{Ext}^i(E, F)$.

- For $A^\bullet \in \mathrm{D}(Y)$, set $[A^\bullet] := \sum (-1)^i [\mathcal{H}^i(A^\bullet)] \in \mathrm{K}(Y)$.

$\rightsquigarrow$ Commutative diagram:

$$\begin{array}{ccc} \mathrm{D}(Y) \times \mathrm{D}(Y) & \xrightarrow{\mathrm{Hom}^*(\_,\_)} & \mathrm{D}(\text{VectorSpaces}) \\ \big\downarrow [\_]\times[\_] & & \big\downarrow [\_] \\ \mathrm{K}(Y) \times \mathrm{K}(Y) & \xrightarrow{\langle \_,\_ \rangle} & \mathbb{Z} \end{array}$$

Slogan: $\mathrm{Hom}^*(\_, \_)$ categorifies $\langle \_, \_ \rangle$.

Fourier–Mukai Transforms

Definition (Fourier–Mukai Transforms)

Given $P \in \mathrm{D}(Y \times Z)$ **'Fourier–Mukai kernel'**

$\rightsquigarrow$ $\mathrm{FM}_P \colon D(Y) \to \mathrm{D}(Z) \quad E \mapsto R\,\mathrm{pr}_{Z*}(P \otimes^L \mathrm{pr}_Y^* E)$.

$\rightsquigarrow$ induced correspondence operators:

$$\begin{array}{ccc} \mathrm{D}(Y) & \xrightarrow{\mathrm{FM}_P} & \mathrm{D}(Z) \\ \big\downarrow{\scriptstyle [\_]} & & \big\downarrow{\scriptstyle [\_]} \\ \mathrm{K}(Y) & \xrightarrow{\Phi_{[P]}} & \mathrm{K}(Z) \\ \big\downarrow{\scriptstyle \mathrm{ch}} & & \big\downarrow{\scriptstyle \mathrm{ch}} \\ \mathrm{H}^*(Y) & \xrightarrow{\Phi_{\mathrm{ch}(P)}} & \mathrm{H}^*(Z) \end{array}$$

Approach to Categorification of Heisenberg Action

First Approach

Set $P = \mathcal{O}_{Z^{\ell,n}}$ (structure sheaf of Nakajima correspondence) and consider $A(\ell, n) := \mathrm{FM}_P \colon \mathrm{D}(X \times X^{[\ell]}) \to \mathrm{D}(X^{[\ell+n]})$. Let $A(\ell, -n)$ be the (right) adjoint functor.

Problem

It is too hard to compute the compositions $A(\ell, -n) \circ A(\ell, n)$ since $Z^{n,\ell}$ is badly singular.

Solution: Derived McKay Correspondence

Translate the categorification question to an easier equivariant problem using the **McKay correspondence**

$$\mathrm{D}(X^{[n]}) \cong \mathrm{D}_{\mathfrak{S}_n}(X^n)\,.$$

Outline

General Crepant Resolution Principle

Set-up: Let M smooth quasi-projective variety, $G \subset \mathrm{Aut}(M)$ finite subgroup such that ω_M descends to a line bundle $\omega_{M/G}$.

Definition (Crepant Resolution)

A resolution of singularities $\mu\colon Y \to M/G$ is **crepant** $:\iff$ $\mu^*\omega_{M/G} \stackrel{\cong}{=} \omega_Y$

Crepant Resolution Principle (Conjecture)

The geometry of Y should reflect the G-equivariant geometry of M.

More concretely: All invariants of Y should agree with the corresponding invariants of the **stack (orbifold)** $[M/G]$.

Derived McKay Correspondence ($M = X^n$ case)

The **Hilbert–Chow morphism** $\mu\colon X^{[n]} \to X^{(n)} = X^n/\mathfrak{S}_n$ is a crepant resolution.

Theorem (Bridgeland–King–Reid + Haiman 2001)

X smooth projective surface, $n \in \mathbb{N}$. Then:

$$\mathrm{D}(X^{[n]}) \cong \mathrm{D}_{\mathfrak{S}_n}(X^n) \cong \mathrm{D}([X^n/\mathfrak{S}_n])\,.$$

$\mathrm{D}_{\mathfrak{S}_n}(X^n) := \mathrm{D}(\mathrm{Coh}_{\mathfrak{S}_n}(X^n))$ **Equivariant derived category.**
$\mathrm{Coh}_{\mathfrak{S}_n}(X^n) = \mathrm{Coh}([X^n/\mathfrak{S}_n])$: Abelian category of $\mathfrak{S}_n$**-equivariant sheaves.**
Objects: Pairs (E, λ) with $E \in \mathrm{Coh}(X^n)$ and $\lambda = (\lambda_\sigma \colon E \xrightarrow{\cong} \sigma^* E)_{\sigma \in \mathfrak{S}_n}$ a $\mathfrak{S}_n$**-linearisation.**
Morphisms: $\mathrm{Hom}_{\mathrm{Coh}_{\mathfrak{S}_n}(X^n)}\big((E,\lambda),(F,\nu)\big) = \mathrm{Hom}_{\mathrm{Coh}(X^n)}\big(E,F\big)$.

Outline

Idea of Construction

Construct $A(\ell, n)\colon \mathrm{D}_{\mathfrak{S}_n}(X \times X^\ell) \to \mathrm{D}_{\mathfrak{S}_{\ell+n}}(X^{\ell+n})$ on equivariant side and use McKay correspondence.

Nakajima Correspondences in $X \times X^{[\ell]} \times X^{[\ell+n]}$

$$Z^{\ell,n} = \{(x, Z, Z') \mid Z \subset Z',\ Z \text{ and } Z' \text{ only differ in } x\}$$

$\updownarrow$ McKay correspondence

Partial Diagonals in $X \times X^\ell \times X^{n+\ell}$

$$\Delta_0 = \{(x; x_1, \ldots, x_\ell; x_1, \ldots, x_\ell, x, \ldots, x)\} \cong X \times X^\ell$$

Example ($\ell = 0$)

$A(0, n) = \delta_*\colon \mathrm{D}(X) \to \mathrm{D}_{\mathfrak{S}_n}(X^n) \cong \mathrm{D}(X^{[n]})$ is (equivariant) push-forward along embedding of small diagonal $\delta\colon X \hookrightarrow X^n$.

$\mathbb{P}$-functor versions of the Nakajima operators

Let X be a smooth projective surface.

Theorem (_)

There exists a series $A(\ell, n) = \mathrm{FM}_{P^{\ell,n}} \colon \mathrm{D}(X \times X^{[\ell]}) \to \mathrm{D}(X^{[\ell+n]})$ of Fourier–Mukai transforms with $\mathrm{supp}(P^{\ell,n}) = Z^{\ell,n}$. For $n > \max\{\ell, 1\}$, the $A(\ell, n)$ are **$\mathbb{P}$-functors.** In particular, (for $\omega_X = \mathcal{O}_X$)

$$A(\ell, -n) \circ A(\ell, n) \cong \mathrm{id} \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-2(n-1)]$$

where $A(\ell, -n) \colon \mathrm{D}(X^{[n+\ell]}) \to \mathrm{D}(X \times X^{[\ell]})$ is the right-adjoint.

Addington (2011) defined $\mathbb{P}$-functors in order to construct **non-standard autoequivalences** of derived categories.

Comparison with Heisenberg Commutator Relations

- **Heisenberg relations:** $[a_\alpha(m), a_\beta(n)] = \delta_{m,-n} n\langle\alpha,\beta\rangle \cdot \mathrm{id}_{\mathbb{H}}$.
- Set $m = -n$ and consider degree $\ell < n$:

$$a_\alpha(\ell,-n)a_\beta(\ell,n) = n\langle\alpha,\beta\rangle \cdot \mathrm{id}_{\mathrm{H}^*(X^{[\ell]})}\ .$$

- **Fix** $E \in \mathrm{D}(X)$: $A_E(\ell,n) := A(\ell,n) \circ i_E \colon \mathrm{D}(X^{[\ell]}) \to \mathrm{D}(X^{[n+\ell]})$ with $i_E \colon \mathrm{D}(X^{[\ell]}) \to \mathrm{D}(X \times X^{[\ell]})$, $B \mapsto E \boxtimes B$.

$$\begin{aligned} A_E(\ell,-n)A_F(\ell,n) &= i_E^R A(\ell,-n)A(\ell,n) i_F \\ &= i_E^R i_F([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)]) \\ &= \mathrm{Hom}^*(E,F)([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)]) \end{aligned}$$

$\downarrow$ Descend to $\mathrm{K}(X^{[\ell]})$, set $\alpha = [E]$, $\beta = [F]$, $a(\ell,n) = [A(\ell,n)]$

$$a_\alpha(\ell,-n)a_\beta(\ell,n) = n\langle\alpha,\beta\rangle \cdot \mathrm{id}_{\mathrm{K}(X^{[\ell]})}\ .$$

Induced Categorical Structures

No Categorification!

$A(n, \ell)$ do not fulfil analogues of Heisenberg relations for $n \neq m$.

Features of the Construction

- Get interesting **autoequivalences** of $\mathrm{D}_{\mathfrak{S}_m}(X^m) \cong \mathrm{D}(X^{[m]})$ **'characteristic functions of the stacky loci'**.
- Construction makes sense for smooth X of **arbitrary dimension** (forget about $X^{[m]}$ and only consider $[X^m/\mathfrak{S}_m]$).
- Curve case: $A(\ell, n)\colon \mathrm{D}(C \times [C^\ell/\mathfrak{S}_\ell]) \hookrightarrow \mathrm{D}([C^{\ell+n}/\mathfrak{S}_{\ell+n}])$ **fully faithful.**
  - $\rightsquigarrow$ Characteristic autoequivalences of the stacky loci.
  - $\rightsquigarrow$ **Semi-orthogonal decomposition** which categorifies decomposition of **orbifold cohomology**.
- $\exists$ analogous $A(\ell, n)$ for **generalised Kummer varieties.**

Outline

Categorical Heisenberg Action

Theorem (_)

For every smooth projective variety (stack) X there exists a categorical Heisenberg action on $\mathbb{D} := \bigoplus_{\ell \geq 0} \mathrm{D}([X^\ell/\mathfrak{S}_\ell])$.

Lifts other generators $p_\beta^{(n)}$, $q_\beta^{(n)}$ (and other relations) of Heisenberg algebra: **halves of the vertex operators**

$$\sum_{n \geq 0} p_\beta^{(n)} z^n = \exp\Big(\sum_{\ell \geq 1} \frac{a_\beta(-\ell)}{\ell} z^\ell\Big), \sum_{n \geq 0} q_\beta^{(n)} z^n = \exp\Big(\sum_{\ell \geq 1} \frac{a_\beta(\ell)}{\ell} z^\ell\Big)$$

Non-Integer Coefficients

Cannot reconstruct lifts of Nakajima operators from this.

Straight-forward generalisation of parts of constructions of Cautis–Licata ($X \to \mathbb{C}^2/G$ resolution of Kleinian singularity) and Khovanov (X a point).

Categorical Heisenberg Action

Definition/Lemma

For $\chi \in \mathbb{C}$ and k a non-negative integer, we set
$s^k\chi := \binom{\chi+k-1}{k} := \frac{1}{k!}(\chi+k-1)(\chi+k-2)\cdots(\chi+1)\chi$.
For V^* a graded vector space: $\chi(S^k V^*) = s^k(\chi(V^*))$

$$[q_\alpha^{(m)}, q_\beta^{(n)}] = 0 = [p_\alpha^{(m)}, p_\beta^{(n)}]$$
$$q_\alpha^{(m)} p_\beta^{(n)} = \sum_{k=0}^{\min\{m,n\}} s^k\langle\alpha,\beta\rangle \cdot p_\beta^{(n-k)} q_\alpha^{(m-k)} .$$

Definition (Categorical Heisenberg Action)

A family of adjoint functors $P^{(m)} : \mathbb{D} \leftrightarrows \mathbb{D} : Q_E^{(m)}$, for $E \in \mathrm{D}(X)$ and $m \in \mathbb{Z}$, such that

$$Q_E^{(m)} Q_F^{(n)} \cong Q_F^{(n)} Q_E^{(m)} \quad , \quad P_E^{(m)} P_F^{(n)} \cong P_F^{(n)} P_E^{(m)}$$
$$Q_E^{(m)} P_F^{(n)} \cong \bigoplus_{k=0}^{\min\{m,n\}} S^k \operatorname{Hom}^*(E,F) \otimes_{\mathbb{C}} P_F^{(n-k)} Q_E^{(m-k)}$$

$\rightsquigarrow$ Heisenberg module structure on $\mathbb{K}$.

Outline

1. Preliminaries
   - Symmetric Products and Hilbert Schemes of Points on Surfaces
   - Cohomology of Hilbert Schemes and the Heisenberg Algebra
   - Derived Categories and Grothendieck Groups
   - McKay Correspondence

2. Three Constructions
   - Nakajima $\mathbb{P}$-functors
   - Lift of the Heisenberg Module Structure
   - Categorical Hopf Algebras

The Case that X is a Point

Note: $\mathrm{Coh}_{\mathfrak{S}_n}(\mathrm{pt}) \cong \mathrm{Rep}(\mathfrak{S}_n)$, $\mathrm{K}([\mathrm{pt}/\mathfrak{S}_n]) \cong R(\mathfrak{S}_n)$.
Consider: Graded vector space $R := \bigoplus_{n \geq 0} R(\mathfrak{S}_n)$.

Theorem (…, Zelevinsky)

R is a **positive self-adjoint Hopf algebra (PSH)**. This means:

- **Bilinear form** $\langle V, W \rangle := \hom(V, W)^G$.
- **Multiplication** $m = \mathrm{Ind}\colon R \otimes R \to R$, on graded pieces: $R_a \otimes R_b \to R_{a+b}, \quad V \otimes W \mapsto \bigoplus_{\mathfrak{S}_{a+b}/(\mathfrak{S}_a \times \mathfrak{S}_b)} (V \boxtimes W)$
- **Comultiplication** $\nabla = \mathrm{Res}\colon R \to R \otimes R$ adjoint to m.

$$
\begin{array}{ccccc}
R \otimes R & & \xrightarrow{\quad m \quad} & & R \\
\downarrow{\scriptstyle \nabla \otimes \nabla} & & & & \downarrow{\scriptstyle \nabla} \\
R \otimes R \otimes R \otimes R & \xrightarrow{R \otimes \leftrightarrows \otimes R} & R \otimes R \otimes R \otimes R & \xrightarrow{m \otimes m} & R \otimes R
\end{array}
$$

The General Case

- **Want:** $\mathbb{D} := \bigoplus_{n\geq 0} \mathrm{D}([X^n/\mathfrak{S}_n]) = \bigoplus_{n\geq 0} \mathrm{D}_{\mathfrak{S}_n}(X^n)$ as a **PSH category.**
- **Idea:** For stacks $\mathcal{Y}$, $\mathcal{Z}$ have '$\mathrm{D}(\mathcal{Y}\times\mathcal{Z}) \cong \mathrm{D}(\mathcal{Y}) \otimes \mathrm{D}(\mathcal{Z})$' (can be made precise on level of dg-categories).
- $[X^a/\mathfrak{S}_a] \times [X^b/\mathfrak{S}_b] = [X^{a+b}/(\mathfrak{S}_a \times \mathfrak{S}_b)]$.
- $E \in \mathrm{D}_{\mathfrak{S}_{a+b}}(X^{a+b})$: $\quad \mathrm{Ind}_{\mathfrak{S}_a\times\mathfrak{S}_b}^{\mathfrak{S}_{a+b}}(E) = \bigoplus_{\mathfrak{S}_{a+b}/(\mathfrak{S}_a\times\mathfrak{S}_b)} \sigma^* E$.
  $\rightsquigarrow$ **Adjoint pair:**
  $m = \mathrm{Ind}\colon\ \mathrm{D}_{\mathfrak{S}_a\times\mathfrak{S}_b}(X^a \times X^b) \leftrightarrows \mathrm{D}_{\mathfrak{S}_{a+b}}(X^{a+b}) : \mathrm{Res} = \nabla$

$$\begin{array}{ccccc}
\mathbb{D}\otimes\mathbb{D} & & \xrightarrow{\quad m \quad} & & \mathbb{D} \\
\downarrow{\scriptstyle \nabla\otimes\nabla} & & & & \downarrow{\scriptstyle \nabla} \\
\mathbb{D}\otimes\mathbb{D}\otimes\mathbb{D}\otimes\mathbb{D} & \xrightarrow{\mathbb{D}\otimes\leftrightarrows\otimes\mathbb{D}} & \mathbb{D}\otimes\mathbb{D}\otimes\mathbb{D}\otimes\mathbb{D} & \xrightarrow{m\otimes m} & \mathbb{D}\otimes\mathbb{D}
\end{array}$$

Slogan: $\mathbb{D}$ is a **geometric PSH category.**

Work in Progress

- A. Gal and E. Gal define **Heisenberg double** associated to every PSH category (in a stricter sense). Would like to do the same for geometric PSH category.
- There exists the notion of **symmetric product of a (dg) category** by Ganter and Kapranov such that $\mathrm{Sym}^n(\mathrm{D}(X)) \cong \mathrm{D}([X^n/\mathfrak{S}_n])$.
  Question: To which extend do the above constructions generalise to symmetric products of categories?