Invariant Einstein Metrics on Stiefel Manifolds

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Stiefel manifolds

Stiefel manifolds $V_k \mathbb{F}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are the set of all orthonormal $k$-frames in $\mathbb{F}^n$. It can be shown that $V_k \mathbb{F}^n$ is diffeomorphic to a homogeneous space $G/H$. In particular:

- **In case $\mathbb{F} = \mathbb{R}$**
  
  $V_k \mathbb{R}^n \cong \text{SO}(n)/\text{SO}(n - k)$

- **In case $\mathbb{F} = \mathbb{C}$**
  
  $V_k \mathbb{C}^n \cong \text{SU}(n)/\text{SU}(n - k)$

- **In case $\mathbb{F} = \mathbb{H}$**
  
  $V_k \mathbb{H}^n \cong \text{Sp}(n)/\text{Sp}(n - k)$

In all cases the Stiefel manifolds are *reductive homogeneous spaces*, with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ and $\mathfrak{m} \cong T_o(G/H)$, with respect to negative of Killing form of $\mathfrak{g}$.

If $H$ is connected then $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m} \iff [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. 
A $G$-invariant metric $g$ on homogeneous space $G/H$ is the metric for which the diffeomorphism $\tau_\alpha : G/H \to G/H$, $gH \mapsto \alpha gH$ is an isometry. It can be shown that

**Proposition 1**

There exists a one-to-one correspondence between:

1. $G$-invariant metrics $g$ on $G/H$
2. $\text{Ad}^{G/H}$-invariant inner products $\langle \cdot, \cdot \rangle$ on $m$, that is

   $$\langle \text{Ad}^{G/H}(h)X, \text{Ad}^{G/H}(h)Y \rangle = \langle X, Y \rangle \quad \text{for all } X, Y \in m, h \in H$$

3. (if $H$ is compact and $m = \mathfrak{h} \perp$ with respect to the negative of the Killing form $B$ of $G$) $\text{Ad}^{G/H}$-equivariant, $B$-symmetric and positive definite operators $A : m \to m$ such that $\langle X, Y \rangle = B(A(X), Y)$.

We call such an inner product $\text{Ad}^G(H)$-invariant, or simply $\text{Ad}(H)$-invariant
Isotropy irreducible homogeneous space

In the case where the isotropy representation of a reductive homogeneous space $G/H$

$$\text{Ad}^{G/H} : H \rightarrow \text{Aut}(m)$$

$$h \mapsto (d\tau_h)_o : m \rightarrow m$$

is irreducible, then $G/H$ admits a unique (up to scalar) $G$-invariant metric $g$, which is also Einstein $\Rightarrow \text{Ric}_g = \lambda \cdot g$.
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▶ These spaces have been studied in 1968 by J. Wolf.

Some examples of such spaces are the following:

- $\text{SO}(n + 1)/\text{SO}(n) \cong S^n$
- $\text{Spin}(7)/\text{G}_2 \cong S^7$
- $\text{G}_2/\text{SU}(3) \cong S^6$
- $\text{SU}(n)/\text{S(U(1) \times U(n))} \cong \mathbb{C}P^n$.  

Isotropy reducible homogeneous space

In the case where the isotropy representation is a direct sum of irreducible representations $\varphi_i : H \to \text{Aut}(m_i), i = 1, 2, \ldots s$, that is

$$\text{Ad}^{G/H} \cong \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_s \to \text{Aut}(m_1 \oplus m_2 \oplus \cdots \oplus m_s),$$

then we have the following two cases:

(A)

- The representations $\varphi_i$ are non equivalent.

In 2004 Böhm-Wang-Ziller conjectured the following: Let $G/H$ be a compact homogeneous space whose isotropy representation splits into a finite sum of non-equivalent and irreducible, subrepresentations. Then the number of $G$-invariant Einstein metrics on $G/H$ is finite.
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(B)  
- Some of the representations $\varphi_i$ are equivalent, that is $\varphi_i \approx \varphi_j (i \neq j)$. 
Isotropy reducible homogeneous space, case (A)

When the representations $\varphi_i$ are non equivalent then the decomposition of $m$

$$m = m_1 \oplus m_2 \oplus \cdots \oplus m_s$$

is unique and $m_i, m_j \ i \neq j$ are perpendicular.

▶ In this case all $\text{Ad}(H)$-invariant inner products on $m$ are described as follows:

$$\langle \cdot, \cdot \rangle = x_1(-B)|_{m_1} + x_2(-B)|_{m_2} + \cdots + x_s(-B)|_{m_s} \quad x_i \in \mathbb{R}^+, \ i = 1, 2, \ldots, s$$

▶ The matrix of the operator $A : m \rightarrow m$ with respect to $(-B)$-orthonormal basis is:

$$
\begin{pmatrix}
    x_1 \text{Id}_{m_1} & 0 \\
    \vdots & \ddots \\
    0 & x_s \text{Id}_{m_s}
\end{pmatrix}
$$
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- The matrix of the operator $A: m \to m$ with respect to $(-B)$-orthonormal basis is:

$$ \begin{pmatrix}
    x_1 \text{Id}_{m_1} & 0 \\
    \cdot & \cdot \\
    0 & x_s \text{Id}_{m_s}
\end{pmatrix} $$

The $G$-invariant metrics that correspond to these inner products are called diagonal.
Now for the Ricci tensor of diagonal $G$-invariant metrics we have the following:

We set $d_i := \dim m_i$ and let $\{e^i_\alpha\}^{d_i}_{\alpha=1}$ be a $(-B)$-orthonormal basis adapted to the above decomposition of $m$, i.e. $e^i_\alpha \in m_i$ $i = 1, 2, \ldots, s$.

Consider the numbers $A^\gamma_{\alpha\beta} = (-B)([e^i_\alpha, e^j_\beta], e^k_\gamma)$ such that

$$[e^i_\alpha, e^j_\beta] = \sum_\gamma A^\gamma_{\alpha\beta} e^k_\gamma$$

and set

$$A_{ijk} := \begin{bmatrix} k \\ i, j \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$$

where the sum taken over all three indices $\alpha, \beta, \gamma$ with $e^i_\alpha \in m_i, e^j_\beta \in m_j, e^k_\gamma \in m_k$.

The numbers $A_{ijk}$ are non-negative, independent of the $(-B)$-orthonormal bases chosen for $m_i, m_j, m_k$, and are symmetric in all three indices:

$$A_{ijk} = A_{jik} = A_{kij}.$$
The Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ of a $G$-invariant Riemannian metric on $G/H$ has also a diagonal form, i.e.

$$\text{Ric}_{\langle \cdot, \cdot \rangle} = \sum_{k=0}^{s} r_k x_k (-B)|_{m_k}.$$ 

We have the following proposition due to Park and Sakane (1997).

**Proposition 2**

The components $r_1, \ldots, r_q$ of the Ricci tensor $\text{Ric}_{\langle \cdot, \cdot \rangle}$ on $G/H$ are given by

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1, \ldots, q), \quad (1)$$

where the sum is taken over $i, j = 1, \ldots, q$. In particular for each $k$ it holds that

$$\sum_{i,j} \begin{bmatrix} j \\ ki \end{bmatrix} = \sum_{i,j} A_{kij} = d_k := \dim m_k. \quad (2)$$
When some of the $\varphi_i, \varphi_j$ in the isotropy representation of $G/H$ are equivalent, then

- the diagonal $G$-invariant metrics is not unique, and
- the submodules $m_i, m_j$ does not necessarily perpendicular.

In this case the matrix of the operator $(\cdot, \cdot) = \langle A \cdot, \cdot \rangle$ has some non zero non diagonal elements.

▶ Also the Ricci tensor is not easy to describe
For the real Stiefel manifolds $V_k\mathbb{R}^n \cong SO(n)/SO(n-k)$ the isotropy representation is given as follows:

$$\text{Ad}^{SO(n)}|_{SO(n-k)} = \cdots = \bigwedge^2 \lambda_{n-k} \oplus 1 \oplus \cdots \oplus 1 \oplus \lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}$$

$$\text{Ad}^{SO(n-k)} \oplus \lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}$$

$$\bigwedge^{(k/2)} - \text{times} \quad k - \text{times}$$
Isotropy reducible homogeneous space, case (B)--Examples

• For the real Stiefel manifolds $V_k \mathbb{R}^n \cong SO(n)/SO(n - k)$ the isotropy representation is given as follows:

$$Ad^{SO(n)}_{SO(n-k)} = \cdots = \bigoplus_{k \text{-times}} \lambda_{n-k} \bigoplus \bigoplus_{\binom{k}{2}} \lambda_{n-k} \bigoplus \bigoplus_{(k-1) \text{-times}} \lambda_{n-k} \bigoplus \cdots \bigoplus \lambda_{n-k}$$

For $n = 4$ and $k = 2$ the matrix of the operator $A : m \to m$ has the following form:

$$\begin{pmatrix}
    x_0 & 0 & 0 & 0 & 0 \\
    0 & x_1 & 0 & \lambda & 0 \\
    0 & 0 & x_1 & 0 & \lambda \\
    0 & \lambda & 0 & x_2 & 0 \\
    0 & 0 & \lambda & 0 & x_2
\end{pmatrix} \ \ \ \ \lambda \in \mathbb{R}, \ x_i \in \mathbb{R}^+, \ i = 0, 1, 2.$$

• For the quaternionic Stiefel manifolds $V_k \mathbb{H}^n$ the isotropy representation is given as follows:

$$Ad^{Sp(n)}_{Sp(n-k)} \otimes \mathbb{C} = \cdots = \bigoplus_{\binom{2k}{2} \text{-times}} S^2 \nu_{n-k} \bigoplus \bigoplus_{2k \text{-times}} \nu_{n-k} \bigoplus \bigoplus_{\text{Ad}^{Sp(n-k)}} \nu_{n-k} \bigoplus \cdots \bigoplus \nu_{n-k}.$$
Some history

- **Kobayashi (1963):** Proved the existence of an $SO(n)$-invariant Einstein metric on the unit tangent bundle $T_1S^n \cong SO(n)/SO(n - 2)$.

- **Sagle (1970) - Jensen (1973):** Proved the existence of $SO(n)$-invariant Einstein metrics on the Stiefel manifolds $V_k\mathbb{R}^n \cong SO(n)/SO(n - k)$, for $k \geq 3$

  metrics of the form: $\leftrightarrow \langle \cdot , \cdot \rangle = \begin{pmatrix} 0 & a & 1 \\ a & a & 1 \\ 1 & 1 & * \end{pmatrix}$.

- **Back - Hsiang (1987) and Kerr (1998):** Proved that for $n \geq 5$ the Stiefel manifolds $V_2\mathbb{R}^n \cong SO(n)/SO(n - 2)$ admit exactly one (diagonal) $SO(n)$-invariant Einstein metric.

- **Arvanitoyeorgos-Dzhepko-Nikonov (2009):** Showed that for $s > 1$ and $l > k > 3$ the Stiefel manifolds $V_{sk}\mathbb{R}^{sk+l} \cong G(sk + l)/G(l)$ admit at least four $G(sk + l)$-invariant Einstein metrics which are also $Ad(\{SO(l), Sp(l)\})$-invariant (two of these are Jensen’s metrics) where $G(l) \in \{SO(l), Sp(l)\}$.

  metrics of the form: $\leftrightarrow \langle \cdot , \cdot \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & * \end{pmatrix}$. 
As seen before, the $G$-invariant metrics $\mathcal{M}^G$ on $G/H \cong V_k F^n$, $F \in \{\mathbb{R}, \mathbb{H}\}$ are not only diagonal. For this reason the complete description of $G$-invariant Einstein metrics is difficult, because the Ricci tensor is not easy to describe. So we search for a subset of these metrics which are diagonal.

**General construction**

Let $G/H$ a homogeneous spaces with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We consider the operator

$$\text{Ad}(n) : \mathfrak{g} \to \mathfrak{g}$$

where $n \in N_G(H) = \{g \in G : gHg^{-1} = H\}$. Then

**Proposition 3**

The operator $\text{Ad}(n)|_m : m \to g$ takes values in $m$, that is $\varphi = \text{Ad}(n)|_m \in \text{Aut}(m)$. Also, $(\text{Ad}(n)|_m)^{-1} = (\text{Ad}(n)|_m)^t$. 

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General construction

We define the isometric action

$$\Phi \times \mathcal{M}^G \to \mathcal{M}^G, \quad (\varphi, A) \mapsto \varphi \circ A \circ \varphi^{-1} \equiv \tilde{A},$$

where $\Phi$ is the set $\{\varphi = \text{Ad}(n)|_m : n \in N_G(H)\} \subset \text{Aut}(m)$.

**Proposition 4**

The action of $\Phi$ on $\mathcal{M}^G$ is well defined, i.e. $\tilde{A}$ is $\text{Ad}(H)$-equivariant, symmetric and positive definite.

**Remark:** Metrics corresponding to the operator $A$ are equivalent, up to automorphism $\text{Ad}(n) : m \to m$, to the metrics corresponding to the operator $\tilde{A}$. 
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**Proposition 4**

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**Remark:** Metrics corresponding to the operator $A$ are equivalent, up to automorphism $\text{Ad}(n) : m \to m$, to the metrics corresponding to the operator $\tilde{A}$.

From the above action we consider the set of all fixed points (subset of $\mathcal{M}^G$):

$$
(\mathcal{M}^G)^\Phi = \{ A \in \mathcal{M}^G : \varphi \circ A \circ \varphi^{-1} = A \text{ for all } \varphi \in \Phi \}
$$

- Any element of $(\mathcal{M}^G)^\Phi$ parametrizes all $\text{Ad}(N_G(H))$-invariant inner products of $m$ and thus it defines a subset of all inner products on $m$. 

Since $H \subset N_G(H)$ we have:

**Proposition 5**

Let $G/H$ be a homogeneous space. Then there exists a one-to-one correspondence between:

1. $G$-invariant metrics on $G/H$,
2. $\text{Ad}(H)$-invariant inner products on $m$,
3. Fixed points

$$\left(\mathcal{M}^G\right)^{\Phi_H} = \left\{ A \in \mathcal{M}^G : \psi \circ A \circ \psi^{-1} = A, \text{ for all } \psi \in \Phi_H \right\}$$

of the action $\Phi_H = \{ \phi = \text{Ad}(h)|_m : h \in H \} \subset \Phi$ on $\mathcal{M}^G$.

- $(\mathcal{M}^G)^{\Phi} \subset (\mathcal{M}^G)^{\Phi_H}$. 
We work with some closed subgroup $K$ of $G$ such that

$$H \subset K \subset N_G(H) \subset G.$$  

Then the fixed point set of the non trivial action of

$$\Phi_K = \{ \varphi = \text{Ad}(k)|_m : k \in K \} \subset \Phi$$ on $M^G$ is

$$(M^G)^{\Phi_K} = \{ A \in M^G : \varphi \circ A \circ \varphi^{-1} = A \text{ for all } \varphi \in \Phi_K \},$$

and this set determines a subset of all $\text{Ad}(K)$-invariant inner products of $m$.

We have the inclusions

$$(M^G)^{\Phi} \subset (M^G)^{\Phi_K} \subset (M^G)^{\Phi_H}.$$
By Proposition 5 the subset \((\mathcal{M}^G)_{\Phi K}\) is in one-to-one correspondence with a subset \(\mathcal{M}^G, K\) of all \(G\)-invariant metrics, call it \(\text{Ad}(K)\)-invariant, as shown in the following figure:
By Proposition 5 the subset \((\mathcal{M}^G)^\Phi_K\) is in one-to-one correspondence with a subset \(\mathcal{M}^{G,K}\) of all \(G\)-invariant metrics, call it \(\text{Ad}(K)\)-invariant, as shown in the following figure:

**Proposition 6**

Let \(K\) be a subgroup of \(G\) with \(H \subset K \subset G\) and such that \(K = L \times H\), for some subgroup \(L\) of \(G\). Then \(K\) is contained in \(N_G(H)\).
We apply the previous proposition for the Stiefel manifolds

\[ V_{k_1+k_2}F^{k_1+k_2+k_3} \cong G_{k_1+k_2+k_3}/G_3, \]

\[ G_{k_1+k_2+k_3} \in \{ \text{SO}(k_1 + k_2 + k_3), \text{Sp}(k_1 + k_2 + k_3) \}, \]

\[ G_i \in \{ \text{SO}(k_i), \text{Sp}(k_i) \} \ (i = 1, 2, 3) \] and \[ F \in \{ \mathbb{R}, \mathbb{H} \}, \] where we take the following two cases for the subgroup \( K = L \times G_3 \):

(A) \( K = (G_1 \times G_2) \times G_3 \), and search for

\[ \text{Ad}(K) \equiv \text{Ad} \left( (G_1 \times G_2) \times G_3 \right) \text{-invariant metrics.} \]
We apply the previous proposition for the Stiefel manifolds

\[ V_{k_1+k_2} \mathbb{F}^{k_1+k_2+k_3} \cong G_{k_1+k_2+k_3}/G_3, \]

where \( G_{k_1+k_2+k_3} \in \{ \text{SO}(k_1 + k_2 + k_3), \text{Sp}(k_1 + k_2 + k_3) \} \), \( G_i \in \{ \text{SO}(k_i), \text{Sp}(k_i) \} \) (\( i = 1, 2, 3 \)) and \( \mathbb{F} \in \{ \mathbb{R}, \mathbb{H} \} \), where we take the following two cases for the subgroup \( K = L \times G_3 \):

(A) \( K = (G_1 \times G_2) \times G_3 \), and search for

\[ \text{Ad}(K) \equiv \text{Ad}((G_1 \times G_2) \times G_3) \text{-invariant metrics.} \]

(B) \( K = U(k_1 + k_2) \times \text{Sp}(k_3) \), and search for

\[ \text{Ad}(K) \equiv \text{Ad}(U(k_1 + k_2) \times \text{Sp}(k_3)) \text{-invariant metrics.} \]

The benefit for such metrics is that they are diagonal metrics on the homogeneous space.
We study the case (A)

\[ K = (G_1 \times G_2) \times G_3 \text{ where } G_i \in \{\text{SO}(k_i), \text{Sp}(k_i)\}, \ i = 1, 2, 3 \]

that is

\[ K = \text{SO}(k_1) \times \text{SO}(k_2) \times \text{SO}(k_3) \rightarrow V_{k_1+k_2} \mathbb{R}^n \]

\[ K = \text{Sp}(k_1) \times \text{Sp}(k_2) \times \text{Sp}(k_3) \rightarrow V_{k_1+k_2} \mathbb{H}^n \]
Case (A) \[ K = (G_1 \times G_2) \times G_3, \quad G_i \in \{\text{SO}(k_i), \text{Sp}(k_i)\} \]

We view the Stiefel manifold \( V_{k_1+k_2} \mathbb{F}^n \), where \( n = k_1 + k_2 + k_3 \) as total space over the \textit{generalized Wallach space}, i.e:

\[
\frac{G_1 \times G_2 \times G_3}{G_3} \rightarrow \frac{G_n}{G_3} \rightarrow \frac{G_n}{G_1 \times G_2 \times G_3}
\]

The tangent space \( \mathfrak{p} \) of the generalized Wallach space has three non equivalent \( \text{Ad}(K) \)-invariant, irreducible isotropy summands, that is

\[
\mathfrak{p} = \mathfrak{p}_{12} \oplus \mathfrak{p}_{13} \oplus \mathfrak{p}_{23},
\]

and the tangent space of the fiber is the Lie algebra

\[
\mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \text{where} \quad \mathfrak{g}_i \in \{\text{so}(k_i), \text{sp}(k_i)\}, \quad i = 1, 2.
\]
We view the Stiefel manifold $V_{k_1+k_2} \mathbb{F}^n$, where $n = k_1 + k_2 + k_3$ as total space over the generalized Wallach space, i.e:

$$
\frac{G_1 \times G_2 \times G_3}{G_3} \rightarrow \frac{G_n}{G_3} \rightarrow \frac{G_n}{G_1 \times G_2 \times G_3}
$$

The tangent space $p$ of the generalized Wallach space has three non equivalent $Ad(K)$-invariant, irreducible isotropy summands, that is

$$p = p_{12} \oplus p_{13} \oplus p_{23},$$

and the tangent space of the fiber is the Lie algebra

$$g_1 \oplus g_2 \text{ where } g_i \in \{so(k_i), sp(k_i)\}, \, i = 1, 2.$$

Therefore, the tangent space $m$ of the total space can be written as a direct sum of five non equivalent $Ad(K)$-invariant, irreducible components:

$$m = g_1 \oplus g_2 \oplus p_{12} \oplus p_{13} \oplus p_{23}$$

$$= \begin{pmatrix}
g_1 & p_{12} & p_{13} \\
t p_{12} & g_2 & p_{23} \\
t p_{13} & -t p_{23} & 0
\end{pmatrix}$$
Case (A) $K = (G_1 \times G_2) \times G_3, \ G_i \in \{\text{SO}(k_i), \text{Sp}(k_i)\}$

From the previous decomposition any $\text{Ad}(K)$-invariant metric is diagonal and is determined by $\text{Ad}(K)$-invariant inner products of the form:

$$\langle \cdot, \cdot \rangle = x_1 (-B)|_{g_1} + x_2 (-B)|_{g_2} + x_{12} (-B)|_{p_{12}} + x_{13} (-B)|_{p_{13}} + x_{23} (-B)|_{p_{23}}$$

$$\leftrightarrow$$

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} x_1 & x_{12} & x_{13} \\ x_{12} & x_2 & x_{23} \\ x_{13} & x_{23} & * \end{pmatrix}.$$  Here $k_1 \geq 2, k_2 \geq 2$ and $k_3 \geq 1$.

In the case where we have $k_1 = 1$, then for the real Stiefel manifold $V_{1+k_2} \mathbb{R}^{1+k_2+k_3}$ the above inner products take the form

$$\langle \cdot, \cdot \rangle = x_2 (-B)|_{\text{so}(k_2)} + x_{12} (-B)|_{m_{12}} + x_{13} (-B)|_{m_{13}} + x_{23} (-B)|_{m_{23}}$$

$$\leftrightarrow$$

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_2 & x_{23} \\ x_{13} & x_{23} & * \end{pmatrix}.$$  Here $k_1 = 1, k_2 \geq 2$ and $k_3 \geq 1$. 
$K = (G_1 \times G_2) \times G_3, \ G_i \in \{\text{SO}(k_i), \text{Sp}(k_i)\}$

We need to determine the Ricci components $r_1, r_2, r_{ij} \ (1 \leq i < j \leq 3$ for the metric that correspond to the inner products (3) and (4). We first need to identify the non zero numbers $A_{ijk} := \begin{bmatrix} k_i \\
ij \end{bmatrix}$. From some Lie bracket relations of $g_i$ and $p_{ij}$ we have:


From the Lemma below (due to Arvanitoyeorgos, Dzhepko and Nikonorov) we have,

**Lemma 5**

For $a, b, c = 1, 2, 3$ and $(a-b)(b-c)(c-a) \neq 0$ the following relations hold:

<table>
<thead>
<tr>
<th>Real case</th>
<th>Quaternionic case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{aaa} = \frac{k_a(k_a-1)(k_a-2)}{2(n-2)}$</td>
<td>$A_{aaa} = \frac{k_a(k_a+1)(2k_a+1)}{n+1}$</td>
</tr>
<tr>
<td>$A_{(ab)(ab)a} = \frac{k_a k_b (k_a-1)}{2(n-2)}$</td>
<td>$A_{(ab)(ab)a} = \frac{k_a k_b (2k_a+1)}{(n+1)}$</td>
</tr>
<tr>
<td>$A_{(ab)(bc)(ac)} = \frac{k_a k_b k_c}{2(n-2)}$</td>
<td>$A_{(ab)(bc)(ac)} = \frac{2k_a k_b k_c}{n+1}$</td>
</tr>
</tbody>
</table>
$K = (G_1 \times G_2) \times G_3, \ G_i \in \{\text{SO}(k_i), \text{Sp}(k_i)\}$

Lemma 6

The components of the Ricci tensor for the $\text{Ad}(K)$-invariant metric determined by (3) for the \textbf{real case} are given as follows:

\[
\begin{align*}
  r_1 &= \frac{k_1 - 2}{4(n - 2)x_1} + \frac{1}{4(n - 2)} \left( k_2 \frac{x_1}{x_{12}^2} + k_3 \frac{x_1}{x_{13}^2} \right), \\
  r_2 &= \frac{k_2 - 2}{4(n - 2)x_2} + \frac{1}{4(n - 2)} \left( k_1 \frac{x_2}{x_{12}^2} + k_3 \frac{x_2}{x_{23}^2} \right), \\
  r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n - 2)} \left( \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}} \right) \\
  &\quad - \frac{1}{4(n - 2)} \left( (k_1 - 1) \frac{x_1}{x_{12}^2} + (k_2 - 1) \frac{x_2}{x_{12}^2} \right), \\
  r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n - 2)} \left( \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}} \right) - \frac{1}{4(n - 2)} \left( (k_1 - 1) \frac{x_1}{x_{13}^2} \right) \\
  r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n - 2)} \left( \frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}} \right) - \frac{1}{4(n - 2)} \left( (k_2 - 1) \frac{x_2}{x_{23}^2} \right)
\end{align*}
\]

where $n = k_1 + k_2 + k_3$. 
$K = (G_1 \times G_2) \times G_3, \ G_i \in \{\text{SO}(k_i), \text{Sp}(k_i)\}$

**Lemma 7**

The components of the Ricci tensor for the $\text{Ad}(K)$-invariant metric determined by (3) for the quaternionic case are given as follows:

\[
\begin{align*}
    r_1 &= \frac{k_1 + 1}{4(n + 1)x_1} + \frac{k_2}{4(n + 1)x_{12}^2} + \frac{k_3}{4(n + 1)x_{13}^2}, \\
    r_2 &= \frac{k_2 + 1}{4(n + 1)x_2} + \frac{k_1}{4(n + 1)x_{12}^2} + \frac{k_3}{4(n + 1)x_{23}^2}, \\
    r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n + 1)} \left( \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}} \right) \\
    &- \frac{2k_1 + 1}{8(n + 1)x_{12}^2} - \frac{2k_2 + 1}{8(n + 1)x_{12}^2}, \\
    r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n + 1)} \left( \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}} \right) - \frac{2k_1 + 1}{8(n + 1)x_{13}^2} \\
    r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n + 1)} \left( \frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}} \right) - \frac{2k_2 + 1}{8(n + 1)x_{23}^2}.
\end{align*}
\]
The components of the Ricci tensor for the $\text{Ad}(K)$-invariant metric determined by (4) (real case only), are given as follows:

$$r_2 = \frac{k_2 - 2}{4(n - 2)x_2} + \frac{1}{4(n - 2)} \left(\frac{x_2}{x_{12}^2} + k_3 \frac{x_2}{x_{23}^2}\right),$$

$$r_{12} = \frac{1}{2x_{12}} + \frac{k_3}{4(n - 2)} \left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{4(n - 2)} \left((k_2 - 1) \frac{x_2}{x_{12}^2}\right),$$

$$r_{23} = \frac{1}{2x_{23}} + \frac{1}{4(n - 2)} \left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{1}{4(n - 2)} \left((k_2 - 1) \frac{x_2}{x_{23}^2}\right),$$

$$r_{13} = \frac{1}{2x_{13}} + \frac{k_2}{4(n - 2)} \left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right),$$

where $n = 1 + k_2 + k_3$. 
## Einstein metrics on $V_{1+k_2}R^n$

For the Stiefel manifolds $V_4R^n \cong SO(n)/SO(n-4)$, where $k_2 = 3$ and $k_3 = n - 4$, the

$$Ad(SO(3) \times SO(n-4))$$-invariant Einstein metrics

are the solutions of the system

$$r_2 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23},$$

and we set $x_{23} = 1$. Then we have

$$f_1 = -(n-4)x_{12}x_3^2x_2 + (n-4)x_{12}x_2^2x_{13}^2 + (n-4)x_{12}x_{13}^2x_2$$
$$-2(n-2)x_{12}x_{13}x_2 + (n-4)x_{12}x_2 + x_{12}^2x_{13} + 3x_{13}x_2^2 = 0,$$

$$f_2 = (n-3)x_{12}x_3^2 - 2(n-2)x_{12}x_2^2x_{13} - (n-5)x_{12}x_{13}^2$$
$$+ 2(n-2)x_{12}x_{13} + (3-n)x_{12} + 2x_{12}x_{13}^2x_2 - 2x_{13}x_2 = 0,$$

$$f_3 = (n-2)x_{12}x_{13} - (n-2)x_{12} + x_{12}^2 - x_{12}x_{13}x_2$$
$$-2x_{13}^2 + 2 = 0.$$  \hspace{1cm} (5)

We take a Gröbner basis for the ideal $I$ of the polynomial ring $R = \mathbb{Q}[z, x_2, x_{12}, x_{13}]$ which is generated by

$$\{f_1, f_2, f_3, z x_2 x_{12} x_{13} - 1\},$$

to find non zero solutions of the above system.
Einstein metrics on Real Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^n$

By the aid of computer programs for symbolic computations we obtain the following results:

**Theorem 1 (A. Arvanitoyeorgos-Y. Sakane-M.S.)**

The Stiefel manifolds $V_4\mathbb{R}^n = SO(n)/SO(n-4)$ ($n \geq 6$) admit at least four invariant Einstein metrics. Two of them are Jensen’s metrics and the other two are given by the $\text{Ad}(SO(3) \times SO(n-4))$-invariant inner products of the form (4).

In the same way, for the Stiefel manifolds $V_5\mathbb{R}^7$, we consider the cases

$$k_1 = 2, k_2 = 3, k_3 = 2 \quad k_1 = 1, k_2 = 4, k_3 = 2$$

Then we have:

**Theorem 2 (A. Arvanitoyeorgos-Y. Sakane-M.S.)**

The Stiefel manifold $V_5\mathbb{R}^7 = SO(7)/SO(2)$ admits at least six invariant Einstein metrics. Two of them are Jensen’s metrics, the other two are given by $\text{Ad}(SO(2) \times SO(3) \times SO(2))$-invariant inner products of the form (3), and the rest two are given by $\text{Ad}(SO(4) \times SO(2))$-invariant inner products of the form (4).
Einstein metrics on quaternionic Stiefel manifolds $V_{k_1+k_2} \mathbb{H}^n$

For the quaternionic Stiefel manifolds we solve the system

$$r_1 = r_2, \quad r_2 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23}$$

and we obtain the following results:

- For the case $k_1 = 1, k_2 = 1, k_3 = 1$ the Ad(Sp(1) $\times$ Sp(1) $\times$ Sp(1))-invariant Einstein metrics on $V_2 \mathbb{H}^3$ are

  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (0.276281, 0.251266, 0.460887, 0.568722, 1)$$
  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (1.112249, 0.417937, 1.598741, 0.595776, 1)$$
  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (0.701500, 1.866891, 2.683459, 1.678482, 1)$$
  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (0.441809, 0.485793, 0.810389, 1.758325, 1).$$

Two are Jensen’s metrics:

  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (0.472797, 0.472797, 0.472797, 1, 1)$$
  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (1.812916, 1.812916, 1.812916, 1, 1),$$

and the other two are Arvanitoyeorgos-Dzhepko-Nikonorov metrics:

  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (0.3448897, 0.3448897, 0.800199, 1, 1)$$
  $$(x_1, x_2, x_{12}, x_{13}, x_{23}) \approx (0.483972, 0.483972, 2.585187, 1, 1).$$
Einstein metrics on Quaternionic Stiefel manifolds $V_{k_1+k_2} \mathbb{H}^n$

- In the same way for $k_1 = n - 2$, $k_2 = 1$, $k_3 = 1$ the $\text{Ad}(\text{Sp}(n-2) \times \text{Sp}(1) \times \text{Sp}(1))$-invariant Einstein metrics on $V_{n-1} \mathbb{H}^n$ are

1. $3 < n < 8$ there are 8 metrics, 2 of Jensen’s metrics and 6 are new.
2. $7 < n < 30$ there are 10 metrics, 2 of Jensen’s and 8 are new.
3. $n > 29$ there are 12 metrics, 2 Jensen’s and the rest 10 are new.
Einstein metrics on Quaternionic Stiefel manifolds $V_{k_1+k_2} \mathbb{H}^n$

- In the same way for $k_1 = n - 2$, $k_2 = 1$, $k_3 = 1$ the Ad(Sp($n - 2$) $\times$ Sp(1) $\times$ Sp(1))-invariant Einstein metrics on $V_{n-1} \mathbb{H}^n$ are
  1. $3 < n < 8$ there are 8 metrics, 2 of Jensen’s metrics and 6 are new.
  2. $7 < n < 30$ there are 10 metrics, 2 of Jensen’s and 8 are new.
  3. $n > 29$ there are 12 metrics, 2 Jensen’s and the rest 10 are new.

- In case where $k_1 = n - 3$, $k_2 = 1$, $k_3 = 2$ the Ad(Sp($n - 3$) $\times$ Sp(1) $\times$ Sp(2))-invariant Einstein metrics on $V_{n-2} \mathbb{H}^n$ are
  1. $n = 4$ there are 8 metrics, 2 Jensen’s, two Nikonorov-Arvanitoyeorgos-Dzhepko and 4 are new.
  2. $4 < n < 10$ there are 8 metrics, 2 Jensen’s and 6 new.
  3. $n = 10$ there are 10 metrics, 2 Jensen’s and 8 new.
  4. $11 < n < 28$ there are 8 metrics, 2 Jensen’s and 6 new.
  5. $27 < n < 41$ there are 10 metrics, 2 Jensen’s and 8 new.
  6. $n > 40$ there are 12 metrics, 2 Jensen’s and 10 are new.
We now study the case (B)

\[ K = U(k_1 + k_2) \times Sp(k_3) \]

for the quaternionic Stiefel manifolds \( V_{k_1+k_2} \mathbb{H}^n \), where \( n = k_1 + k_2 + k_3 \).

We set \( p = k_1 + k_2 \), so \( k_3 = n - p \).
\[ K = U(p) \times \text{Sp}(n-p) \]

In this case we view the Stiefel manifold \( V_p \mathbb{H}^n \), where \( n = k_1 + k_2 + k_3 \), as a total space over the flag manifold with two isotropy summands i.e:

\[
\begin{align*}
\frac{U(p) \times \text{Sp}(n-p)}{\text{Sp}(n-p)} & \to \frac{\text{Sp}(n)}{\text{Sp}(n-p)} \to \frac{\text{Sp}(n)}{U(p) \times \text{Sp}(n-p)} \\
\end{align*}
\]

▶ The tangent space \( \mathfrak{m} \) of the base space is written as a direct sum of two non equivalent \( \text{Ad}(K) \)-invariant irreducible isotropy summands \( m_1, m_2 \) of dimension \( d_2 = \dim(m_1) = 4p(n-p) \) and \( d_3 = \dim(m_2) = p(p+1) \).

Also, the tangent space of the fiber \( U(p) \cong U(1) \times \text{SU}(p) \) is the Lie algebra \( \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \) where \( \mathfrak{h}_0 \) is the center of \( u(p) \) and \( \mathfrak{h}_1 = \mathfrak{su}(p) \), with \( d_0 = \dim(\mathfrak{h}_0) = 1 \) and \( d_1 = \dim(\mathfrak{h}_1) = p^2 - 1 \).

▶ Therefore the tangent space \( \mathfrak{p} \) of Stiefel manifold can be written as direct sum of four non equivalent \( \text{Ad}(K) \)-invariant irreducible submodules:

\[ \mathfrak{p} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus m_1 \oplus m_2. \]
$K = U(p) \times \text{Sp}(n - p)$

The diagonal $\text{Ad}(K)$-invariant metrics on $V_p \mathbb{H}^n$ are determined by the following $\text{Ad}(K)$-invariant inner products on $p$

$$\langle \cdot, \cdot \rangle = u_0(-B)|_{\mathfrak{h}_0} + u_1(-B)|_{\mathfrak{h}_1} + x_1(-B)|_{\mathfrak{m}_1} + x_2(-B)|_{\mathfrak{m}_2}. \quad (6)$$

We know that $[m_1, m_1] \subset \mathfrak{h} \oplus m_2, [m_2, m_2] \subset \mathfrak{h}, [m_1, m_2] \subset m_1$, hence the only non zero numbers $A_{ijk} = \begin{bmatrix} k \\ i \\ j \end{bmatrix}$ are

$$A_{220}, A_{330}, A_{111}, A_{122}, A_{133}, A_{322}.$$ 

From Arvanitoyeorgos-Mori-Sakane we obtain the following:

**Lemma 9**

For the metric $\langle \cdot, \cdot \rangle$ on $\text{Sp}(n)/\text{Sp}(n - p)$, the non-zero numbers $A_{ijk}$ are given as follows:

$$A_{220} = \frac{d_2}{d_2 + 4d_3} \quad A_{330} = \frac{4d_3}{d_2 + 4d_3} \quad A_{111} = \frac{2d_3(2d_1 + 2 - d_3)}{d_2 + 4d_3}$$

$$A_{122} = \frac{d_1 d_2}{d_2 + 4d_3} \quad A_{133} = \frac{2d_3(d_3 - 2)}{d_2 + 4d_3} \quad A_{322} = \frac{d_2 d_3}{d_2 + 4d_3}$$
Lemma 10

The components of the Ricci tensor for the $Ad(K)$-invariant metric determined by (6) are given as follows:

$$r_0 = \frac{u_0}{4x_1^2} \frac{d_2}{(d_2 + 4d_3)} + \frac{u_0}{4x_2^2} \frac{4d_3}{(d_2 + 4d_3)}$$

$$r_1 = \frac{1}{4d_1 u_1} \frac{2d_3(2d_1 + 2 - d_3)}{(d_2 + 4d_3)} + \frac{u_1}{4x_1^2} \frac{d_2}{(d_2 + 4d_3)} + \frac{u_1}{2d_1 x_2^2} \frac{d_3(d_3 - 2)}{(d_2 + 4d_3)}$$

$$r_2 = \frac{1}{2x_1} - \frac{x_2}{2x_1^2} \frac{d_3}{(d_2 + 4d_3)} - \frac{1}{2x_1^2} \left( u_0 \frac{1}{(d_2 + 4d_3)} + u_1 \frac{d_1}{(d_2 + 4d_3)} \right)$$

$$r_3 = \frac{1}{x_2} \left( \frac{1}{2} - \frac{1}{2} \frac{d_2}{(d_2 + 4d_3)} \right) + \frac{x_2}{4x_1^2} \frac{d_2}{(d_2 + 4d_3)} - \frac{1}{x_2^2} \left( u_0 \frac{2}{(d_2 + 4d_3)} + u_1 \frac{d_3 - 2}{(d_2 + 4d_3)} \right)$$

where $d_1 = p^2 - 1$, $d_2 = 4p(n - p)$, $d_3 = p(p + 1)$.

Next, we solve the Einstein equation for the Stiefel manifold $V_2 \mathbb{H}^n$. In this case we have $d_0 = 1$, $d_1 = 3$, $d_2 = 8(n - 2)$, $d_3 = 6$. 

$$K = U(p) \times Sp(n - p)$$
Theorem 3 (A. Arvanitoyeorgos-Y. Sakane-M.S.)

The Stiefel manifold $V_2 \mathbb{H}^n \cong \text{Sp}(n)/\text{Sp}(n-2)$ admits four invariant Einstein metrics. Two of them are Jensen’s metrics and the other two are given by the $\text{Ad}(U(2) \times \text{Sp}(n-2))$-invariant inner products of the form (6).
### Theorem 3 (A. Arvanitoyeorgos-Y. Sakane-M.S.)

The Stiefel manifold $V_2\mathbb{H}^n \cong \text{Sp}(n)/\text{Sp}(n-2)$ admits **four** invariant Einstein metrics. Two of them are Jensen’s metrics and the other two are given by the $\text{Ad}(U(2) \times \text{Sp}(n-2))$-invariant inner products of the form (6).

**Proof**

We consider the system of equation

$$r_0 = r_1, \quad r_1 = r_2, \quad r_2 = r_3.$$  \hspace{1cm} (7)

We set $x_2 = 1$ and then system (7) reduces to

\[
\begin{align*}
    f_1 &= 2n u_0 u_1 - 2n u_1^2 + 6u_0 u_1 x_1^2 - 4u_0 u_1 - 4u_1^2 x_1^2 + 4u_1^2 - 2x_1^2 = 0 \\
    f_2 &= 4nu_1^2 - 8nu_1 x_1 + u_0 u_1 + 8u_1^2 x_1^2 - 5u_1^2 - 8u_1 x_1 + 6u_1 + 4x_1^2 = 0 \\
    f_3 &= 8nx_1 - 4n + 4u_0 x_1^2 - u_0 + 8u_1 x_1^2 - 3u_1 - 24x_1^2 + 8x_1 + 2 = 0.
\end{align*}
\]  \hspace{1cm} (8)
We consider a polynomial ring $R = \mathbb{Q}[z, u_0, u_1, x_1]$ and an ideal $I$ generated by \{f_1, f_2, f_3, zu_0 u_1 x_1 - 1\} to find non zero solutions for the system (8). We take a lexicographic order $>$ with $z > u_0 > x_1 > u_1$ for a monomial ordering on $R$. Then, the Gröbner basis for the ideal $I$ contains the polynomial $(u_1 - 1)U_1(u_1)$ where $U_1$ is given by:

$$U_1(u_1) = (4n - 1)^4 u_1^8 - 2(4n - 55)(4n - 1)^3 u_1^7 + (4n - 1)^2 (512n^3 - 48n^2 - 2040n + 2903) u_1^6 - 4(4n - 1)(288n^4 - 3224n^3 + 216n^2 + 10419n - 6076) u_1^5 + (14336n^6 - 5120n^5 - 103168n^4 + 78208n^3 + 104608n^2 - 104280n + 30583) u_1^4 - 2(2048n^6 - 1536n^5 + 3840n^4 - 11408n^3 - 28320n^2 + 59088n - 22489) u_1^3 + (2048n^5 + 832n^4 - 10848n^3 + 17924n^2 - 23472n + 13237) u_1^2 - 4(n - 1)(64n^4 - 96n^3 + 336n^2 - 374n + 205) u_1 + 4(n - 1)^2 (4n - 1)^2$$
Case (B)

\[ K = U(2) \times Sp(n - 2) \]

\[ - - - (u_1 - 1)U_1(u_1) - - - \]

**Case A: \( u_1 \neq 1 \)**

We prove that the equation \( U_1(u_1) = 0 \) has two positive solutions. Observe that
Case (B)

\[ K = U(2) \times \text{Sp}(n-2) \]

\[ - - - (u_1 - 1)U_1(u_1) - - - \]

**Case A:** \( u_1 \neq 1 \)

We prove that the equation \( U_1(u_1) = 0 \) has two positive solutions. Observe that

- For \( u_1 = 0 \)
  \[
  U_1(0) = 68112 - 133344n + 73744n^2 + 47360n^3 - 61696n^4 + 3328n^5 + 10240n^6 \]
  is positive for all \( n \geq 3 \).

- For \( u_1 = 1/5 \)
  \[
  U_1(1/5) = 1098.64 - 2511.49n + 1988.33n^2 - 639.029n^3 + 15.3295n^4 + 46.1537n^5 - 9.8304n^6 \]
  is negative for \( n \geq 3 \).

So we have one solution \( u_1 = \alpha_1 \) between \( 0 < \alpha_1 < 1/5 \).
We prove that the equation $U_1(u_1) = 0$ has two positive solutions. Observe that

Case A: $u_1 \neq 1$

- For $u_1 = 0$
  \[
  U_1(0) = 68112 - 133344n + 73744n^2 + 47360n^3 - 61696n^4 + 3328n^5 + 10240n^6 \text{ is positive for all } n \geq 3.
  \]

- For $u_1 = 1/5$
  \[
  U_1(1/5) = 1098.64 - 2511.49n + 1988.33n^2 - 639.029n^3 + 15.3295n^4 + 46.1537n^5 - 9.8304n^6 \text{ is negative for } n \geq 3.
  \]

So we have one solution $u_1 = \alpha_1$ between $0 < \alpha_1 < 1/5$.

- For $u_1 = 1$
  \[
  U_1(1) = 68112 - 133344n + 73744n^2 + 47360n^3 - 61696n^4 + 3328n^5 + 10240n^6 \text{ is always positive for } n \geq 3,
  \]

Hence we have a second solution $u_1 = \beta_1$ between $1/5 < \beta_1 < 1$. 

\[
K = U(2) \times \text{Sp}(n - 2)
\]
Next, we consider the ideal $J$ generated by the polynomials
\[
\{ f_1, f_2, f_3, z u_0 u_1 x_1 (u_1 - 1) - 1 \}.
\]

We take the lexicographic orders $>$ with

1. $z > u_0 > x_1 > u_1$. Then the Gröbner basis of $J$ contains the polynomial $U_1(u_1)$ and the polynomial $a_1(n) x_1 + W_1(u_1, n)$

2. $z > x_1 > u_0 > u_1$. Then the Gröbner basis of $J$ contains the polynomial $U_1(u_1)$ and the polynomial $a_2(n) u_0 + W_2(u_1, n)$

where $a_i(n) i = 1, 2$ is a polynomial of $n$ of degree 17 for $i = 1$, and of degree 16 for $i = 2$. For $n \geq 3$ the polynomial $a_i(n) i = 1, 2$ is positive. Thus for positive values $u_1 = \alpha_1, \beta_1$ found above we obtain real values $x_1 = \gamma_1, \gamma_2$ and $u_0 = \alpha_0, \beta_0$ as solutions of system (8).
Now we prove that the solutions $x_1 = \gamma_1, \gamma_2$ and $u_0 = \alpha_0, \beta_0$ are positive. We consider the ideal $J$ with the lexicographic order $> \cdot$ with

1. $z > u_0 > u_1 > x_1$ then the Gröbner basis of $J$ contains the $U_1(u_1)$ and the polynomial

$$X_1(x_1) = \sum_{k=0}^{8} b_k(n) x_1^k$$

2. $z > x_1 > u_1 > u_0$ then the Gröbner basis of $J$ contains the $U_1(u_1)$ and the polynomial

$$U_0(u_0) = \sum_{k=0}^{8} c_k(n) u_0^k$$

for $n \geq 3$ the coefficients of the polynomials $b_k(n), c_k(n)$ are positive when the $k$ is even degree and negative for odd degree. Thus if the equations $X_1(x_1) = 0$ and $U_0(u_0) = 0$ has real solutions, then these are all positive. So the solutions $x_1 = \gamma_1, \gamma_2$ and $u_0 = \alpha_0, \beta_0$ are positive.
Case (B): \( u_1 = 1 \)

Then from the system (8) we get the solutions:

\[
\begin{aligned}
\{ u_0 = 1, u_1 = 1, x_1 = & \frac{2 + 2n - \sqrt{-2 - 4n + 4n^2}}{6}, x_2 = 1 \} \\
\text{and} \\
\{ u_0 = 1, u_1 = 1, x_1 = & \frac{2 + 2n + \sqrt{-2 - 4n + 4n^2}}{6}, x_2 = 1 \}
\end{aligned}
\]

which are Jensen’s metrics.

So the new Einstein metrics on \( V_2 \mathbb{H}^n \) are of the form

\[
\begin{aligned}
\{ u_0 = \alpha_0, u_1 = \alpha_1, x_1 = \gamma_1, x_2 = 1 \} \\
\{ u_0 = \beta_0, u_1 = \beta_1, x_1 = \gamma_2, x_2 = 1 \}
\end{aligned}
\]
Comparison of the metrics on $V_4\mathbb{R}^n = \text{SO}(n)/ \text{SO}(n - 4)$

- Jensen’s metrics on Stiefel manifold $V_4\mathbb{R}^n = \text{SO}(n)/ \text{SO}(n - 4)$

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & a & 1 \\ a & a & 1 \\ 1 & 1 & * \end{pmatrix}, \ \text{Ad}(\text{SO}(4) \times \text{SO}(n - 4))$$-invariant.

- Our Einstein metrics

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & \beta & \gamma \\ \beta & \alpha & 1 \\ \gamma & 1 & * \end{pmatrix}, \ \text{Ad}(\text{SO}(3) \times \text{SO}(n - 4))$$-invariant $(\alpha, \beta, \gamma \neq 1$ are all different $)$.

- For the Stiefel manifolds $V_\ell\mathbb{R}^{k+k+\ell} = \text{SO}(2k + \ell)/ \text{SO}(\ell)$ $(\ell > k \geq 3)$

Einstein metrics of Arvanitoyeorgos, Dzhepko and Nikonorov

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & * \end{pmatrix} \ (\alpha, \beta \text{ are different })$$.
New Einstein metrics on complex Stiefel manifold $V_3\mathbb{C}^{n+3}$

**Theorem**

On a complex Stiefel manifold $V_3\mathbb{C}^{n+3} \cong SU(n+3)/SU(n)$ for $n \geq 2$, there exist **new invariant Einstein metrics** which are different from Jensen’s metrics.

- In this case we view the Stiefel manifold $V_3\mathbb{C}^{n+3}$ as a total space over the generalized flag manifold

\[
SU(1+2+n)/S(U(1) \times U(2) \times U(n)) \quad n \geq 2
\]


