ADAPTIVE QUARKONIAL DOMAIN DECOMPOSITION METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with new discretization methods for the numerical treatment of elliptic partial differential equations. We derive an adaptive approximation scheme that is based on frames of quarkonial type. These new frames are in turn constructed from a finite set of functions via translation, dilation and multiplication by monomials. By using non-overlapping domain decomposition ideas, we establish quarkonial frames on domains that can be decomposed into the union of parametric images of unit cubes. We also show that these new representation systems are stable in a certain range of Sobolev spaces. The construction is performed in such a way that, similar to the wavelet setting, the frame elements, the so-called *quarklets*, possess a certain number of vanishing moments. This enables us to generalize the basic building blocks of adaptive wavelet algorithms to the quarklet case. The applicability of the new approach is demonstrated by numerical experiments for the Poisson equation on L-shaped domains.

Key words: Adaptive numerical algorithms, domain decomposition, frames, quarkonial decompositions, Sobolev spaces,

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1. Introduction

Many problems in science and engineering are modeled by partial differential equations. Very often, an analytic expression of the unknown solution is not available, so that efficient numerical schemes for its constructive approximation are needed. During the last decades, many different approaches have been developed, such as finite differences, finite elements, and spectral methods, just to name a few. The amount of literature is overwhelming and can of course not be discussed in detail here. When it comes to real-life problems, systems with hundreds of thousands or even millions of degrees of freedom have to be handled, so that *adaptive* strategies are essential to increase the overall efficiency. In principle, an adaptive algorithm is an updating strategy in the sense that additional degrees of freedom are only spent in regions where the numerical approximation is still far away from the exact solution. To realize such a scheme, efficient and reliable a posteriori error estimators and associated refinement strategies have to be derived. In particular, in the realm of finite element methods (FEM), many impressive results in this direction have been obtained. Once again, the amount of literature cannot be discussed in detail here; we refer, e.g., to the book of Verfürth [35] for an overview. In principle, the following general strategies have been developed: the h-FEM which is based on adaptive space refinement, the p-FEM which corresponds to polynomial enrichments, and a combination of both, the hp-FEM. It is observed in practice that adaptive hp-methods are often very efficient; sometimes they even have exponential convergence [1]. However, rigorous convergence and complexity proofs of FEM schemes could be derived only recently. In the last years, some results have been obtained for h- and hp-FEM of second-order elliptic equations, we mention the timely reviews [2–5, 26] and the references therein.

At this point, another recently developed approach, i.e., adaptive numerical schemes based on wavelets, comes into play. The strong analytic properties of wavelets can be used to derive adaptive schemes that are guaranteed to converge with optimal order in the sense that they realize the convergence order of the best N-term wavelet approximation. We refer to [9, 33] for details. So far, these schemes are based on adaptive space refinements, i.e., they correspond to the h-methods. Then, a very natural question shows up: is it possible to construct wavelet versions of adaptive hp-methods? And if so, can this in the long run pave the way to new convergence proofs for hp-methods?

This paper can be interpreted as one step in this direction. The first challenge is of course the question how to incorporate polynomial enrichment into a wavelet system. It turns out that this is possible when working with *frames*, i.e. redundant generating systems $\mathcal{F} = \{f_\lambda\}_{\lambda\in\mathcal{I}}$ of a Hilbert space H that are still numerically stable in the sense of the equivalence

$$\|f\|_H^2 \eqsim \inf_{\{\boldsymbol{c}\in\ell_2(\mathcal{I}):f=\sum_{\lambda\in\mathcal{I}} c_\lambda f_\lambda\}} \sum_{\lambda\in\mathbb{N}} |c_\lambda|^2, \quad \text{for all } f \in H,$$

see Appendix A.1 for equivalent definitions and basic facts from frame theory. The frame concept has the advantage that it provides much more flexibility than a basis setting. As a consequence, on complicated computational domains, frames are much easier to construct than Riesz bases. In particular, our work is inspired by an approach of H. Triebel. In [34], he derived highly redundant frames for Sobolev and Besov function spaces where the frame elements are constructed via dilation and polynomial enrichment of a partition of unity. These frames can be interpreted as subatomic, i.e., *quarkonial* decompositions, and this concept is of course very close to the idea of an hp-finite element system.

However, to design an adaptive numerical scheme directly based on these quarkonial decompositions is highly non-trivial since the frame elements do not possess vanishing moments, and the vanishing moment property is essential for the convergence and optimality proofs of adaptive wavelet schemes, see again [9] for details. Therefore, we proceed in a different way. We start with a biorthogonal wavelet basis as, e.g., constructed in [11], and implement the polynomial enrichment by multiplying the wavelets with monomials. In the predecessor paper [16], we have shown that this approach indeed works for problems on the real line. It has turned out that the resulting highly redundant system has the frame property in scales on Sobolev spaces. Moreover, the whole construction has been designed in such a way that the vanishing moments of the underlying wavelet basis are preserved which implies that the basic building blocks of adaptive wavelet schemes can still be constructed.

In this paper, we generalize this concept to quite general domains contained in $\mathbb{R}^d$. Even in the classical wavelet setting, the construction of wavelets on domains is a non-trivial task. Usually, this is performed by some kind of domain decomposition approach. One possible way could be to use an overlapping domain decomposition approach as outlined in [15]. Indeed, it has been shown that the resulting adaptive wavelet frame schemes again converge with optimal order. However, in practice, one is very often faced with non-trivial quadrature problems that hamper the overall performance of the scheme. Therefore, we proceed in a different way and use a non overlapping domain decomposition similarly to the earlier work [18]. It has been shown in [7] that this approach gives rise to generalized tensor product wavelet bases on quadranglelizable domains. In general, tensor product wavelets can be interpreted as a wavelet version of sparse grids. Therefore, related approximation schemes can attain dimension-independent convergence rates. In [7], it has been shown that these properties carry over to the case of more general domains. In this paper, we show that a combination of these ideas with quarkonial decompositions indeed works and gives rise to a generalized tensor product quarklet frame on computational domains which can be quadrangulated.

To carry out this program, several steps have to be performed. First of all, the quarkonial frame construction in [16] has to be adapted to problems on bounded intervals. In particular, Dirichlet boundary conditions have to be incorporated. Once this is done, a quarkonial frame on unit cubes can be designed by taking tensor products. Then, one has to establish that the new systems are again stable in scales of Sobolev spaces. This is by no means obvious because the underlying Sobolev spaces are usually not of tensor product type. Finally, this construction has to be generalized to arbitrary domains by using non-overlapping domain decomposition strategies and suitable extension operators. In this paper, we show that all these steps can indeed be carried out. Moreover we prove that several very important properties such as vanishing moments are preserved, which again implies that the basic building blocks of adaptive algorithms still can be derived.

This paper is organized as follows. In Section 2, we show that the construction in [16] can be adapted to bounded intervals in such a way that, e.g., Dirichlet boundary conditions can be incorporated without destroying the vanishing moment properties of the underlying biorthogonal wavelet basis. In Section 3, we generalize the construction to bounded domains contained in $\mathbb{R}^d$. We start with the case of unit cubes. We show that by tensorizing quarkonial frames on intervals one obtains frames for the Sobolev spaces $H^s((0,1)^d)$. In contrast to the basis case, this does not follow by general arguments; it seems that additional conditions on the underlying frames are necessary. Fortunately, these conditions are satisfied in our case. Then the case of quadranglelizable domains is studied. We show that, given reference frames on the unit cube, these frames can be extended to the whole domain in such a way that their union once again provides a frame for scales of Sobolev spaces. Having shown this, in principle one can run the general machinery of adaptive frame algorithms as outlined in [15]. To this end, several building blocks have to be established. In particular, a routine **APPLY** is needed which approximates products of the infinite stiffness matrix with finitely supported vectors. This can be performed provided that the stiffness matrix is *compressible* which is usually implied by the vanishing moment property, see again [9, 33]. In Section 4 we establish a first compression result related with the new generalized quarkonial tensor frames. Finally, in Section 5, we conduct first numerical experiments. In particular, the Poisson equation on an L-shaped domain is studied. It turns out that for natural test examples adaptive quarklet schemes outperform the well-established wavelet (frame) methods. Therefore, the higher redundancy induced by the polynomial enrichment really pays off in practice.

2. Quarkonial decomposition on the interval

In this section, we present an explicit construction of quarkonial decompositions on bounded intervals. It turns out that properly rescaled versions of the resulting representation systems form frames for L_2 as well as for the Sobolev spaces H^s. The construction is based on a generalization of the approach outlined in [16]. There quarkonial decompositions in $L_2(\mathbb{R})$ have been constructed by polynomial enrichment of a given biorthogonal wavelet basis. For readers convenience, in Subsection 2.1 we briefly recall the main results of [16]. To generalize this approach to bounded intervals, first of all a *multiresolution analysis* (MRA) and a biorthogonal wavelet basis on the interval are needed. This topic has been intensively studied within the last years, see e.g.. [6, 17]. In this paper, we particularly employ the construction of M. Primbs [28], since several numerical experiments have shown that this basis produces the best constants. In Subsection 2.2, we briefly recall this construction. Quite surprisingly, it turns out that a direct application of the ideas in [16] does not work since this would destroy the vanishing moment property at the boundary. The necessary modifications will be described in Subsection 2.4. Finally, in Subsection 2.5, we prove that the frame properties established in [16] indeed carry over to the boundary adapted setting.

2.1. **The shift-invariant case.** As a starting point, let us recall some techniques from the design of quarklet frames on the real line, see [16] for details. The core ingredient is a shift-invariant wavelet Riesz basis for $L_2(\mathbb{R})$ with sufficient regularity, approximation and compression properties. Usually, wavelets are constructed by means of a multiresolution analysis (MRA) which is a nested sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed linear subspaces whose union is dense in $L_2(\mathbb{R})$ while their intersection is zero. Moreover, we assume that all spaces are related by dyadic dilation, i.e., $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$, and that V_0 is shift-invariant, i.e., $f \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}$.

In addition, one asks for a function φ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz basis for its closed span V_0 with respect to $\|\cdot\|_{L_2(\mathbb{R})}$. Then φ is called the *generator* of the multiresolution analysis.

The properties of a multiresolution analysis imply that φ is refinable, i.e., there exist refinement coefficients $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, such that φ admits the expansion

$$\varphi = \sum_{k\in\mathbb{Z}} a_k \varphi(2\cdot - k).$$

For our purposes, we further assume that φ fulfills the additional properties:

(i) φ is compactly supported and $\mathbf{a}$ is a finite sequence;
(ii) $\varphi \in H^s(\mathbb{R})$ for all $0 \le s < m - \frac{1}{2}$ and some $m \in \mathbb{N}$;
(iii) $\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = 1$;
(iv) the system $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is able to reproduce polynomials $\mathbb{P}_{m-1}$ of degree at most $m-1$, i.e., for each $q \in \mathbb{P}_{m-1}$, there exist coefficients $\mathbf{c} = \{c_k\}_{k\in\mathbb{Z}} \in \ell_0(\mathbb{Z})$ with the locally finite expansion

$$q = \sum_{k\in\mathbb{Z}} c_k \varphi(\cdot - k).$$

These assumptions are fulfilled, e.g., by cardinal B-splines N_m of order m. Moreover, the properties (i)-(iv) imply that the integer translates of φ form a *partition of unity*,

$$\sum_{k\in\mathbb{Z}} \varphi(\cdot - k) \equiv 1.$$

Given a (MRA), a wavelet basis can be constructed by finding one function ψ whose translates span a complement W_0 of V_0 in V_1, $V_1 = V_0 \oplus W_0$, see, e.g., [20] for details. We assume that the wavelet $\psi \in L_2(\mathbb{R})$ has the following properties:

(i) ψ is compactly supported and fulfills

$$\psi = \sum_{k\in\mathbb{N}} b_k \varphi(2\cdot -k), \tag{2.1}$$

with expansion coefficients $b_k \in \mathbb{R}$, only finitely many of them being non-zero;

(ii) ψ has $\tilde{m}$ vanishing moments, i.e. $\int_{-\infty}^{\infty} q(x)\psi(x)\,\mathrm{d}x = 0$ for all $q \in \mathbb{P}_{\tilde{m}-1}$;

(iii) the system

$$\Sigma := \big\{\varphi(\cdot - k) : k \in \mathbb{Z}\big\} \cup \big\{2^{j/2}\psi(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\big\}$$

is a Riesz basis for $L_2(\mathbb{R})$.

Although constructions with other kinds of generators are possible, we confine ourselves in this subsection to the case of a shifted cardinal B-spline generator $\varphi = N_m(\cdot + \lfloor \frac{m}{2} \rfloor)$. Associated spline wavelets ψ with these properties (i)-(iii) have been constructed in [11], where $\mathbb{N} \ni \tilde{m} \geq m$ and $m + \tilde{m} \in 2\mathbb{N}$. What is more, by simple rescaling, the system

$$\Sigma^s := \big\{\varphi(\cdot - k) : k \in \mathbb{Z}\big\} \cup \big\{2^{-j(s-1/2)}\psi(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\big\}$$

is a Riesz basis for $H^s(\mathbb{R})$, where $0 \leq s < m - \frac{1}{2}$.

In the construction of quarkonial frames for $L_2(\mathbb{R})$ from [16], we obtain *quarks* φ_p by enrichment of φ with polynomials of degree $p \in \mathbb{N}_0$ via pointwise multiplication with monomials,

$$\varphi_p := \left(\frac{\cdot}{\lceil m/2 \rceil}\right)^p \varphi,\ p \in \mathbb{N}_0.$$

Replacing φ by φ_p in (2.1) gives rise to the compactly supported *quarklets*

$$\psi_p := \sum_{k\in\mathbb{Z}} b_k \varphi_p(2\cdot -k),\ p \in \mathbb{N}_0. \tag{2.2}$$

It has been shown in [16] that there exist weights $w_p > 0$ which decrease inverse-polynomially in $p \in \mathbb{N}_0$, such that

$$\Psi := \big\{\varphi_p(\cdot - k) : p \in \mathbb{N}_0, k \in \mathbb{Z}\big\} \cup \big\{w_p 2^{j/2}\psi_p(2^j \cdot -k) : p, j \in \mathbb{N}_0, k \in \mathbb{Z}\big\} \supset \Sigma \tag{2.3}$$

is a frame for $L_2(\mathbb{R})$, cf. [16, Theorem 3]. The main result considering the case of the whole real line is the following ([16, Theorem 4]).

Theorem 2.1. *Let $w_{p,j,s} := 2^{-js}(p+1)^{-2s-\delta}$ with $\delta > 1$. Then the system*

$$\Psi^s := \{w_{p,0,s}\varphi_p(\cdot - k), w_{p,j,s}2^{j/2}\psi_p(2^j \cdot -k) : p, j \in \mathbb{N}_0, k \in \mathbb{Z}\} \tag{2.4}$$

is a frame for $H^s(\mathbb{R})$, $0 \leq s < m - \frac{1}{2}$.

2.2. **Wavelets on the interval.** The construction principles of quarklet frames on the interval are very similar to the real axis case. Again, a (MRA) and the associated wavelet basis on the interval $I := (0,1)$ are needed. Moreover, we would like the system to satisfy certain boundary conditions. Systems that fulfill these requirements are, e.g., the ones constructed in [18] or [27, 28]. In view of their good numerical properties, we decided to take the wavelet bases constructed by Primbs [27, 28] as the fundament of our construction. Up to modifications of the respective index sets, however, it is possible to derive quarklet frames also from other spline wavelet bases on the interval.

In this subsection we summarize the construction principles and the most important properties of the Primbs wavelet basis. First let us fix some notation.

Let $m, \tilde{m} \in \mathbb{N}_0$, so that $\tilde{m} \geq m \geq 2$, $m+\tilde{m} \in 2\mathbb{N}$. Let $\vec{\sigma} = (\sigma_l, \sigma_r) \in \{0, \lfloor s+1/2 \rfloor\}^2$ denote the order of boundary conditions. Furthermore, let $j \in \mathbb{N}$, $j \geq j_0 \in \mathbb{N}$ with j_0 sufficiently large. With $\Delta_{j,\vec{\sigma}} \subset \mathbb{Z}$ we denote the index set

$$\Delta_{j,\vec{\sigma}} := \{-m+1+\operatorname{sgn}\sigma_l, \dots, 2^j - 1 - \operatorname{sgn}\sigma_r\}. \tag{2.5}$$

Dealing with free boundary conditions, we set $\Delta_j := \Delta_{j,(0,0)}$. As in the shift-invariant case the construction of a wavelet basis on the unit interval commonly is based upon a multiresolution analysis. The first step is again the construction of a boundary-adapted generator function. Given the knots

$$t_k^j := \begin{cases} 0, & k = -m+1, \dots, 0, \\ 2^{-j}k, & k = 1, \dots, 2^j - 1, \\ 1, & k = 2^j, \dots, 2^j + m - 1, \end{cases}$$

with boundary knots of multiplicity m and single inner knots, the *Schoenberg B-Splines* are defined by

$$B_{j,k}^m(x) := (t_{k+m}^j - t_k^j)\,(\cdot - x)_+^{m-1}[t_k^j, \dots, t_{k+m}^j], \quad k \in \Delta_j, x \in I,$$

using *divided differences* of the function $t_+ := \max(t,0)$, $t \in \mathbb{R}$. The generating functions of the Primbs basis are defined by

$$\varphi_{j,k} := 2^{j/2} B_{j,k}^m, \quad k \in \Delta_j. \tag{2.6}$$

For further information on B-Splines we refer to [30, § 4.3] and [21]. As the Primbs basis is a biorthogonal wavelet basis, a *dual* multiresolution analysis with *dual generators* $\tilde{\varphi}_{j,k}$ is necessary for the construction. If the generators are represented as column vectors $\Phi_j := \{\varphi_{j,k} : k \in \Delta_{j,\vec{\sigma}}\}$, $\tilde{\Phi}_j := \{\tilde{\varphi}_{j,k'} : k' \in \Delta_{j,\vec{\sigma}}\}$, they fulfill the duality relation

$$\langle \Phi_j, \tilde{\Phi}_j \rangle := \big(\langle \varphi_{j,k}, \tilde{\varphi}_{j,k'} \rangle_{L_2(I)}\big)_{k,k' \in \Delta_{j,\vec{\sigma}}} = I_{\Delta_{j,\vec{\sigma}}}.$$

For construction details on the dual MRA we refer to [28, Chapter 4]. One of the main properties of the dual generators is polynomial exactness of order $\tilde{m}-1$, cf. [28, Section 2.2]. To yield stable systems, the MRA has to fulfill certain additional *Jackson* and *Bernstein inequalities*. On the primal side they have the following form, cf. [28, Lemma 5.2]:

$$\inf_{v\in V_j} ||u-v||_{L_2(I)} \leq C2^{-js}||u||_{H^s(I)} \text{ for all } u \in H^s(I),\ 0 \leq s < m, \tag{2.7}$$

$$||u||_{H^s(I)} \leq C'2^{js}||u||_{L_2(I)} \text{ for all } u \in V_j,\ 0 \leq s < m-\frac{1}{2}. \tag{2.8}$$

In the construction of the Primbs basis, it is possible to choose different boundary conditions on the primal and the dual side, cf. [27, Section 4.7]. On the one hand, we will need primal wavelets both with free and zero boundary conditions. On the other hand, the corresponding dual wavelets always should be of free boundary type. This choice is motivated as follows: As already stated above, in numerical applications the vanishing moment property is essential, and with free boundary conditions the dual generators have polynomial exactness $\tilde{m}-1$, which allows to construct primal wavelets with $\tilde{m}$ vanishing moments.

For the wavelets the following index sets are defined:

$$\nabla_{j,\vec{\sigma}} := \begin{cases} \{0,\cdots,2^j-1\}, & j \geq j_0, \\ \Delta_{j,\vec{\sigma}}, & j = j_0-1. \end{cases} \tag{2.9}$$

One main step carried out in [28] is the *stable completion*, i.e., the construction of suitable matrices $M^{\vec{\sigma}}_{j,1}$, $\tilde{M}^{\vec{\sigma}}_{j,1}$, which contain the two-scale coefficients of $\Psi_j = \{\psi^{\vec{\sigma}}_{j,k} : k \in \nabla_{j,\vec{\sigma}}\}$:

$$\begin{aligned} \Psi_j &:= M^{\vec{\sigma},T}_{j,1}\Phi_{j+1}, \quad j \geq j_0, \\ M^{\vec{\sigma},T}_{j,1} &:= \left(b^{j,\vec{\sigma}}_{k,l}\right)_{k\in\nabla_{j,\vec{\sigma}},l\in\Delta_{j+1,\vec{\sigma}}} \in \mathbb{R}^{|\nabla_{j,\vec{\sigma}}|\times|\Delta_{j+1,\vec{\sigma}}|}, \end{aligned} \tag{2.10}$$

and analogously for $\tilde{\Psi}_j$. Then, the duality relations $\langle \Psi_j, \tilde{\Phi}_j\rangle = \mathbf{0}$, $\langle \Phi_j, \tilde{\Psi}_j\rangle = \mathbf{0}$, $\langle \Psi_j, \tilde{\Psi}_j\rangle = I_{\nabla_{j,\vec{\sigma}}}$ are fulfilled. Furthermore the weighted system

$$\Sigma^s_{\vec{\sigma}} := 2^{-j_0 s}\Phi_{j_0} \cup \bigcup_{j\geq j_0} 2^{-js}\Psi_j, \quad 0 \leq s < m-\frac{1}{2}, \tag{2.11}$$

is a Riesz basis of $H^s_{\vec{\sigma}}(I)$ with dual basis $\tilde{\Sigma}^s_{\vec{\sigma}}$, which is defined analogously.

2.3. **Construction of boundary quarks.** In this section we construct quarks on the interval and derive crucial Bernstein and norm estimates. Since it is well known that the inner Schoenberg splines are dilated and translated copies of the cardinal B-spline N_m, we may define the inner quarks as the ones constructed in [16]. Hence, all important estimates for the inner quarks are already available and it is sufficient to focus on the boundary quarks, as they differ from the construction in [16].

We define a quark as the product of a generator with a certain monomial. As before, let $p \in \mathbb{N}_0$. Then, the (Schoenberg B-spline) quarks $\varphi_{p,j,k}$ are defined by

$$\varphi_{p,j,k} := \begin{cases} \left(\frac{2^j \cdot}{k+m}\right)^p \varphi_{j,k}, & k = -m+1, \dots, -1, \\ \left(\frac{2^j \cdot - k - \lfloor \frac{m}{2} \rfloor}{\lceil \frac{m}{2} \rceil}\right)^p \varphi_{j,k}, & k = 0, \dots, 2^j - m, \\ \varphi_{p,j,2^j-m-k}(1 - \cdot), & k = 2^j - m + 1, \dots, 2^j - 1. \end{cases} \tag{2.12}$$

The quarks form subspaces $V_{p,j}$ of $L_2(I)$:

$$V_{p,j} := \operatorname{clos}_{L_2(I)} \operatorname{span}\{\varphi_{q,j,k} : q = 0, \dots, p, k \in \Delta_j\}. \tag{2.13}$$

In the sequel it is helpful to study the properties of the quarks independently of the level. For this purpose we introduce quarks on level zero on the interval $[0, \infty)$:

$$\varphi_{p,0,k}(x) := \begin{cases} \left(\frac{x}{k+m}\right)^p B^m_{0,k}, & k = -m+1, \dots, -1, \\ \left(\frac{x - k - \lfloor \frac{m}{2} \rfloor}{\lceil \frac{m}{2} \rceil}\right)^p B^m_{0,k}, & k = 0, 1, \dots, \end{cases}$$

where the knot sequence $\{t^0_k\}_{k \geq -m+1}$ is given by

$$t^0_k := \begin{cases} 0, & k = -m+1, \dots, -1, \\ k, & k = 0, 1, \dots. \end{cases}$$

First we will show a two-scale-relation for the boundary quarks. This will be necessary in Subsection 2.5 to derive frame properties of boundary adapted quarklet systems. For the inner quarks a two-scale-relation was already shown in [16, Appendix]. Because of the symmetry we restrict our discussion to left boundary quarks.

Proposition 2.2. *For every $p \in \mathbb{N}_0$ there exist coefficients $a^j_{q,k,l} \in \mathbb{R}$, so that the left boundary quarks fulfill a two-scale-relation of the form*

$$\varphi_{p,j,k} = \sum_{l=-m+1}^{m-2} \sum_{q=0}^{p} a^j_{q,k,l} \varphi_{p,j+1,l}, \quad k = -m+1, \dots, -1. \tag{2.14}$$

Proof. Let $k \in \{-m+1, \dots, -1\}$ be fixed. We use the corresponding two-scale-relation for the Primbs wavelet generators, cf. [28, Lemma 3.3]:

$$\varphi_{j,k} = \sum_{l=-m+1}^{m-2} a^j_{k,l} \varphi_{j+1,l}.$$

Inserting this relation into the definition of the left boundary quarks, we obtain:

$$\begin{aligned}
\varphi_{p,j,k} &= \left(\frac{2^j\cdot}{k+m}\right)^p \varphi_{j,k} = \left(\frac{2^j\cdot}{k+m}\right)^p \sum_{l=-m+1}^{m-2} a^j_{k,l}\varphi_{j+1,l} \\
&= \left(\frac{2^j\cdot}{k+m}\right)^p \left(\sum_{l=-m+1}^{-1} a^j_{k,l}\varphi_{j+1,l} + \sum_{l=0}^{m-2} a^j_{k,l}\varphi_{j+1,l}\right). \qquad (2.15)
\end{aligned}$$

The first sum can be converted into a sum of left boundary quarks of degree p:

$$\begin{aligned}
\left(\frac{2^j\cdot}{k+m}\right)^p \sum_{l=-m+1}^{-1} a^j_{k,l}\varphi_{j+1,l} &= \sum_{l=-m+1}^{-1} a^j_{k,l}\left(\frac{l+m}{2(k+m)}\right)^p \frac{(2^{j+1}\cdot)^p}{(l+m)^p}\varphi_{j+1,l} \\
&= \sum_{l=-m+1}^{-1} a^j_{k,l}\left(\frac{l+m}{2(k+m)}\right)^p \varphi_{p,j+1,l}. \qquad (2.16)
\end{aligned}$$

For the second sum we obtain by an application of the binomial theorem:

$$\begin{aligned}
(2^j\cdot)^p \sum_{l=0}^{m-2} a^j_{k,l}\varphi_{j+1,l} =& 2^{-p}\sum_{l=0}^{m-2} a^j_{k,l}\left(2^{j+1}\cdot - l - \lfloor\frac{m}{2}\rfloor + l + \lfloor\frac{m}{2}\rfloor\right)^p \varphi_{j+1,l} \\
=& 2^{-p}\sum_{l=0}^{m-2} a^j_{k,l}\sum_{q=0}^{p}\binom{p}{q}\left(2^{j+1}\cdot - l - \lfloor\frac{m}{2}\rfloor\right)^q \left(l + \lfloor\frac{m}{2}\rfloor\right)^{p-q}\varphi_{j+1,l}.
\end{aligned}$$

Putting the monomials and wavelet generators together, we get:

$$\begin{aligned}
(2^j\cdot)^p \sum_{l=0}^{m-2} a^j_{k,l}\varphi_{j+1,l} &= \sum_{l=0}^{m-2}\sum_{q=0}^{p} a^j_{k,l}2^{-p}\binom{p}{q}\left(l+\lfloor\frac{m}{2}\rfloor\right)^{p-q}\lceil\frac{m}{2}\rceil^q \\
&\quad\cdot \frac{\left(2^{j+1}\cdot - l - \lfloor\frac{m}{2}\rfloor\right)^q}{\lceil\frac{m}{2}\rceil^q}\varphi_{j+1,l} \\
&= \sum_{l=0}^{m-2}\sum_{q=0}^{p} a^j_{k,l}2^{-p}\binom{p}{q}\left(l+\lfloor\frac{m}{2}\rfloor\right)^{p-q}\lceil\frac{m}{2}\rceil^q\varphi_{q,j+1,l}. \qquad (2.17)
\end{aligned}$$

Combining (2.15)-(2.17) leads to the coefficients of the two-scale-relation

$$a^j_{q,k,l} = \begin{cases} a^j_{k,l}\left(\frac{l+m}{2(k+m)}\right)^p \delta_{p,q}, & l=-m+1,\dots,-1, \\ a^j_{k,l}\left(\frac{1}{2(k+m)}\right)^p \binom{p}{q}\left(l+\lfloor\frac{m}{2}\rfloor\right)^{p-q}\lceil\frac{m}{2}\rceil^q, & l=0,\dots,m-2. \end{cases}$$

□

To be able to show the stability of the quarklet systems, bounds for the L_q-norm of the boundary quarks are necessary. In Proposition 2.3 we formulate such a statement. Analogous properties for the inner quarks have been discussed in [16]. The quite technical proof of Proposition 2.3 can be found in the appendix.

Proposition 2.3. *Let $1 \leq k \leq m-1$ and $\varphi_{p,0,-m+k}$ a left boundary quark. For every $1 \leq q \leq \infty$ there exist constants $c = c(m,k,q) > 0$, $C = C(m,k,q) > 0$, such that for all $p \geq (m-1)(k-1)$:*

$$c(p+1)^{-(m-1+1/q)} \leq ||\varphi_{p,0,-m+k}||_{L_q(\mathbb{R})} \leq C(p+1)^{-(m-1+1/q)}. \tag{2.18}$$

As mentioned in Subsection 2.2, Jackson and Bernstein inequalities play a key role to obtain stable systems not only in $L_2(I)$, but also in scales of Sobolev spaces $H^s(I)$. In [16, Theorem 1] a Jackson inequality for general polynomials and generating functions has been proven, but in our case it suffices to rely on the Jackson inequalities for $p = 0$ as given in (2.7), since the inclusion of a Riesz basis in our frame construction already assures the lower frame inequality, cf. (A.1). In the quark case, the classical Bernstein inequality (2.8) reads as follows:

Proposition 2.4. *Let $p \in \mathbb{N}_0$, $j \geq j_0$ and the spaces $V_{p,j}$ be given by (2.13). Then the following Bernstein inequalities hold true: For $1 \leq q \leq \infty$ and $r \in \mathbb{N}_0$, $r \leq m-1$ there exist constants $C = C(m,q) > 0$, such that for all $f \in V_{p,j}$:*

$$||f^{(r)}||_{L_q(I)} \leq C(p+1)^{2r} 2^{jr} ||f||_{L_q(I)}. \tag{2.19}$$

For $0 \leq s < m - \frac{1}{2}$ there exist constants $C' = C'(m,s) > 0$ so that for all $f \in V_{p,j}$:

$$|f|_{H^s(I)} \leq C'(p+1)^{2s} 2^{js} ||f||_{L_2(I)}. \tag{2.20}$$

Proof. The proof can be performed by following the lines of [16, Corollary 1, 2]. ☐

2.4. **Construction of boundary quarklets.** Now we discuss the construction of quarklets on the interval. For the inner quarklets, we proceed as in Subsection 2.1 and assign the two-scale relation of the underlying wavelet to the quarks with the same coefficients. Quite surprisingly, a similar approach does not work for the boundary quarklets since this would destroy the vanishing moment properties. It turns out that in order to preserve the vanishing moment properties of the underlying wavelet Riesz basis for the full quarklet system, it is necessary to define the two-scale relation of the boundary quarklets appropriately.

In any case, analogously to the shift-invariant case, cf. Subsection 2.1, quarklets are defined as linear combinations of quark generators on the next higher level. Then, the relation (2.2) for one quarklet becomes

$$\psi^{\vec{\sigma}}_{p,j,k} := \sum_{l \in \Delta_{j+1,\vec{\sigma}}} b^{p,j,\vec{\sigma}}_{k,l} \varphi_{p,j+1,l}, \quad k \in \nabla_{j,\vec{\sigma}}. \tag{2.21}$$

We already notice that in contrast to (2.2) the coefficients $b_{k,l}^{p,j,\vec{\sigma}}$ in (2.21) do not only depend on l.

At first, let us discuss the construction of the inner quarklets. For $p, j \in \mathbb{N}_0$, $j \geq j_0$, $k \in \nabla_{j,\vec{\sigma}}$ with $m-1 \leq k \leq 2^j - m$ the inner wavelets of the Primbs basis are given by $\psi_{j,k}^{\vec{\sigma}} = \sum_{l \in \Delta_{j+1,\vec{\sigma}}} b_{k,l}^{j,\vec{\sigma}} \varphi_{j+1,l}$, cf. (2.10). We construct an inner quarklet by keeping these coefficients and inserting them into (2.21):

$$b_{k,l}^{p,j,\vec{\sigma}} := b_{k,l}^{j,\vec{\sigma}}, \quad m-1 \leq k \leq 2^j - m,\ l \in \Delta_{j+1,\vec{\sigma}}. \tag{2.22}$$

If the inner Primbs wavelets have $\tilde{m}$ vanishing moments, the inner quarklets defined above have the same number of vanishing moments. This result is shown in [16, Lemma 2] for cardinal B-spline quarks and therefore it holds true for the inner Schoenberg B-spline quarks.

The next step is to construct boundary quarklets. As already mentioned, the coefficients of the boundary wavelets are not suitable for the boundary quarklets, since in general the vanishing moment properties can not be preserved. A simple counter-example is given by

$$\int_{\mathbb{R}} \sum_{l=-1}^{2} b_{0,l}^{2,(0,0)} \varphi_{1,3,l} = \frac{1}{8},$$

where the non-trivial coefficients are $(b_{0,l}^{2,(0,0)})_{l=-1}^{2} = \sqrt{2}(\frac{3}{2}, -\frac{9}{8}, \frac{1}{4}, \frac{1}{8})$.

Therefore, Instead of keeping the coefficients, our approach is to modify the coefficients in that way that the $\tilde{m}$ equations

$$\int_{\mathbb{R}} x^q \psi_{p,j,k}^{\vec{\sigma}}(x)\ \mathrm{d}x = \int_{\mathbb{R}} x^q \sum_{l \in \Delta_{j+1,\vec{\sigma}}} b_{k,l}^{p,j,\vec{\sigma}} \varphi_{p,j+1,l}\ \mathrm{d}x = 0, \quad q = 0, \dots, \tilde{m}-1 \tag{2.23}$$

are fulfilled not only for $p = 0$ but for all $p \in \mathbb{N}_0$. We restrict our discussion to left boundary quarklets, i.e., $k = 0, \dots, m-2$, and assume that they are only composed of left boundary and inner quarks. To get at least one non-trivial solution of (2.23) we further assume that every boundary quarklet consists of $\tilde{m}+1$ quarks. Furthermore the k-th quarklets representation should begin at the leftmost but k quark with respect to boundary conditions. This leads to the $\tilde{m} \times (\tilde{m}+1)$ linear system of equations

$$\sum_{l=-m+1+\operatorname{sgn}\sigma_l+k}^{-m+1+\operatorname{sgn}\sigma_l+k+\tilde{m}} b_{k,l}^{p,j,\vec{\sigma}} \int_{\mathbb{R}} x^q \varphi_{p,j+1,l}(x)\ \mathrm{d}x = 0, \quad q = 0, \dots, \tilde{m}-1. \tag{2.24}$$

Numerical tests indicate that the associated coefficient matrix has full rank so that the solution is unique except for scaling. Hence, we are able to construct quarklets at the boundary with vanishing moments. If $0 \neq \boldsymbol{b}_k^{p,j,\vec{\sigma}} \in \mathbb{R}^{\tilde{m}+1}$ solves (2.24), we define the k-th left boundary quarklet by

$$\psi^{\vec{\sigma}}_{p,j,k} := \sum_{l=-m+1+\operatorname{sgn}\sigma_l+k}^{-m+1+\operatorname{sgn}\sigma_l+k+\tilde{m}} b^{p,j,\vec{\sigma}}_{k,l}\varphi_{p,j+1,l}, \quad k=0,\dots,m-2. \tag{2.25}$$

The vanishing moment property of the quarklets immediately leads to the following cancellation property of the quarklets.

Lemma 2.5. *Let $p,j\in\mathbb{N}_0$, $j\geq j_0$, $k\in\nabla_{j,\vec{\sigma}}$ and $\psi^{\vec{\sigma}}_{p,j,k}$ a quarklet with $\tilde{m}$ vanishing moments. There exists a constant $C(m,\psi)>0$, such that for every $r\in\mathbb{N}_0, r\leq\tilde{m}-1$ and $f\in W^r_\infty(\mathbb{R})$:*

$$|\langle f,\psi^{\vec{\sigma}}_{p,j,k}\rangle_{L_2(\mathbb{R})}| \leq C(p+1)^{-m}2^{-j(r+1/2)}|f|_{W^r_\infty(supp\ \psi^{\vec{\sigma}}_{p,j,k})}. \tag{2.26}$$

Proof. The proof can be performed by following the lines of [16, Lemma 3]. From the vanishing moments of the quarklets, Hölder's inequality and a Whitney type estimate it follows:

$$|\langle f,\psi^{\vec{\sigma}}_{p,j,k}\rangle_{L_2(\mathbb{R})}| \leq C_1|\text{supp}\ \psi_{p,j,k}|^r|f|_{W^r_\infty(\text{supp}\ \psi^{\vec{\sigma}}_{p,j,k})}||\psi^{\vec{\sigma}}_{p,j,k}||_{L_1(\text{supp}\ \psi^{\vec{\sigma}}_{p,j,k})}, \tag{2.27}$$

where $C_1>0$ only depends on r. To further estimate the L_1-norm expression in (2.27) we use the symmetry of the boundary quarks, (2.21) and the relation

$$\varphi_{p,j,k} = 2^{j/2}\varphi_{p,0,k}(2^j\cdot), \quad k=-m+1,\dots,2^j-m.$$

Combining this relation and the norm estimate (2.18) we obtain

$$||\psi^{\vec{\sigma}}_{p,j,k}||_{L_1(\text{supp}\ \psi^{\vec{\sigma}}_{p,j,k})} \leq C_2 2^{-\frac{j+1}{2}}(p+1)^{-m}\sum_{l\in\Delta_{j+1,\vec{\sigma}}}|b^{p,j,\vec{\sigma}}_{k,l}|,$$

where $C_2>0$ only depends on m. The claim finally follows by estimating the asymptotic behaviour of
$|\text{supp}\ \psi^{\vec{\sigma}}_{p,j,k}|$ by 2^{-j}. □

The following proposition transfers the estimates for the Gramian matrices from [16, Proposition 2] to the boundary adapted case. This is the last missing ingredient to show the frame property of the quarklet systems in $L_2(I)$ and $H^s(I)$.

Proposition 2.6. *For fixed $p\in\mathbb{N}_0$, the operators induced by the Gramian matrices, which are given by*

$$G_p := \left(\langle\varphi_{p,j_0,k},\varphi_{p,j_0,k'}\rangle_{L_2(\mathbb{R})}\right)_{k,k'\in\nabla_{j_0-1,\vec{\sigma}}}, \tag{2.28}$$

$$H_p := \left(\langle\psi^{\vec{\sigma}}_{p,j,k},\psi^{\vec{\sigma}}_{p,j',k'}\rangle_{L_2(\mathbb{R})}\right)_{(j,k):j\geq j_0,k\in\nabla_{j,\vec{\sigma}},(j',k'):j'\geq j_0,k'\in\nabla_{j',\vec{\sigma}}} \tag{2.29}$$

are bounded operators on $\ell_2(\{(j_0-1,k) : k \in \nabla_{j_0-1,\vec{\sigma}}\})$ *and* $\ell_2(\{(j,k) : j \geq j_0, k \in \nabla_{j,\vec{\sigma}}\}))$*, respectively, i.e., there exist constants* $C' = C'(m,\varphi) > 0$*,* $C'' = C''(m,\psi) > 0$*, such that*

$$||G_p||_{\mathscr{L}(\ell_2(\{(j_0-1,k):k\in\nabla_{j_0-1,\vec{\sigma}}\}))} \leq C'(p+1)^{-(2m-1)}, \tag{2.30}$$

$$||H_p||_{\mathscr{L}(\ell_2(\{(j,k):j\geq j_0,k\in\nabla_{j,\vec{\sigma}}\}))} \leq C''(p+1)^{-1}. \tag{2.31}$$

Proof. The proof is based upon the cancellation property (2.26) and can be performed by following the lines of [16, Proposition 2]. □

2.5. **Frames for $L_2(I)$ and $H^s(I)$.** After introducing the construction of quarks and quarklets on the interval and proving some crucial estimates in the Subsections 2.3 and 2.4 we are finally able to transfer the frame properties of the shift-invariant quarklets, cf. Subsection 2.1, to the case of boundary adapted quarklets. We define the index set for the whole quarklet system by

$$\nabla_{\vec{\sigma}} := \{(p,j,k) : p,j \in \mathbb{N}_0, j \geq j_0-1, k \in \nabla_{j,\vec{\sigma}}\}, \tag{2.32}$$

which contains the Primbs basis index set

$$\nabla^R_{\vec{\sigma}} := \{(0,j,k) : j \in \mathbb{N}_0, j \geq j_0-1, k \in \nabla_{j,\vec{\sigma}}\}. \tag{2.33}$$

With these index sets at hand we can formulate the following theorem, which states the frame property in $L_2(I)$.

Theorem 2.7. *The weighted quarklet system*

$$\Psi_{\vec{\sigma}} := \{(p+1)^{-\delta/2}\psi^{\vec{\sigma}}_\lambda : \lambda \in \nabla_{\vec{\sigma}}\}, \quad \delta > 1, \tag{2.34}$$

is a frame for $L_2(I)$*.*

Proof. The used weights $\omega_p := (p+1)^{-\delta/2}$ fulfill $\omega_0 = 1$, so the quarklet system contains an underlying Riesz basis which implies the lower frame estimate, cf. [16, Theorem 3]. The convergence of the sum $\sum_p \omega_p(p+1)^{-1/2} < \infty$ implies the upper frame estimate, cf. [16, Theorem 3]. □

Choosing suitable weights we even derive frames for Sobolev spaces $H^s_{\vec{\sigma}}(I)$, $0 < s < m - \frac{1}{2}$. First we study finite subsets of the full quarklet system $\Psi_{\vec{\sigma}}$ and the subspaces they span:

$$\Psi_{p,j} := \{\psi_{p,i,k} : i = j_0-1,...,j-1, k \in \nabla_{i,\vec{\sigma}}\}, \tag{2.35}$$

$$U_{p,j} := \overline{\mathrm{span}\{\bigcup_{q=0}^{p} \Psi_{q,j}\}}. \tag{2.36}$$

From the two-scale-relation (2.14) it follows $U_{p,j} \subset V_{p,j}$. This subset-relation is necessary to transfer the technical results of [16, Proposition 3] to the interval case.

It follows immediately that the Bernstein inequality (2.20) holds true for all $f \in U_{p,j}$. In addition, a *Bessel* property, cf. (A.8), can be shown:

Lemma 2.8. *Let $p, j \in \mathbb{N}_0$ be fixed. Then, for all $f \in U_{p,j}$ the following holds true:*

$$C^{-1}(p+1)^{-\delta}||f||_{L_2(I)} \leq \inf_{\boldsymbol{c}: \sum_{q=0}^{p} \sum_{i=j_0-1}^{j-1} \sum_{k \in \nabla_{i,\vec{\sigma}}} c_{q,i,k}\psi_{q,i,k}=f} \sum_{q=0}^{p} \sum_{i=j_0-1}^{j-1} \sum_{k \in \nabla_i} |c_{q,i,k}|^2. \tag{2.37}$$

Hence, the system $\bigcup_{q=0}^{p} \Psi_{q,j}$ forms a Bessel system, cf. (A.8)*, in $U_{p,j}$ with Bessel bound $B = C(m, \psi)(p+1)^{\delta} > 0$, $\delta > 1$.*

Proof. For a proof we refer to [16, Proposition 3]. □

The following theorem is the main result of this section. It states that the construction ideas of frames for scales of Sobolev spaces in the shift-invariant case, cf. Theorem 2.1, can be carried over to the boundary adapted quarklets. These frames serve as a starting point for the construction of multivariate tensor frames on cubes and more general domains, as it will be outlined in Section 3.

Theorem 2.9. *For $0 \leq s < m - \frac{1}{2}$ the weighted quarklet system*

$$\Psi^s_{\vec{\sigma}} := \{(p+1)^{-2s-\delta} 2^{-sj} \psi^{\vec{\sigma}}_{\lambda} : \lambda \in \nabla_{\vec{\sigma}}\}, \quad \delta > 1, \tag{2.38}$$

is a frame for $H^s_{\vec{\sigma}}(I)$.

Proof. The proof can be carried out by showing the Bessel property of quarklet systems with increasing cardinality of the index sets, and can again be performed by following the lines of [16, Theorem 4]. □

3. Quarklets on domains

The course of the section is as follows: in Subsection 3.1 we introduce the domains of interest as the union of parametric images of the unit cube and recall some ideas of [7] concerning extension operators and isomorphisms between Sobolev spaces on different domains. In Subsection 3.2 we describe in a general setting how a combination of frames on cubes, Bessel systems which include the image of an extension operator and simple extensions lead to frames for Sobolev spaces on our target domain $\Omega \subset \mathbb{R}^d$. In Subsection 3.3 we show how the univariate frames of Section 2 can be used to obtain frames on cubes. Finally, in Subsection 3.4 we show that the general machinery outlined in Subsection 3.2 can be applied to our setting and present explicit constructions.

3.1. **Preliminaries.** In this subsection we collect the basic tools which are needed to generalize Riesz bases on cubes to Riesz bases on general domains. Further information can be found in [7]. This approach can be used as a starting point of the new frame construction on general domains. The final construction can be found in Subsection 3.4.

Let us first describe the types of domains we will be concerned with in the sequel. Let $\square := I^d$. Let $\{\square_0, \ldots, \square_N\}$ with $\square_j := \tau_j + \square$, $\tau_j \in \mathbb{Z}^d$, $j = 0, \ldots, N$ be a fixed finite set of hypercubes. We assume $\cup_{j=0}^N \square_j \subset \Omega \subset (\cup_{j=0}^N \overline{\square}_j)^{\mathrm{int}}$ and such that $\partial\Omega$ is the union of (closed) facets of the $\square_j$'s. Later on we will present a construction of frames for Sobolev spaces on Ω from frames for corresponding Sobolev spaces on the subdomains $\square_k$ by using extension operators. These extension operators form a crucial ingredient in the final construction, see again Subsection 3.4. The following conditions $(\mathcal{D}_1)$–$(\mathcal{D}_5)$ are taken from [7] and ensure the existence of suitable extension operators.

We set $\Omega_i^{(0)} := \square_i$, $i = 0, \ldots, N$ and create a sequence $(\{\Omega_i^{(q)} : q \leq i \leq N\})_{0 \leq q \leq N}$ of sets of polytopes, where each next entry in this sequence is created by joining two polytopes from the previous entry whose joint interface is part of a hyperplane. More precisely, we assume that for any $1 \leq q \leq N$, there exists a $q \leq \bar{i} = \bar{i}^{(q)} \leq N$ and $q - 1 \leq i_1 = i_1^{(q)} \neq i_2 = i_2^{(q)} \leq N$ such that

$(\mathcal{D}_1)$ $\Omega_{\bar{i}}^{(q)} = \left(\overline{\Omega_{i_1}^{(q-1)} \cup \Omega_{i_2}^{(q-1)}} \setminus \partial\Omega\right)^{\mathrm{int}}$ is connected, and the interface $J := \Omega_{\bar{i}}^{(q)} \setminus (\Omega_{i_1}^{(q-1)} \cup \Omega_{i_2}^{(q-1)})$ is part of a hyperplane,

$(\mathcal{D}_2)$ $\{\Omega_i^{(q)} : q \leq i \leq N, i \neq \bar{i}\} = \left\{\Omega_i^{(q-1)} : q - 1 \leq i \leq N, i \neq \{i_1, i_2\}\right\}$,

$(\mathcal{D}_3)$ $\Omega_N^{(N)} = \Omega$.

By construction, the boundary of each $\Omega_i^{(q)}$ is a union of facets of hypercubes $\square_j$. We define $\mathring{H}^s(\Omega_i^{(q)})$ to be the closure in $H^s(\Omega_i^{(q)})$ of the smooth functions that are supported in the interior of $\Omega_i^{(q)}$. In particular, homogeneous boundary conditions are imposed on those facets of $\square_j$ that lie inside $\partial\Omega$. Hence we have $\mathring{H}^s(\Omega_N^{(N)}) = H_0^s(\Omega)$ and for some $\boldsymbol{\sigma}(j) \in (\{0, \lfloor s + 1/2 \rfloor\}^2)^d$,

$$\mathring{H}^s(\Omega_j^{(0)}) = \mathring{H}^s(\square_j) = H^s_{\boldsymbol{\sigma}(j)}(\square_j).$$

The boundary conditions on the hypercubes that determine the spaces $\mathring{H}^s(\Omega_i^{(q)})$, and the order in which polytopes are joined should be chosen such that

$(\mathcal{D}_4)$ on the $\Omega_{i_1}^{(q-1)}$ and $\Omega_{i_2}^{(q-1)}$ sides of J, the boundary conditions are of order 0 and $\lfloor t + \frac{1}{2} \rfloor$, respectively,

and, w.l.o.g. assuming that $J = \{0\} \times \breve{J}$ and $I \times \breve{J} \subset \Omega_{i_1}^{(q-1)}$,

($\mathcal{D}_5$) for any function in $\mathring{H}^s(\Omega_{i_1}^{(q-1)})$ that vanishes near $\{0,1\} \times \breve{J}$, its reflection in $\{0\}\times\mathbb{R}^{n-1}$ (extended with zero, and then restricted to $\Omega_{i_2}^{(q-1)}$) is in $\mathring{H}^s(\Omega_{i_2}^{(q-1)})$.

The condition ($\mathcal{D}_5$) can be formulated by saying that the order of the boundary condition at any subfacet of $\Omega_{i_1}^{(q-1)}$ adjacent to J should not be less than this order at its reflection in J, where in case this reflection is not part of $\partial\Omega_{i_2}^{(q-1)}$, the latter order should be read as the highest possible one $\lfloor s+\frac{1}{2}\rfloor$; and furthermore, that the order of the boundary condition at any subfacet of $\Omega_{i_2}^{(q-1)}$ adjacent to J should not be larger than this order at its reflection in J, where in case this reflection is not part of $\partial\Omega_{i_1}^{(q-1)}$, the latter order should be read as the lowest possible one 0.

Given $1 \le q \le N$, for $l \in \{1,2\}$, let $R_l^{(q)}$ be the *restriction* of functions on $\Omega_{\bar{i}}^{(q)}$ to $\Omega_{i_l}^{(q-1)}$, let $\eta_2^{(q)}$ be the *extension* of functions on $\Omega_{i_2}^{(q-1)}$ to $\Omega_{\bar{i}}^{(q)}$ by zero, and let $E_1^{(q)}$ be any *extension* that is well defined on Sobolev spaces on $\Omega_{i_1}^{(q-1)}$ to Sobolev spaces $\Omega_{\bar{i}}^{(q)}$.

Roughly speaking, in every step of our construction we will glue together two adjacent domains. One ingredient in such a step will be a bijective operator between Sobolev spaces on those domains. In the following proposition, which is taken from [7, Proposition 2.1], we consider a more general framework and give conditions under which a class of mappings between a Banach space and the Cartesian product of two other Banach spaces consists of isomorphisms. In Proposition 3.2, cf. [7, Proposition 4.2], we apply these statements to our special case.

Proposition 3.1. *For normed linear spaces V and V_i ($i=1,2$), let $E_1 \in B(V_1,V)$, $\eta_2 \in B(V_2,V)$, $R_1 \in B(V,V_1)$, and $R_2 \in B(\operatorname{Ran}(\eta_2),V_2)$ be such that*

$$R_1E_1 = \mathrm{Id}, \quad R_2\eta_2 = \mathrm{Id}, \quad R_1\eta_2 = 0, \quad \operatorname{Ran}(\mathrm{Id}-E_1R_1) \subset \operatorname{Ran}(\eta_2).$$

Then

$$E = [E_1 \quad \eta_2] \in B(V_1 \times V_2, V) \text{ is boundedly invertible,}$$

with inverse

$$E^{-1} = \begin{bmatrix} R_1 \\ R_2(\mathrm{Id}-E_1R_1) \end{bmatrix}.$$

Proposition 3.2. *Assume that $E_1^{(q)} \in B(\mathring{H}^s(\Omega_{i_1}^{(q-1)}), \mathring{H}^s(\Omega_{\bar{i}}^{(q)}))$, $\eta_2^{(q)} \in B(\mathring{H}^s(\Omega_{i_2}^{(q-1)}), \mathring{H}^s(\Omega_{\bar{i}}^{(q)}))$. Then,*

$$E^{(q)} := [E_1^{(q)} \quad \eta_2^{(q)}] \in B\big(\prod_{l=1}^{2} \mathring{H}^s(\Omega_{i_l}^{(q-1)}), \mathring{H}^s(\Omega_{\bar{i}}^{(q)})\big)$$

is boundedly invertible.

Proof. This is a direct application of Proposition 3.1 with $V_1 = \mathring{H}^s(\Omega_{i_1}^{(q-1)})$, $V_2 = \mathring{H}^s(\Omega_{i_2}^{(q-1)})$, $V = \mathring{H}^s(\Omega_{\bar{i}}^{(q)})$, $E_1 = E_1^{(q)}$, $\eta_2 = \eta_2^{(q)}$ and $R_l = R_l^{(q)}$, for $l \in \{1, 2\}$. □

Sequential execution of extensions as in Proposition 3.2 induces an isomorphism from the Cartesian product of Sobolev spaces on the cubes $\square_j$ onto the Sobolev spaces on the target domain Ω.

Corollary 3.3. *For F being the composition for $q = 1, \dots, N$ of the mappings $E^{(q)}$ from Proposition 3.2, trivially extended with identity operators in coordinates $i \in \{q-1, \dots, N\} \setminus \{i_1^{(q)}, i_2^{(q)}\}$, it holds that*

$$F \in B\Big(\prod_{j=0}^{N} \mathring{H}^s(\square_j), H_0^s(\Omega)\Big). \tag{3.1}$$

is boundedly invertible.

Remark 3.4. *If we apply F to Riesz bases on cubes $\square_j$ we end up with a Riesz basis on Ω. While this is also true for the case of frames, the way for the construction of frames in this paper will be a bit different, mainly to preserve the vanishing moments of the frames on cubes. Nevertheless, the operators $E^{(q)}$ as defined in Proposition 3.2 will play an important role in the construction process.*

The next proposition provides the link between the extension approach and tensor products. It states that under the conditions $(\mathcal{D}_1)$–$(\mathcal{D}_5)$, the extensions $E_1^{(q)}$ can be constructed (essentially) as tensor products of *univariate extensions* with identity operators in the other Cartesian directions.

Proposition 3.5. *In the setting of $(\mathcal{D}_1)$, w.l.o.g. let $J = \{0\} \times \breve{J}$ and $I \times \breve{J} \subset \Omega_{i_1}^{(q-1)}$. Let G_1 be an* extension *operator of functions on I to functions on $(-1, 1)$ such that*

$$G_1 \in B(L_2(I), L_2(-1, 1)), \quad G_1 \in B(H^s(I), H^s_{(\lfloor s+\frac{1}{2} \rfloor, 0)}(-1, 1)).$$

Let $T \in B(\mathring{H}^s(\Omega_{i_1}^{(q-1)}), \mathring{H}^s(\Omega_{i_2}^{(q-1)}))$ be defined as the composition of the restriction to $I \times \breve{J}$, followed by an application of

$$G_1 \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id},$$

an extension by 0 *to $\Omega_{i_2}^{(q-1)} \setminus (-1, 0) \times \breve{J}$ and a restriction to $\Omega_{i_2}^{(q-1)}$. Then, we define $E^{(q)} \in B(\prod_{l=1}^{2} \mathring{H}^s(\Omega_{i_l}^{(q-1)}), \mathring{H}^s(\Omega_{\bar{i}}^{(q)}))$ as the operator which is the identity operator if restricted to $\mathring{H}^s(\Omega_{i_1}^{(q-1)})$ and T if restricted to $\mathring{H}^s(\Omega_{i_2}^{(q-1)})$. By proceeding this way, $E^{(q)}$ is well-defined and boundedly invertible by Proposition 3.2.*

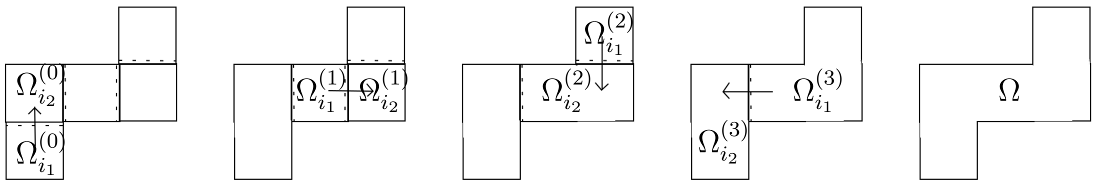

FIGURE 1. Example of a domain decomposition such that $\mathcal{D}_1 - \mathcal{D}_5$ are fulfilled. The arrows indicate the direction of the non-trivial extension. Dotted lines and solid lines indicate free and zero boundary conditions, respectively.

3.2. **Construction of frames by extension.** Based on the setting outlined in Subsection 3.1, we will now describe a general procedure to construct frames for the Sobolev space $H^s(\Omega)$, provided that suitable frames and Riesz-bases, respectively, on the cubes $\square_j$ are given. Suitable frames and bases on cubes will be constructed in Subsection 3.3. Finally, a combination of the results of Subsection 3.2 and 3.3 will provide us with the desired quarklet frame, cf. Subsection 3.4.

For $j = 0, \ldots, N$, let $\mathbf{\Psi}_j$ be a frame for $L_2(\square_j)$, that renormalized in $H^s(\square_j)$, is a frame for $\overset{\circ}{H}{}^s(\square_j)$. Furthermore assume that there exists a Riesz basis $\mathbf{\Sigma}_j \subset \mathbf{\Psi}_j$ for $L_2(\square_j)$, that renormalized in $H^s(\square_j)$, is a Riesz basis for $\overset{\circ}{H}{}^s(\square_j)$. Renormalized versions of all sets will be indicated with an upper s. For $q = 0, \ldots, N$, $i = q, \ldots, N$ and $s \geq 0$ we define recursively

$$\mathbf{\Sigma}_i^{s,(q)} := \begin{cases} \mathbf{\Sigma}_i^s, & q = 0, \\ \mathbf{\Sigma}_{\hat{i}}^{s,(q-1)}, & 1 \leq q \leq N,\ i \neq \bar{i},\ \Omega_i^{(q)} = \Omega_{\hat{i}}^{(q-1)}, \\ E_1^{(q)}(\mathbf{\Sigma}_{i_1}^{s,(q-1)}) \cup \eta_2^{(q)}(\mathbf{\Sigma}_{i_2}^{s,(q-1)}), & 1 \leq q \leq N,\ i = \bar{i}. \end{cases} \tag{3.2}$$

We observe that $\mathbf{\Sigma}_N^{s,(N)}$ is exactly $F(\mathbf{\Sigma}_0^s, \ldots, \mathbf{\Sigma}_N^s)$, with F defined as in Corollary 3.3. Thus, it is a Riesz basis for $H_0^s(\Omega)$, cf. Proposition A.4 (iii). For the frame construction, we have to assume the existence of an additional family $\mathbf{\Xi}^{s,(q)}$ which forms a Bessel system in $\overset{\circ}{H}{}^s(\Omega_i^{(q)})$, cf. (A.8), and satisfies $E_i^{(q)}(\mathbf{\Sigma}_i^{s,(q-1)}) \subset \mathbf{\Xi}^{s,(q)}$. Then for $q = 0, \ldots, N$, $i = q, \ldots, N$ and $s \geq 0$ we set

$$\mathbf{\Psi}_i^{s,(q)} := \begin{cases} \mathbf{\Psi}_i^s, & q = 0, \\ \mathbf{\Psi}_i^{s,(q-1)}, & 1 \leq q \leq N,\ i \neq \bar{i},\ \Omega_i^{(q)} = \Omega_{\hat{i}}^{(q-1)}, \\ \mathbf{\Xi}^{s,(q)} \cup \eta_2^{(q)}(\mathbf{\Psi}_{i_2}^{s,(q-1)}), & 1 \leq q \leq N, i = \bar{i}. \end{cases} \tag{3.3}$$

The next proposition implies that, by proceeding this way, we indeed obtain suitable frames for $H_0^s(\Omega)$. Further information concerning the additional Bessel system as well as construction details can be found in Subsection 3.4, Remark 3.15.

Proposition 3.6. *For $q = 0, \dots, N$, $i = q, \dots, N$ and $s \geq 0$, let $\mathbf{\Psi}_i^{s,(q)}$ be defined as in (3.3). Then, $\mathbf{\Psi}^s := \mathbf{\Psi}_N^{s,(N)}$, is a frame for $H_0^s(\Omega)$.*

Proof. Let $1 \leq q \leq N$. Since $\mathbf{\Psi}_{i_2}^{s,(q-1)}$ is a Bessel system in $\mathring{H}^s(\Omega_{i_2}^{(q-1)})$ and $\eta_2^{(q)} \in B(\mathring{H}^s(\Omega_{i_2}^{(q-1)}), \mathring{H}^s(\Omega_{\bar{i}}^{(q)}))$, we can conclude that $\eta_2^{(q)}(\mathbf{\Psi}_{i_2}^{s,(q-1)})$ is a Bessel system in $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$, cf. Proposition A.4 (i). Hence, $\mathbf{\Psi}_{\bar{i}}^{s,(q)} = \mathbf{\Xi}^{s,(q)} \cup \eta_2^{(q)}(\mathbf{\Psi}_{i_2}^{s,(q-1)})$ is a union of two Bessel systems and therefore a Bessel system in $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$, cf. Proposition A.3 (i).

Since $E_1^{(q)}(\mathbf{\Sigma}_{i_1}^{s,(q-1)}) \subset \mathbf{\Xi}^{s,(q)}$ and $\mathbf{\Sigma}_{i_2}^{s,(q-1)} \subset \mathbf{\Psi}_{i_2}^{s,(q-1)}$, we conclude that $\mathbf{\Sigma}_{\bar{i}}^{s,(q)} \subset \mathbf{\Psi}_{\bar{i}}^{s,(q)}$. For $0 \leq i \leq N$, $\mathbf{\Sigma}_i^{s,(0)}$ is a Riesz basis for $\mathring{H}^s(\Omega_i^{(0)})$. Furthermore $E^{(q)} = [E_1^{(q)} \quad \eta_2^{(q)}] \in B\big(\prod_{l=1}^2 \mathring{H}^s(\Omega_{i_l}^{(q-1)}), \mathring{H}^s(\Omega_{\bar{i}}^{(q)})\big)$ as defined in Proposition 3.2 is boundedly invertible. Thus, we can conclude inductively that $\mathbf{\Sigma}_{\bar{i}}^{s,(q)} = E^{(q)}(\mathbf{\Sigma}_{i_1}^{s,(q-1)}, \mathbf{\Sigma}_{i_2}^{s,(q-1)})$ is a Riesz basis for $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$, cf. Proposition A.4 (iii) . Hence, $\mathbf{\Psi}_{\bar{i}}^{s,(q)}$ as a Bessel system which contains a Riesz basis is a frame for $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$, cf. Proposition A.3 (iii). Especially $\mathbf{\Psi}^s = \mathbf{\Psi}_N^{s,(N)}$ is a frame for $H_0^s(\Omega) = \mathring{H}^s(\Omega_N^{(N)})$. $\square$

3.3. **Frames on cubes.** To carry out our program, we have to construct Riesz bases and frames on the cubes $\square_i$. It is sufficient to consider the case $\square_i = \square = I^d$, since all other cubes can be simply handled by translation. For reasons already outlined in the introduction, it is our goal to construct the desired representation system by means of tensor products of the univariate, boundary adapted quarklet frames introduced in Section 2. However, then an additional difficulty comes into play, namely that the spaces $H^s_{\boldsymbol{\sigma}}(\square), \boldsymbol{\sigma} = (\vec{\sigma}_1, \dots, \vec{\sigma}_d), \vec{\sigma}_i \in \{0, \lfloor s + \frac{1}{2} \rfloor\}$, are usually *not* of tensor product type. Fortunately, the following relations hold for $s \in [0, \infty) \setminus (\mathbb{N}_0 + \frac{1}{2})$, cf. [7]:

$$H^s_{\boldsymbol{\sigma}}(\square) := \bigcap_{i=1}^{d} H_i^s(\square), \tag{3.4}$$

where

$$H_i^s(\square) := L_2(I) \otimes \cdots \otimes L_2(I) \otimes H^s_{\vec{\sigma}_i}(I) \otimes L_2(I) \otimes \cdots \otimes L_2(I) \subset L_2(\square), \tag{3.5}$$

with $H^s_{\vec{\sigma}_i}(I)$ at the i-th spot. For the definitions of inner products and norms on tensor product spaces we refer to [24, Section 2].

The intersection of Hilbert spaces $H^{(i)}$ which all have to be contained in a Hilbert space H is defined as

$$\bigcap_{i=1}^{d} H^{(i)} := \{f : \|f\|_{\bigcap_{i=1}^{d} H^{(i)}} < \infty\}, \quad \|f\|_{\bigcap_{i=1}^{d} H^{(i)}} := \left(\sum_{i=1}^{d} ||f||_{H^{(i)}}^2\right)^{1/2}.$$

Therefore, we have to construct tensor quarklet frames for the spaces (3.5) and to check to which extent the frame property carries over to the intersection (3.4).

The following two lemmas give rise to the construction of frames on tensor-product spaces and on intersections of Hilbert spaces, respectively. They generalize Lemma 3.1.5 and Lemma 3.1.8 of [23] from the case of Riesz bases to the case of frames.

We assume that $\mathcal{F}_{L_2(I)} = \{f_\lambda\}_{\lambda \in \mathcal{J}}$ is a frame for $L_2(I)$ with frame constants $A, B > 0$, such that, for certain scalar weights $w_\lambda > 0$ and an integer $s \geq 0$, the set $\{w_\lambda^{-1} f_\lambda\}_{\lambda \in \mathcal{J}}$ is a frame for $H^s_{\vec{\sigma}}(I)$ with frame constants $A_s, B_s > 0$.

Lemma 3.7. *The system*

$$\mathcal{F}_{H_i^s(\square)} := \left\{w_{\lambda_i}^{-1} f_{\lambda_1} \otimes \cdots \otimes f_{\lambda_d}\right\}_{\boldsymbol{\lambda} \in \mathcal{J}^d}, \quad \boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_d\},$$

is a frame for the tensor product Sobolev space $H_i^s(\square)$ with frame constants $A_s A^{d-1}$ and $B_s B^{d-1}$, i.e.,

$$A_s A^{d-1} \|f\|_{H_i^s(\square)}^2 \leq \sum_{\boldsymbol{\lambda} \in \mathcal{J}^d} \left|\langle f, w_{\lambda_i}^{-1} f_{\lambda_1} \otimes \cdots \otimes f_{\lambda_d} \rangle_{H_i^s(\square)}\right|^2 \leq B_s B^{d-1} \|f\|_{H_i^s(\square)}^2, \tag{3.6}$$

for all $f \in H_i^s(\square)$.

Proof. Without loss of generality, we may assume that $i = 1$. Moreover, it is sufficient to show (3.6) on a dense subset of $H_1^s(\square)$, cf. [8, Lemma 5.1.9] e.g., for all finite sums of tensor product functions like

$$f = \sum_{k=1}^{K} g_k^{(1)} \otimes \cdots \otimes g_k^{(d)}, \quad g_k^{(j)} \in \begin{cases} H^s_{\vec{\sigma}_1}(I) & , \ j = 1, \\ L_2(I) & , \ 2 \leq j \leq d. \end{cases} \tag{3.7}$$

Assume that f has this form, and let $U = \{u_j\}_{j \in \mathbb{N}}$ and $V = \{v_j\}_{j \in \mathbb{N}}$ be orthonormal bases for $H^s(I)$ and $L_2(I)$, respectively. Then obviously, the system $\{u_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_d}\}_{j_l \in \mathbb{N}, 1 \leq l \leq d}$ is an orthonormal basis for $H_1^s(\square)$. By consequence, an application of the Parseval identity in $H_1^s(\Box)$ and in $H_{\vec{\sigma}_1}^s(I)$ yields

$$
\begin{aligned}
\|f\|_{H_1^s(\Box)}^2 &= \sum_{\substack{j_l\in\mathbb{N}\\ 1\le l\le d}} \left|\langle f, u_{j_1}\otimes v_{j_2}\otimes\cdots\otimes v_{j_d}\rangle_{H_1^s(\Box)}\right|^2 \\
&= \sum_{\substack{j_l\in\mathbb{N}\\ 1\le l\le d}} \Big|\sum_{k=1}^K \langle g_k^{(1)}\otimes\cdots\otimes g_k^{(d)}, u_{j_1}\otimes v_{j_2}\otimes\cdots\cdots\otimes v_{j_d}\rangle_{H_1^s(\Box)}\Big|^2 \\
&= \sum_{\substack{j_l\in\mathbb{N}\\ 1\le l\le d}} \Big|\sum_{k=1}^K \langle g_k^{(1)}, u_{j_1}\rangle_{H_{\vec{\sigma}_1}^s(I)} \prod_{\nu=2}^d \langle g_k^{(\nu)}, v_{j_\nu}\rangle_{L_2(I)}\Big|^2 \\
&= \sum_{\substack{j_l\in\mathbb{N}\\ 2\le l\le d}} \sum_{j_1\in\mathbb{N}} \Big|\Big\langle \sum_{k=1}^K \prod_{\nu=2}^d \langle g_k^{(\nu)}, v_{j_\nu}\rangle_{L_2(I)} g_k^{(1)}, u_{j_1}\Big\rangle_{H_{\vec{\sigma}_1}^s(I)}\Big|^2 \\
&= \sum_{\substack{j_l\in\mathbb{N}\\ 2\le l\le d}} \Big\|\sum_{k=1}^K \prod_{\nu=2}^d \langle g_k^{(\nu)}, v_{j_\nu}\rangle_{L_2(I)} g_k^{(1)}\Big\|_{H_{\vec{\sigma}_1}^s(I)}^2 .
\end{aligned}
$$

The $H_{\vec{\sigma}_1}^s(I)$-norms can be estimated from above and from below by using the frame property of $\{w_{\lambda_1}^{-1} f_{\lambda_1}\}_{\lambda_1\in\mathcal{J}}$ in $H_{\vec{\sigma}_1}^s(I)$, resulting in the auxiliary estimate
(3.8)

$$
A_s\|f\|_{H_1^s(\Box)}^2 \le \sum_{\lambda_1\in\mathcal{J}} w_{\lambda_1}^{-2} \sum_{\substack{j_l\in\mathbb{N}\\ 2\le l\le d}} \Big|\sum_{k=1}^K \langle g_k^{(1)}, f_{\lambda_1}\rangle_{H_{\vec{\sigma}_1}^s(I)} \prod_{\nu=2}^d \langle g_k^{(\nu)}, v_{j_\nu}\rangle_{L_2(I)}\Big|^2 \le B_s\|f\|_{H_1^s(\Box)}^2 .
$$

It remains to bound the middle sum in (3.8) from above and from below. For fixed $\lambda_1,\dots,\lambda_d\in\mathcal{J}$, we can transform

$$
\begin{aligned}
&\sum_{k=1}^K \langle g_k^{(1)}, f_{\lambda_1}\rangle_{H_{\vec{\sigma}_1}^s(I)} \prod_{\nu=2}^d \langle g_k^{(\nu)}, v_{j_\nu}\rangle_{L_2(I)} \\
&= \Big\langle \sum_{k=1}^K \langle g_k^{(1)}, f_{\lambda_1}\rangle_{H_{\vec{\sigma}_1}^s(I)} \prod_{\nu=3}^d \langle g_k^{(\nu)}, v_{j_\nu}\rangle_{L_2(I)} g_k^{(2)}, v_{j_2}\Big\rangle_{L_2(I)} .
\end{aligned}
$$

By using the Parseval identity in $L_2(I)$, we deduce

$$\begin{aligned}\sum_{j_2\in\mathcal{J}}\Big|\sum_{k=1}^{K}\langle g_k^{(1)},f_{\lambda_1}\rangle_{H^s_{\vec{\sigma}_1}(I)}\prod_{\nu=2}^{d}\langle g_k^{(\nu)},v_{j_\nu}\rangle_{L_2(I)}\Big|^2\\ =\Big\|\sum_{k=1}^{K}\langle g_k^{(1)},f_{\lambda_1}\rangle_{H^s_{\vec{\sigma}_1}(I)}\prod_{\nu=3}^{d}\langle g_k^{(\nu)},v_{j_\nu}\rangle_{L_2(I)}g_k^{(2)}\Big\|^2_{L_2(I)},\end{aligned}$$

so that the frame property of $\mathcal{F}$ in $L_2(I)$ yields

$$\begin{aligned}&A\sum_{j_2\in\mathcal{J}}\Big|\sum_{k=1}^{K}\langle g_k^{(1)},f_{\lambda_1}\rangle_{H^s_{\vec{\sigma}_1}(I)}\prod_{\nu=2}^{d}\langle g_k^{(\nu)},v_{j_\nu}\rangle_{L_2(I)}\Big|^2\\ &\le\sum_{\lambda_2\in\mathcal{J}}\Big|\Big\langle\sum_{k=1}^{K}\langle g_k^{(1)},f_{\lambda_1}\rangle_{H^s_{\vec{\sigma}_1}(I)}\prod_{\nu=3}^{d}\langle g_k^{(\nu)},v_{j_\nu}\rangle_{L_2(I)}g_k^{(2)},f_{\lambda_2}\Big\rangle_{L_2(I)}\Big|^2\\ &\le B\sum_{j_2\in\mathcal{J}}\Big|\sum_{k=1}^{K}\langle g_k^{(1)},f_{\lambda_1}\rangle_{H^s_{\vec{\sigma}_1}(I)}\prod_{\nu=2}^{d}\langle g_k^{(\nu)},v_{j_\nu}\rangle_{L_2(I)}\Big|^2.\end{aligned}$$

In view of (3.8), this implies

$$\begin{aligned}A_sA\|f\|^2_{H^s_1(\square)}&\le\sum_{\lambda_1,\lambda_2\in\mathcal{J}}w_{\lambda_1}^{-2}\sum_{\substack{j_l\in\mathbb{N}\\3\le l\le d}}\Big|\sum_{k=1}^{K}\langle g_k^{(1)},f_{\lambda_1}\rangle_{H^s_{\vec{\sigma}_1}(I)}\langle g_k^{(2)},f_{\lambda_2}\rangle_{L_2(I)}\prod_{\nu=3}^{d}\langle g_k^{(\nu)},v_{j_\nu}\rangle_{L_2(I)}\Big|^2\\ &\le B_sB\|f\|^2_{H^s_1(\square)}.\end{aligned}$$

The claim (3.6) follows by repeating the aforementioned calculations and estimates in each of the remaining modes $3\le\nu\le d$. $\square$

Remark 3.8. *By following the lines of the proof of Lemma 3.7, one can also show that the system $\mathcal{F}_{L_2(\square)}=\big\{f_{\lambda_1}\otimes\cdots\otimes f_{\lambda_d}\big\}_{\boldsymbol{\lambda}\in\mathcal{J}^d}$ is a frame for $L_2(\square)$ with frame bounds A^d, B^d.*

An application of Lemma 3.7 provides us with tensor frames for all the spaces $H^s_i(\square)$ defined in (3.5). It remains to check under which conditions these frames also give rise to suitable systems in the intersection space $H^s_{\boldsymbol{\sigma}}(\square)$ in (3.4). Quite surprisingly, to perform our proof, it is not sufficient that the individual system possesses the frame property. In addition, each of the frames must contain a Riesz basis. Although this assumption is in a certain sense restrictive, it is always satisfied since our quarkonial frames by construction contain a wavelet Riesz basis.

Lemma 3.9. *Let $\mathcal{F}_{\mathcal{H}}=\{\boldsymbol{f}_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in\mathcal{I}}$ be a frame for a Hilbert space $\mathcal{H}$ such that for $i\in\{1,\ldots,d\}$ and some non-zero scalars $w^{(i)}_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda}\in\mathcal{I}$, the sets $\mathcal{F}_{\mathcal{H}^{(i)}}:=\{(w^{(i)}_{\boldsymbol{\lambda}})^{-1}\boldsymbol{f}_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in\mathcal{I}}$*

form frames for Hilbert spaces $\mathcal{H}^{(i)} \subset \mathcal{H}$*. Furthermore we assume that there exists a Riesz basis* $\mathcal{R}_{\mathcal{H}} := \{\boldsymbol{f}_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in\mathcal{I}_{\mathcal{R}}} \subset \mathcal{F}_{\mathcal{H}}$ *for* $\mathcal{H}$ *such that the sets* $\mathcal{R}_{\mathcal{H}^{(i)}} := \{(w_{\boldsymbol{\lambda}}^{(i)})^{-1}\boldsymbol{f}_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in\mathcal{I}_{\mathcal{R}}}$ *form Riesz bases for* $\mathcal{H}^{(i)} \subset \mathcal{H}$*. Then the collection*

$$\left\{ \left(\sum_{i=1}^{d} (w_{\boldsymbol{\lambda}}^{(i)})^2 \right)^{-1/2} \boldsymbol{f}_{\boldsymbol{\lambda}} \right\}_{\boldsymbol{\lambda}\in\mathcal{I}}$$

is a frame for $\bigcap_{i=1}^{d} \mathcal{H}^{(i)} \subset \mathcal{H}$*.*

Proof. It is sufficient to prove the lemma for the case $d = 2$. Then, the general result follows by induction. Let $\boldsymbol{f} \in \mathcal{H}^{(1)} \cap \mathcal{H}^{(2)}$. Since $\mathcal{R}_{\mathcal{H}}$ is a Riesz basis for $\mathcal{H}$ we have a unique representation $\boldsymbol{f} = \sum_{\boldsymbol{\lambda}\in\mathcal{I}_{\mathcal{R}}} \hat{c}_{\boldsymbol{\lambda}} \boldsymbol{f}_{\boldsymbol{\lambda}}$. Let B_i be the optimal upper frame bounds and $B_{\max} = \max\{B_1, B_2\}$. Then the frame property of $\mathcal{F}_{\mathcal{H}^{(i)}}$ in $\mathcal{H}^{(i)}$, $i \in \{1, 2\}$ implies

$$B_{\max}^{-1} \|\boldsymbol{f}\|_{\mathcal{H}^{(i)}}^2 \leq B_i^{-1} \|\boldsymbol{f}\|_{\mathcal{H}^{(i)}}^2 \leq \inf_{\boldsymbol{c}^{(i)}\in\ell_2(\mathcal{I}):(\boldsymbol{c}^{(i)})^T\mathcal{F}_{\mathcal{H}}=\boldsymbol{f}} \sum_{\boldsymbol{\lambda}\in\mathcal{I}} (w_{\boldsymbol{\lambda}}^{(i)})^2 (c_{\boldsymbol{\lambda}}^{(i)})^2 \tag{3.9}$$

The definition of $\|\cdot\|_{\mathcal{H}^{(1)}\cap\mathcal{H}^{(2)}}$ and (3.9) lead to

$$\begin{aligned} B_{\max}^{-1} \|f\|_{\mathcal{H}^{(1)}\cap\mathcal{H}^{(2)}}^2 &\leq \inf_{(\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)})\in\ell_2(\mathcal{I})^2:(\boldsymbol{c}^{(i)})^T\mathcal{F}_{\mathcal{H}}=\boldsymbol{f}} \sum_{\boldsymbol{\lambda}\in\mathcal{I}} (w_{\boldsymbol{\lambda}}^{(1)})^2 (c_{\boldsymbol{\lambda}}^{(1)})^2 + (w_{\boldsymbol{\lambda}}^{(2)})^2 (c_{\boldsymbol{\lambda}}^{(2)})^2 \\ &\leq \inf_{\boldsymbol{c}\in\ell_2(\mathcal{I}):\boldsymbol{c}^T\mathcal{F}_{\mathcal{H}}=\boldsymbol{f}} \sum_{\boldsymbol{\lambda}\in\mathcal{I}} \left((w_{\boldsymbol{\lambda}}^{(1)})^2 + (w_{\boldsymbol{\lambda}}^{(2)})^2 \right) c_{\boldsymbol{\lambda}}^2, \end{aligned} \tag{3.10}$$

showing the lower frame inequality. Let $A_i^{\mathcal{R}}$, $i \in \{1, 2\}$ be the optimal lower Riesz constants and $A_{\min}^{\mathcal{R}} = \min\{A_1^{\mathcal{R}}, A_2^{\mathcal{R}}\}$. For the upper frame inequality we use the unique representation and the Riesz basis properties of $\mathcal{R}_{\mathcal{H}^{(i)}}$ in $\mathcal{H}^{(i)}$, $i \in \{1, 2\}$ to estimate

$$\begin{aligned} \inf_{\boldsymbol{c}\in\ell_2(\mathcal{I}):\boldsymbol{c}^T\mathcal{F}_{\mathcal{H}}=\boldsymbol{f}} \sum_{\boldsymbol{\lambda}\in\mathcal{I}} \left((w_{\boldsymbol{\lambda}}^{(1)})^2 + (w_{\boldsymbol{\lambda}}^{(2)})^2 \right) c_{\boldsymbol{\lambda}}^2 &\leq \sum_{\boldsymbol{\lambda}\in\mathcal{I}_{\mathcal{R}}} \left((w_{\boldsymbol{\lambda}}^{(1)})^2 + (w_{\boldsymbol{\lambda}}^{(2)})^2 \right) \hat{c}_{\boldsymbol{\lambda}}^2 \\ &= \sum_{\boldsymbol{\lambda}\in\mathcal{I}_{\mathcal{R}}} (w_{\boldsymbol{\lambda}}^{(1)})^2 \hat{c}_{\boldsymbol{\lambda}}^2 + \sum_{\boldsymbol{\lambda}\in\mathcal{I}_{\mathcal{R}}} (w_{\boldsymbol{\lambda}}^{(2)})^2 \hat{c}_{\boldsymbol{\lambda}}^2 \\ &\leq (A_1^{\mathcal{R}})^{-1} \|f\|_{\mathcal{H}^{(1)}}^2 + (A_2^{\mathcal{R}})^{-1} \|f\|_{\mathcal{H}^{(2)}}^2 \\ &\leq (A_{\min}^{\mathcal{R}})^{-1} \|f\|_{\mathcal{H}^{(1)}\cap\mathcal{H}^{(2)}}^2, \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) proves the claim. □

An application of Remark 3.8 and Theorem 2.7 yields the following theorem, which is one of the main results of this paper.

Theorem 3.10. *Let* $\{\Psi^{\vec{\sigma_i}}_{\lambda_i}\}, i = 1, \dots, d$, *be a family of univariate boundary adapted quarklet frames of order* $m \geq 2$, *with* $\tilde{m}$ *vanishing moments,* $\tilde{m} \geq m$, *according to Theorem 2.7. Then the family*

$$\boldsymbol{\Psi}_{\boldsymbol{\sigma}} := \bigotimes_{i=1}^{d} \Psi_{\vec{\sigma_i}} = \left\{ (w^{L_2}_{\boldsymbol{\lambda}})^{-1} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}} := \prod_{i=1}^{d} \nabla_{\vec{\sigma_i}} \right\}, \tag{3.12}$$

$$\boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}} := \bigotimes_{i=1}^{d} \psi^{\vec{\sigma_i}}_{\lambda_i}, \tag{3.13}$$

with the weights

$$w^{L_2}_{\boldsymbol{\lambda}} := \prod_{i=1}^{d} (p_i + 1)^{\delta/2}, \quad \delta > 1, \tag{3.14}$$

is a quarkonial tensor frame for $L_2(\square)$.

By means of Lemma 3.7, Lemma 3.9 and Theorem 2.9 we also obtain quarkonial frames for the Sobolev space $H^s_{\boldsymbol{\sigma}}(\square)$, which is a second main result.

Theorem 3.11. *Let* $\{\Psi^{\vec{\sigma_i}}_{\lambda_i}\}, i = 1, \dots, d$, *be a family of univariate boundary adapted quarklet frames of order* $m \geq 2$, *with* $\tilde{m}$ *vanishing moments,* $\tilde{m} \geq m$, *according to Theorem 2.7. Then the family*

$$\boldsymbol{\Psi}^s_{\boldsymbol{\sigma}} := \left\{ (w^{H^s}_{\boldsymbol{\lambda}})^{-1} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}} \right\}, \tag{3.15}$$

with the weights

$$w^{H^s}_{\boldsymbol{\lambda}} := \left(\sum_{i=1}^{d} (p_i + 1)^{4s+\delta_1} 4^{sj_i} \right)^{1/2} \prod_{i=1}^{d} (p_i + 1)^{\delta_2/2}, \quad \delta_1 > 1,\ \delta_1 + \delta_2 > 2, \tag{3.16}$$

is a frame for $H^s_{\boldsymbol{\sigma}}(\square)$, $0 \leq s < m - \frac{1}{2}$, $s \notin \mathbb{N}_0 + \frac{1}{2}$.

Remark 3.12. *Let us also introduce the notation*

$$\boldsymbol{\Sigma}_{\boldsymbol{\sigma}} := \bigotimes_{i=1}^{d} \Sigma_{\vec{\sigma_i}} = \left\{ \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}^R_{\boldsymbol{\sigma}} := \prod_{i=1}^{d} \nabla^R_{\vec{\sigma_i}} \right\} \tag{3.17}$$

for the $L_2(\square)$ *Riesz basis. Accordingly,*

$$\boldsymbol{\Sigma}^s_{\boldsymbol{\sigma}} := \left\{ \left(\sum_{i=1}^{d} 4^{sj_i} \right)^{-1/2} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}^R_{\boldsymbol{\sigma}} \right\} \tag{3.18}$$

denotes a Riesz basis for $H^s_{\boldsymbol{\sigma}}(\square)$.

Clearly, Riesz bases and frames for $L_2(\square_j)$, $j = \{0, \ldots, N\}$, can be chosen as

$$\boldsymbol{\Sigma}_j := \boldsymbol{\Sigma}_{\boldsymbol{\sigma}(j)}(\cdot - \tau_j), \quad \boldsymbol{\Psi}_j := \boldsymbol{\Psi}_{\boldsymbol{\sigma}(j)}(\cdot - \tau_j),$$

whose renormalized versions are Riesz bases and frames for $H^s_{\boldsymbol{\sigma}(j)}(\square_j)$.

3.4. **Frames on general domains.** Once we have constructed quarkonial tensor frames for scales of Sobolev spaces on cubes, the next step is clearly the generalization to arbitrary domains as described in Subsection 3.1. To this end, we want to apply the general machinery as outlined in Subsection 3.2. Then, two basic ingredients have to be provided: suitable extension operators $E_1^{(q)}$, cf. (3.2), and the additional Bessel systems $\Xi^{(q)}$, cf. (3.3).

3.4.1. *Construction of scale-dependent extension operators.* For $\vec{\sigma} = (\sigma_l, \sigma_r) \in \{0, \lfloor s + 1/2 \rfloor\}^2$, the index set $\nabla^R_{\vec{\sigma}}$, cf. (2.33), and with $\vec{0} := (0,0)$, the functions in the univariate wavelet Riesz basis $\Sigma_{\vec{\sigma}}$, cf. (2.11), and its dual Riesz basis $\tilde{\Sigma}_{\vec{\sigma}}$ satisfy the following technical properties, cf. [7, Section 2]:

($\mathcal{W}_1$) $|\langle \tilde{\psi}^{\vec{\sigma}}_\lambda, u \rangle_{L_2(\mathcal{I})}| \lesssim 2^{-jt} \|u\|_{H^t(\operatorname{supp} \tilde{\psi}^{\vec{\sigma}})}$ $(u \in H^t(\mathcal{I}) \cap H^s_{\vec{\sigma}}(\mathcal{I}),\ \lambda \in \nabla^R_{\vec{\sigma}})$, for some $\mathbb{N} \ni t > s$,

($\mathcal{W}_2$) $1 > \rho \ := \ \sup_{\lambda \in \nabla^R_{\vec{\sigma}}} 2^j \max(\operatorname{diam} \operatorname{supp} \tilde{\psi}^{\vec{\sigma}}_\lambda, \operatorname{diam} \operatorname{supp} \psi^{\vec{\sigma}}_\lambda)$
$\eqsim \ \inf_{\lambda \in \nabla^R_{\vec{\sigma}}} 2^j \max(\operatorname{diam} \operatorname{supp} \tilde{\psi}^{\vec{\sigma}}_\lambda, \operatorname{diam} \operatorname{supp} \psi^{\vec{\sigma}}_\lambda)$,

($\mathcal{W}_3$) $\sup_{i,k \in \mathbb{N}_0} \#\{\lambda \in \nabla^R_{\vec{\sigma}} : j = i \wedge [k2^{-i}, (k+1)2^{-i}] \cap (\operatorname{supp} \tilde{\psi}^{\vec{\sigma}}_\lambda \cup \operatorname{supp} \psi^{\vec{\sigma}}_\lambda) \neq \emptyset\} < \infty.$

($\mathcal{W}_4$) $V^{\vec{\sigma}}_i := \operatorname{span}\{\psi^{\vec{\sigma}}_\lambda : \lambda \in \nabla^R_{\vec{\sigma}}, j \le i\} = V^{\vec{0}}_i \cap H^s_{\vec{\sigma}}(\mathcal{I})$,

($\mathcal{W}_5$) $\nabla^R_{\vec{\sigma}}$ is the disjoint union of $\nabla^{R,(\ell)}_{\sigma_\ell}$, $\nabla^{R,(I)}$, $\nabla^{R,(r)}_{\sigma_r,}$ such that
  (i) $\sup_{\lambda \in \nabla^{R,(\ell)}_{\sigma_\ell}, x \in \operatorname{supp} \psi^{\vec{\sigma}}_\lambda} 2^j |x| \lesssim \rho, \qquad \sup_{\lambda \in \nabla^{R,(r)}_{\sigma_r}, x \in \operatorname{supp} \psi^{\vec{\sigma}}_\lambda} 2^j |1 - x| \lesssim \rho,$
  (ii) for $\lambda \in \nabla^{R,(I)}$, $\psi^{\vec{\sigma}}_\lambda = \psi^{\vec{0}}_\lambda$, $\tilde{\psi}^{\vec{\sigma}}_\lambda = \tilde{\psi}^{\vec{0}}_\lambda$, and the extensions of $\psi^{\vec{0}}_\lambda$ and $\tilde{\psi}^{\vec{0}}_\lambda$ by zero are in $H^s(\mathbb{R})$ and $L_2(\mathbb{R})$, respectively.

($\mathcal{W}_6$) $\begin{cases} \operatorname{span}\{\psi^{\vec{0}}_\lambda(1 - \cdot) : \lambda \in \nabla^{R,(I)},\ j = i\} = \operatorname{span}\{\psi^{\vec{0}}_\lambda : \lambda \in \nabla^{R,(I)},\ j = i\}, \\ \operatorname{span}\{\psi^{(\sigma_\ell, \sigma_r)}_\lambda(1 - \cdot) : \lambda \in \nabla^{R,(\ell)}_{\sigma_\ell},\ j = i\} = \operatorname{span}\{\psi^{(\sigma_r, \sigma_\ell)}_\lambda : \lambda \in \nabla^{R,(r)}_{\sigma_r},\ j = i\}, \end{cases}$

($\mathcal{W}_7$) $\begin{cases} \psi^{\vec{\sigma}}_\lambda(2^l \cdot) \in \operatorname{span}\{\psi^{\vec{\sigma}}_\mu : \mu \in \nabla^{R,(\ell)}_{\sigma_\ell}\} \quad (l \in \mathbb{N}_0,\ \lambda \in \nabla^{R,(\ell)}_{\sigma_\ell}), \\ \psi^{\vec{0}}_\lambda(2^l \cdot) \in \operatorname{span}\{\psi^{\vec{0}}_\mu : \mu \in \nabla^{R,(I)}\} \quad (l \in \mathbb{N}_0,\ \lambda \in \nabla^{R,(I)}). \end{cases}$

Let us first consider the simple *reflection*

$$(3.19) \qquad \begin{aligned} (\breve{G}_1 v)(x) &:= v(x) \quad & x \in I \\ (\breve{G}_1 v)(-x) &:= v(x) \quad & x \in I, \end{aligned}$$

for any $v \in L_2(I)$. Obviously, we have

$$\begin{aligned} \breve{G}_1 &\in B(L_2(I), L_2(-1,1)) \\ \breve{G}_1 &\in B(H^s(I), H^s(-1,1)), \end{aligned} \tag{3.20}$$

for $s < 3/2$.

Remark 3.13. *The use of the reflection operator has certain advantages and drawbacks. On the one hand, the reflection preserves the vanishing moment properties of the underlying frame elements which is a central ingredient for compression estimates, see Subsection 4.2. Moreover, the reflection possesses a moderate operator norm.*

On the other hand, it is clear that the reflection idea only works for Sobolev spaces H^s, $s < 3/2$, i.e., the resulting numerical schemes are restricted to second order elliptic equations. This bottleneck could be clearly avoided by using, e.g., higher order Hestenes extension operators. However, in recent studies, it has turned out that the norm of a Hestenes extension operator grows fast with respect to its order parameter. Moreover, it is not a priori clear if the vanishing moments are preserved. For this reason, in this paper we stick with the simple reflection operator.

Let η_1 and η_2 denote the extensions by zero of functions on I or on $(-1,0)$ to functions on $(-1,1)$, with R_1 and R_2 denoting their adjoints. With a univariate extension $\breve{G}_1$ as in (3.19) at hand, the obvious approach is to define $E_1^{(q)}$ according to Proposition 3.5 with $G_1 = \breve{G}_1$. A problem with the choice $G_1 = \breve{G}_1$ is that generally it does *not* imply the desirable property $\operatorname{diam}(\operatorname{supp} G_1 u) \lesssim \operatorname{diam}(\operatorname{supp} u)$. Indeed, think of the application of the reflection to a function u with a small support that is not located near the interface.

To solve this and the corresponding problem for the adjoint extension, following [7] we will apply our construction using the modified, *scale-dependent* univariate extension operator

$$G_1 : u \mapsto \sum_{\lambda \in \nabla_0^{R,(\ell)}} \langle u, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} \breve{G}_1 \psi_\lambda^{\vec{0}} + \sum_{\lambda \in \nabla^{R,(I)} \cup \nabla_0^{R,(r)}} \langle u, \tilde{\psi}_\lambda^{\vec{0}} \rangle_{L_2(\mathcal{I})} \eta_1 \psi_\lambda^{\vec{0}}. \tag{3.21}$$

So this operator reflects only wavelets that are supported near the interface. A proof of the following proposition can be found in [7, Proposition 5.2].

Proposition 3.14. *For $\vec{\sigma} \in \{0, \lfloor s + \frac{1}{2} \rfloor\}^2$, the scale-dependent extension G_1 from* (3.21) *satisfies*

$$G_1 \psi_\mu^{\vec{\sigma}} = \begin{cases} \eta_1 \psi_\mu^{\vec{\sigma}} & \text{when } \mu \in \nabla^{R,(I)} \cup \nabla_{\sigma_r}^{R,(r)}, \\ \breve{G}_1 \psi_\mu^{\vec{\sigma}} & \text{when } \mu \in \nabla_{\sigma_\ell}^{R,(\ell)}. \end{cases} \tag{3.22}$$

The resulting adjoint extension $G_2 := (\mathrm{Id} - \eta_1 G_1^*)\eta_2$ *satisfies*

$$G_2(\tilde{\psi}_\mu^{\vec{\sigma}}(1+\cdot)) = \begin{cases} \eta_2(\tilde{\psi}_\mu^{\vec{\sigma}}(1+\cdot)) & \textit{when } \mu \in \nabla^{R,(I)} \cup \nabla_{\sigma_\ell}^{R,(\ell)}, \\ \breve{G}_2(\tilde{\psi}_\mu^{\vec{\sigma}}(1+\cdot)) & \textit{when } \mu \in \nabla_{\sigma_r}^{R,(r)}. \end{cases} \tag{3.23}$$

We have $G_1 \in B(L_2(I), L_2(-1,1))$, *and* $G_1 \in B(H^s(I), H^s_{(\lfloor s+\frac{1}{2}\rfloor,0)}(-1,1))$, *for* $s < 3/2$.

Finally, for $\mu \in \nabla_{\vec{\sigma}}$, *it holds that*

$$\operatorname{diam}(\operatorname{supp} \breve{G}_1 \psi_\mu^{\vec{\sigma}}) \lesssim \operatorname{diam}(\operatorname{supp} \psi_\mu^{\vec{\sigma}}),$$
$$\operatorname{diam}(\operatorname{supp} \breve{G}_2 \tilde{\psi}_\mu^{\vec{\sigma}}) \lesssim \operatorname{diam}(\operatorname{supp} \tilde{\psi}_\mu^{\vec{\sigma}}).$$

Remark 3.15. *In general, it is not possible to divide the univariate quarklet sets in such parts that statements similar to* (3.22) *and* (3.23) *hold. This can be explained as follows: since the univariate wavelets build a Riesz basis for a Sobolev space on the unit interval, every quarklet can be decomposed into wavelet elements. For quarklets near the boundary, it is not guaranteed that the participating wavelets of these decomposition lie exclusively in* $\nabla^{R,(I)} \cup \nabla_{\sigma_r}^{R,(r)}$ *or in* $\nabla_{\sigma_\ell}^{R,(\ell)}$. *Thus, it could happen that one part of the decomposition will be reflected and another part will be extended by zero. This would destroy the vanishing moments of the extended quarklets. Moreover, the wavelet decompositions of the quarklets have to be computed for every single quarklet, which is possible in theory but in practice very time-consuming. This is the reason why we use another approach with Bessel systems, which was already introduced in Section 3.2 and will be carried out further in the next subsubsection.*

3.4.2. *The Bessel systems* $\Xi^{s,(q)}$. For the univariate quarklet frame $\Psi_{\vec{\sigma}}$ we can specify a *non-canonical* dual frame, cf. (A.6), if we augment the dual Riesz basis of the univariate wavelet basis $\Sigma_{\vec{\sigma}}$, cf. (2.11), with zero functions:

$$\Theta_{\vec{\sigma}} := \{\theta_\lambda^{\vec{\sigma}} : \lambda \in \nabla_{\vec{\sigma}}\}, \quad \theta_\lambda^{\vec{\sigma}} := \tilde{\psi}_\lambda^{\vec{\sigma}}, \text{ for } \lambda \in \nabla_{\vec{\sigma}}^R, \quad \theta_\lambda^{\vec{\sigma}} :\equiv 0, \text{ for } \lambda \in \nabla_{\vec{\sigma}} \setminus \nabla_{\vec{\sigma}}^R. \tag{3.24}$$

It is obvious that $\Theta_{\vec{\sigma}}$ is a dual frame of $\Psi_{\vec{\sigma}}$, since

$$\sum_{\lambda \in \nabla_{\vec{\sigma}}} \langle f, \theta_\lambda^{\vec{\sigma}} \rangle_{L_2(I)} \psi_\lambda^{\vec{\sigma}} = \sum_{\lambda \in \nabla_{\vec{\sigma}}^R} \langle f, \tilde{\psi}_\lambda^{\vec{\sigma}} \rangle_{L_2(I)} \psi_\lambda^{\vec{\sigma}} = f, \quad \text{for all } f \in L_2(I).$$

With this dual frame at hand, $(\mathcal{W}_1)$-$(\mathcal{W}_3)$ also hold true if we replace $\nabla_{\vec{\sigma}}^R$ with $\nabla_{\vec{\sigma}}$ and $\tilde{\psi}_\lambda^{\vec{\sigma}}$ with $\theta_\lambda^{\vec{\sigma}}$. Also, it is possible to construct $\nabla_{\sigma_\ell}^{(\ell)} \supset \nabla_{\sigma_\ell}^{R,(\ell)}$, $\nabla^{(I)} \supset \nabla^{R,(I)}$, $\nabla_{\sigma_r,}^{(r)} \supset \nabla_{\sigma_r,}^{R,(r)}$, such that $\nabla_{\vec{\sigma}} = \nabla_{\sigma_\ell} \,\dot{\cup}\, \nabla^{(I)} \,\dot{\cup}\, \nabla_{\sigma_r}$, and

(1) $\displaystyle\sup_{\lambda \in \nabla_{\sigma_\ell}^{(\ell)},\, x \in \operatorname{supp} \psi_\lambda^{\vec{\sigma}}} 2^j |x| \lesssim \rho, \qquad \sup_{\lambda \in \nabla_{\sigma_r}^{(r)},\, x \in \operatorname{supp} \psi_\lambda^{\vec{\sigma}}} 2^j |1-x| \lesssim \rho,$

(2) for $\lambda \in \nabla^{(I)}$, $\psi_\lambda^{\vec{\sigma}} = \psi_\lambda^{\vec{0}}$, $\theta_\lambda^{\vec{\sigma}} = \theta_\lambda^{\vec{0}}$, and the extensions of $\psi_\lambda^{\vec{0}}$ and $\theta_\lambda^{\vec{0}}$ by zero are in $H^s(\mathbb{R})$ and $L_2(\mathbb{R})$, respectively,

cf. $(\mathcal{W}_5)$. For $q \in \{1, \dots, N\}$, $s \geq 0$, we define $\mathbf{\Psi}_{i_1,\ell}^{s,(q-1)}$ as the subset of functions $f \in \mathbf{\Psi}_{i_1}^{s,(q-1)}$ with the following properties:

(i) the support of f intersected with $I \times \breve{J}$ is not empty,
(ii) the cube of origin $\square_i$ of f lies in the neighborhood of $\{0\} \times \breve{J}$, i.e., for all $\varepsilon > 0$: $\operatorname{diam}(\square_i, \{0\} \times \breve{J}) < \varepsilon$,
(iii) the first Cartesian index of f restricted to its cube of origin is contained in $\nabla_0^{(\ell)}$.

With $\mathbf{\Psi}_{i_1,r}^{s,(q-1)} := \mathbf{\Psi}_{i_1}^{s,(q-1)} \setminus \mathbf{\Psi}_{i_1,\ell}^{s,(q-1)}$ we denote the complementary subset. Now we are ready to define the sets $\mathbf{\Xi}^{s,(q)}$ from (3.3) as

$$\mathbf{\Xi}^{s,(q)} := \breve{E}_1^{(q)}(\mathbf{\Psi}_{i_1,\ell}^{s,(q-1)}) \cup \eta_1^{(q)}(\mathbf{\Psi}_{i_1,r}^{s,(q-1)}), \tag{3.25}$$

where $\breve{E}_1^{(q)}$, $q \in \{1, \dots, N\}$, are the operators corresponding to the simple reflection $\breve{G}_1$.

Proposition 3.16. *For $q \in \{1, \dots, N\}$, the set $\mathbf{\Xi}^{0,(q)}$ defined in (3.25) is a Bessel system for $L_2(\Omega_{\bar{i}}^{(q)})$ and $\mathbf{\Xi}^{s,(q)}$ a Bessel system for $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$, $0 < s < 3/2$, $s \neq \frac{1}{2}$. Also, we have $E_1^{(q)}(\mathbf{\Sigma}_{i_1}^{s,(q-1)}) \subset \mathbf{\Xi}^{s,(q)}$.*

Proof. Both $\mathbf{\Psi}_{i_1,\ell}^{0,(q-1)}$ and $\mathbf{\Psi}_{i_1,r}^{0,(q-1)}$ are subsets of the frame $\mathbf{\Psi}_{i_1}^{0,(q-1)}$ for $L_2(\Omega_{i_1}^{(q-1)})$. Hence, they are Bessel systems for $L_2(\Omega_{i_1}^{(q-1)})$. Since both $\breve{E}_1^{(q)}$ and $\eta_1^{(q)}$ are bounded operators from $L_2(\Omega_{i_1}^{(q-1)})$ to $L_2(\Omega_{\bar{i}}^{(q)})$, the images $\breve{E}_1^{(q)}(\mathbf{\Psi}_{i_1,\ell}^{0,(q-1)})$ and $\eta_1^{(q)}(\mathbf{\Psi}_{i_1,r}^{0,(q-1)})$ are Bessel systems for $L_2(\Omega_{\bar{i}}^{(q)})$, cf. Proposition A.4 (i). For the renormalized versions we have to take care of the boundary conditions and the smoothness of the functions. For $s < 3/2$, it is $\breve{G}_1 \in B(H^s_{(0,\lfloor s+\frac{1}{2}\rfloor)}(I), H^s_0(-1,1))$. Since the first Cartesian component of $\mathbf{\Psi}_{i_1,\ell}^{s,(q-1)}$ is in $H^s_{(0,\lfloor s+\frac{1}{2}\rfloor)}(I)$ the image of $\mathbf{\Psi}_{i_1,\ell}^{s,(q-1)}$ under $\breve{E}_1^{(q)}$ is bounded in $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$ and therefore a Bessel system in $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$, cf. Proposition A.4 (i). For the zero extension part we have $\eta_1 \in B(H^s_{(\lfloor s+\frac{1}{2}\rfloor,0)}(I), H^s_{(\lfloor s+\frac{1}{2}\rfloor,0)}(-1,1))$. The first Cartesian component of $\mathbf{\Psi}_{i_1,r}^{s,(q-1)}$ is in $H^s_{(\lfloor s+\frac{1}{2}\rfloor,0)}(I)$ and therefore the image of $\mathbf{\Psi}_{i_1,r}^{s,(q-1)}$ under $\eta_1^{(q)}$ is also a Bessel system for $\mathring{H}^s(\Omega_{\bar{i}}^{(q)})$. The relation $E_1^{(q)}(\mathbf{\Sigma}_{i_1}^{s,(q-1)}) \subset \mathbf{\Xi}^{s,(q)}$ follows directly from (3.22) and (3.25) and the way how the sets $\mathbf{\Psi}_{i_1,\ell}^{s,(q-1)}$ and $\mathbf{\Psi}_{i_1,r}^{s,(q-1)}$ are defined. $\square$

It remains to choose the index sets $\nabla_{\sigma_\ell}^{R,(\ell)}, \nabla^{R,(I)}, \nabla_{\sigma_r,}^{R,(r)}$ and $\nabla_{\sigma_\ell}^{(\ell)}, \nabla^{(I)}, \nabla_{\sigma_r,}^{(r)}$ appropriately. Let us assume that $m \geq 3$. From [28] we deduce that the index sets for which either the primal or dual wavelets depend on the incorporated boundary conditions are

$$\nabla_{\sigma_\ell}^{R,(\ell)} = \{(0,j,k) \in \nabla_{\vec{\sigma}} : k \in \nabla_{j,\sigma_\ell}^{(\ell)}\}, \quad \nabla_{\sigma_r}^{(r)} = \{(0,j,k) \in \nabla_{\vec{\sigma}} : k \in \nabla_{j,\sigma_r}^{(r)}\},$$

with

$$\nabla_{j,\sigma_\ell}^{(\ell)} = \begin{cases} \{0,\dots,\frac{m+\tilde{m}-4}{2}\}, & j \geq j_0, \\ \{-m+1+\operatorname{sgn}\sigma_l,\cdots,\tilde{m}-2\}, & j = j_0-1, \end{cases}$$

and

$$\nabla_{j,\sigma_r}^{(r)} = \begin{cases} \{2^j - \frac{m+\tilde{m}-2}{2},\dots,2^j-1\}, & j \geq j_0, \\ \{2^j - m - \tilde{m} + 2,\dots,2^j - 1 - \operatorname{sgn}\sigma_r\}, & j = j_0-1. \end{cases}$$

The quarklet index sets are

$$\nabla_{\sigma_\ell}^{(\ell)} = \{(p,j,k) \in \nabla_{\vec{\sigma}} : k \in \nabla_{p,j,\sigma_\ell}^{(\ell)}\}, \quad \nabla_{\sigma_r}^{(r)} = \{(p,j,k) \in \nabla_{\vec{\sigma}} : k \in \nabla_{p,j,\sigma_r}^{(r)}\}$$

with

$$\nabla_{p,j,\sigma_\ell}^{(\ell)} = \begin{cases} \nabla_{j,\sigma_\ell}^{(\ell)}, & p = 0, \\ \{0+\operatorname{sgn}\sigma_l,\dots,0\}, & p > 0, j \geq j_0, \\ \{-m+1+\operatorname{sgn}\sigma_l,\cdots,-m+1\}, & p > 0, j = j_0-1, \end{cases}$$

and

$$\nabla_{p,j,\sigma_r}^{(r)} = \begin{cases} \nabla_{j,\sigma_r}^{(r)}, & p = 0, \\ \{2^j-1,\dots,2^j-1-\operatorname{sgn}\sigma_r\} & p > 0, \end{cases} \quad j \geq j_0.$$

In order to identify individual quarklets from the collections constructed by the applications of the extension operators, we have to introduce some more notations. For $0 \leq q \leq N$, we set the index sets

$$\boldsymbol{\nabla}_i^{(0)} := \boldsymbol{\nabla}_{\boldsymbol{\sigma}(i)} \times \{i\} \text{ and, for } q > 0,$$

$$\boldsymbol{\nabla}_i^{(q)} := \begin{cases} \boldsymbol{\nabla}_{i_1}^{(q-1)} \cup \boldsymbol{\nabla}_{i_2}^{(q-1)} & \text{if } i = \bar{i}, \\ \boldsymbol{\nabla}_{\hat{i}}^{(q-1)} & \text{if } i \in \{q,\dots,N\} \setminus \{\bar{i}\} \text{ and } \Omega_i^{(q)} = \Omega_{\hat{i}}^{(q-1)}. \end{cases} \tag{3.26}$$

We define the quarklets on the domains $\Omega_i^{(q)}$ as

$$\boldsymbol{\psi}_{\boldsymbol{\lambda},i}^{(0,i)} := \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\boldsymbol{\sigma}(i)}(\cdot - \tau_i), \tag{3.27}$$

and, for $q > 0$,

$$\boldsymbol{\psi}_{\boldsymbol{\lambda},n}^{(q,i)} := \begin{cases} \left.\begin{cases} \breve{E}_1^{(q)} \boldsymbol{\psi}_{\boldsymbol{\lambda},n}^{(q-1,i_1)} & (\boldsymbol{\lambda},n) \in \boldsymbol{\nabla}_{i_1,\ell}^{(q-1)} \\ \eta_1^{(q)} \boldsymbol{\psi}_{\boldsymbol{\lambda},n}^{(q-1,i_1)} & (\boldsymbol{\lambda},n) \in \boldsymbol{\nabla}_{i_1,r}^{(q-1)} \\ \eta_2^{(q)} \boldsymbol{\psi}_{\boldsymbol{\lambda},n}^{(q-1,i_2)} & (\boldsymbol{\lambda},n) \in \boldsymbol{\nabla}_{i_2}^{(q-1)} \end{cases}\right\} & \text{if } i = \bar{i}, \\ \boldsymbol{\psi}_{\boldsymbol{\lambda},n}^{(q-1,\hat{i})} & \text{if } i \in \{q,\dots,N\} \setminus \{\bar{i}\} \text{ and } \Omega_i^{(q)} = \Omega_{\hat{i}}^{(q-1)}, \end{cases} \tag{3.28}$$

The index $n \in \{0,\dots,N\}$ indicates the cube $\square_n$ where the quarklet stems from. The subsets $\boldsymbol{\nabla}_{i_1,\ell}^{(q-1)}$ and $\boldsymbol{\nabla}_{i_1,r}^{(q-1)}$ are defined according to $\boldsymbol{\Psi}_{i_1,\ell}^{s,(q-1)}$ and $\boldsymbol{\Psi}_{i_1,r}^{s,(q-1)}$. With this notations at hand we are now able to formulate the main theorem of this paper.

Theorem 3.17. *Let $\Psi_{\vec{\sigma}}$ denote a quarklet system of order $m \geq 2$, $\tilde{m}$ vanishing moments, $\tilde{m} \geq m$ and $m + \tilde{m}$ even, as constructed in Theorem 2.7. Furthermore, let $\Omega \in \mathbb{R}^d$ be a bounded domain that can be decomposed into cubes $\square_i$, $i = 0, \ldots, N$. If we choose weights $\boldsymbol{w}_{\boldsymbol{\lambda}}^{H^s}$ as in* (3.16)*, the system*

$$\boldsymbol{\Psi}^s := \left\{ (\boldsymbol{w}_{\boldsymbol{\lambda}}^{H^s})^{-1} \boldsymbol{\psi}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} = (\boldsymbol{\lambda}, n) \in \boldsymbol{\nabla} \right\}, \quad \delta_1 > 1,\, \delta_1 + \delta_2 > 2, \tag{3.29}$$

with $\boldsymbol{\psi}_{\boldsymbol{\alpha}} := \boldsymbol{\psi}_{\boldsymbol{\lambda},n}^{(N,N)}$, cf. (3.28)*, $\boldsymbol{\nabla} := \boldsymbol{\nabla}_N^{(N)}$, cf.* (3.26)*, is a frame for $H_0^s(\Omega)$, $0 \leq s < \frac{3}{2}$, $s \neq \frac{1}{2}$.*

4. Adaptive quarklet schemes

4.1. **Adaptive frame schemes for elliptic operator equations.** As already mentioned in the introduction, the stability of weighted quarkonial frames in Sobolev spaces and the compression properties of the individual quarklets can be used to derive adaptive discretization schemes for linear elliptic operator equations in a quite systematic way, see [10, 14, 15, 31, 33] for the general reasoning.

In order to briefly illustrate the main ideas of such schemes, let us consider a linear elliptic variational problem of the form

$$a(u, v) = F(v), \quad \text{for all } v \in \mathcal{H}, \tag{4.1}$$

where $\mathcal{H}$ is the solution Hilbert space and $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ a symmetric, elliptic bilinear form and $F : \mathcal{H} \to \mathbb{R}$ a continuous functional. Given a frame $\mathcal{F} = \{f_\lambda\}_{\lambda \in \mathcal{I}}$ for $\mathcal{H}$ with countable index set $\mathcal{I}$, it is well-known [10, 14, 31] that (4.1) is equivalent to the linear system of equations

$$\mathbf{A}\mathbf{u} = \mathbf{F}, \tag{4.2}$$

where $\mathbf{A} := (a(f_\mu, f_\lambda))_{\mu,\lambda \in \mathcal{I}} \in \mathcal{L}(\ell_2(\mathcal{I}))$ is the biinfinite stiffness matrix, $\mathbf{u} := (u_\lambda)_{\lambda \in \mathcal{I}}$ is a coefficient array of the unknown solution $u = \sum_{\lambda \in \mathcal{I}} u_\lambda f_\lambda$ with respect to the frame $\mathcal{F}$, and $\mathbf{F} := (F(f_\lambda)_{\lambda \in \mathcal{I}})$ contains the values of the right-hand side F at individual frame elements. Due to the redundancy of the frame $\mathcal{F}$, the system matrix $\mathbf{A}$ may have a non-trivial kernel, so that (4.2) is not uniquely solvable. Straightforward Galerkin-type approaches might hence run into problems, since the stiffness matrix might be singular or arbitrarily ill-conditioned.

Nonetheless, classical iterative schemes like the damped Richardson iteration

$$\mathbf{u}^{(j+1)} := \mathbf{u}^{(j)} + \omega(\mathbf{F} - \mathbf{A}\mathbf{u}^{(j)}), \quad 0 < \omega < \frac{2}{\|\mathbf{A}\|_{\mathcal{L}(\ell_2(\mathcal{I}))}}, \quad j = 0, 1, \ldots \tag{4.3}$$

or variations thereof, like steepest descent or conjugate gradient iterations, can still be applied in a numerically stable way, and the associated expansions $u^{(j)} := \sum_{\lambda \in \mathcal{I}} u_\lambda^{(j)} f_\lambda \in \mathcal{H}$ will converge to the solution u under quite general assumptions. By judiciously choosing the respective tolerances, convergence can even be preserved under perturbation of the exact iterations when, e.g., each evaluation of the infinite-dimensional right-hand side $\mathbf{F}$ and each matrix-vector product $\mathbf{Av}$ are replaced by suitable numerical approximations [9, 10, 14, 15, 19, 31, 33]. The efficient inexact evaluation of $\mathbf{F}$ is closely related with the rate of best N-term approximation of the continuous right-hand-side F from the given dictionary. Since our frame contains a wavelet Riesz basis, sufficient approximation rates can be inferred via Besov regularity estimates, cf. [22].

Therefore, inexact matrix-vector multiplications play a key role within adaptive wavelet methods. In order to realize them in a computationally efficient way, it is essential to exploit that the system matrix $\mathbf{A}$ is not arbitrarily structured but features certain compressibility properties. By this we mean that $\mathbf{A}$ can be approximated well by sparse matrices with a finite number of entries per row and column. To be precise, we call a matrix $\mathbf{M} : \ell_2(\mathcal{I}) \to \ell_2(\mathcal{I})$ s^*-*compressible*, if there exist $C > 0$ and, for every $J \in \mathbb{N}_0$, matrices $\mathbf{M}_J$ with at most $C2^J$ non-trivial entries per row and column, which fulfill

$$||\mathbf{M} - \mathbf{M}_J||_{\mathcal{L}(\ell_2(\mathcal{I}))} \lesssim 2^{-Js^*}.$$

If the entries of $\mathbf{A}$ have a sufficiently fast off-diagonal decay, such approximations can be constructed in a quite generic way, see [9, 31, 32], and are the central ingredient in a so-called **APPLY** routine which realizes an inexact version of the matrix-vector-multiplication $\mathbf{Au}^{(j)}$ in each iteration of (4.3).

In the sequel, we will show that similar to wavelet systems, quarklet frames can induce compressible stiffness matrices in the aforementioned sense. The most important example of a second order elliptic PDE which serves as the standard test case for numerical algorithms is the Poisson-equation on polygonal or polyhedral domains. Therefore, in the sequel we will derive detailed compression results in particular for this case.

Let the domain Ω satisfy the assumptions of Section 3. For a fixed right-hand side $F \in H^{-1}(\Omega)$ we want to compute the solution $u \in H_0^1(\Omega)$ to (4.1), where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \sum_{k=1}^{d} \int_\Omega \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} \, \mathrm{d}x. \tag{4.4}$$

In the setting of Section 3, the domain Ω is a hypercube or a union of finitely many translated copies thereof, and the frame elements $\boldsymbol{\psi}_{\boldsymbol{\lambda}}$ are sums of tensor products of univariate functions. Therefore, the individual entries $a(\boldsymbol{\psi}_{\boldsymbol{\mu}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}})$ of the stiffness matrix $\mathbf{A}$ are sums of products of univariate integrals. Let, for example, $\Omega = I^2$, and $\{f_\lambda : \lambda \in \mathcal{I}\}$ be a frame for $L_2(I)$ such that $\{w_\lambda^{-1} f_\lambda : \lambda \in \mathcal{I}\}$ is a frame for $H_0^1(I)$. Then,

$$\boldsymbol{\mathcal{F}} := \left\{ (w_{\lambda_1}^2 + w_{\lambda_2}^2)^{-1/2} f_{\lambda_1} \otimes f_{\lambda_2} : \lambda_1, \lambda_2 \in \mathcal{I} \right\}$$

is a frame for

$$H_0^1(I^2) = H_0^1(I) \otimes L_2(I) \cap L_2(I) \otimes H_0^1(I),$$

and the stiffness matrix $\mathbf{A}$ with respect to $\boldsymbol{\mathcal{F}}$ is a sum of Kronecker products,

$$\mathbf{A} = \mathbf{D}_2^{-1}(\mathbf{B} \otimes \mathbf{G} + \mathbf{G} \otimes \mathbf{B})\mathbf{D}_2^{-1},$$

where $\mathbf{B} = (\int_0^1 f_\lambda' f_\mu' \, \mathrm{d}x)_{\lambda,\mu\in\mathcal{I}}$ and $\mathbf{G} = (\int_0^1 f_\lambda f_\mu \, \mathrm{d}x)_{\lambda,\mu\in\mathcal{I}}$ are one-dimensional stiffness and Gramian matrices, respectively and $\mathbf{D}_2 = (w_{\boldsymbol{\lambda}})_{\boldsymbol{\lambda}\in\mathcal{I}^2}$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $w_{\boldsymbol{\lambda}} := (w_{\lambda_1}^2 + w_{\lambda_2}^2)^{1/2}$.

In the light of these tensor product techniques, we will first derive compression estimates for quarklet discretizations of one-dimensional elliptic equations. After that, we will show how to generalize them to the multivariate setting discussed in Section 3.

4.2. **Compression.** As we have seen in the two-dimensional case, the stiffness matrix of the Poisson equation (4.4) is a sum of Kronecker products of one-dimensional Laplacian and Gramian matrices. For $d \in \mathbb{N}$ dimensions this can be generalized easily to

$$\mathbf{A} = \mathbf{D}_d^{-1}(\mathbf{B} \otimes \mathbf{G} \otimes \ldots \otimes \mathbf{G} + \ldots + \mathbf{G} \otimes \ldots \otimes \mathbf{G} \otimes \mathbf{B})\mathbf{D}_d^{-1}.$$

Hence, to estimate the compressibility properties of the resulting stiffness matrix of the Laplacian (4.4), we need estimates for the inner products of the basic univariate quarks and quarklets.

Proposition 4.1. *Let $m \geq 3$. There exists $C = C(m)$, such that the unweighted quarks and quarklets satisfy*

$$\big|\langle \psi_{p,j,k}, \psi_{p',j',k'} \rangle_{L_2(I)}\big| \leq C\big((p+1)(p'+1)\big)^{m-1} 2^{-|j-j'|(m-\frac{1}{2})}. \tag{4.5}$$

Proof. The combination of Lemma 2.5, Proposition 2.4, the definitions (2.21), (2.12), and (2.2), and for the last step Proposition 2.3 yields

$$\begin{aligned}
\big|\langle \psi_{p,j,k}, \psi_{p',j',k'} \rangle_{L_2(I)}\big| &\lesssim (p+1)^{-m} 2^{-j(m-\frac{1}{2})} |\psi_{p',j',k'}|_{W_\infty^{m-1}(\operatorname{supp} \psi_{p,j,k}^{\tilde{\sigma}})} \\
&\lesssim (p+1)^{-m} 2^{-j(m-\frac{1}{2})} (p'+1)^{2(m-1)} 2^{j'(m-1)} \|\psi_{p',j',k'}\|_{L_\infty(I)} \\
&= (p+1)^{-m} 2^{-j(m-\frac{1}{2})} (p'+1)^{2(m-1)} 2^{j'(m-\frac{1}{2})} \|\varphi_{p',0}\|_{L_\infty(I)} \\
&\lesssim (p+1)^{-m} (p'+1)^{m-1} 2^{(j'-j)(m-\frac{1}{2})}.
\end{aligned}$$

The analogous result holds with interchanged roles of (p, j, k) and (p', j', k'). The minimum over both estimates yields (4.5). ☐

By following the lines of the proof of Proposition 4.1, a similar estimate for the derivatives of quarks and quarklets can be derived. We also refer to [16, Proposition 6.1], where an analogous result for the whole real line has been proven.

Proposition 4.2. *Let $m \geq 3$. There exists $C = C(m)$, such that the unweighted quarks and quarklets satisfy*

$$\left|\langle \psi'_{p,j,k}, \psi'_{p',j',k'} \rangle_{L_2(I)}\right| \leq C 2^{j+j'} \big((p+1)(p'+1)\big)^{m-1} 2^{-|j-j'|(m-\frac{3}{2})}. \tag{4.6}$$

For the readers' convenience, we consider the multivariate compression estimates only on the unit cube, i.e. $\Omega = \square$. But let us mention that the results carry over to the case of general domains, since in this case the amount of cubes where quarklets have non-trivial support is uniformly bounded by a finite number which only depends on the space dimension d.

The combination of the last two propositions yields the desired estimate for the entries of the stiffness matrix of (4.4).

Proposition 4.3. *Let $m \geq 3$, $d \geq 2$. Let the weighted quarklets $(w^{H^1}_{\boldsymbol{\lambda}})^{-1} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}$, $(w^{H^1}_{\boldsymbol{\lambda}'})^{-1} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}'}$, $\boldsymbol{\lambda} := (\boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k})$, $\boldsymbol{\lambda}' := (\boldsymbol{p}', \boldsymbol{j}', \boldsymbol{k}')$ be defined as in* (3.15)*, and the bilinear form a as in* (4.4)*. Then it holds*

$$|a((w^{H^1}_{\boldsymbol{\lambda}})^{-1} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}, (w^{H^1}_{\boldsymbol{\lambda}'})^{-1} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}'})| \lesssim \prod_{i=1}^{d} \big(1 + |p_i - p'_i|\big)^{m-1-\delta_2/2} 2^{-|j-j'|(m-3/2)}, \tag{4.7}$$

with $\delta_2 > 2m - 2$.

Proof. There is nothing to prove if $\operatorname{supp} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}} \cap \operatorname{supp} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}'} = \emptyset$. Otherwise we use the tensor product structure of the quarklets to obtain

$$a(\boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}, \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}'}) = \sum_{i=1}^{d} \prod_{r=1}^{d} \left\langle \left(\psi^{\sigma_r}_{p_r,j_r,k_r}\right)^{(\delta_{ir})}, \left(\psi^{\sigma_r}_{p'_r,j'_r,k'_r}\right)^{(\delta_{ir})} \right\rangle_{L_2(I)},$$

where the Kronecker deltas indicate whether the quarklet itself or its first derivative is concerned. Applying the estimates (4.5) and (4.6) leads to

$$\begin{aligned}
|a(\boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}, \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}'})| &\leq \sum_{i=1}^{d} \prod_{r=1}^{d} \big((p_r+1)(p'_r+1)\big)^{m-1+\delta_{ir}} 2^{\delta_{ir}(j_r+j'_r)} 2^{-|j_r-j'_r|(m-1/2-\delta_{ir})} \\
&\leq \sum_{i=1}^{d} \big((p_i+1)(p'_i+1)\big) 2^{j_i+j'_i} \prod_{r=1}^{d} \big((p_r+1)(p'_r+1)\big)^{m-1} 2^{-|j_r-j'_r|(m-3/2)}.
\end{aligned}$$

Estimating the weights $w_{\boldsymbol{\lambda}}, w_{\boldsymbol{\lambda}'}$ defined in (3.16) by the Cauchy-Schwarz inequality, we obtain

$$w^{-1}_{\boldsymbol{\lambda}} w^{-1}_{\boldsymbol{\lambda}'} \leq \left(\sum_{i=1}^{d} ((p_i+1)(p'_i+1))^{2+\delta_1/2} 2^{(j_i+j'_i)} \right)^{-1} \prod_{r=1}^{d} ((p_r+1)(p'_r+1))^{-\delta_2/2}.$$

Combining the previous estimates, we obtain

$$|a(w_{\boldsymbol{\lambda}}^{-1}\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\boldsymbol{\sigma}}, w_{\boldsymbol{\lambda}'}^{-1}\boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\boldsymbol{\sigma}})| \leq \prod_{r=1}^{d}((p_r+1)(p_r'+1))^{m-1-\delta_2/2}2^{-|j_r-j_r'|(m-3/2)}.$$

Choosing $\delta_2 > 2m-2$ and using the relation

$$(p+1)(p'+1) \geq 1 + |p-p'|,$$

we finally get the claim. $\square$

Theorem 4.4. *Let $m \geq 3$. Let $\mathbf{A}$, defined by (4.2), be the stiffness matrix of the Poisson equation (4.4) discretized by $\mathbf{\Psi}_{\boldsymbol{\sigma}}^1$, defined in (3.15). Further, for $J \in \mathbb{N}_0$, with $\boldsymbol{\lambda} = (\boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k}), \boldsymbol{\lambda}' = (\boldsymbol{p}', \boldsymbol{j}', \boldsymbol{k}') \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}}$, define $\mathbf{A}_J$ by setting all entries from $\mathbf{A}$ to zero that satisfy*

$$a\log_2(\prod_{i=1}^{d} 1 + |p_i - p_i'|) + b|\boldsymbol{j} - \boldsymbol{j}'| > J, \tag{4.8}$$

where $a, b > 0$. Then, for $\delta_2 > 2m-2$, the maximal number of non-zero entries in each row and column of $\mathbf{A}_J$ is of the order

$$\left(J^{2d-2}2^{\frac{J}{a}} + J^{d-1}2^{\frac{J}{b}}\right)\begin{cases} J, & a = b, \\ 1, & \textit{otherwise}. \end{cases} \tag{4.9}$$

Furthermore, with $\tau := m - 1 - \frac{\delta_2}{2}$ it holds that

$$\|\mathbf{A} - \mathbf{A}_J\|_{\mathcal{L}(\ell_2(\boldsymbol{\nabla}_{\boldsymbol{\sigma}}))} \lesssim \left(J^{d-1}2^{-(m-2)\frac{J}{b}} + J^{2d-2}2^{(1+\tau)\frac{J}{a}}\right)\begin{cases} J, & \frac{a}{b} = -\frac{1+\tau}{m-2}, \\ 1, & \textit{otherwise}. \end{cases} \tag{4.10}$$

In particular, $\mathbf{A}$ is s^*-compressible with

$$s^* := \min\{a,b\}\min\{\frac{-1-\tau}{a}, \frac{m-2}{b}\}. \tag{4.11}$$

Remark 4.5. *In the compression estimate (4.11), the exponential factors do not depend on the spatial dimension d. In this sense, quarklet frames provide dimension independent compression rates. For fixed m, τ, in (4.11), the optimal choices of a, b yield rates*

$$s^* = \begin{cases} -(1+\tau), & \frac{a}{b} \in [-\frac{1+\tau}{m-2}, 1), \\ m-2, & \frac{a}{b} \in [1, -\frac{1+\tau}{m-2}]. \end{cases}$$

The proof of Theorem 4.4 is quite technical. In the course of the proof, we will use the following facts:

(i) Let $K \in \mathbb{N}$, $t \in \mathbb{R}_+$. Then,

$$\sum_{n=1}^{K} n^{-t} \le 1 + \int_1^K x^{-t} \mathrm{d}x \lesssim \begin{cases} K^{1-t}, & t < 1, \\ 1 + \ln(K), & t = 1, \\ 1, & t > 1. \end{cases} \tag{4.12}$$

(ii) Let $K \in \mathbb{N}$, $t > 1$. Then,

$$\sum_{n=K}^{\infty} n^{-t} \le K^{-t} + \int_K^{\infty} x^{-t} \mathrm{d}x \lesssim K^{1-t}. \tag{4.13}$$

(iii) Let $r \in \mathbb{N}$, $t \in \mathbb{R}_+$, $L_0 \in \mathbb{N}_0$ and $L_1 := \max\{L_0, r/t - 1\}$. Then,

$$\begin{aligned} \sum_{n=L_0}^{\infty} (1+n)^r e^{-tn} &\lesssim (1+L_1)^r e^{-tL_1} + \int_{L_1}^{\infty} (1+x)^r e^{-tx} \mathrm{d}x \\ &\lesssim (1+L_1)^r e^{-tL_1}. \end{aligned} \tag{4.14}$$

Proof of Theorem 4.4. First we are going to estimate the number of non-trivial entries, i.e., (4.9). To simplify the notation we assume $j_0 = 0$ for the minimal level in each coordinate of the quarklet frame $\boldsymbol{\Psi}^1_{\boldsymbol{\sigma}}$.

Let $\boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}}$ be fixed. The number of $\boldsymbol{\lambda}' \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}}$ with fixed $\boldsymbol{p}'$ that fulfill $\operatorname{supp} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}} \cap \operatorname{supp} \boldsymbol{\psi}^{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}'} \neq \emptyset$ is of the order $\prod_{i=1}^d \max\{1, 2^{j_i' - j_i}\} \le 2^{|\boldsymbol{j}-\boldsymbol{j}'|}$. Further, $|\{\boldsymbol{j} \in \mathbb{N}_0^d : |\boldsymbol{j}| = l\}| = \binom{l+d-1}{l} \lesssim (1+l)^{d-1}$ with a constant depending on d holds. Together, this implies that the number of entries in the λ-th row of $\mathbf{A}_J$ is bounded by

$$\begin{aligned} &\sum_{\substack{\boldsymbol{p}' \in \mathbb{N}_0^d \\ \prod_{i=1}^d 1 + |p_i - p_i'| \le 2^{\frac{J}{a}}}} \; \sum_{l=0}^{\lfloor \frac{J}{b} - \frac{a}{b} \log_2(\prod_{i=1}^d 1 + |p_i - p_i'|) \rfloor} \; \sum_{\substack{\boldsymbol{j}' \in \mathbb{N}_0^d \\ |\boldsymbol{j}-\boldsymbol{j}'| = l}} 2^{|\boldsymbol{j}-\boldsymbol{j}'|} \\ &\qquad \le \sum_{\substack{\boldsymbol{p}'' \in \mathbb{N}^d \\ \prod_{i=1}^d p_i'' \le 2^{\frac{J}{a}}}} \; \sum_{l=0}^{\lfloor \frac{J}{b} - \frac{a}{b} \log_2(\prod_{i=1}^d p_i'') \rfloor} \binom{l+d-1}{l} 2^l. \end{aligned}$$

In the latter term, $\binom{l+d-1}{l}$ can be estimated from above by $\left(1+\frac{J}{b}\right)^{d-1}$. Hence,

$$\begin{aligned}\sum_{\substack{\boldsymbol{p}'\in\mathbb{N}_0^d\\ \prod_{i=1}^d 1+|p_i-p_i'|\le 2^{\frac{J}{a}}}} \sum_{l=0}^{\lfloor \frac{J}{b}-\frac{a}{b}\log_2(\prod_{i=1}^d 1+|p_i-p_i'|)\rfloor} \sum_{\substack{\boldsymbol{j}'\in\mathbb{N}_0^d\\ |\boldsymbol{j}-\boldsymbol{j}'|=l}} 2^{|\boldsymbol{j}-\boldsymbol{j}'|} \\ \lesssim \left(\frac{J}{b}\right)^{d-1} 2^{\frac{J}{b}} \sum_{\substack{\boldsymbol{p}''\in\mathbb{N}^d\\ \prod_{i=1}^d p_i''\le 2^{\frac{J}{a}}}} \left(\prod_{i=1}^d p_i''\right)^{-\frac{a}{b}}.\end{aligned} \tag{4.15}$$

We separate the last component of $\boldsymbol{p}''$ to obtain

$$\sum_{\substack{\boldsymbol{p}''\in\mathbb{N}^d\\ \prod_{i=1}^d p_i''\le 2^{\frac{J}{a}}}} \left(\prod_{i=1}^d p_i''\right)^{-\frac{a}{b}} = \sum_{\substack{\boldsymbol{p}''\in\mathbb{N}^{d-1}\\ \prod_{i=1}^{d-1} p_i''\le 2^{\frac{J}{a}}}} \sum_{p_d''=1}^{2^{\frac{J}{a}}\left(\prod_{i=1}^{d-1} p_i''\right)^{-1}} \left(\prod_{i=1}^d p_i''\right)^{-\frac{a}{b}}.$$

Applying (4.12) d times with $K=2^{J/a}$, $t=\frac{a}{b}$ leads to

$$\begin{aligned}\sum_{\substack{\boldsymbol{p}''\in\mathbb{N}^d\\ \prod_{i=1}^d p_i''\le 2^{\frac{J}{a}}}} \left(\prod_{i=1}^d p_i''\right)^{-\frac{a}{b}} &\lesssim \sum_{\substack{\boldsymbol{p}''\in\mathbb{N}^{d-1}\\ \prod_{i=1}^{d-1} p_i''\le 2^{\frac{J}{a}}}} \begin{cases} 2^{\frac{J}{a}(1-\frac{a}{b})}\left(\prod_{i=1}^{d-1} p_i''\right)^{-1}, & a<b,\\ \left(1+\frac{J}{a}-\ln(\prod_{i=1}^{d-1} p_i'')\right)\left(\prod_{i=1}^{d-1} p_i''\right)^{-1}, & a=b,\\ \left(\prod_{i=1}^{d-1} p_i''\right)^{-1}, & a>b,\end{cases} \\ &\lesssim \begin{cases} 2^{\frac{J}{a}(1-\frac{a}{b})}\left(1+\frac{J}{a}\right)^{d-1}, & a<b,\\ \left(1+\frac{J}{a}\right)^{d}, & a=b,\\ 1, & a>b.\end{cases}\end{aligned} \tag{4.16}$$

Finally, by the last estimate, (4.15) can be further estimated by

$$\sum_{\substack{\boldsymbol{p}''\in\mathbb{N}^d\\ \prod_{i=1}^d p_i''\le 2^{\frac{J}{a}}}} \left(\frac{J}{b}\right)^{d-1} 2^{\frac{J}{b}} \left(\prod_{i=1}^d p_i''\right)^{-\frac{a}{b}} \lesssim \begin{cases} \left(\frac{J}{b}\right)^{d-1} 2^{\frac{J}{a}} \left(1+\frac{J}{a}\right)^{d-1}, & a<b,\\ \left(\frac{J}{b}\right)^{d-1} 2^{\frac{J}{b}} \left(1+\frac{J}{a}\right)^{d}, & a=b,\\ \left(\frac{J}{b}\right)^{d-1} 2^{\frac{J}{b}}, & a>b,\end{cases}$$

which implies (4.9).

Next we will derive the compression result (4.10). As a standard tool for such estimates we will employ the Schur lemma. It states that

$$\sup_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{\boldsymbol{\sigma}}} w_{\boldsymbol{\lambda}}^{-1} \sum_{\boldsymbol{\lambda}'\in\boldsymbol{\nabla}_{\boldsymbol{\sigma}}} |(\mathbf{A})_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} - (\mathbf{A}_J)_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}| w_{\boldsymbol{\lambda}'} \le C,$$
$$\sup_{\boldsymbol{\lambda}'\in\boldsymbol{\nabla}_{\boldsymbol{\sigma}}} w_{\boldsymbol{\lambda}'}^{-1} \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{\boldsymbol{\sigma}}} |(\mathbf{A})_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} - (\mathbf{A}_J)_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}| w_{\boldsymbol{\lambda}} \le C$$

with weights $w_{\boldsymbol{\lambda}} > 0$, $\boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma}}$ and $C > 0$ implies $\|\mathbf{A} - \mathbf{A}_J\|_{\mathcal{L}(\ell_2(\boldsymbol{\nabla}_{\boldsymbol{\sigma}}))} \le C$. The symmetry of $\mathbf{A} - \mathbf{A}_J$ implies that it is sufficient to estimate $\sup_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{\boldsymbol{\sigma}}} \alpha_{\boldsymbol{\lambda}}$, where

$$\alpha_{\boldsymbol{\lambda}} := w_{\boldsymbol{\lambda}}^{-1} \sum_{\boldsymbol{\lambda}'\in\boldsymbol{\nabla}_{\boldsymbol{\sigma}}} |(\mathbf{A})_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} - (\mathbf{A}_J)_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}| w_{\boldsymbol{\lambda}'}.$$

We choose weights of the form $w_{\boldsymbol{\lambda}} = 2^{-|\boldsymbol{j}|/2}$. In particular, it holds that

$$\prod_{i=1}^{d} \max\{1, 2^{j_i'-j_i}\}(2^{-|\boldsymbol{j}|/2})^{-1} 2^{-|\boldsymbol{j}'|/2} = 2^{|\boldsymbol{j}-\boldsymbol{j}'|/2}.$$

Therefore, our choice for $w_{\boldsymbol{\lambda}}$, the cut-off rule (4.8), together with the decay of the bilinear form (4.7), the definition of $x_0(\boldsymbol{p}') := \lceil b^{-1}(J - a\log_2(\prod_{i=1}^{d} 1 + |p_i - p_i'|))\rceil$ and $\tau = m - 1 - \frac{\delta_2}{2}$ yields

$$\alpha_\lambda \lesssim \sum_{\boldsymbol{p}'\in\mathbb{N}_0^d} \Big(\prod_{i=1}^{d} \big(1 + |p_i - p_i'|\big)^{\tau}\Big) \sum_{l=\max\{0,x_0(\boldsymbol{p}')\}}^{\infty} \sum_{\substack{\boldsymbol{j}'\in\mathbb{N}_0^d \\ |\boldsymbol{j}-\boldsymbol{j}'|=l}} 2^{-|\boldsymbol{j}-\boldsymbol{j}'|(m-2)}.$$

Estimating the sum involving $\boldsymbol{j}'$ leads to

$$\alpha_\lambda \lesssim \sum_{\boldsymbol{p}'\in\mathbb{N}_0^d} \Big(\prod_{i=1}^{d} \big(1 + |p_i - p_i'|\big)^{\tau}\Big) \sum_{l=\max\{0,x_0(\boldsymbol{p}')\}}^{\infty} 2^{-l(m-2)}(1 + l)^{d-1}. \tag{4.17}$$

Applying (4.14) with $L_0 = \max\{0, x_0(\boldsymbol{p'})\}$, $r = d-1$, $t = \ln(2)(m-2)$ and $L_1 = x_1(\boldsymbol{p'}) := \max\{0, x_0(\boldsymbol{p'}), \frac{d-1}{\ln(2)(m-2)} - 1\}$, yields

$$
\begin{aligned}
\alpha_\lambda &\lesssim \sum_{\boldsymbol{p'} \in \mathbb{N}_0^d} \Big(\prod_{i=1}^d \big(1 + |p_i - p_i'|\big)^\tau \Big) (1 + x_1(\boldsymbol{p'}))^{d-1} 2^{-(m-2)x_1(\boldsymbol{p'})} \\
&\lesssim \sum_{\substack{\boldsymbol{p'} \in \mathbb{N}_0^d \\ x_0(\boldsymbol{p'}) \le \max\{0, -1 + \frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^d \big(1 + |p_i - p_i'|\big)^\tau \\
&\quad + \sum_{\substack{\boldsymbol{p'} \in \mathbb{N}_0^d \\ x_0(\boldsymbol{p'}) > \max\{0, -1 + \frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^d \big(1 + |p_i - p_i'|\big)^\tau (1 + x_0(\boldsymbol{p'}))^{d-1} 2^{-(m-2)x_0(\boldsymbol{p'})}.
\end{aligned}
\tag{4.18}
$$

First we have a closer look at the first sum of (4.18). By splitting the sum and setting $x := (J - b\max\{0, \frac{d-1}{\ln(2)(m-2)} - 1\})/a$, we get

$$
\begin{aligned}
\sum_{\substack{\boldsymbol{p'} \in \mathbb{N}_0^d \\ x_0(\boldsymbol{p'}) \le \max\{0, -1 + \frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^d \big(1 + |p_i - p_i'|\big)^\tau &= \sum_{\substack{\boldsymbol{p'} \in \mathbb{N}_0^d \\ \log_2(\prod_{i=1}^d 1 + |p_i - p_i'|) \ge x}} \prod_{i=1}^d (1 + |p_i - p_i'|)^\tau \\
= \sum_{\boldsymbol{p'} \in \mathbb{N}_0^{d-1}} \prod_{i=1}^{d-1} (1 + |p_i - p_i'|)^\tau & \sum_{\substack{p_d' \in \mathbb{N}_0 \\ \log_2(1 + |p_d - p_d'|) \ge x - \log_2(\prod_{i=1}^{d-1} 1 + |p_i - p_i'|)}} (1 + |p_d - p_d'|)^\tau.
\end{aligned}
$$

Consequently, with (4.13) with $t = -\tau$, $K = 2^{x - \log_2(\prod_{i=1}^d 1 + |p_i - p_i'|)}$ we get

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{p}' \in \mathbb{N}_0^d \\ x_0(\boldsymbol{p}') \le \max\{0, -1 + \frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^{d} (1 + |p_i - p_i'|)^{\tau} \\
&\lesssim \sum_{\boldsymbol{p}' \in \mathbb{N}_0^{d-1}} \prod_{i=1}^{d-1} (1 + |p_i - p_i'|)^{\tau} \min\{1, 2^{(1+\tau)(x - \log_2(\prod_{i=1}^{d-1} 1 + |p_i - p_i'|))}\} \\
&\lesssim \sum_{\substack{\boldsymbol{p}' \in \mathbb{N}_0^{d-1} \\ \log_2(\prod_{i=1}^{d-1} 1 + |p_i - p_i'|) \ge x}} \prod_{i=1}^{d-1} (1 + |p_i - p_i'|)^{\tau} \\
&\quad + \sum_{\substack{\boldsymbol{p}' \in \mathbb{N}_0^{d-1} \\ \log_2(\prod_{i=1}^{d-1} 1 + |p_i - p_i'|) < x}} \prod_{i=1}^{d-1} (1 + |p_i - p_i'|)^{-1} 2^{(1+\tau)x}.
\end{aligned}$$

It follows by induction and with an estimate similar as in (4.16), that

$$\sum_{\substack{\boldsymbol{p}' \in \mathbb{N}_0^d \\ x_0(\boldsymbol{p}') \le \max\{0, -1 + \frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^{d} (1 + |p_i - p_i'|)^{\tau} \lesssim 2^{(1+\tau)x} (1 + x)^{d-1}. \tag{4.19}$$

For the second sum, with the definition of $x_0(\boldsymbol{p}')$ we obtain

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{p}' \in \mathbb{N}_0^d \\ x_0(\boldsymbol{p}') > \max\{0, -1 + \frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^{d} \big(1 + |p_i - p_i'|\big)^{\tau} (1 + x_0(\boldsymbol{p}'))^{d-1} 2^{-(m-2)x_0(\boldsymbol{p}')} \\
\lesssim &\sum_{\substack{\boldsymbol{p}' \in \mathbb{N}_0^d \\ x_0(\boldsymbol{p}') > \max\{0, -1 + \frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^{d} \big(1 + |p_i - p_i'|\big)^{\tau} \Big(1 + \frac{J}{b} - \frac{a}{b} \log_2 \big(\prod_{i=1}^{d} 1 + |p_i - p_i'|\big)\Big)^{d-1} \\
&\cdot 2^{-(m-2)\left(\frac{J}{b} - \frac{a}{b} \log_2(\prod_{i=1}^{d} 1 + |p_i - p_i'|)\right)}.
\end{aligned}$$

We further estimate

$$\sum_{\substack{\boldsymbol{p}'\in\mathbb{N}_0^d \\ x_0(\boldsymbol{p}')>\max\{0,-1+\frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^{d}\big(1+|p_i-p_i'|\big)^{\tau}(1+x_0(\boldsymbol{p}'))^{d-1}2^{-(m-2)x_0(\boldsymbol{p}')}$$

$$\lesssim \Big(1+\frac{J}{b}\Big)^{d-1}2^{-(m-2)\frac{J}{b}} \sum_{\substack{\boldsymbol{p}'\in\mathbb{N}_0^d \\ \log_2(\prod_{i=1}^d 1+|p_i-p_i'|)\le x}} \prod_{i=1}^{d}\big(1+|p_i-p_i'|\big)^{\tau+(m-2)\frac{a}{b}}.$$

Similar estimates as in (4.16) imply

$$\sum_{\substack{\boldsymbol{p}'\in\mathbb{N}_0^d \\ x_0(\boldsymbol{p}')>\max\{0,-1+\frac{d-1}{\ln(2)(m-2)}\}}} \prod_{i=1}^{d}\big(1+|p_i-p_i'|\big)^{\tau}(1+x_0(\boldsymbol{p}'))^{d-1}2^{-(m-2)x_0(\boldsymbol{p}')}$$

$$\lesssim\Big(1+\frac{J}{b}\Big)^{d-1}2^{-(m-2)\frac{J}{b}}\begin{cases}2^{(1+\tau+(m-2)\frac{a}{b})x}(1+x)^{d-1}, & \tau+(m-2)\frac{a}{b}>-1,\\ (1+x)^d, & \tau+(m-2)\frac{a}{b}=-1,\\ 1, & \tau+(m-2)\frac{a}{b}<-1,\end{cases}$$

$$\lesssim\Big(\Big(1+\frac{J}{b}\Big)^{d-1}2^{-(m-2)\frac{J}{b}}+\Big(1+\frac{J}{b}\Big)^{d-1}\Big(1+\frac{J}{a}\Big)^{d-1}2^{(1+\tau)\frac{J}{a}}\Big) \cdot\begin{cases}(1+\frac{J}{a}), & \tau+(m-2)\frac{a}{b}=-1,\\ 1, & \text{otherwise.}\end{cases} \tag{4.20}$$

Finally, combining (4.18) - (4.20) yields (4.10). □

5. Numerical experiments

For the numerical experiments we consider the Poisson equation with homogeneous Dirichlet boundary conditions on the L-shaped domain $\Omega=(-1,1)^2\backslash[0,1)^2$. In this case the bilinear form $a: H_0^1(\Omega)\times H_0^1(\Omega)\mapsto\mathbb{R}$ in (4.1) is given by

$$a(u,v)=\sum_{k=1}^{2}\int_\Omega \frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k}\,\mathrm{d}x.$$

This example is a standard test case for adaptive algorithms, since the reentrant corner induces certain singular solutions, see, e.g., [25], that have to be resolved by the numerical method under investigation. To obtain a quarklet frame for Ω we split the domain as explained in Section 3, into the subdomains $\Omega_0^{(0)}=\{(-1,0)\}+(0,1)^2$, $\Omega_1^{(0)}=\{(-1,-1)\}+(0,1)^2$ and $\Omega_2^{(0)}=\{(0,-1)\}+(0,1)^2$. These subdomains with their incorporated boundary conditions are depicted in Figure 2. The arrows indicate the direction of the non-trivial extension. By proceeding this way, conditions $(\mathcal{D}_1)$-$(\mathcal{D}_5)$ are fulfilled.

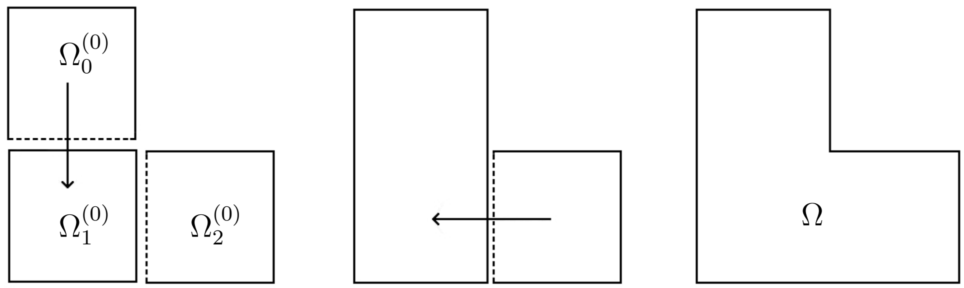

FIGURE 2. Dotted lines indicate free boundary conditions, straight lines indicate zero boundary conditions.

We equip $\Omega_0^{(0)}$ with $\mathbf{\Psi}_0^1 = \Psi^1_{(1,1)}(\cdot{+}1) \times \Psi^1_{(0,1)}$, $\Omega_1^{(0)}$ with $\mathbf{\Psi}_1^1 = \Psi^1_{(1,1)}(\cdot{+}1) \times \Psi^1_{(1,1)}(\cdot + 1)$ and $\Omega_2^{(0)}$ with $\mathbf{\Psi}_2 = \Psi^1_{(0,1)} \times \Psi^1_{(1,1)}(\cdot + 1)$. To obtain a quarklet frame for $H_0^1(\Omega)$ we extend $\mathbf{\Psi}_0^1$ and $\mathbf{\Psi}_2^1$ as described in Section 3. Essentially this corresponds to reflecting those quarklets that do not vanish at the boundaries at the dotted lines in Figure 2. After that, we take the union of the two resulting sets of functions with $\mathbf{\Psi}_1^1$. For the one-dimensional reference frame Ψ_σ^1 in $(0,1)$ we choose the biorthogonal spline wavelets of order $m = 3$ and $\tilde{m} = 3$ vanishing moments. We choose the right-hand side in (4.1) in such a way that the exact solution is the sum of $\sin(2\pi x)\sin(2\pi y)$, $(x, y) \in \Omega$ and the singularity function

$$\mathcal{S}(r,\theta) := 5\zeta(r) r^{2/3} \sin\left(\frac{2}{3}\theta\right), \tag{5.1}$$

with (r, θ) denoting polar coordinates with respect to the re-entrant corner at the origin, and where ζ is a smooth truncation function on $[0, 1]$, which is identically 1 on $[0, r_0]$ and 0 on $[r_1, 1]$, for some $0 < r_0 < r_1 < 1$, see again [25] for details. Singularity functions of the form (5.1) are typical examples of functions that have a very high Besov regularity but a very limited L_2-Sobolev smoothness due to the strong gradient at the reentrant corner. Therefore, for this kind of solution it can be expected that adaptive (h-)algorithms outperform classical uniform schemes. We refer, e.g., to [12, 13] for a detailed discussion of these relationships.

We also expect that the very smooth sinusoidal part of the solution can be very well approximated by piecewise polynomials of high order. Therefore, our test example is contained in the class of problems for which we expect a strong performance of adaptive quarklet schemes.

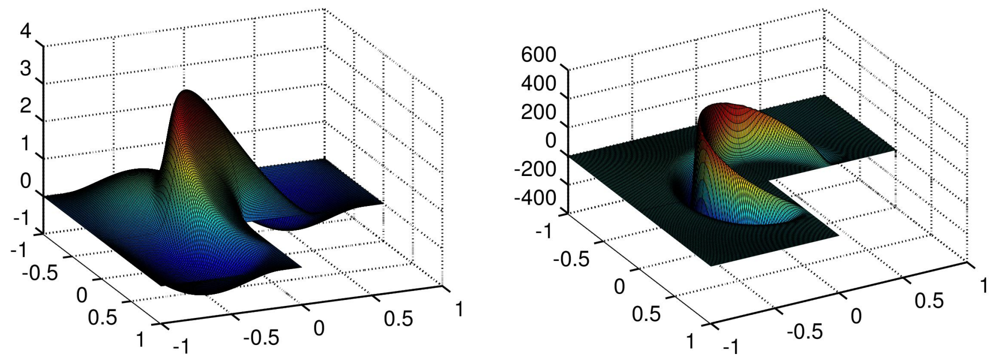

FIGURE 3. Exact solution and right-hand-side.

To solve the problem numerically we utilise an adaptive version of the damped Richardson iteration as described in (4.3). For details we refer to [10, 14, 31]. There, wavelet frames are used to discretize the PDE. But as long as we have compressible matrices all kinds of frames fit into this framework. Hence, we may apply this method also in the quarklet setting. In Figures 4-7 one can see approximate solutions produced by the adaptive scheme after successive iteration steps. In Figure 8 one can observe the ℓ_2-norm of the residual $\mathbf{A}\mathbf{u}^{(j)} - \mathbf{F}$ plotted against the degrees of freedom of the approximants $\mathbf{u}^{(j)}$ and against the spent CPU time. We see that the algorithm is convergent with convergence order $\mathcal{O}(N^{-2})$. In [13] an adaptive wavelet frame approach based on overlapping domain decompositions was used to solve a similar problem. Since the singularity function (5.1) has arbitrary high Besov regularity, the convergence order of adaptive wavelet schemes only depend on the order of the underlying spline system. For $m = 3$, one gets the approximation rate $\mathcal{O}(N^{-1})$, see again [13, Subsection 6.2] for details. If we compare this to our approach we see that the adaptive quarklet schemes outperform the adaptive wavelet schemes in terms of degrees of freedom.

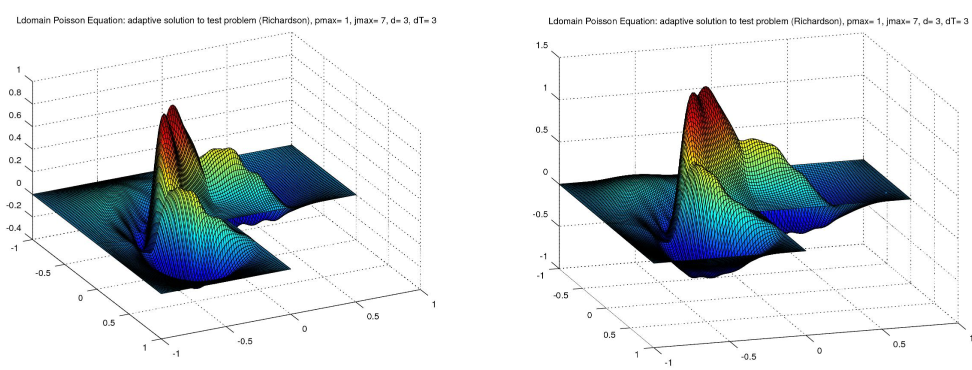

FIGURE 4. Adaptive solutions after 5, 10 iterations, respectively.

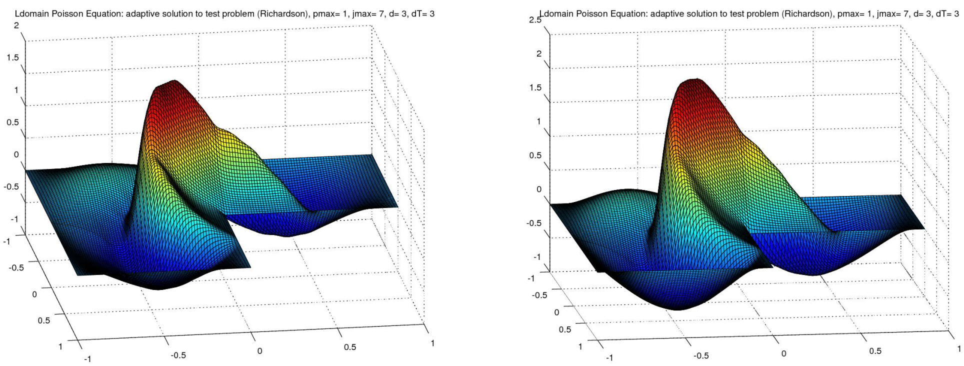

Figure 5. Adaptive solutions after 20, 30 iterations, respectively.

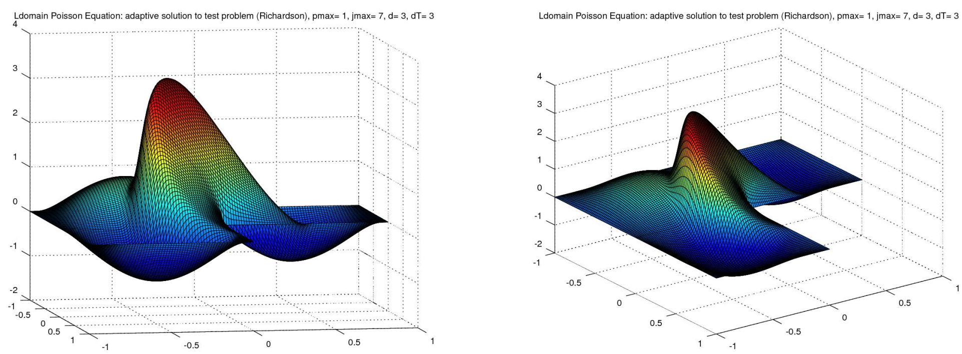

Figure 6. Adaptive solutions after 50, 100 iterations, respectively.

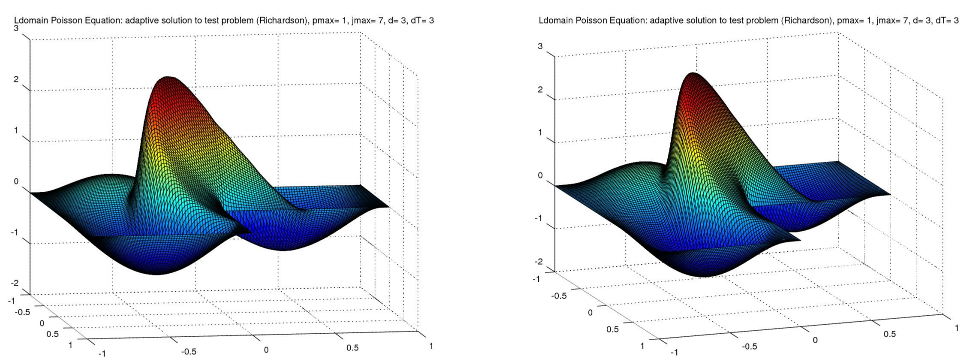

Figure 7. Adaptive solutions after 200, 359 iterations, respectively.

Figure 8 also shows that the CPU time that is currently needed might be improved. This observation indicates that maybe the compression estimates outlined in Section 4 are still suboptimal. Refined compression estimates based, e.g., on second compression ideas [29] will be the topic of further research.

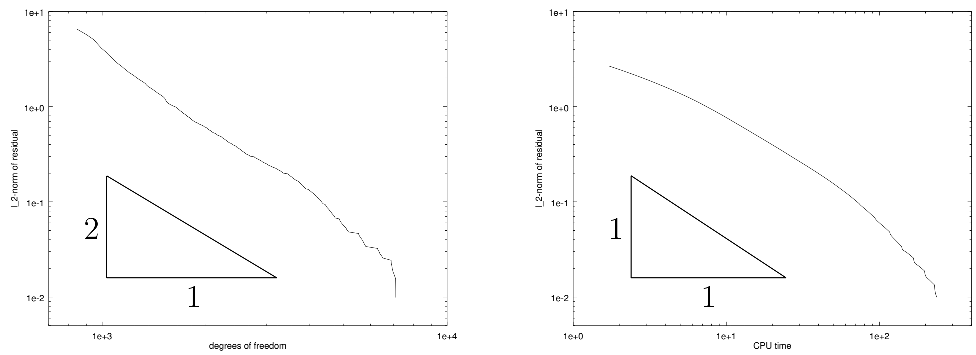

FIGURE 8. Adaptive error asymptotics

In Figures 9–11 the distribution of the coefficients $\mathbf{u} = \{u_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in\nabla}$ of the approximate solution $\sum_{\boldsymbol{\alpha}\in\nabla} u_{\boldsymbol{\alpha}}(\boldsymbol{w}_{\boldsymbol{\alpha}}^{H^1})^{-1}\boldsymbol{\psi}_{\boldsymbol{\alpha}}$ are plotted. In every single figure the coefficients for one fixed polynomial degree $\boldsymbol{p}$ are plotted, with j_1 and j_2 increasing in horizontal and vertical direction, respectively. We can see that qualitatively the distribution of the coefficients behaves as expected in the sense that frame elements with higher polynomial degree are more used in regions where the solution is very smooth.

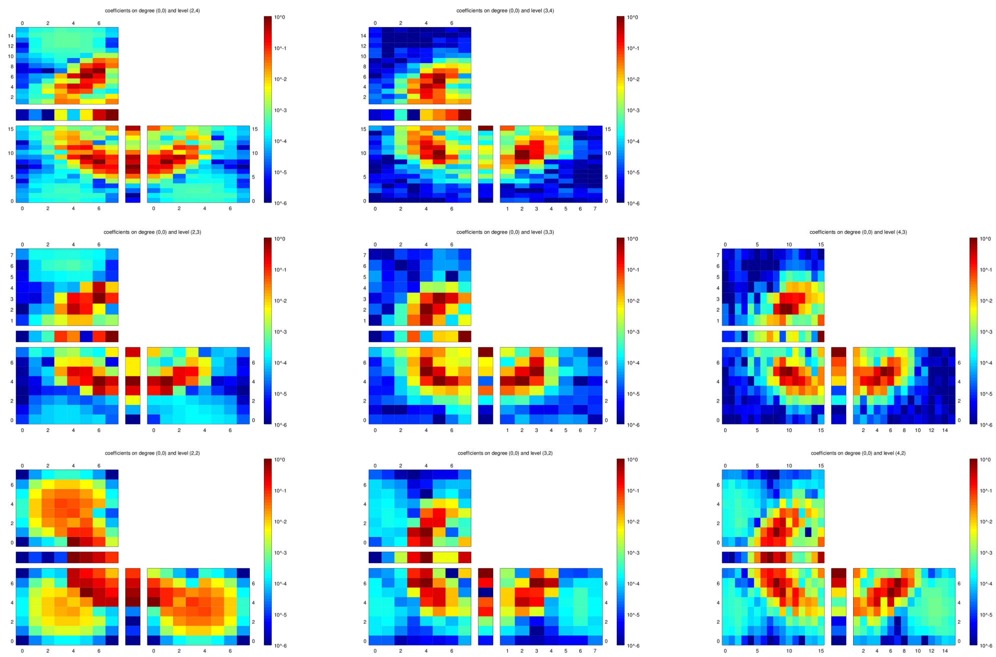

FIGURE 9. Polynomial degree $\boldsymbol{p} = (0, 0)$.

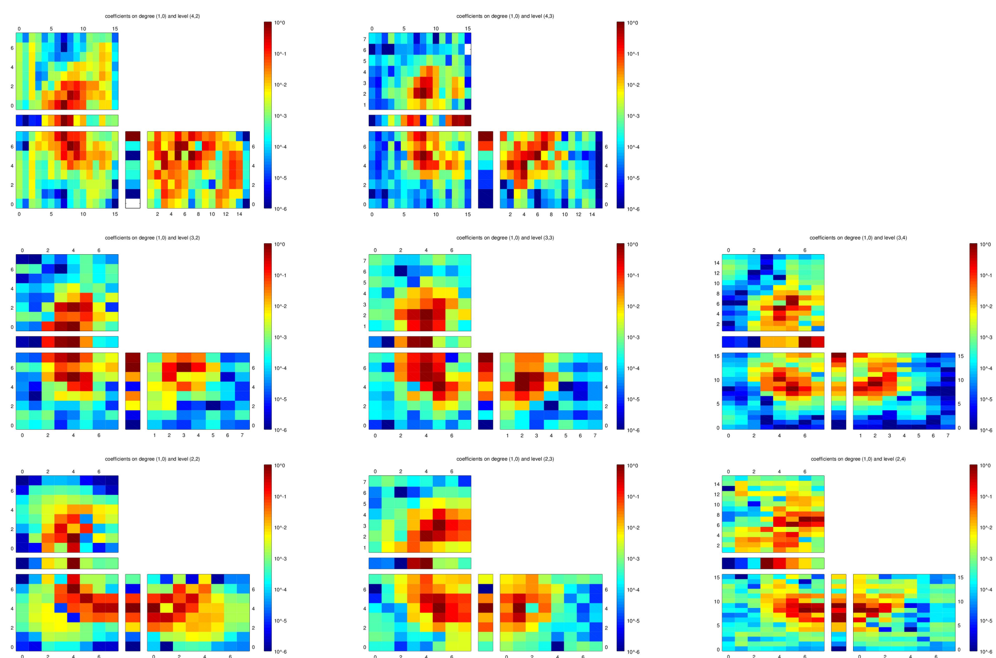

FIGURE 10. Polynomial degree $\boldsymbol{p} = (1, 0)$.

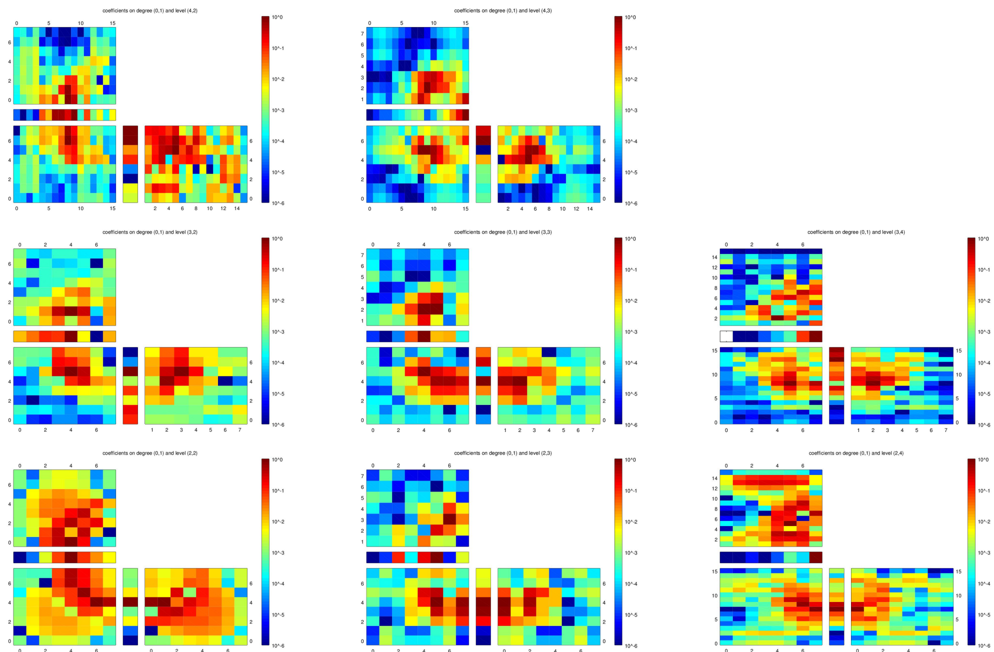

FIGURE 11. Polynomial degree $\boldsymbol{p} = (0, 1)$.

APPENDIX A

A.1. **Basic frame theory.** For the readers' convenience we collect some basic facts about frame theory that have been used throughout the paper. For a comprehensive overview of the topic of frames we refer to [8].

A frame is a stable representation system in a Hilbert space. In contrast to a Riesz basis it allows for redundancy.

Definition A.1. *Let $\mathcal{I}$ be a countable index set. A system $\mathcal{F} = \{f_\lambda\}_{\lambda \in \mathcal{I}} \subset \mathcal{H}$ is a* (Hilbert) frame *for a Hilbert space $\mathcal{H}$ if there exist constants $A, B > 0$ such that it holds*

$$A\|f\|_{\mathcal{H}}^2 \leq \|\{\langle f, f_\lambda\rangle_{\mathcal{H}}\}_{\lambda \in \mathcal{I}}\|_{\ell_2(\mathcal{I})}^2 \leq B\|f\|_{\mathcal{H}}^2 \tag{A.1}$$

for all $f \in \mathcal{H}$. The constants A and B are called lower and upper frame bound, respectively.

The constant B in (A.1) also is referred to as *Bessel bound*. To represent a function via a frame, we introduce the *synthesis operator*

$$F : \ell_2(\mathcal{I}) \to \mathcal{H}, \quad \mathbf{c} \mapsto \sum_{\lambda \in \mathcal{I}} c_\lambda f_\lambda \tag{A.2}$$

and its adjoint

$$F^* : \mathcal{H} \to \ell_2(\mathcal{I}), \quad g \mapsto \{\langle f, f_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}}, \tag{A.3}$$

called the *analysis operator*. Composing both leads to the so-called *frame operator*

$$S : \mathcal{H} \to \mathcal{H}, \quad f \mapsto Sf := FF^*g = \sum_{\lambda\in\mathcal{I}} \langle f, f_\lambda\rangle_{\mathcal{H}} f_\lambda. \tag{A.4}$$

One can show that the frame operator is bounded and invertible and that the system $\tilde{\mathcal{F}} := \{\tilde{f}_\lambda\}_{\lambda\in\mathcal{I}} := \{S^{-1}f_\lambda\}_{\lambda\in\mathcal{I}}$ is also a frame for $\mathcal{H}$, called the *canonical dual frame*. The canonical dual frame puts us into the position to introduce a *frame decomposition*. Due to the fact that $SS^{-1} = S^{-1}S = \mathrm{Id}_{\mathcal{H}}$, we have

$$f = \sum_{\lambda\in\mathcal{I}} \langle f, \tilde{f}_\lambda\rangle_{\mathcal{H}}\, f_\lambda = \sum_{\lambda\in\mathcal{I}} \langle f, f_\lambda\rangle_{\mathcal{H}} \tilde{f}_\lambda, \quad \text{for all } f \in \mathcal{H}. \tag{A.5}$$

In general, (A.5) is not the only possible decomposition. If there exist other decompositions than (A.5) we say that a frame is *redundant*. Systems $\mathcal{G} = \{g_\lambda\}_{\lambda\in\mathcal{I}} \neq \tilde{\mathcal{F}}$ in $\mathcal{H}$, for which

$$f = \sum_{\lambda\in\mathcal{I}} \langle f, g_\lambda\rangle_{\mathcal{H}}\, f_\lambda, \quad \text{for all } f \in \mathcal{H}, \tag{A.6}$$

are called *non-canonical dual frames* or just *dual frames*. As the name suggests, they are indeed frames for $\mathcal{H}$.

An alternative characterization of a frame which makes use of the synthesis operator is given in the next proposition. It is applied throughout the paper as a proof technique. A proof of it can be found in [36, Proposition 2.2].

Proposition A.2. *A system $\mathcal{F} = \{f_\lambda\}_{\lambda\in\mathcal{I}} \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if* $\mathrm{clos}_{\mathcal{H}}(\mathrm{span}(\mathcal{F})) = \mathcal{H}$ *and*

$$B^{-1}\|f\|^2_{\mathcal{H}} \le \inf_{\{\mathbf{c}\in\ell_2(\mathcal{I}),\ F\mathbf{c}=f\}} \|\mathbf{c}\|^2_{\ell_2(\mathcal{I})} \le A^{-1}\|f\|^2_{\mathcal{H}}, \quad \textit{for all } f \in \mathcal{H}. \tag{A.7}$$

The constants A and B in (A.7) coincide with the ones used in (A.1). Let us mention that another criterion for a system $\mathcal{F} = \{f_\lambda\}_{\lambda\in\mathcal{I}} \subset \mathcal{H}$ to be a frame for $\mathcal{H}$ is that its synthesis operator as defined in (A.2) is a well-defined mapping of $\ell_2(\mathcal{I})$ onto $\mathcal{H}$, cf. [8, Theorem 5.5.1].

A slightly weaker concept than a frame is a *Bessel system* for $\mathcal{H}$. We call a system $\mathcal{B} = \{b_\lambda\}_{\lambda\in\mathcal{I}} \subset \mathcal{H}$ a Bessel system for $\mathcal{H}$ if the right hand side inequality in (A.1) holds, i.e., there exists a constant $B > 0$ such that

$$\| \{\langle f, b_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}} \|^2_{\ell_2(\mathcal{I})} \le B\|f\|^2_{\mathcal{H}}, \tag{A.8}$$

for all $f \in \mathcal{H}$. Equivalently, a system is a Bessel system if it fulfills the left hand side inequality in (A.7). The following propositions state some facts about the union of Bessel systems, frames and Riesz bases.

Proposition A.3. *Let $\mathcal{H}$ be a Hilbert space. Then, it holds:*

(i) *The union of finitely many Bessel systems for $\mathcal{H}$ is a Bessel system for $\mathcal{H}$.*
(ii) *A frame for $\mathcal{H}$ united with a Bessel system for $\mathcal{H}$ is a frame for $\mathcal{H}$.*
(iii) *A Bessel system for $\mathcal{H}$ which includes a Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$.*

Proof. To prove (i), we assume $\mathcal{B}_i = \{b_\lambda\}_{\lambda\in\mathcal{I}_i} \subset \mathcal{H}$, $i = 1, \ldots, n$, to be Bessel systems for $\mathcal{H}$ with Bessel bounds $B_i > 0$, $i = 1, \ldots, n$. Let $\mathcal{B} = \bigcup_{i=1}^n \mathcal{B}_i$ and $\mathcal{I} = \bigcup_{i=1}^n \mathcal{I}_i$. Then, for a constant $C > 0$ we have

$$\| \{\langle f, b_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}} \|^2_{\ell_2(\mathcal{I})} \leq C \sum_{i=1}^n \| \{\langle f, b_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}_i} \|^2_{\ell_2(\mathcal{I}_i)} \leq C \sum_{i=1}^n B_i \|f\|^2_{\mathcal{H}}.$$

For (ii) we assume $\mathcal{F} = \{f_\lambda\}_{\lambda\in\mathcal{I}_1}$ and $\mathcal{B} = \{f_\lambda\}_{\lambda\in\mathcal{I}_2}$ to be a frame and a Bessel system for $\mathcal{H}$, respectively. As every frame is a Bessel system, the right hand inequality follows immediately from (i). For the left hand inequality we write

$$\| \{\langle f, f_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}_1\cup\mathcal{I}_2} \|^2_{\ell_2(\mathcal{I}_1\cup\mathcal{I}_2)} \geq \| \{\langle f, f_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}_1} \|^2_{\ell_2(\mathcal{I}_1)} \geq A\|f\|^2_{\mathcal{H}},$$

with $A > 0$ a lower frame bound of $\mathcal{F}$. For the proof of part (iii) we consider a Bessel system $\mathcal{B} = \{b_\lambda\}_{\lambda\in\mathcal{I}}$ for $\mathcal{H}$ which contains a Riesz basis $\mathcal{R} = \{b_\lambda\}_{\lambda\in\mathcal{I}_R}$ for $\mathcal{H}$. We only have to show the left-hand side inequality in (A.1). We write

$$\| \{\langle f, b_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}} \|^2_{\ell_2(\mathcal{I})} \geq \| \{\langle f, b_\lambda\rangle_{\mathcal{H}}\}_{\lambda\in\mathcal{I}_R} \|^2_{\ell_2(\mathcal{I}_R)} \geq A\|f\|^2_{\mathcal{H}},$$

with $A > 0$ a lower Riesz bound of $\mathcal{R}$. To perform the last estimate we used the fact that every Riesz basis is also a frame, c.f. [8, Theorem 5.4.1]. $\square$

To conclude this subsection we state a proposition which considers the image of frames, Bessel systems and Riesz bases under certain operators.

Proposition A.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $U : \mathcal{H}_1 \mapsto \mathcal{H}_2$ an operator. Then, it holds:*

(i) *If $\mathcal{B}$ is a Bessel system for $\mathcal{H}_1$ and U is bounded, then $U\mathcal{B}$ is a Bessel system for $\mathcal{H}_2$.*
(ii) *If $\mathcal{F}$ is a frame for $\mathcal{H}_1$ and U is bounded and surjective, then $U\mathcal{F}$ is a frame for $\mathcal{H}_2$.*
(iii) *If $\mathcal{R}$ is a Riesz bases for $\mathcal{H}_1$ and U is bounded and invertible, then $U\mathcal{R}$ is a Riesz basis for $\mathcal{H}_2$.*

Proof. At first, we assume that U is bounded and $\mathcal{B} = \{b_\lambda\}_{\lambda\in\mathcal{I}}$ is a Bessel system for $\mathcal{H}_1$. For $g \in H_2$, it is

$$\| \{\langle g, Ub_\lambda\rangle_{\mathcal{H}_2}\}_{\lambda\in\mathcal{I}} \|^2_{\ell_2(\mathcal{I})} = \| \{\langle U^*g, b_\lambda\rangle_{\mathcal{H}_1}\}_{\lambda\in\mathcal{I}} \|^2_{\ell_2(\mathcal{I})} \leq B\|U^*g\|^2_{\mathcal{H}_1} \leq B\|U\|^2_{\mathcal{H}_1\mapsto\mathcal{H}_2}\|g\|^2_{\mathcal{H}_2}.$$

For the last inequality we used $\|U\|_{\mathcal{H}_1\mapsto\mathcal{H}_2} = \|U^*\|_{\mathcal{H}_2\mapsto\mathcal{H}_1}$. For a proof of part (ii) we refer to [8, Corollary 5.3.2]. To show (iii) we use the fact, that a system $\mathcal{R} = \{r_\lambda\}_{\lambda\in\mathcal{I}}$

is a Riesz bases for a Hilbert space $\mathcal{H}$ if and even if there exists a Hilbert space $\mathcal{K}$ with an orthonormal basis $\{e_\lambda\}_{\lambda\in\mathcal{I}}$ and a bounded and invertible operator $V : \mathcal{K} \mapsto \mathcal{H}$, such that $\mathcal{R} = \{Ve_\lambda\}_{\lambda\in\mathcal{I}}$, cf. [8, Definition 3.6.1]. So let $\mathcal{R} = \{r_\lambda\}_{\lambda\in\mathcal{I}}$ be a Riesz basis for $\mathcal{H}_1$ and U bounded and invertible. As mentioned above, $\mathcal{R}$ can be written as $\{Ve_\lambda\}_{\lambda\in\mathcal{I}}$, with $V : \mathcal{K} \mapsto \mathcal{H}_1$ bounded and invertible. The composition $UV : \mathcal{K} \mapsto \mathcal{H}_2$ is bounded and invertible as well. Thus, the system $U\mathcal{R} = \{UVe_\lambda\}_{\lambda\in\mathcal{I}}$ is a Riesz basis for $\mathcal{H}_2$. □

A.2. **Proofs.** In this subsection, we present the proofs of two technical results stated in Section 2.

Lemma A.5. *Let $1 \le k \le m-1$ and $\varphi_{p,0,-m+k}$ a left boundary quark. For every $p \ge (m-1)(k-1)$ the unique extremal point of $\varphi_{p,0,-m+k}$ is located at*

$$\hat{x} = \frac{kp}{p+m-1}. \tag{A.9}$$

Proof. Let $x \in \mathbb{R}$. At First we have a look at the leftmost quark, i.e. $k = 1$:

$$\varphi_{p,0,-m+1}(x) = \left(\frac{x}{-m+1+m}\right)^p B^m_{0,-m+1}(x) = x^p B^m_{0,-m+1}(x).$$

Using the differentiation rules and the recursive form of the B-splines, cf. [30, Thm. 4.15, 4.16], we obtain

$$\begin{aligned}
\varphi'_{p,0,-m+1}(x) &= px^{p-1}B^m_{0,-m+1}(x) + x^p B^{m\prime}_{0,-m+1}(x)\\
&= px^{p-1}B^m_{0,-m+1}(x) - x^p(m-1)B^{m-1}_{0,-m+2}(x)\\
&= px^{p-1}\frac{t_1 - x}{t_1 - t_{-m+2}}B^{m-1}_{0,-m+2}(x) - x^p(m-1)B^{m-1}_{0,-m+2}(x)\\
&= x^{p-1}\left(p(1-x) - x(m-1)\right)B^{m-1}_{0,-m+2}(x).
\end{aligned}$$

We obtain the critical points $x = 0$, where the B-spline and also the quark is zero, and $\hat{x} = \frac{p}{p+m-1}$, where $|\varphi_{p,0,-m+1}|$ attends its maximum. Now assume $m \ge 3$, $k \ge 2$ and $\varphi_{p,0,-m+k}$ is the k-th left boundary quark.:

$$\varphi_{p,0,-m+k}(x) = \left(\frac{x}{-m+k+m}\right)^p B^m_{0,-m+k}(x) = k^{-p}x^p B^m_{0,-m+k}(x).$$

The support of $\varphi_{p,0,-m+k}$ is the interval $[0,k]$. In the first step we show that $\varphi_{p,0,-m+k}$ is monotonically increasing on $[0,k-1]$. For the first derivative we estimate

$$\begin{aligned}
\varphi'_{p,0,-m+k}(x) &= k^{-p}px^{p-1}B^m_{0,-m+k}(x) + k^{-p}x^p B^{m\prime}_{0,-m+k}(x)\\
&= k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) + xB^{m\prime}_{0,-m+k}(x)\right)\\
&\ge k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - \left|xB^{m\prime}_{0,-m+k}(x)\right|\right).
\end{aligned}$$

Again we use the differentiation rules and recursion to derive

$$\begin{aligned}\varphi'_{p,0,-m+k}(x) &\geq k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - \left|x(m-1)\left(\frac{B^{m-1}_{0,-m+k}(x)}{k-1} - \frac{B^{m-1}_{0,-m+k+1}(x)}{k}\right)\right|\right)\\ &\geq k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - x(m-1)\left(\frac{B^{m-1}_{0,-m+k}(x)}{k-1} + \frac{B^{m-1}_{0,-m+k+1}(x)}{k}\right)\right).\end{aligned}$$

For $x \in [0,1]$ it holds $k - x \geq x$, which yields

$$\begin{aligned}\varphi'_{p,0,-m+k}(x) &\geq k^{-p}x^{p-1}\\ &\cdot\left(pB^m_{0,-m+k}(x) - (m-1)\left(\frac{x}{k-1}B^{m-1}_{0,-m+k}(x) + \frac{k-x}{k}B^{m-1}_{0,-m+k+1}(x)\right)\right)\\ &= k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - (m-1)B^m_{0,-m+k}(x)\right)\\ &= k^{-p}x^{p-1}\left(p-(m-1)\right)B^m_{0,-m+k}(x).\end{aligned}$$

Hence the derivative is non-negative on $[0,1]$ if $p \geq m-1$. For $x \in [1, k-1]$, it trivially holds $x \geq 1$ and $k - x \geq 1$. It follows

$$\begin{aligned}\varphi'_{p,0,-m+k}(x) &\geq k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - x\left|B^{m\prime}_{0,-m+k}(x)\right|\right)\\ &\geq k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - (k-1)\left|B^{m\prime}_{0,-m+k}(x)\right|\right)\\ &= k^{-p}x^{p-1}\\ &\cdot\left(pB^m_{0,-m+k}(x) - (k-1)\left|(m-1)\left(\frac{B^{m-1}_{0,-m+k}(x)}{k-1} - \frac{B^{m-1}_{0,-m+k+1}(x)}{k}\right)\right|\right).\end{aligned}$$

By the above considerations we can further estimate

$$\begin{aligned}&\varphi'_{p,0,-m+k}(x)\\ &\geq k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - (k-1)(m-1)\left|\frac{1}{k-1}B^{m-1}_{0,-m+k}(x) - \frac{1}{k}B^{m-1}_{0,-m+k+1}(x)\right|\right)\\ &\geq k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - (k-1)(m-1)\left(\frac{1}{k-1}B^{m-1}_{0,-m+k}(x) + \frac{1}{k}B^{m-1}_{0,-m+k+1}(x)\right)\right)\\ &\geq k^{-p}x^{p-1}\\ &\cdot\left(pB^m_{0,-m+k}(x) - (k-1)(m-1)\left(\frac{x}{k-1}B^{m-1}_{0,-m+k}(x) + \frac{k-x}{k}B^{m-1}_{0,-m+k+1}(x)\right)\right).\end{aligned}$$

By the recursive relation of B-splines we get

$$\begin{aligned}\varphi'_{p,0,-m+k}(x) &\geq k^{-p}x^{p-1}\left(pB^m_{0,-m+k}(x) - (k-1)(m-1)B^m_{0,-m+k}(x)\right)\\ &= k^{-p}x^{p-1}\left(p-(k-1)(m-1)\right)B^m_{0,-m+k}(x).\end{aligned}$$

Finally we can conclude that for $p \geq (m-1)(k-1)$ the derivative is non-negative on $[1, k-1]$. So all extremal points are located in $[k-1, k]$, where we can compute an explicit form of $\varphi_{p,0,-d+k}$. To do this, we first compute the explicit form of $B^d_{0,-d+k}$. By definition and the recursion for divided differences we get:

$$\begin{aligned} B^m_{0,-m+k}(x) &= (t^0_k - t^0_{-m+k})\,(\cdot - x)^{m-1}_+[t^0_{-m+k}, \dots, t^0_k] \\ &= \frac{k}{k}\left((\cdot - x)^{m-1}_+[t^0_{-m+k+1}, \dots, t^0_k] - (\cdot - x)^{m-1}_+[t^0_{-m+k}, \dots, t^0_{k-1}]\right) \\ &= (\cdot - x)^{m-1}_+[t^0_{-m+k+1}, \dots, t^0_k]. \end{aligned}$$

The latter divided difference vanishes, because of $x \geq k-1$. On the interval $[0, k-1]$ the truncated polynomial $(\cdot - x)^{m-1}_+$ is zero. Hence all of the coefficients of the interpolating polynomial are zero. By repeating this argument $m-k-1$ times we obtain

$$B^m_{0,-m+k}(x) = k^{-1-(m-k-1)}(\cdot - x)^{m-1}_+[1, \dots, k].$$

Further $k-1$ times iteration gives

$$B^m_{0,-m+k}(x) = k^{-m+k}\frac{1}{(k-1)!}(\cdot - x)^{m-1}_+[k].$$

We end up with

$$B^d_{0,-m+k}|_{[k-1,k]}(x) = k^{-m+k}\frac{1}{(k-1)!}(k-x)^{m-1}.$$

With this representation we compute the derivative $\varphi'_{p,0,-m+k}$ on $[k-1, k]$:

$$\begin{aligned} \varphi'_{p,0,-m+k}(x) &= k^{-p}px^{p-1}B^m_{0,-m+k}(x) + k^{-p}x^p B^{m\prime}_{0,-m+k}(x) \\ &= k^{-p}x^{p-1} \\ &\cdot \left(pk^{-m+k}\frac{1}{(k-1)!}(k-x)^{m-1} - xk^{-m+k}\frac{1}{(k-1)!}(m-1)(k-x)^{m-2}\right) \\ &= k^{-p-m+k}x^{p-1}\frac{1}{(k-1)!}\left((k-x)^{m-2}(p(k-x) - x(m-1)\right). \end{aligned}$$

We obtain the critical points $x = 0$, $x = k$, where $B^m_{0,-m+k}$ is zero, and $\hat{x} = \frac{kp}{p+m-1}$, where $|\varphi_{p,0,-m+k}|$ attains its maximum. Indeed $\hat{x}$ lies in $[k-1, k]$, because on the one hand we have

$$\hat{x} = \frac{kp}{p+d-1} \leq \frac{kp + k(d-1)}{p+d-1} = \frac{k(p+d-1)}{p+d-1} = k.$$

On the other hand it holds true that

$$k-1 = k - \frac{k(d-1)}{k(d-1)} = k - \frac{k(d-1)}{(d-1)(k-1)+d-1} \leq k - \frac{k(d-1)}{p+d-1} = \frac{kp}{p+d-1} = \hat{x}.$$

□

Proposition A.6. *Let $1 \leq k \leq m-1$ and $\varphi_{p,0,-m+k}$ a left boundary quark. For every $1 \leq q \leq \infty$ there exist constants $c = c(m,k,q) > 0$, $C = C(m,k,q) > 0$, so that for all $p \geq (m-1)(k-1)$:*

$$c(p+1)^{-(m-1+1/q)} \leq ||\varphi_{p,0,-m+k}||_{L_q(\mathbb{R})} \leq C(p+1)^{-(m-1+1/q)}. \tag{A.10}$$

Proof. We show (A.10) for the extremal cases $q \in \{1, \infty\}$ and conclude by the Hölder inequality. To derive the upper bound for $q = 1$ we use an integration formula for general B-splines and functions $f \in C^m([t^0_{-m+k}, t^0_k])$, cf. [30, Thm. 4.23]:

$$\int_{t^0_{-m+k}}^{t^0_k} B^m_{0,-m+k}(x) f^{(m)}(x)\ \mathrm{d}x = (t^0_k - t^0_{-m+k})(m-1)! f[t^0_{-m+k}, \ldots, t^0_k].$$

Choosing $f(x) := x^{p+m} \frac{1}{(p+m)\cdots(p+1)}$ we obtain

$$\begin{aligned}
||\varphi_{p,0,-m+k}||_{L_1(\mathbb{R})} &= \left(\frac{1}{k}\right)^p \int_{t^0_{-m+k}}^{t^0_k} B^m_{0,-m+k}(x) x^p\ \mathrm{d}x \\
&= \left(\frac{1}{k}\right)^p (k-0)(m-1)!\, (\cdot)^{p+m}[t^0_{-m+k}, \ldots, t^0_k] \frac{1}{(p+m)\cdots(p+1)} \\
&\leq \left(\frac{1}{k}\right)^{p-1} (m-1)!\, (\cdot)^{p+m}[t^0_{-m+k}, \ldots, t^0_k](p+1)^{-m}.
\end{aligned}$$

To estimate the divided difference we use a Leibniz rule with $x^{p+m} = x x^{p+m-1}$, cf. [30, Thm. 2.52]:

$$(\cdot)^{p+m}[t^0_{-m+k}, \ldots, t^0_k] = \sum_{i=-k+m}^{k} (\cdot)^1[t^0_{-m+k}, \ldots, t^0_i]\ (\cdot)^{p+m-1}[t^0_i, \ldots, t^0_k].$$

For the first order polynomial there remains just one non-trivial summand:

$$\begin{aligned}
(\cdot)^{p+m}[t^0_{-m+k}, \ldots, t^0_k] &= (\cdot)^1[t^0_{-m+k}]\ (\cdot)^{p+m-1}[t^0_{-m+k}, \ldots, t^0_k] \\
&\quad + (\cdot)^1[t^0_{-m+k}, t^0_{-m+k+1}]\ (\cdot)^{p+m-1}[t^0_{-m+k+1}, \ldots, t^0_k] \\
&= (\cdot)^{p+m-1}[t^0_{-m+k+1}, \ldots, t^0_k].
\end{aligned}$$

Repeating this argument $d-k$ times we get

$$(\cdot)^{p+m}[t_{-m+k}, \ldots, t_k] = (\cdot)^{p+k}[t^0_0, \ldots, t^0_k].$$

By eliminating the leading zeros we get equidistant knots and can replace the divided difference by a forward difference, cf. [30, Thm. 2.57]:

$$(\cdot)^{p+m}[t^0_{-m+k},...,t^0_k] = \frac{1}{k!}(\Delta^k(\cdot)^{p+k})(0) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^{p+k} \le \frac{1}{k!}k^p\sum_{j=0}^{k}\binom{k}{j}j^k.$$

Finally we get the upper estimate with $C(m,k) = \frac{(m-1)!}{(k-1)!}\sum_{j=0}^{k}\binom{k}{j}j^k$:

$$||\varphi_{p,0,-m+k}||_{L_1(\mathbb{R})} \le C(p+1)^{-m}. \tag{A.11}$$

Now let $q=\infty$. We directly compute

$$\begin{aligned}||\varphi_{p,0,-m+k}||_{L_\infty(\mathbb{R})} &= |\varphi_{p,0,-m+k}(\hat{x})| = k^{-p}\hat{x}^p k^{-m+k}\frac{1}{(k-1)!}(k-\hat{x})^{m-1}\\ &= \frac{k^{-m+k}}{(k-1)!}\left(\frac{p}{p+m-1}\right)^p\left(\frac{k(m-1)}{p+m-1}\right)^{m-1}.\end{aligned}$$

We get the upper estimate with some constant $C(m,k) = \frac{k^{-m+k}}{(k-1)!}(k(m-1))^{m-1}$:

$$||\varphi_{p,0,-m+k}||_{L_\infty(\mathbb{R})} \le C(p+1)^{-(m-1)}. \tag{A.12}$$

For $1<q<\infty$ an application of the Hölder inequality and (A.11), (A.12) yield

$$\begin{aligned}||\varphi_{p,0,-m+k}||^q_{L_q(\mathbb{R})} &\le ||\varphi_{p,0,-m+k}||^{1/q}_{L_1(\mathbb{R})}||\varphi_{p,0,-m+k}||^{1-1/q}_{L_\infty(\mathbb{R})}\\ &\le C(p+1)^{-(m-1-1/q)},\end{aligned}$$

which proves the upper estimate. Now we turn over to the lower estimate. Let $q=\infty$. From our previous calculations we directly get the lower estimate with $c(m,k) = \tilde{c}e^{1-m}\frac{k^{-m+k}}{(k-1)!}(k(m-1))^{m-1}$, where $\tilde{c}>0$ just depends on m:

$$c(p+1)^{-(m-1)} \le ||\varphi_{p,0,-m+k}||_{L_\infty(\mathbb{R})}. \tag{A.13}$$

It remains to show the lower estimate for $q\in\mathbb{N}$. An elementary estimate leads to

$$\begin{aligned}||\varphi_{p,0,-m+k}||^q_{L_q(\mathbb{R})} &= \int_0^k |\varphi_{p,0,-m+k}(x)|^q \,\mathrm{d}x \ge \int_{k-1}^k |\varphi_{p,0,-m+k}(x)|^q \,\mathrm{d}x\\ &= \int_{k-1}^k \left(\frac{x}{k}\right)^{pq}\left(B^m_{0,-m+k}(x)\right)^q \,\mathrm{d}x\\ &= \int_{k-1}^k \left(\frac{x}{k}\right)^{pq}\left(k^{-m+k}\frac{1}{(k-1)!}(k-x)^{m-1}\right)^q \,\mathrm{d}x.\end{aligned}$$

Substitution leads to

$$\begin{aligned}||\varphi_{p,0,-m+k}||^q_{L_q(\mathbb{R})} &\geq \frac{1}{((k-1)!)^q} k^{(-m+k)q} \int_{k-1}^{k} \left(\frac{x}{k}\right)^{pq} (k-x)^{(m-1)q} \,\mathrm{d}x \\ &= \frac{1}{((k-1)!)^q} k^{(-m+k)q} \int_0^1 \left(\frac{k-y}{k}\right)^{pq} y^{(m-1)q} \,\mathrm{d}y \\ &\geq \frac{1}{((k-1)!)^q} k^{(-m+k)q} \int_0^1 (1-y)^{pq}\, y^{(m-1)q} \,\mathrm{d}y.\end{aligned}$$

By $m(q-1)$ times partial integration we obtain

$$\begin{aligned}\int_0^1 (1-y)^{pq} y^{(m-1)q} \,\mathrm{d}y &= \frac{(m-1)q}{pq+1} \int_0^1 (1-y)^{pq+1} y^{(m-1)q-1} \,\mathrm{d}y \\ &= \frac{((m-1)q)!}{(pq+1)(pq+2)\cdots(pq+mq-q)} \frac{1}{pq+mq-q+1}.\end{aligned}$$

We go on estimating by

$$\int_0^1 (1-y)^{pq} y^{(m-1)q} \,\mathrm{d}y \geq \frac{((m-1)q)!}{(\tilde{c}(p+1))^{mq-q+1}},$$

where $\tilde{c} > 0$ just depends on m and q. Finally we get the lower estimate with $c(m,k,q) = \frac{k^{-m+k}}{(k-1)!} \left(\frac{((m-1)q)!}{\tilde{c}}\right)^{1/q}$:

$$||\varphi_{p,0,-m+k}||_{L_q(\mathbb{R})} \geq c(p+1)^{-(m-1+1/q)}. \tag{A.14}$$

For $1 < q < \infty$ we again use Hölder's inequality. First let $1 < q \leq 2$, then by (A.13),(A.14) it follows

$$\begin{aligned}||\varphi_{p,0,-m+k}||^q_{L_q(\mathbb{R})} &\geq ||\varphi_{p,0,-m+k}||^{2/q}_{L_2(\mathbb{R})} ||\varphi_{p,0,-m+k}||^{1-2/q}_{L_\infty(\mathbb{R})} \\ &\geq c(p+1)^{-(m-1-1/q)}.\end{aligned}$$

For $2 \leq q < \infty$, using (A.14) we have

$$\begin{aligned}||\varphi_{p,0,-m+k}||^q_{L_q(\mathbb{R})} &\geq ||\varphi_{p,0,-m+k}||^{2-2/q}_{L_2(\mathbb{R})} ||\varphi_{p,0,-m+k}||^{2/q-1}_{L_1(\mathbb{R})} \\ &\geq c(p+1)^{-(m-1-1/q)},\end{aligned}$$

which completes the proof. $\square$

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