Describing the singular behaviour of parabolic equations on cones in fractional Sobolev spaces

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Abstract

In this paper, the Dirichlet problem for parabolic equations in a wedge is considered. In particular, we study the smoothness of the solutions in the fractional Sobolev scale H^s, $s \in \mathbb{R}$. The regularity in these spaces is related with the approximation order that can be achieved by numerical schemes based on uniform grid refinements. Our results provide a first attempt to generalize the well-known $H^{3/2}$-Theorem of [13] to parabolic PDEs. As a special case the heat equation on radial-symmetric cones is investigated.

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1 Introduction

In this paper we are concerned with the fractional Sobolev regularity of solutions to parabolic equations of the form

$$\frac{\partial u}{\partial t}(x,t) - \sum_{i,j=1}^{d} A_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = f(x,t) \quad \text{in} \quad \mathcal{K} \times \mathbb{R}, \qquad u\big|_{\partial\mathcal{K}\times\mathbb{R}} = 0, \tag{1.1}$$

where $\mathcal{K} = K \times \mathbb{R}^{d-m}$ is a wedge in $\mathbb{R}^d$ and $K \subset \mathbb{R}^m$ a smooth cone, cf. Definition 2.1. Moreover, $A = (A_{ij}(t))_{i,j=1}^d$ is symmetric and the coefficients $A_{ij}(t)$ are assumed to be real valued measurable functions of t satisfying an ellipticity condition in $\mathbb{R}^d$, see (3.3).
Our main regularity result is stated in Theorem 3.5. For convenience let us

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consider here the special case of $\mathcal{K} = K$ being a smooth cone in $\mathbb{R}^3$. Then we obtain from Theorem 3.5 that for right hand sides $f \in L_2(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(K))$ where $\mathcal{K}^0_{-\mu,2}(K)$ is a specific Kondratiev space, see Section 2.2, and φ being a suitable cut-off function with support in the truncated cone K_0, the solution u of problem (1.1) satisfies

$$\varphi u \in L_2(\mathbb{R}, H^s(K_0)) \quad \text{for any} \quad s < 2 - \mu,$$

where μ is bounded from above and below by $\max\left(\frac{1}{2} - \lambda_c^+, 0\right) < \mu < \frac{3}{2} + \lambda_c^-$ ($\mu = 0$ is allowed if $\frac{1}{2} < \lambda_c^+$). Here $\lambda_c^{\pm} > 0$ are the so-called critical exponents introduced in [20]. We see that optimal regularity results are achieved for small μ and the best we can hope for in terms of fractional Sobolev regularity is

$$s < \min\left(\frac{3}{2} + \lambda_c^+, 2\right), \tag{1.2}$$

additionally assuming that the right hand side f belongs to the approriate subspace of $L_2(K \times \mathbb{R})$. In Example 3.8 we furthermore present a result from [19], which indicates that our results are optimal in the sense that we cannot expect $s > \frac{3}{2} + \lambda_c^+$ in general.

Our regularity results are in good agreement with the corresponding theory of elliptic equations: concerning the Sobolev regularity of solutions to elliptic problems it is well-known that for smooth coefficients and smooth boundaries we have $u \in H^{s+2}(\Omega)$ for $f \in H^s(\Omega)$, but this becomes false for more general non-smooth domains. In particular, in [13] it has been shown that for general Lipschitz domains Ω for the solution of the Poisson equation we only have $u \in H^s$ for all $s \leq 3/2$, even for smooth right-hand side f due to singularities of the solution near the boundary. This $H^{3/2}$–Theorem has some important consequences for the numerical treatment of the Poisson equation. Indeed, it implies that the order of convergence that can be achieved by numerical methods based on uniform grid refinement is limited by $3/2d$,as long as we do not impose further properties on the domain see, e.g. [3, 11] for details. Moreover, for the Poisson equation with Dirichlet boundary conditions considered on corner domains, in [8] the same upper bound (1.2) we derive here for the spacial regularity of our parabolic problem is given.

The authors are aware of the fact that regularity theory for partial differential equations is a well-established field of research with a long history, and a lot of spectacular results have been achieved. In particular, for elliptic PDEs, the amount of literature is enormous and cannot be completely discussed here. Let us just refer to [7, 9, 10, 13, 16, 23] for an (extremely uncomplete) overview. Also for parabolic equations many results are known so far, let us e.g. refer to [14, 15, 22]. Nevertheless, to the authors surprise, concerning the generalization of the $H^{3/2}$-Theorem to parabolic equations not much seems to be known so far. In [10] it is shown that for the heat equation on a polygonal domain $\Omega \subset \mathbb{R}^2$ with right hand side $f \in L_2((0, \infty) \times \Omega)$ there is a solution $u \in L_2((0,\infty), H^s(\Omega))$ with $s < \frac{\pi}{\max_j \omega_j} + 1$, where ω_j denote the inner angles of the polygon. Thus, for $\omega_j \to 2\pi$ we see that $s < \frac{3}{2}$ is optimal. The 3-dimensional case is not treated there at all. Our results now can be seen as a first step to generalize the $H^{3/2}$-Theorem to other parabolic problems. However, concerning the type of the domain, we are not as general as [13], since the domains we consider are restricted to wedges (or cones) according to Definition 2.1 instead of general Lipschitz domains. Furthermore, our right-hand sides f are chosen to be (only) subsets of $L_2(\mathcal{K} \times \mathbb{R})$ unless $\mu = 0$ is allowed.

To substantiate our findings, in Section 3.3 we discuss in detail their implications for the classical case of the heat equation on a smooth cone $K \subset \mathbb{R}^3$, i.e., we consider

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in} \quad K \times \mathbb{R}, \qquad u\big|_{\partial K \times \mathbb{R}} = 0. \tag{1.3}$$

In this case the critical exponents $\lambda_c^{\pm}$ are known to both coincide with $\lambda_1^+ = -\frac{1}{2} + \sqrt{\Lambda_1 + \frac{1}{4}}$, where Λ_1 is the first eigenvalue of the Dirichlet problem of the Laplace-Beltrami operator in $\Omega = K \cap S^2$. For the particular case of $\Omega = \Omega_{\theta_0}$ being a spherical cap, i.e., K being a radial symmetric cone with opening angle $\theta_0 \in (0,\pi)$, the critical exponents $\lambda_c^{\pm}$ can be determined precisely (they coincide in this case with the minimal root $\nu_{\min}$ of the Legendre function with respect to the angle θ_0). Exact values for λ_1^+ and certain angles θ_0 are listed in Figure 7. If, additionally, K is a convex cone we obtain (similar to the elliptic case) a shift in the spacial regularity by 2, i.e., for a right–hand side $f \in L_2(\mathbb{R} \times K)$ and cutoff function φ with support in the truncated cone K_0, there is a solution u of the heat equation (1.3) satisfying

$$\varphi u \in L_2(\mathbb{R}, H^s(K_0)) \quad \text{for any} \quad s < 2,$$

see Corollary 3.11.

The paper is organized as follows. In Section 2 we collect the background material related to the function spaces we need for our investigations. Especially the weighted Sobolev spaces will play an important role. Then in Section 3 we present our regularity results for the general parabolic problem (1.1) as well as more specific results for the heat equation on a spherical cap.

2 Function spaces

2.1 Classical function spaces and types of domains

We start by collecting some general notation used throughout the paper.

As usual, $\mathbb{N}$ stands for the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers, and $\mathbb{R}^d$, $d \in \mathbb{N}$, is the d-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^d$, denoting the Euclidean norm of x. For $a \in \mathbb{R}$, let $\lfloor a \rfloor := \max\{k \in \mathbb{Z} : k \leq a\}$. Let $\mathbb{N}_0^d$, where $d \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \mathbb{N}_0$ and $|\alpha| := \sum_{j=1}^d \alpha_j$.
Furthermore, $B_\varepsilon(x)$ is the open ball of radius $\varepsilon > 0$ centered at x.
We denote by c a generic positive constant which is independent of the main parameters, but its value may change from line to line. The expression $A \lesssim B$ means that $A \leq c\,B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.
Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded.
By $\operatorname{supp} f$ we denote the support of the function f. A domain Ω is a connected open set in $\mathbb{R}^d$ (which can be unbounded). Let $L_p(\Omega)$, $1 \leq p \leq \infty$, be the Lebesque spaces on Ω as usual. For $r \in \mathbb{N} \cup \{\infty\}$ we write $C^r(\Omega)$ for the space of all real-valued r-times continuously differentiable functions, whereas $C(\Omega)$ is the space of bounded continuous functions. Moreover, $\mathcal{D}(\Omega)$ denotes the set of test functions, i.e., the collection of all infinitely differentiable functions with support compactly contained in Ω. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with with $|\alpha| := \alpha_1 + \ldots + \alpha_d = r$, $r \in \mathbb{N}_0$, and an r-times differentiable function $u : \Omega \to \mathbb{R}$, we write

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial (x_1)^{\alpha_1} \ldots \partial (x_d)^{\alpha_d}}$$

for the corresponding partial derivative or $\partial^\alpha u = D^\alpha u$. Hence, the space $C^r(\Omega)$ is normed by

$$\|u|C^r(\Omega)\| := \max_{|\alpha| \leq r} \sup_{x \in \Omega} |D^\alpha u(x)| < \infty.$$

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing functions, whereas $\mathcal{S}'(\mathbb{R}^d)$ is the set of tempered distributions on $\mathbb{R}^d$. The set of distributions on Ω will be denoted by $\mathcal{D}'(\Omega)$. For the application of a distribution $u \in \mathcal{D}'(\Omega)$ to a test function $\varphi \in \mathcal{D}(\Omega)$ we write (u, φ).
Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then $W_p^m(\mathbb{R}^d)$ denotes the standard Sobolev space which contains all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| u \,|W_p^m(\mathbb{R}^d)\| := \Big(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^d} |D^\alpha u(x)|^p \, dx \Big)^{1/p} < \infty$$

(with the usual modification if $p = \infty$). Derivatives must be understood in the sense of distributions. Concerning the definition of generalized Lebesgue and Sobolev spaces $L_p(I, X)$ and $W_p^k(\mathbb{R}, X)$ for X-valued functions, where X denotes some Banach space, we refer e.g. to [6], Section 2.2.
Furthermore, let $\mathcal{F}$ stand for the Fourier-transform on $\mathcal{S}'(\mathbb{R}^d)$ with inverse

$\mathcal{F}^{-1}$. For $s \in \mathbb{R}$ and $1 < p < \infty$ we define fractional Sobolev spaces $H^s(\mathbb{R}^d)$ as the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|H^s(\mathbb{R}^d)\| := \|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}f)|L_2(\mathbb{R}^d)\| < \infty.$$

These spaces partially coincide with the classical Sobolev spaces, i.e., we have $H^s(\mathbb{R}^d) = W_2^m(\mathbb{R}^d)$ for $s = m$ with $m \in \mathbb{N}_0$.
Corresponding spaces on domains can be defined via restriction, i.e., we put

$$\begin{aligned} H^s(\Omega) &= \left\{ f \in \mathcal{D}'(\Omega) :\ \exists g \in H^s(\mathbb{R}^d),\ g\big|_{\Omega} = f \right\}, \\ \|f|H^s(\Omega)\| &= \inf_{g|_\Omega = f} \|f|H^s(\mathbb{R}^d)\|. \end{aligned}$$

Now we introduce Besov and Triebel-Lizorkin spaces via the Fourier-analytical version in terms of dyadic Littlewood-Paley decompositions. For further information on these function spaces we refer to [28] and the references therein. We start with a function $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ with $\varphi_0 = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq \frac{3}{2}$. Define $\varphi_1(x) = \varphi_0(2x) - \varphi_0(x)$ and put $\varphi_j(x) = \varphi_1(2^{-j+1}x)$. Then $\{\varphi_j\}_{j\in\mathbb{N}_0}$ forms a so-called dyadic resolution of unity; in particular, we have $\sum_{j\geq 0}\varphi_j(x) = 1$ for every $x \in \mathbb{R}^d$. Based on such resolutions of unity, we can decompose every tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ into a series of entire analytical functions,

$$f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\varphi_j \mathcal{F}f),$$

converging in $\mathcal{S}'(\mathbb{R}^d)$. Then, for $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ ($p < \infty$ in case of Triebel-Lizorkin spaces), the Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ are defined as the collection of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|B^s_{p,q}(\mathbb{R}^d)\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)|L_p(\mathbb{R}^d)\|^q \right)^{1/q} < \infty$$

(with a supremum instead of a sum if $q = \infty$). Moreover, the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ are defined in a similar way by interchanging the order in which the norms are taken. In particular, they contain all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|F^s_{p,q}(\mathbb{R}^d)\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\cdot)|^q \right)^{1/q} \Big| L_p(\mathbb{R}^d) \right\| < \infty.$$

Corresponding spaces on domains $\Omega \subset \mathbb{R}^d$ are defined via restriction. Let $A \in \{B, F\}$. Then

$$A^s_{p,q}(\Omega) := \{ f \in \mathcal{D}'(\Omega) :\ \exists g \in A^s_{p,q}(\mathbb{R}^d),\ g\big|_{\Omega} = f \},$$

normed by

$$\|f|A^s_{p,q}(\Omega)\| := \inf_{g|_\Omega = f} \|g|A^s_{p,q}(\mathbb{R}^d)\|.$$

Within the scales we have Sobolev-type embeddings, i.e., for $\sigma < s$ and $p < \tau$ it holds

$$A^s_{p,q}(\Omega) \hookrightarrow A^\sigma_{\tau,r}(\Omega) \qquad \text{if} \qquad s - \frac{d}{p} \geq \sigma - \frac{d}{\tau}, \tag{2.1}$$

where $0 < r \leq \infty$ and, additionally, $q \leq r$ if $A = B$. A final important aspect of Triebel-Lizorkin spaces is their close relation to many classical function spaces. For our purposes, we especially mention the identities

$$F^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d) \quad \text{and} \quad F^m_{2,2}(\mathbb{R}^d) = H^m(\mathbb{R}^d) = W^m_2(\mathbb{R}^d), \tag{2.2}$$

where $m \in \mathbb{N}_0$, $s \in \mathbb{R}$, and $1 < p < \infty$.

In our later considerations we will mainly deal with wedges $\mathcal{K} = K \times \mathbb{R}^{d-m}$ where $K \subset \mathbb{R}^m$ is a cone defined as follows.

Definition 2.1 *Let $2 \leq m \leq d$. We define a cone K in $\mathbb{R}^m$ via*

$$K := \{x \in \mathbb{R}^m : \ x/|x| = \omega \in \Omega\},$$

where we assume that $\Omega = K \cap S^{m-1}$ is of class $\mathcal{C}^{1,1}$. Moreover, let

$$\mathcal{K} = K \times \mathbb{R}^{d-m}$$

be a wedge in $\mathbb{R}^d$.

Remark 2.2 Note that in our considerations the case $\mathcal{K} = K$, i.e., $m = d$, is not excluded. Two important examples for domains which are covered by Definition 2.1 are given in Figures 1 and 2.

We need some further notation. Let $x = (x', x'') = (x_1, \ldots, x_d)$ be a point in $\mathbb{R}^d$. In particular, $x' = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $x'' = (x_{m+1}, \ldots, x_d) \in \mathbb{R}^{d-m}$. Moreover, (x, t) is a point in $\mathbb{R}^{d+1}$.
In the sequel we will also deal with the truncated cone

$$K_0 := \{B^m_{r_0}(0) \cap K\} = \{x \in K : \ |x| < r_0\}, \tag{2.3}$$

the bounded wedge

$$\mathcal{K}_0 := K_0 \times B^{d-m}_{r_0}(0), \tag{2.4}$$

and the truncated cylinder

$$Q_{r_0}(0, t_0) := \mathcal{K}_0 \times (t_0 - r_0^2, t_0]$$

with constants $r_0 > 0$ and $t_0 \in \mathbb{R}$.

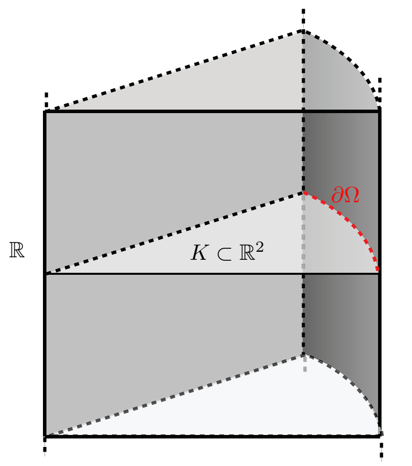

Figure 1: $m = 2 < 3 = d$,
$\mathcal{K} = K \times \mathbb{R}$

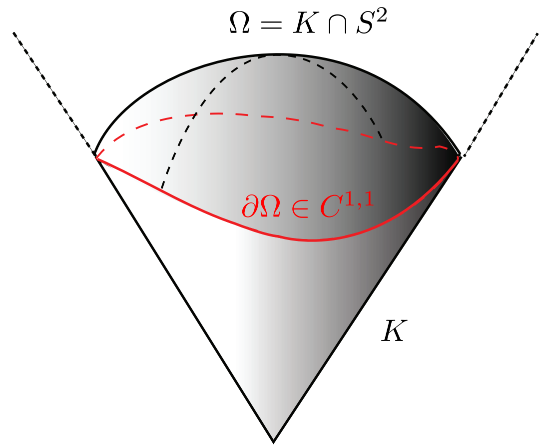

Figure 2: $m = 3 = d$,
$\mathcal{K} = K$

2.2 Weighted Sobolev spaces

In this subsection, we introduce a family of weighted Sobobev spaces, the so–called *Kondratiev* spaces. Regularity estimates in these spaces will be the major tool to establish regularity in fractional Sobolev spaces.

Definition 2.3 *Let Ω be a domain of $\mathbb{R}^d$ and let M be a nontrivial closed subset of its boundary $\partial\Omega$. Furthermore, let $1 \leq p \leq \infty$, $m \in \mathbb{N}_0$, and $a \in \mathbb{R}$. We define the space $\mathcal{K}^m_{a,p}(\Omega, M)$ as the collection of all measurable functions, which admit m weak derivatives in Ω satisfying*

$$\|u|\mathcal{K}^m_{a,p}(\Omega, M)\| := \Big(\sum_{|\alpha| \leq m} \int_\Omega |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p \, dx \Big)^{1/p} < \infty$$

if $p < \infty$, modified by

$$\|u|\mathcal{K}^m_{a,\infty}(\Omega, M)\| := \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)| < \infty$$

if $p = \infty$. Therein, the weight function ρ is defined by

$$\rho(x) := \min\{1, \operatorname{dist}(x, M)\}\,, \qquad x \in \Omega\,.$$

Remark 2.4 In our applications the set M will usually be the *singularity set* S of the domain Ω, i.e., the set of all points $x \in \partial\Omega$ for which for any $\varepsilon > 0$ the set $\partial\Omega \cap B_\varepsilon(x)$ is not smooth. In this case, we simply abbreviate

$$\mathcal{K}^m_{a,p}(\Omega) := \mathcal{K}^m_{a,p}(\Omega, S).$$

In particular, for a wedge $\mathcal{K} = K \times \mathbb{R}^{d-m}$ according to Definition 2.1 the singularity set is $S = \{0\} \times \mathbb{R}^{d-m}$ with dimension $\delta = d - m$. In this case we put $\rho(x) = \min(|x'|, 1)$ and get

$$\|u|\mathcal{K}^m_{a,p}(\mathcal{K})\| := \Big(\sum_{|\alpha|\leq m} \int_{\mathcal{K}} \big| \min(|x'|,1)^{|\alpha|-a} \partial^\alpha u(x) \big|^p \, dx \Big)^{1/p}.$$

We collect some properties of the Kondratiev spaces:

(i) $\mathcal{K}^m_{a,p}(\Omega, M)$ is a Banach space, see [17], [18], and [27, Thm. 3.2.2(a)].

(ii) The scale of Kondratiev spaces is monotone in m and a, i.e.,

$$\mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow \mathcal{K}^{m'}_{a,p}(\Omega, M) \quad \text{and} \quad \mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow \mathcal{K}^m_{a',p}(\Omega, M) \tag{2.5}$$

if $m > m'$ and $a > a'$.

(iii) Let $a \geq 0$. Then $\mathcal{K}^m_{a,p}(\Omega, M) \hookrightarrow L_p(\Omega)$.

(iv) A function $\psi \in C^m(\Omega)$ is a pointwise multiplier for $\mathcal{K}^m_{a,p}(\Omega, M)$, i.e., $\psi\, u \in \mathcal{K}^m_{a,p}(\Omega, M)$ for all $u \in \mathcal{K}^m_{a,p}(\Omega, M)$. This follows directly from the definition of the spaces.

3 Regularity results in fractional Sobolev spaces

In this section, we present our new regularity results for parabolic PDEs. First of all, in Subsection 3.1, we discuss the problem we will be concerned with. Then, in Subsection 3.2, we recall a well-known regularity result in weighted Sobolev spaces, cf. Theorem 3.2. This result is one of the central ingredients to establish our new regularity result in Theorem 3.5. We also briefly discuss the regularity in time. Finally, in Subsection 3.3, we apply our findings to the classical heat equation.

3.1 Non-divergence parabolic equations

3.1.1 The general problem

We study parabolic equations of the form

$$\mathcal{L}u = f \quad \text{in} \quad \mathcal{K} \times \mathbb{R}, \qquad u\big|_{\partial\mathcal{K}\times\mathbb{R}} = 0, \tag{3.1}$$

where

$$\mathcal{L}u(x,t) := \frac{\partial}{\partial t} u(x,t) - \sum_{i,j=1}^{d} A_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t). \tag{3.2}$$

We only assume that the coefficients A_{ij} are real valued measurable functions of t satisfying $A_{ij} = A_{ji}$ and that for some constant $\nu > 0$ we have

$$\nu|\xi|^2 \leq A_{ij}(t)\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^d. \tag{3.3}$$

Equation (3.1) is understood in the distributional sense (i.e., in the sense of generalized functions) only with resprect to x. By a solution of (3.1) we mean a function $u(t)$, $t \in \mathbb{R}$, taking values in the set of generalized functions on $\mathcal{K}$ (i.e., in $\mathcal{D}'(\mathcal{K})$) such that, for any $t, s \in \mathbb{R}$ satisfying $t \geq s$ and $\varphi \in \mathcal{D}(\mathcal{K})$, we have

$$(u(t), \varphi) = (u(s), \varphi) + \int_s^t \left[\sum_{i,j=1}^d A_{ij}(r) \Big(u(r), \frac{\partial^2 u}{\partial x_i \partial x_j} \varphi \Big) + (f(r), \varphi) \right] \mathrm{d}r.$$

3.1.2 Critical exponents

We define the critical exponent $\lambda_c^+ \equiv \lambda_c^+(\mathcal{K}, \mathcal{L})$ for the operator $\mathcal{L}$ and the wedge $\mathcal{K}$ as the supremum of all λ such that

$$|u(x,t)| \leq C(\lambda, \kappa) \left(\frac{|x'|}{R} \right)^\lambda \sup_{Q_{\kappa R}(0,t_0)} |u| \qquad \text{for} \quad (x,t) \in Q_{R/2}(0, t_0) \tag{3.4}$$

for a certain $\kappa \in (1/2, 1)$ independent of t_0, R, and u. This inequality must hold for all $t_0 \in \mathbb{R}$, $R > 0$ and $u \in \mathcal{V}_{\text{loc}}(Q_R(0,t_0))$, i.e., the space containing all functions having finite norm

$$\sup_{t \in (t_0 - R'^2, t_0]} \|u(t, \cdot)|L_2(B_{R'}(0))\| + \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} u \Big| L_2(Q_{R'}(0, t_0)) \right\|$$

for all $R' \in (0, R)$, which additionally satisfy

$$\mathcal{L}u = 0 \quad \text{in} \quad Q_R(0, t_0), \qquad u\big|_{x \in \partial \mathcal{K}} = 0.$$

Furthermore, we define

$$\lambda_c^- \equiv \lambda_c^-(\mathcal{K}, \mathcal{L}) := \lambda_c^+(\mathcal{K}, \hat{\mathcal{L}}),$$

where $\hat{\mathcal{L}}$ is defined as $\mathcal{L}$ in (3.2) with $A_{ij}(t)$ replaced by $A_{ij}(-t)$.

Remark 3.1 We list some important properties and estimates for the critical exponents $\lambda_c^\pm$ for various geometries of $\mathcal{K}$, see [20, Sect. 2] for details.

(i) The definition (3.4) does not depend on $\kappa \in (1/2, 1)$.

(ii) It can be shown that $\lambda_c^\pm > 0$.

(iii) Let $\mathcal{K}_1 \subset \mathcal{K}_2$. Then $\lambda_c^+(\mathcal{K}_1,\mathcal{L}) \geq \lambda_c^+(\mathcal{K}_2,\mathcal{L})$ and $\lambda_c^-(\mathcal{K}_1,\mathcal{L}) \geq \lambda_c^-(\mathcal{K}_2,\mathcal{L})$.

(iv) If $A_{ij}(t) = \delta_{ij}$ then

$$\lambda_c^{\pm} \equiv \lambda_1^+ := -\frac{m-2}{2} + \sqrt{\Lambda_1 + \frac{(m-2)^2}{4}}, \tag{3.5}$$

where Λ_1 is the first eigenvalue of the Dirichlet boundary value problem of the Laplace-Beltrami operator on Ω.

(v) If K is an acute cone, i.e.,

$$\overline{K} \setminus \{0\} \subset \mathbb{R}^m_+ = \{x' \in \mathbb{R}^m : \ x_1 > 0\},$$

then $\lambda_c^{\pm} > 1$.

(vi) If $K \to \mathbb{R}^m_+$, then $\lambda_c^{\pm} \to 1$.

(vii) $\lambda_c^{\pm} \geq -\frac{m-2}{2} + \nu\sqrt{\Lambda_1 + \frac{(m-2)^2}{4}}$, where ν is the constant from (3.3). In particular, if $\Lambda_1 \to \infty$, then $\lambda_c^{\pm} \to \infty$.

3.2 Regularity results

Our aim is to study the regularity of the solution u of (3.1) in generalized fractional Sobolev spaces. We rely on [20, Thm. 1.1], were the authors obtained results concerning the regularity of the solution u of (3.1) in generalized weighted Sobolev spaces. We reformulate their results in terms of our spaces from Definition 2.3 as follows.

Theorem 3.2 (Weighted Sobolev regularity) *Let $2 \leq m \leq d$, $1 < p, q < \infty$, and let $\mathcal{K} = K \times \mathbb{R}^{d-m}$ be a wedge according to Definition 2.1. Suppose that*

$$2 - \frac{m}{p} - \lambda_c^+ < \mu < m - \frac{m}{p} + \lambda_c^-,$$

where $\lambda_c^{\pm}$ are the critical exponents defined in Section 3.1.2. Furthermore, assume $f \in L_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K}))$. Then there is a solution u of problem (3.1) satisfying

$$\|u|L_q(\mathbb{R}, \mathcal{K}^2_{2-\mu,p}(\mathcal{K})) \cap W^1_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K}))\| \lesssim \|f|L_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K}))\|.$$

In order to investigate the fractional Sobolev regularity of the solution u of (3.1) we need the following embedding result from [12, Thm. 4.9].

Theorem 3.3 *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with piecewise smooth boundary and singularity set S of dimension δ. Moreover, let $1 < p < \infty$, $0 < \tau < p$, $m \in \mathbb{N}_0$, and $a > 0$. Then for*

$$m - a < (d-\delta)\left(\frac{1}{\tau} - \frac{1}{p}\right)$$

we have an embedding

$$\mathcal{K}^m_{a,p}(D) \hookrightarrow F^m_{\tau,2}(D). \tag{3.6}$$

Remark 3.4 The fact that Theorem 3.3 also holds when $m = 0$ was observed in [5, Rem. 15].

We wish to combine the results from [20] as stated in Theorem 3.2 with the embedding result from Theorem 3.3. The problem arises that Theorem 3.2 holds for unbounded wedges $\mathcal{K} \subset \mathbb{R}^d$ whereas the embedding result in Theorem 3.3 is true for bounded Lipschitz domains $D \subset \mathbb{R}^d$. In order to avoid this problem we consider the truncated wedge $\mathcal{K}_0$ as defined in (2.4). Then, the additional difficulty occurs that the Kondratiev norm on the truncated wedge is not just defined by restriction. Instead, the distance to the new corners produced by the truncation from considering $\mathcal{K}_0$ instead of $\mathcal{K}$ have to be taken into account. We solve this problem by multiplying u with a radial cut-off function $\varphi \in C_0^\infty(\mathcal{K}_0)$ satisfying

$$\varphi(x) \equiv \varphi(x', x'') = \begin{cases} 1 & \text{on} \quad (B^m_{r_0-\varepsilon}(0) \cap K) \times B^{d-m}_{r_0-\varepsilon}(0), \\ 0 & \text{on} \quad \mathcal{K}_0 \setminus \big((B^m_{r_0-\frac{\varepsilon}{2}}(0) \cap K) \times B^{d-m}_{r_0-\frac{\varepsilon}{2}}(0)\big). \end{cases} \tag{3.7}$$

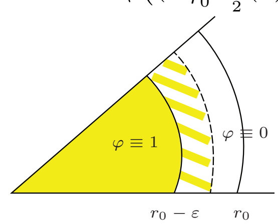

Figure 3: Illustration of cut-off function φ when $m = d = 2$

This truncation process does not induce serious restrictions for when it comes to practical applications it is clear that only truncated wedges can be considered. Then the regularity of φu corresponds to the regularity of u as stated in Theorem 3.2 and we obtain

$$\begin{aligned} &\|\varphi u | L_q(\mathbb{R}, \mathcal{K}^2_{2-\mu,2}(\mathcal{K}_0)) \cap W^1_q(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(\mathcal{K}_0))\| \\ &\qquad \lesssim \|\varphi u | L_q(\mathbb{R}, \mathcal{K}^2_{2-\mu,2}(\mathcal{K})) \cap W^1_q(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(\mathcal{K}))\| \\ &\qquad \le c_\varphi \|u | L_q(\mathbb{R}, \mathcal{K}^2_{2-\mu,2}(\mathcal{K})) \cap W^1_q(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(\mathcal{K}))\|, \end{aligned}$$

cf. Remark 2.4(iv). We are now in a position to apply the embedding result from Theorem 3.3 to the function φu. This together with the regularity results for weighted Sobolev spaces from Theorem 3.2 gives the following fractional Sobolev regularity.

Theorem 3.5 (Fractional Sobolev regularity) *Let $2 \leq m \leq d$, $1 < q < \infty$, $1 < p \leq 2$, and let $\mathcal{K} = K \times \mathbb{R}^{d-m}$ be a wedge according to Definition 2.1 with singularity set of dimension $\delta = d - m$. Suppose that*

$$\max\left(2 - \frac{m}{p} - \lambda_c^+, 0\right) < \mu < m - \frac{m}{p} + \lambda_c^-, \tag{3.8}$$

where $\mu = 0$ is allowed if $2 - \frac{m}{p} - \lambda_c^+ < 0$. In particular, $\lambda_c^\pm$ are the critical exponents defined in Section 3.1.2. Furthermore, assume $f \in L_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K}))$ and let φ be the cut-off function from (3.7). Then there is a solution u of problem (3.1) satisfying

$$\varphi u \in L_q(\mathbb{R}, H^s(\mathcal{K}_0)) \quad \textit{for any} \quad s < 2 - \frac{d\mu}{d-\delta} - \frac{d}{p} + \frac{d}{2}.$$

Proof : We make use of Theorem 3.3. Here we consider the truncated wedge $\mathcal{K}_0$, which is a bounded Lipschitz domain. By Theorem 3.2 and our assumptions we know that $\varphi u(\cdot, t) \in \mathcal{K}^2_{2-\mu,p}(\mathcal{K}_0)$. But then from (3.6) we obtain

$$\mathcal{K}^2_{2-\mu,p}(\mathcal{K}_0) \hookrightarrow F^2_{\tau,2}(\mathcal{K}_0) \tag{3.9}$$

if

$$2 - (2-\mu) < (d-\delta)\left(\frac{1}{\tau} - \frac{1}{p}\right) \quad \text{i.e.,} \quad \frac{1}{\tau} > \frac{\mu}{d-\delta} + \frac{1}{p} =: \frac{1}{\tau^*}.$$

Note that by the support properties of φu we may smoothen the boundary of $\mathcal{K}_0$ at the new corners and edges of the truncated wedge without any problem such that the singular set of the smoothed truncated wedge is the same as the singular set of the unbounded wedge $\mathcal{K}$. In particular, in this case we have $\delta = d - m$. Now Sobolev's embedding for the Triebel-Lizorkin spaces from (2.1) yields for $s < 2$ and $\tau < 2$ that

$$F^2_{\tau,2}(\mathcal{K}_0) \hookrightarrow F^s_{2,2}(\mathcal{K}_0) = H^s(\mathcal{K}_0), \tag{3.10}$$

if $2 - \frac{d}{\tau} \geq s - \frac{d}{2}$. Inserting τ^* gives

$$2 - \frac{d\mu}{d-\delta} - \frac{d}{p} > s - \frac{d}{2}, \quad \text{i.e.,} \quad s < 2 - \frac{d\mu}{d-\delta} - \frac{d}{p} + \frac{d}{2},$$

and we deduce that $\tau < 2$ and $s < 2$ are satisfied if $\mu \geq 0$ and $\tau < p \leq 2$. Combining the above results (3.9) and (3.10) yields

$$\varphi u \in L_q(\mathbb{R}, \mathcal{K}^2_{2-\mu,p}(\mathcal{K}_0)) \hookrightarrow L_q(\mathbb{R}, H^s(\mathcal{K}_0))$$

for all $s < 2 - \frac{d\mu}{d-\delta} - \frac{d}{p} + \frac{d}{2}$, which completes the proof. □

Remark 3.6 Since $\delta = d - m$, the restriction on s in Theorem 3.5 is

$$s < 2 - \frac{d\mu}{m} - \frac{d}{p} + \frac{d}{2}. \tag{3.11}$$

In particular, for $d = m$ and $p = q = 2$ we obtain $s < 2 - \mu$.

Remark 3.7 (i) From Theorem 3.5 we obtain (similar as in the elliptic case) a shift by 2 in the fractional Sobolev scale when we consider equations of the form $\mathcal{L}u = f$. In particular, let (3.8) be satisfied. Then for right hand side $f \in L_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K}))$ we have $\varphi f(\cdot, t) \in \mathcal{K}^0_{-\mu,p}(\mathcal{K}_0)$, where φ is the cut-off function defined in (3.7). Using (3.6) we obtain

$$\mathcal{K}^0_{-\mu,p}(\mathcal{K}_0) \hookrightarrow F^0_{\tau,2}(\mathcal{K}_0) \tag{3.12}$$

if

$$\mu < (d-\delta)\left(\frac{1}{\tau} - \frac{1}{p}\right), \qquad \text{i.e.,} \qquad \frac{1}{\tau} > \frac{\mu}{d-\delta} + \frac{1}{p} =: \frac{1}{\tau^*}.$$

Sobolev's embedding for the Triebel-Lizorkin spaces, cf. (2.1), yields for $s < 0$ and $\tau < 2$ that

$$F^0_{\tau,2}(\mathcal{K}_0) \hookrightarrow F^s_{2,2}(\mathcal{K}_0) = H^s(\mathcal{K}_0), \tag{3.13}$$

if $0 - \frac{d}{\tau} \geq s - \frac{d}{2}$. Inserting τ^* gives

$$-\frac{d\mu}{d-\delta} - \frac{d}{p} > s - \frac{d}{2}, \qquad \text{i.e.,} \qquad s < -\frac{d\mu}{d-\delta} - \frac{d}{p} + \frac{d}{2},$$

and we deduce that $\tau < 2$ and $s < 0$ are satisfied if $\mu \geq 0$ and $\tau < p \leq 2$. Combining the above results (3.12) and (3.13) yields

$$\varphi f \in L_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K}_0)) \hookrightarrow L_q(\mathbb{R}, H^s(\mathcal{K}_0))$$

for all $s < -\frac{d\mu}{d-\delta} - \frac{d}{p} + \frac{d}{2}$. The results in Theorem 3.5 then imply that under these conditions we obtain a solution satisfying $\varphi u \in L_q(\mathbb{R}, H^{s+2}(\mathcal{K}_0))$.

(ii) For the special case when $p = 2$, $d = m = 3$, and $\delta = 0$ (which we may assume by the support of φ since in that case the singular set of $\mathcal{K} = K$ contains only the vertex of the cone), we obtain fractional Sobolev smoothness $s < -\mu$ for f and $s < 2 - \mu$ for the solution u. The situation is illustrated in Figure 4. We obtain the maximal Sobolev regularity for the smallest possible value of μ. The condition (3.8) in this case reads as $\mu > \max\left(\frac{1}{2} - \lambda_c^+, 0\right)$ with $\lambda_c^+ > 0$.

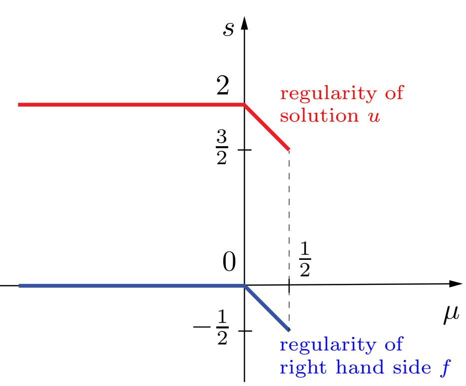

Figure 4: Shift from s to $s + 2$

(iii) We compare our regularity results with related ones. Concerning the solutions to elliptic problems it is nowadays classical knowledge that their Sobolev regularity depends not only on the properties of the coefficients and the right-hand side, but also on the regularity/roughness of the boundary of the underlying domain. While for smooth coefficients and smooth boundaries we have $u \in H^{s+2}(\Omega)$ for $f \in H^s(\Omega)$, it is well-known that this becomes false for more general domains. In particular, if we only assume Ω to be a Lipschitz domain, then it was shown by Jerison and Kenig [13] that in general we only have $u \in H^s$ for all $s \leq 3/2$ for the solution of the Poisson equation, even for smooth right-hand side f. This behaviour is caused by singularities near the boundary. Moreover, this famous $H^{3/2}$-Theorem implies that the optimal rate of convergence for nonadaptive methods of approximation is just $3/2d$ as long as we do not impose further properties on Ω.
If, additionally, $\Omega \subset \mathbb{R}^2$ is a polygonal domain then it was shown in Grisvard [10, Rem. 2.4.6] that the Poisson equation with Dirichlet boundary conditions, i.e.,

$$\Delta u = f \quad \text{in} \quad \Omega, \qquad u\big|_{\partial\Omega} = 0, \tag{3.14}$$

for $f \in L_2(\Omega)$ has a solution $u \in H^s(\Omega)$ for every $s < \frac{\pi}{\max_j \omega_j} + 1$, where ω_j denote the inner angles of the polygon. Furthermore, if we consider bounded polyhedral domains $\Omega \subset \mathbb{R}^3$, the results in Grisvard [10, Sect. 2.6] imply that (3.14) has a solution $u \in H^s(\Omega)$ with $s \in (\frac{3}{2}, 2]$, see [10, Cor. 2.6.7]. As in the two dimensional case again the upper bound of s depends on the inner angles of the polygon.
In particular, for (3.14) considered on corner domains in $\mathbb{R}^3$ in [8] the upper bound is precisely determined by $s < \lambda_c^+ + \frac{3}{2}$.
Turning towards parabolic problems we now see that Theorem 3.5 gives corresponding results. Also in this situation μ depends on the inner angle of the cone and the smaller the angles the higher is the regularity. In this context we refer to the results on specific cones derived in Theorem 3.10, where this dependence becomes clearer. For the special case that $p = 2$, $d = m = 3$, and $\delta = 0$ already mentioned above, we see from Theorem 3.5 that for right hand side $f \in L_2(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(\mathcal{K}))$ equation (3.1) has a solution

$$\varphi u \in L_2(\mathbb{R}, H^s(\mathcal{K}_0)) \qquad \text{with} \qquad s < 2 - \mu,$$

where $\mu > \max(\frac{1}{2} - \lambda_c^+)$ for $\lambda_c^+ \in \left(0, \frac{1}{2}\right]$, i.e., the best we can hope for in this case in general is $s < \min\left(\frac{3}{2} + \lambda_c^+, 2\right)$. We give an example below which demonstrates that these results are optimal, i.e., $s > \frac{3}{2} + \lambda_c^+$ is impossible even for smooth right hand sides f. In this sense, our results can be interpreted as a first step towards the generalization of the $H^{3/2}$-Theorem to parabolic problems. However, we remark that our results are not as general as the $H^{3/2}$-Theorem from [13]. Firstly, the bounded wedges $\mathcal{K}_0$ we consider here are (only) special Lipschitz domains. Secondly, if $\mu > 0$ we have $f \in L_2(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(\mathcal{K})) \subset L_2(\mathbb{R}, \mathcal{K}^0_{0,2}(\mathcal{K})) = L_2(\mathbb{R} \times \mathcal{K})$, so we do not cover all right hand sides $f \in L_2(\mathbb{R} \times \mathcal{K})$. Our results are also in good agreement with Grisvard [10, Thm. 5.2.1]. There it was shown for the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in} \quad \Omega \times (0, \infty), \qquad u\big|_{\partial\Omega\times(0,\infty)} = 0,$$

when $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, that for right hand sides $f \in L_2((0, \infty) \times \Omega)$ there is a solution u satisfying

$$u \in L_2((0, \infty), H^s(\Omega)) \quad \text{with} \quad s < \frac{\pi}{\max_j \omega_j} + 1,$$

where ω_j again denote the inner angles of the polygon.

Example 3.8 The following example was considered in [19] and indicates that our results are optimal. The authors consider the heat equation on a cone $K \subset \mathbb{R}^m$ as in Definition 2.1, where $\Omega = K \cap S^{m-1}$ is of class C^∞ and find representations for the coefficients in the asymptotic expansions of the solutions near the conical point. To be more precise, the following boundary value problem is considered:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 \quad \text{in} \quad K \times (0, \infty), \\ u\big|_{\partial K\times(0,\infty)} &= 0, \\ u\big|_{t=0} &= \varphi \quad \text{in} \quad K, \end{aligned}$$

where $\varphi \in \mathcal{D}(\overline{K}\backslash\{0\})$, satisfying $\varphi(x) = 0$ for $x \in \partial K$. Moreover, let $\{\Lambda_j\}_{j\in\mathbb{N}}$ be the nondecreasing sequence of eigenvalues of the Laplace-Beltrami operator on Ω (with Dirichlet boundary condition) counted with their multiplicities, and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthonormal (in $L_2(\Omega)$) sequence of eigenfunctions corresponding to the eigenvalues Λ_j. Furthermore, by $\lambda_j^\pm$ we denote the solutions of the quadratic equation $\lambda(\lambda + d - 2) = \Lambda_j$. In particular, it follows that $\lambda_1^+ \equiv \lambda_c^\pm$ from (3.5) in this case. Let M be the largest integer such that $\lambda_M^+ < \frac{1}{2}$. Then the authors show in [19] that the solution u is given by

$$u(t, x) = \frac{1}{2} \sum_{j=1}^{M} r^{\lambda_j^+} \varphi_j(\omega) \sum_{k=0}^{m_j} \frac{\left(r^2 \frac{\partial}{\partial t}\right)^k h_j(t)}{k!\Gamma(k + \sigma_j + 1)} + \mathcal{O}(r^{\lambda_{M+1}^+}), \tag{3.15}$$

where $\sigma_j = (\lambda_j^+ - \lambda_j^-)/2$, $r < 1$, $m_j = \lfloor(\lambda_{M+1}^+ - \lambda_j^+)/2\rfloor$, and

$$h_j(t) = (2\sqrt{t})^{-\lambda_j^+} \int_K \varphi(2\sqrt{t}x) r^{\lambda_j^+} e^{-r^2} \varphi_j(\omega) \mathrm{d}x.$$

From the representation (3.15) we see that the singularity of the solution u near the vertex behaves like $r^{\lambda_1^+} = r^{\lambda_c^+}$. Therefore, $u(\cdot, t)$ belongs locally to $H^s(K)$ if for some constant $c > 0$ we have

$$\int_0^c r^{2(\lambda_c^+ - s)} r^{m-1} \mathrm{d}r \sim \left[r^{2(\lambda_c^+ - s)+m} \right]_{r=0}^{c} < \infty,$$

which holds for $2(\lambda_c^+ - s) + m > 0$, i.e., $s < \frac{m}{2} + \lambda_c^+$ and shows that Theorem 3.5 gives the best result possible in this case. Note that there is the slight discrepancy that the results from [19] hold for time axis $t \in (0, \infty)$ whereas our results are established for $t \in \mathbb{R}$. However, this should be immaterial in the context when regarding spacial regularity of the solution u.

So far we have focused on spacial regularity of the solution u to (3.1). With the help of Theorem 3.2 also the following Hölder regularity in time can be shown.

Theorem 3.9 (Regularity in time) *Let* $2 \leq m \leq d$, $1 < p, q < \infty$, *and let* $\mathcal{K} = K \times \mathbb{R}^{d-m}$ *be a wedge according to Definition* 2.1*. Suppose that*

$$2 - \frac{m}{p} - \lambda_c^+ < \mu < m - \frac{m}{p} + \lambda_c^-,$$

where $\lambda_c^{\pm}$ *are the critical exponents defined in Section* 3.1.2*. Furthermore, assume* $f \in L_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K}))$*. Then there is a solution* u *of problem* (3.1) *satisfying*

$$u \in \mathcal{C}^{0,1-\frac{1}{q}}(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K})).$$

Proof : This is an immediate consequence of Theorem 3.2 together with Sobolev's embedding theorem for Banach-space valued functions, cf. [25, Cor. 26]. In particular, we have the following embedding

$$\begin{aligned} u \in L_q(\mathbb{R}, \mathcal{K}^2_{2-\mu,p}(\mathcal{K})) \cap W^1_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K})) &\hookrightarrow W^1_q(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K})) \\ &\hookrightarrow \mathcal{C}^{0,1-\frac{1}{q}}(\mathbb{R}, \mathcal{K}^0_{-\mu,p}(\mathcal{K})), \end{aligned}$$

which completes the proof. □

3.3 The heat equation

As a special important case of (3.1) we now consider the heat equation

$$\frac{\partial}{\partial t} u - \Delta u = f \quad \text{in } K \times \mathbb{R}, \qquad u\big|_{\partial K \times \mathbb{R}} = 0, \tag{3.16}$$

where $K \subset \mathbb{R}^3$ is the cone in Definition 2.1, $m = d = 3$, and assume that $\Omega = K \cap S^2$ is of class $\mathcal{C}^{1,1}$. Using (3.5) we see that the critical exponents coincide in this case. In particular, we have

$$\lambda_c^{\pm} = \lambda_1^+ := -\frac{1}{2} + \sqrt{\Lambda_1 + \frac{1}{4}}\,, \tag{3.17}$$

where Λ_1 is the first eigenvalue of the Dirichlet problem of the Laplace-Beltrami operator in Ω. In terms of fractional Sobolev regularity of the solution u of (3.16) for $p = q = 2$ Theorem 3.5 now implies the following.

Theorem 3.10 (Fractional Sobolev regularity of heat equation)
Let $K \subset \mathbb{R}^3$ be a cone according to Definition 2.1. *Suppose*

$$\max\left(\frac{1}{2} - \lambda_1^+, 0\right) < \mu < \frac{3}{2} + \lambda_1^+,$$

where $\mu = 0$ is allowed if $\frac{1}{2} - \lambda_1^+ < 0$ and λ_1^+ is defined in (3.17). *Furthermore, assume $f \in L_2(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(K))$ and let φ be the cut-off function from* (3.7). *Then there is a solution u of* (3.16) *satisfying*

$$\varphi u \in L_2(\mathbb{R}, H^s(K_0)) \quad \text{for any} \quad s < 2 - \mu.$$

3.3.1 Eigenvalues of the Laplacian on a spherical cap

We are interested in exact values of λ_1^+ for particular cones $K \subset \mathbb{R}^3$ in Theorem 3.10. In view of (3.17) we need to determine the first eigenvalue Λ_1 of the Dirichlet problem of the Laplace-Beltrami operator on $\Omega = K \cap S^2$. In the particular case of $\Omega = K \cap S^2$ being a spherical cap precise results are known. Therefore, we now investigate the problem

$$\left\{ \begin{array}{rcll} \Delta_{S^2} w + \Lambda w &=& 0 & \text{in} \quad \Omega_{\theta_0} \times \mathbb{R}, \\ w &=& 0 & \text{on } \partial\Omega_{\theta_0} \times \mathbb{R}, \end{array} \right\} \tag{3.18}$$

where $\Omega_{\theta_0} := K \cap S^2$ is a spherical cap, which in polar coordinates is expressed as

$$\begin{cases} y_1 &= \sin\theta \sin\varphi, \\ y_2 &= \sin\theta\cos\varphi, \\ y_3 &= \cos\theta, \end{cases}$$

i.e., $\Omega_{\theta_0} = \{(y_1, y_2, y_3) \in S^2 : \ \theta \in (0, \theta_0),\ \varphi \in [0, 2\pi]\}$. Since Ω_{θ_0} is a compact Riemannian manifold with smooth boundary it follows from [26, Cor. 1.5] that the eigenfunctions w of the Laplace-Beltrami operator in (3.18) are smooth.

The operator $\Delta_{S^2} + \lambda$ in this coordinates reads as

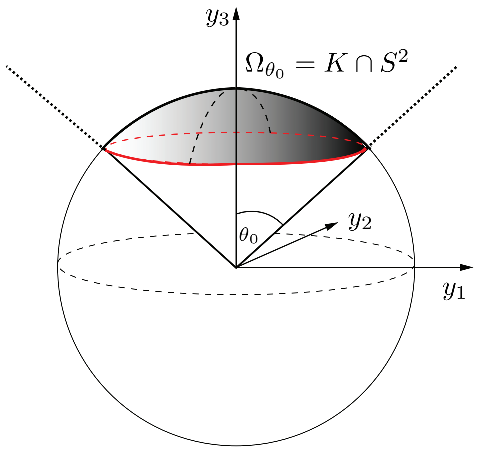

Figure 5: spherical cap,
angle $\theta_0 < \frac{\pi}{2}$

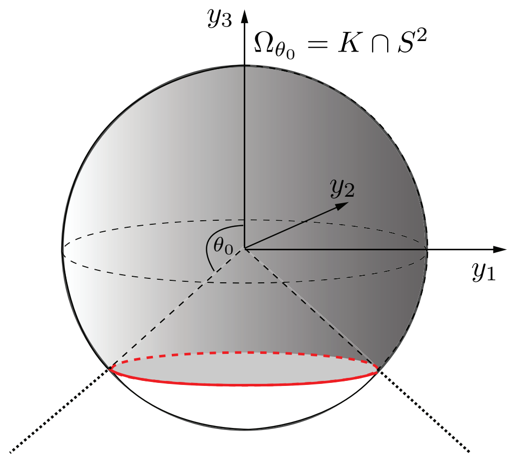

Figure 6: spherical cap,
angle $\theta_0 > \frac{\pi}{2}$

$$\Delta_{S^2} w + \Lambda w = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta}\sin\theta\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} + \Lambda w.$$

Solutions to (3.18) are expressed by using separation of variables,

$$w(\theta,\varphi) = P(x)\Phi(\varphi) \qquad \text{with} \qquad x = \cos\theta.$$

With this we see that

$$\begin{aligned} 0 &= \Delta_{S^2} w + \Lambda w \\ &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial P(\cos\theta)}{\partial\theta}\sin\theta\right)\Phi(\varphi) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}P(\cos\theta) + \Lambda P(\cos\theta)\Phi(\varphi) \\ &= \left((1-x^2)\frac{\partial^2 P}{\partial x^2} - 2x\frac{\partial P}{\partial x}\right)\Phi(\varphi) + \frac{1}{1-x^2}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}P(x) + \Lambda P(x)\Phi(\varphi) \end{aligned} \tag{3.19}$$

Recall that the eigenfunctions $w(\theta,\varphi) = P(x)\Phi(\varphi)$ are smooth (and from [24, Thm. 3] is follows that the eigenfunction corresponding to the smallest positive eigenvalue Λ_1 has no roots). Thus, we can multiply both sides of (3.19) by $(1-x^2)/(P(x)\Phi(\varphi))$ and separating variables yields

$$\frac{1-x^2}{P(x)}\left((1-x^2)\frac{\partial^2 P}{\partial x^2} - 2x\frac{\partial P}{\partial x}\right) + \Lambda(1-x^2) = -\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}\frac{1}{\Phi(\varphi)} =: m^2. \tag{3.20}$$

By the regularity of the solutions, $|P(1)| < \infty$, $\Phi(0) = \Phi(2\pi)$, and $\Phi'(0) = \Phi'(2\pi)$ must be satisfied. For convenience below we define $\nu \geq 0$ satisfying $\Lambda := \nu(\nu+1)$. From (3.20) for $P(x)$ and $\Phi(\varphi)$ we see that

$$(1-x^2)\frac{d^2 P}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}P}{\mathrm{d}x} + \left(\nu(\nu+1) - \frac{m^2}{1-x^2}\right)P = 0, \tag{3.21}$$

and

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} + m^2\Phi = 0. \tag{3.22}$$

From the periodicity of $\Phi(\varphi)$ it follows that m is a non-negative integer and any solutions to (3.22) are expressed as $\Phi(\varphi) = c_1 \cos(m\varphi) + c_2 \sin(m\varphi)$. Furthermore, (3.21) is known as the *associated Legendre equation*, which has two kinds of (linearly independent) solutions $P = P_\nu^m(x)$ and $Q_\nu^m(x)$ such that $|P_\nu^m(1)| < \infty$ and $|Q_\nu^m(x)| \to \infty$ as $x \to 1$, respectively. From the condition $|P(1)| < \infty$, we only have to treat $P = P_\nu^m(x)$ and, in conclusion, $\Lambda = \nu(\nu+1)$ and $c_1 P_\nu^m(\cos\theta)\cos(m\varphi) + c_2 P_\nu^m(\cos\theta)\sin(m\varphi)$ are eigenvalues and eigenfunctions of $\Delta_{S^2} w + \Lambda w = 0$ on S^2, respectively.
In order to solve the eigenvalue problem (3.18), we are required to find solutions to (3.21) satisfying the boundary condition

$$P(\cos\theta_0) = 0. \tag{3.23}$$

For any fixed $m = 0, 1, 2, \ldots$ there exist infinitely many $\Lambda = \nu(\nu+1)$ satisfying (3.21) and (3.23). Since we are interested in the smallest positive eigenvalue Λ_1 of (3.18), we have to look for the smallest root $\nu_{\min}$ of the Legendre function P_ν^m such that $P_\nu^m(\cos\theta_0) = 0$. Moreover, from (3.17) we see that

$$\lambda_1^+ = -\frac{1}{2} + \sqrt{\nu_{\min}(\nu_{\min}+1) + \frac{1}{4}} = -\frac{1}{2} + \sqrt{\left(\nu_{\min} + \frac{1}{2}\right)^2} = \nu_{\min},$$

hence, λ_1^+ in this case is determined by the minimal root $\nu_{\min}$ of the Legendre function P_ν^m with respect to the angle θ_0. Figure 7 is taken from [1], where the roots of the Legendre functions P_ν^m for different values of θ_0 were computed.

Now for the heat equation in the special case that K is a convex cone with $\Omega_\theta = K \cap S$ being a spherical cap we obtain the following regularity result.

Corollary 3.11 *Let $K \subset \mathbb{R}^3$ be a convex cone such that $\Omega_{\theta_0} = K \cap S$ is a spherical cap. Assume $f \in L_2(\mathbb{R} \times K)$. Furthermore, let φ be the cut-off function from* (3.7)*. Then there is a solution u of* (3.16) *satisfying*

$$\varphi u \in L_2(\mathbb{R}, H^s(K_0)) \quad \textit{for any} \quad s < 2.$$

Proof : If K is convex, then $\theta_0 \le \frac{\pi}{2}$. But then the values in Figure 7 imply that $\lambda_1^+ \ge 1$. Therefore, we can choose $\mu = 0$ in Theorem 3.10, which completes the proof. $\square$

θ_0	λ_1^+	θ_0	λ_1^+	θ_0	λ_1^+
5°	27.0558	65°	1.5988	125°	0.5523
10°	13.2756	70°	1.4456	130°	0.5063
15°	8.6812	75°	1.3124	135°	0.4631
20°	6.3832	80°	1.1956	140°	0.4223
25°	5.0038	85°	1.0922	145°	0.3834
30°	4.0837	90°	$\boxed{1.000}$	150°	0.3462
35°	3.4260	95°	0.9172	155°	0.3101
40°	2.9323	100°	0.8423	160°	0.2745
45°	2.5479	105°	0.7741	165°	0.2387
50°	2.2400	110°	0.7118	170°	0.2012
55°	1.9878	115°	0.6545	175°	0.1581
60°	1.7773	120°	0.6015		

Figure 7: Values for $\lambda_1^+ = \nu_{\min}$ when $\Omega_{\theta_0} = K \cap S$ is a spherical cap

Remark 3.12 (i) A closer look at Figure 7 reveals that Corollary 3.11 remains true for cones K with $\Omega_{\theta_0} = K \cap S$ being a spherical cap and angles $\theta_0 \in [0, 130°)$, since we have $\lambda_1^+ > \frac{1}{2}$ in this case and can therefore choose $\mu = 0$ in Theorem 3.10.

(ii) If $0 < \lambda_1^+ \leq 1/2$ Theorem 3.10 yields that $\mu > \frac{1}{2} - \lambda_1^+$. Then for $f \in L_2(\mathbb{R}, \mathcal{K}^0_{-\frac{1}{2}+\lambda_1^+,2}(\mathcal{K}))$ and φ being the cut-off function from (3.7), the solution u of (3.16) satisfies

$$\varphi u \in L_2(\mathbb{R}, H^s(\mathcal{K}_0)) \quad \text{with} \quad s < \frac{3}{2} + \lambda_1^+.$$

(iii) Our results strongly indicate that it might be possible to improve recent regularity results for parabolic equations from [6, Ex. 4.7] (although there the time interval was $[0, T]$ for some $T > 0$ whereas here we consider $\mathbb{R}$).
The results from Theorem 3.10 together with what was stated in [6, Ex. 4.7] suggest that if $f \in L_2(\mathbb{R}, \mathcal{K}^0_{-\mu,2}(K))$ with $\mu > \max\left(\frac{1}{2} - \lambda_1^+, 0\right)$ and φ being the cut-off function from (3.7), then the solution u of (3.16) satisfies

$$\varphi u \in L_2(\mathbb{R}, B^{\gamma}_{\tau,\infty}(K_0)) \qquad \text{for all} \qquad \frac{1}{2} < \frac{1}{\tau} < \frac{\gamma}{3} + \frac{1}{2}, \tag{3.24}$$

where for the Besov regularity γ we derive the upper bound

$$\gamma < 3(2-\mu) < \begin{cases} 6 & \text{if } \lambda_1^+ > \frac{1}{2}, \\ \frac{9}{2} + 3\lambda_1^+ & \text{if } 0 < \lambda_1^+ \leq \frac{1}{2}. \end{cases}$$

This implies that for all cones K with $\Omega_{\theta_0} = K \cap S$ being a spherical cap we might always have $\gamma < \frac{9}{2}$ whereas in [6, Ex. 4.7] we only obtained $\gamma < 3$ in case of non-convex cones.
Moreover, the results from Theorem 3.10 can also help to improve the nonlinear Sobolev regularity results from [6, Thm. 4.10]. There we needed the restriction that $m \geq 2$ for the Sobolev regularity of the solution, i.e., $u(\cdot, t) \in W_2^m(K)$. The reason for this was a multiplier result used in the proof, cf. [6, formula (4.23)]. This multiplier result can now be reformulated with the help of the spaces $H^s(K)$ as long as $s > \frac{3}{2}$.
Finally, let us mention that the obtained results hint, that when it comes to numerical schemes providing constructive approximations of the solutions, adaptive schemes usually would outperform uniform ones. The reason for this lies in the fact that as a role of thumb the convergence order that can be achieved by adaptive algorithms is determined by the regularity γ of the exact solution in the adaptivity scale of Besov spaces (3.24), whereas the convergence order for uniform schemes depends on the classical Sobolev smoothness s. From the considerations above we have $\gamma < 3(2-\mu)$ in contrast to $s < 2 - \mu$.

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