Quadrature Formulas for Refinable Functions and Wavelets II: Error Analysis

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Abstract. This paper is concerned with the construction and the analysis of Gauss quadrature formulas for computing integrals of (smooth) functions against refinable functions and wavelets. The main goal of this paper is to develop rigorous error estimates for these formulas. For the univariate setting, we derive asymptotic error bounds for a huge class of weight functions including spline functions. We also discuss multivariate quadrature rules and present error estimates for specific nonseparable refinable functions, i.e., for some special box splines.

Key Words: Gauss quadrature, scaling functions, wavelets, splines, error estimates.

AMS subject classification: Primary 65D32, secondary 41A30, 42C15.

1 Introduction

The application of wavelets to practical problems requires the computation of inner products of a given function with wavelets or with an associated refinable function. In most cases, this can not be done directly by finding primitives, so that suitable quadrature formulas are needed. However, most of the classical quadrature formulas may not perform very well, because neither the refinable functions nor the wavelets are necessarily very smooth. Moreover, in many cases, these functions are only known implicitly via certain functional equations from which the function values have to be computed or approximated. This is usually expensive and/or inaccurate.

In [BBDK] Gauss quadrature formulas for refinable functions and wavelets were derived using these functions as weight functions. As the weight function only has to be nonnegative, this approach circumvents the difficulties due to the lacking smoothness. The construction presented there is quite general and works very well for both, the univariate and the multivariate case.

This paper can be viewed as a continuation of [BBDK], discussing several questions that remained open there. Especially, we shall be concerned with rigorous error estimates for

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the quadrature formulas constructed in [BBDK]. For the univariate case, we will present asymptotic error estimates for a wide class of weight functions. For the multivariate setting, we will provide and discuss methods to estimate the error for formulas of any order, at least for some specific weight functions.

Another important issue that we will deal with is the following. In practice, many applications require the use of wavelets on bounded domains. Then near the boundary one has to use so–called *boundary adapted* wavelets and scaling functions, see, e.g., [DKU]. Then the question arises how to design suitable Gauss quadrature rules for these boundary adapted functions. The problem is that these functions are no longer refinable in the classical sense. This causes difficulties concerning the computation of their moments which is an essential step in any Gauss quadrature rule. We present one specific solution to this problem here.

The outline of the paper is as follows. In Section 2, we shortly review the wavelet setting as far as we need it for our purposes. Section 3 is devoted to the (univariate) Gauss quadrature. Based on a well–known result given in [Sz], we derive estimates for the leading coefficients of the associated orthogonal polynomials. They can be used to state and to prove the desired error estimates for a huge class of weight functions including spline functions. We also summarize the construction principles for Gauss formulas for wavelets and refinable functions on the real line as introduced in [BBDK]. In Section 4 we apply the machinery developed in Section 3 to the case of B–spline wavelets and refinable functions. We derive Gauss quadrature rules for boundary adapted functions, and investigate the error estimates for both, the usual cardinal B–splines and the boundary adapted versions. In Section 5, we will treat the multidimensional case. We review the construction of quadrature formulas for nonseparable multivariate weight functions. Then, we derive some error estimates for these cases. By using the general theory as outlined in [St], we present rigorous error bounds for some special weight functions, i.e., for specific box splines which are frequently used in practice.

2 The wavelet setting

In this section, we shall briefly recall the basic setting of wavelet analysis as far as it is needed for our purposes. In general, a function ψ is called a *wavelet* if all its scaled, dilated, and integer translated versions

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbb{Z}, \tag{1}$$

form a Riesz basis of $L_2(\mathbb{R})$. Usually, wavelets can be found by means of functional equations

$$\psi(x) = \sum_{k\in\mathbb{Z}} b_k \varphi(2x-k), \tag{2}$$

where φ is a *refinable* functions, i.e., φ satisfies a *two–scale relation*

$$\varphi(x) = \sum_{k\in\mathbb{Z}} a_k \varphi(2x-k) \tag{3}$$

with the *mask* $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$. In the sequel, we shall restrict ourselves to refinable functions and wavelets with compact support and we will always assume that

$$\mathrm{supp}(\mathbf{a}) := \{k \in \mathbb{Z} | a_k \neq 0\} \subseteq [m_1, m_2]. \tag{4}$$

It can be checked that (4) implies $\mathrm{supp}\varphi \subseteq [m_1, m_2]$.

There are several methods to construct wavelets in higher dimensions. The simplest way is to use tensor products. There also exist multivariate wavelet constructions with respect to non–separable refinable functions ϕ satisfying

$$\phi(x) = \sum_{k\in\mathbb{Z}^d} a_k \phi(2x - k), \qquad \{a_k\}_{k\in\mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d), \tag{5}$$

see, e.g., [JM] for details. In any case, a family ψ^i, $i = 1, \ldots, 2^d - 1$, of wavelets is needed. Each ψ^i satisfies a functional equation similar to (2),

$$\psi^i(x) = \sum_{k\in\mathbb{Z}^d} b_k^i \phi(2x - k). \tag{6}$$

As in the univariate case, we shall henceforth assume that the scaling functions and wavelets under consideration are all compactly supported.

It is often very convenient to have access to a suitable *biorthogonal* wavelet basis. For a given (univariate) wavelet basis $\{\psi_{j,k},\ j,k \in \mathbb{Z}\}$, one is interested in finding a second system $\{\tilde{\psi}_{j,k},\ j,k \in \mathbb{Z}\}$ satisfying

$$(\psi_{j,k}(\cdot), \tilde{\psi}_{j',k'}(\cdot)) = \delta_{j,j'}\delta_{k,k'}, \quad j,j',k,k' \in \mathbb{Z}. \tag{7}$$

Here $(\cdot,\cdot)$ clearly denotes the usual L_2–inner product. The construction of such a biorthogonal system essentially relies on a suitable second generator $\tilde{\varphi}$ such that φ and $\tilde{\varphi}$ form a *dual pair*,

$$(\varphi(\cdot), \tilde{\varphi}(\cdot - k)) = \delta_{0,k}. \tag{8}$$

Elegant constructions can be found, e.g., in [CDF]. Generalizations to higher dimensions also exist [CD].

For further information on wavelet analysis, the reader is referred to one of the textbooks on wavelets which have appeared quite recently [Ch, D, Dau, KL, Me, W].

To treat practical problems, it is often necessary to construct wavelet bases on bounded domains. In this paper, we shall focus on the case $\Omega = (0,1)$. The construction of wavelets on the interval is meanwhile well understood, see, e.g., [AHJP, CDV, DKU]. Here we refer to the approach in [DKU] where a biorthogonal wavelet basis is constructed. The common strategy is to start with a pair of dual generators on $\mathbb{R}$. Specifically, we choose here a biorthogonal system from the family constructed in [CDF] where the primal scaling functions consist of cardinal B–splines. For $j \geq j_0$ where j_0 is fixed (sufficiently large to disentangle end point effects) one builds Φ_j by keeping those translates $2^{j/2}\varphi(2^j \cdot -k), k \in \mathbb{Z}$, that are fully supported in $[0,1]$. These will be referred to as *interior basis functions*. For B–splines of order m, at each end of the interval m fixed linear combinations of the $2^{j/2}\varphi(2^j \cdot -k)$ are added in such a way that the resulting collection Φ_j spans all

polynomials of order m on $(0,1)$. One proceeds in the same way with the dual scaling functions restoring the original order of polynomial exactness while keeping $\#\Phi_j = \#\tilde{\Phi}_j$. Then only the interior basis functions inherit the biorthogonality from the line whereas the boundary modifications have perturbed biorthogonality. It can be shown though that in this spline family one can *always* biorthogonalize [DKU], ending up with pairs of generator bases $\Phi_j, \tilde{\Phi}_j$. These bases always consist of three parts signified by the index sets $\Delta_j^L, \Delta_j^I, \Delta_j^R$ (and similarly for the dual collections) identifying the left boundary, interior and right boundary basis functions. Only the size of the interior sets Δ_j^I depends on j. The number of boundary functions stays always the same. Moreover, for each end point one has a fixed finite number (namely m for the primal respectively $\tilde{m}$ for the dual basis) of scaling relations which can be computed a–priorily and stored. The interior basis functions satisfy, of course, the classical stationary refinement rule from the line case. Moreover, the whole set of scaling functions is refinable in the following generalized sense. There exist *refinement matrices* $\boldsymbol{M}_{j,0}$ and $\tilde{\boldsymbol{M}}_{j,0}$ such that

$$\Phi_j^T = \Phi_{j+1}^T \mathbf{M}_{j,0}, \qquad \tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{\mathbf{M}}_{j,0}, \tag{9}$$

where the bases are viewed as vectors whose components are the individual scaling functions. The refinement matrices have fixed upper left and lower right blocks. Only the stationary block A_j changes its size with growing level j, see Figure 1.

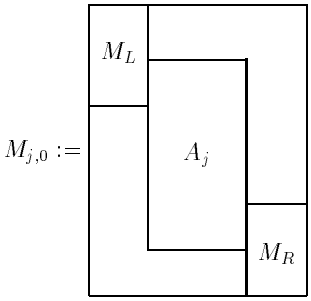

Figure 1: Structure of refinement matrices for spline wavelets on the interval.

3 Gauss quadrature

This section is devoted to the (univariate) Gauss quadrature setting. We start in Subsection 3.1 by recalling the basic facts. In Subsection 3.2, we derive some error estimates and apply them to the important special case of spline functions. Finally, in Subsection 3.3, we shortly review the construction of quadrature rules for refinable functions and wavelets.

3.1 General setting

A univariate Gauss quadrature rule replaces an integral by a weighted sum of point evaluations of f, i.e.,

$$\int_{[a,b]} f(x)w(x)dx \approx I_w^n(f) := \sum_{i=1}^{n} \lambda_i f(x_i), \qquad [a,b] \subset I\!R, \quad n \in I\!N, \tag{10}$$

with *knots* x_i and *weights* λ_i, $i = 1, \ldots n$. Here w is called the *weight function*, which in the most classical case is chosen to be $w \equiv 1$. However, the theory can be developed for a huge class of functions. In fact, w only has to satisfy the following conditions, cf. [Sto], p. 135.

Definition 3.1 *A function w on the (finite or infinite) interval (a,b) is called a* weight function *if it is nonnegative and measurable and if all its moments*

$$\gamma_i(w) := \int_a^b x^i w(x)\, dx \quad i = 0, 1, \ldots \tag{11}$$

exist. Moreover, we always require that $\gamma_0(w) > 0$.

In this paper, we will only consider compactly supported weight functions.

Defining the inner product corresponding to w as

$$(f,g)_w := \int_{[a,b]} f(x)g(x)w(x)dx, \tag{12}$$

the knots x_i of the Gauss rule are the zeros of the n–th orthogonal polynomial with respect to this scalar product. A quadrature rule is said to be of *degree* N if it is exact for all polynomials with order up to N. Gauss quadrature rules with n points are of degree $N = 2n$.

For further information on quadrature, the reader is referred, e.g., to [DR, K, St] and the references therein.

3.2 Error analysis

Once a Gauss quadrature rule of the form (10) is established, one is clearly interested in computing the *quadrature error* which is defined by

$$E_w^n(f) := \int_{[a,b]} f(x)w(x)dx - I_w^n(f). \tag{13}$$

A classical result states that if $f \in C^{2n}(I\!R)$

$$E_w^n(f) = \frac{f^{(2n)}(\xi)}{(2n)!k_{n,0}^2}, \qquad \text{for some } a < \xi < b, \tag{14}$$

see again [DR] for details. Here $k_{n,0}$ denotes the leading coefficient of the n–th orthonormal polynomial with respect to the scalar product $(\cdot,\cdot)_w$.

Hence, if the weight function and the degree is given, it is possible to control the error for any specific Gauss quadrature rule. However, the question remains open, how the error will behave asymptotically as n increases. Fortunately, there exist some results in this direction, at least for weight functions satisfying some additional conditions as we shall now explain.

Definition 3.2 *Let w be a weight function according to Definition* 3.1 *on the interval* $[-1,1]$. *w is said to be in the function class $\mathcal{W}$ if*

$$\int_{-\pi}^{\pi} |\ln w^*(\vartheta)| d\vartheta < \infty, \tag{15}$$

for $w^*(\vartheta) := w(\cos(\vartheta))\,|\sin(\vartheta)|$.

For this special class $\mathcal{W}$, the following theorem holds, see, e.g., [Sz] for details.

Theorem 3.3 *Let $w \in \mathcal{W}$ and*

$$p_n(x) = \sum_{i=0}^{n} k_{n,i} x^{n-i}, \quad n = 0,1,2,\ldots \tag{16}$$

be the system of orthonormal polynomials associated with the weight function w. Then, as n tends to infinity,

$$k_{n,0} \sim 2^n C_w, \qquad \textit{with} \;\; C_w := \pi^{-1/2} \exp\left(\frac{-1}{2\pi} \int_{-1}^{1} \frac{\ln(w(x))}{\sqrt{1-x^2}}\, dx \right). \tag{17}$$

As usual, '$a \sim b$' means that both quantities can be uniformly bounded by some constant multiple of each other. Likewise, '$\lesssim$' indicates inequality up to constant factors.

Theorem 3.3 now sets us in the position to control the asymptotic error of Gauss formulas for $w \in \mathcal{W}$. Let us mention that the above result holds for a somewhat wider class of functions, details can again be found in [Sz], p. 296.

To use Theorem 3.3 in practice, it is clearly desirable to have an easy criterion at hand to check the condition (15). In the following lemma we give one result in this direction which, as we shall see later on, can be applied to any spline function on $[-1,1]$ and therefore also to every spline wavelet and refinable function, respectively.

Lemma 3.4 *Let w be a bounded weight function on $[-1,1]$ with zeros x_i, $i = 1,\ldots,L$. If there exist numbers $r_i, c_i, \beta_i > 0, i = 1,\ldots,L$ and $\delta > 0$ such that*

$$w(x) \;\geq\; c_i |x_i - x|^{\beta_i}, \quad x \in [x_i - r_i, x_i + r_i] \cap [-1,1], \tag{18}$$

$$w(x) \;\geq\; \delta, \quad x \in [-1,1] \setminus \bigcup_{i=1}^{L} [x_i - r_i, x_i + r_i], \tag{19}$$

then w is contained in $\mathcal{W}$.

Proof. We have to check that $\int_{-\pi}^{\pi} |\ln w^*(\vartheta)| d\vartheta$, with $w^* := w(\cos(\vartheta))|\sin(\vartheta)|$ exists. As w is bounded, w^* is also bounded and we may assume without loss of generality that $w^* \leq 1$. Hence

$$\int_{-\pi}^{\pi} |\ln w^*(\vartheta)| \, d\vartheta = -\int_{-\pi}^{\pi} \ln w^*(\vartheta) \, d\vartheta. \tag{20}$$

To show the existence of the integral, let

$$\mathcal{Z} := \{\vartheta_\nu \in [-\pi, \pi] : \ \cos(\vartheta_\nu) = x_i, \ i = 1 \ldots, L\}.$$

For $r > 0$ sufficiently small, we may decompose the expression in (20) as

$$\begin{aligned} \int_{[-\pi,\pi]} \ln w^*(\vartheta) \, d\vartheta &= \int_{[-\pi,\pi] \setminus \bigcup_{\vartheta_\nu \in \mathcal{Z}} [\vartheta_\nu - r, \vartheta_\nu + r]} \ln w^*(\vartheta) \, d\vartheta \\ &+ \sum_{\vartheta_\nu \in \mathcal{Z}} \int_{[\vartheta_\nu - r, \vartheta_\nu + r] \cap [-\pi,\pi]} \ln w^*(\vartheta) \, d\vartheta. \end{aligned} \tag{21}$$

We start by showing that each of the summands of the second expression in (21) exists. To this end, consider for $\vartheta_\nu \in \mathcal{Z}$ the decomposition

$$\begin{aligned} \int_{[\vartheta_\nu - r, \vartheta_\nu + r] \cap [-\pi,\pi]} \ln w^*(\vartheta) \, d\vartheta &= \int_{[\vartheta_\nu, \vartheta_\nu + r] \cap [-\pi,\pi]} \ln w(\cos(\vartheta)) \, d\vartheta \\ &+ \int_{[\vartheta_\nu - r, \vartheta_\nu] \cap [-\pi,\pi]} \ln w(\cos(\vartheta)) \, d\vartheta \\ &+ \int_{[\vartheta_\nu - r, \vartheta_\nu + r] \cap [-\pi,\pi]} \ln |\sin(\vartheta)| \, d\vartheta \ . \end{aligned}$$

We will first show the existence of the first term on the right–hand side of (22). Without loss of generality, we may assume that $\vartheta_\nu \neq \pi$ so that $[\vartheta_\nu, \vartheta_\nu + r] \subset [-\pi, \pi]$ for r sufficiently small. Let x_{i_ν} be such that $\cos(\vartheta_\nu) = x_{i_\nu}$. Then, by assumption (18)

$$\begin{aligned} \int_{\vartheta_\nu}^{\vartheta_\nu + r} \ln w(\cos(\vartheta)) \, d\vartheta &\geq \int_{\vartheta_\nu}^{\vartheta_\nu + r} \ln \left(c_{i_\nu} |x_{i_\nu} - \cos(\vartheta)|^{\beta_{i_\nu}} \right) d\vartheta \\ &= \int_{\vartheta_\nu}^{\vartheta_\nu + r} \ln c_{i_\nu} \, d\vartheta + \beta_{i_\nu} \int_{\vartheta_\nu}^{\vartheta_\nu + r} \ln |x_{i_\nu} - \cos(\vartheta)| \, d\vartheta. \end{aligned} \tag{22}$$

The first integral clearly exists. Concerning the second one, we first assume that $\vartheta_\nu \neq 0, -\pi$ and substitute $y = x_{i_\nu} - \cos(\vartheta)$. This yields with $\tilde{r} := x_{i_\nu} - \cos(\vartheta_\nu + r) \neq 0$

$$\begin{aligned} \int_{\vartheta_\nu}^{\vartheta_\nu + r} \ln|x_{i_\nu} - \cos(\vartheta)|\, d\vartheta &= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\tilde{r}} \ln|y| \left(1 - (x_{i_\nu} - y)^2\right)^{-1/2} dy \\ &\gtrsim \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\tilde{r}} \ln|y|\, d\vartheta. \end{aligned} \tag{23}$$

The existence of the integral in the cases $\theta_\nu = 0, -\pi$ can be shown analogously by using $1 - \cos(x) = 2\sin^2(x/2)$. Hence in all three cases the corresponding integrals are finite, which establishes the existence of the first term in (22). The existence of the second term can be shown in a similar fashion. Hence we are left with the last term. Its existence is clear as long as $\vartheta_\nu \notin \{-\pi, 0, \pi\}$. However, these exceptional cases can be treated by employing similar arguments as for (22). Therefore we have shown that the second expression in (21) exists. So it remains to study the first term. By taking (19) into account, we observe that only the vicinities of the points $-\pi, 0$, and π require some special attention. For that, we may again proceed as above. It finally follows that

$$\int_{-\pi}^{\pi} |\ln w^*(\vartheta)| d\vartheta < \infty,$$

i.e., $w \in \mathcal{W}$. □

Lemma 3.4 and Theorem 3.3 can now be used to derive the following result for spline functions.

Corollary 3.5 *Let w be a nonnegative spline function on $[-1, 1]$ having only finitely many zeros and let $k_{n,0}$ be defined according to (16). Then as n tends to infinity*

$$k_{n,0} \sim 2^n C_w, \qquad \text{with } C_w := \pi^{-1/2} \exp\left(\frac{-1}{2\pi} \int_{-1}^{1} \frac{\ln(w(x))}{\sqrt{1 - x^2}}\, dx \right). \tag{24}$$

Furthermore, if $f \in C^{2n}, n \in \mathbb{N}$, the error of the corresponding Gauss quadrature rule satisfies for n tending to infinity

$$|E_w^n(f)| = \left| \int_{-1}^{1} f(x) w(x) dx - I_w^n(f) \right| \lesssim \frac{\|f^{(2n)}\|_{L_\infty[-1,1]}}{(2n)! 2^{2n} C_w^2}. \tag{25}$$

Proof. As (18) and (19) are obviously fulfilled for all spline functions by Lemma 3.4, any nonnegative spline with finitely many zeros is contained in $\mathcal{W}$. Therefore (24) follows immediately from Theorem 3.3. Then formula (25) is a simple consequence of the classical error estimate (14) for Gauss quadrature errors. □

3.3 Gauss formulas for wavelets and refinable functions

The integrals we have to deal with in the wavelet setting are of the following form

$$(f, \theta_{j,k}) \quad = \int\limits_{\mathrm{supp}(\theta_{j,k})} f(x)\theta_{j,k}(x)\, dx \tag{26}$$

where θ is either a refinable function φ or a wavelet ψ, see Section 2. By means of a substitution, one can write (26) as

$$\int\limits_{\mathrm{supp}(\theta_{j,k})} f(x)\theta_{j,k}(x)\, dx \quad = \quad 2^{-j/2} \int\limits_{\Omega} f(2^{-j}(u+k))\, \theta(u)\, du, \tag{27}$$

with $\Omega := \mathrm{supp}(\theta)$. Hence it suffices to find a quadrature rule for integrals of the form

$$\int\limits_{\Omega} f(x)\theta(x)dx, \tag{28}$$

to be able to compute all integrals in(26).

In [BBDK], one solution to this problem has been given. The general idea was to construct Gaussian quadrature rules using θ as weight function. Then, no smoothness is required for θ, however, it should be nonnegative, which might not always be the case. Yet, this problem can be fixed using the following *lifting trick*. Suppose that for $\theta \not\geq 0, \theta \in \{\varphi, \psi\}$ there exists an appropriate constant $c > 0$ such that on $\mathrm{supp}(\theta)$

$$\theta^c(x) := \theta(x) + c\chi_{[l_1,l_2]}(x) \geq 0, \tag{29}$$

where $\chi_{[l_1,l_2]}$ denotes the characteristic function of $[l_1, l_2] \supseteq \mathrm{supp}(\theta)$. Then one can set up a quadrature rule with weights λ_i^c and knots x_i^c corresponding to the nonnegative function $\theta^c(x)$. After that, a second Gauss rule with weights λ_i^χ and knots x_i^χ for the characteristic function $\chi_{[l_1,l_2]}$ has to be determined. Hence we are left with the task to compute the moments of refinable functions and wavelets, respectively. This can be done using an algorithm by [DM2] and the functional equation (2), see [BBDK].

4 Gauss quadrature for cardinal B–spline systems

Refinable functions and wavelets related to cardinal B–splines are of special interest in practice. Indeed, in recent years, it has turned out that spline wavelets form a powerful tool for many applications of wavelet analysis such as signal/image analysis/compression and the numerical treatment of operator equations. This is also due to the fact that in the context of solving problems on bounded domains, the B–spline approach provides a suitable way to construct appropriate bases on these domains, see Section 2. Consequently, the aim of this section is twofold. Firstly, we want to derive and to analize quadrature formulas for boundary adapted scaling functions and wavelets. Secondly, we will apply the error analysis presented in Section 3.2 to both, the classical and the boundary adapted scaling functions and wavelets.

4.1 Construction of quadrature formulas for boundary adapted scaling functions

We have already mentioned that the construction of a suitable Gauss formula essentially relies on the computation of the moments of the weight function. As outlined in Section 2, the scaling functions near the boundary constructed according to [DKU] are no longer refinable in the classical sense. Therefore the usual way to compute the moments as explained in [DM2] does not work in this case. One possible remedy reads as follows. Let

$$\varphi_{j,k}^L(x), \quad k = 0, \ldots, m-1 \tag{30}$$

denote the m functions adapted to the left side of the interval, compare with Section 2. For the computation of their moments

$$\gamma_{l,j,k}^L := \int_0^1 x^l \varphi_{j,k}^L(x) dx \tag{31}$$

we may use a different representation of $\gamma_{l,j,k}^L$ from [DKU]. Let $\tilde{\varphi}_{j,l}^L$ denote a boundary adapted version of the dual generator before the final biorthogonalization procedure is performed. Then

$$\gamma_{l,j,k}^L = \left(\varphi_{j,k}^L(\cdot),\ (\cdot)^l\right)_{[0,1]} = 2^{-j/2}\ 2^{-lj}\left(\varphi_{j,k}^L(\cdot), \tilde{\varphi}_{j,l}^L(\cdot)\right)_{[0,1]}, \tag{32}$$

provided that the dual system is exact of order greater or equal l. But for the biorthogonal spline system, dual functions of arbitrary order are available, at least in principle. Moreover, the term on the right–hand side in (32) can be computed exactly (up to roundoff), see again [DKU] for details. Consequently, for each of the boundary adapted functions, we may use (32) to derive a Gauss formula by following exactly the lines in [BBDK]. So we end up with $m+1$ formulas, one for the interior functions and m for the boundary adapted functions (the right boundary can be handled by using symmetry arguments). In principle, the values in (32) also depend on the level j. However, this dependency only results in a scaling by powers of two. Therefore the quadrature rules for the boundary adapted functions have to be computed only once and for all on the coarsest level, just like for the interior functions.

We conclude this section with the following example. Let N_2^L be one of the two boundary adapted functions corresponding to the centralized cardinal B–spline of order two, see Figure 2, and let

$$f(x) := \cos(x)\ e^x. \tag{33}$$

We will test the above method for computing

$$\int_{\mathrm{supp}(N_2^L)} f(x) N_2^L(x)\ dx = \sqrt{2}(8e^{1/4}\sin(1/4) - 8e^{1/8}\sin(1/8) - 1) \approx 0.58150188308191.$$

In the following table, the values and the relative errors for the corresponding quadrature rules are listed. We see that the relative error decreases very rapidly as n increases, so that our formulas indeed perform satisfactorily.

n	$I^n_{N_2^L}(f)$	rel. error
1	0.5817194542408	3.742e-04
2	0.58150322100026	2.301e-06
3	0.58150188304296	6.700e-11
4	0.58150188308187	8.126e-14

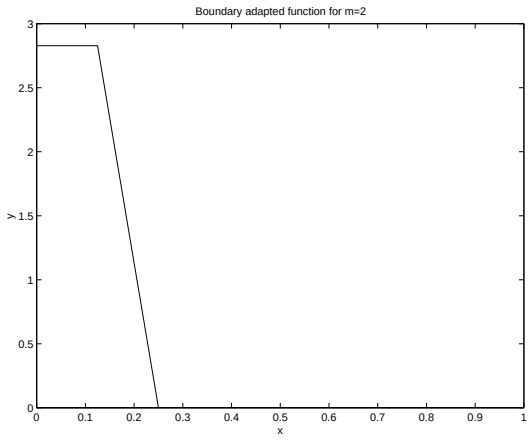

Figure 2: N_2^L, one of the two boundary adapted scaling function of N_2.

4.2 Error analysis for the B–spline case

As already outlined above, cardinal B–splines play an important role in practical applications. Therefore, we want to investigate the error behaviour of Gauss quadrature rules for this case in more detail.

Let us first remark that after a suitable transformation to the interval $[-1, 1]$ the general result stated in Corollary 3.5 also holds for the boundary adapted functions because they are still piecewise polynomials.

In the following table, we list the computed values of C_w according to (24) for both, the B–splines and their boundary adapted versions. Let N_ρ denote the usual centralized cardinal B–spline of order ρ, $\mathfrak{N}_\rho$ its transformed version with $\mathrm{supp}(\mathfrak{N}_\rho) = [-1, 1]$ and let $\mathfrak{N}^L_{\rho,k},\ k = 0, \ldots, \rho - 1$ be the corresponding transformed boundary adapted functions.

For the centralized B–spline $N_2 = \mathfrak{N}_2$, some of the leading coefficients were computed exactly in [BBDK]. In the following table, we compare the estimated and exact leading coefficient k_n in this case.

ρ	$C_{\mathfrak{N}_\rho}$	$C_{\mathfrak{N}^L_{\rho,0}}$	$C_{\mathfrak{N}^L_{\rho,1}}$	$C_{\mathfrak{N}^L_{\rho,2}}$	$C_{\mathfrak{N}^L_{\rho,3}}$
2	1.41	0.89	1.41		
3	3.39	1.12	1.04	0.97	
4	7.98	1.64	1.41	1.22	1.06

n	1	2	3	4	5	6
k_n	2.45	5.07	10.51	21.26	43.17	86.80
$2^n\ C_{\mathfrak{N}_2}$	2.84	5.68	11.36	22.72	45.44	90.24

We see that the estimated k_n deviate from the exact ones only by factors dropping from 1.13 to 1.04, showing that the asymptotic estimate for this case is already very good for small n.

5 The multivariate case

Extensions of the one dimensional wavelet theory to the multivariate case are most easily done by a tensor product ansatz, compare with Section 2. In [BBDK], we have shown that one can derive Gaussian type formulas for these cases from the one dimensional case described above by simple *product formulas*. Error formulas for these product formulas can also be easily deduced from the one dimensional estimates, see, e.g., [StS]. However, in many applications, it is desirable to work with *nonseparable* refinable functions and wavelets. Then one has to proceed in a somewhat different way. In Subsection 5.1, we shall explain the general idea. With regard to the applications we have in mind, we restrict ourselves mostly to the case $d = 2$. Furthermore, we confine the discussion to nonnegative refinable functions because we shall mainly apply the analysis to specific box splines. For the general case, we refer to [BBDK]. In Subsection 5.2, we derive certain multivariate error estimates. After briefly recalling the basic facts, some specific box splines are studied in detail.

5.1 Quadrature rules for the nonseparable case

Let us briefly recall the construction of multivariate quadrature rules which are not of tensor product type. We want to find a quadrature rule of degree N having the form

$$\int\limits_{\mathrm{supp}\phi} f(x,y)\phi(x,y)dxdy \sim \sum_{i=1}^{n(N)} \lambda_i f(x_i,y_i). \tag{34}$$

To this end, one has to choose the points

$$(x_i, y_i), \quad i = 1, \ldots, n, \ n = n(N) = \frac{(1+N)N}{2} \tag{35}$$

in such a way that they do not all lie on a curve $Q_N(x, y) = 0$, where Q_N is a polynomial of order N. Then the coefficients λ_i can be found as solutions of the linear system

$$\lambda_1 x_1^{l_1} y_1^{l_2} + \cdots + \lambda_n x_n^{l_1} y_n^{l_2} = \int\limits_{\mathrm{supp}\phi} x^{l_1} y^{l_2} \phi(x, y) dx dy, \qquad 0 \le |l| \le N - 1, \tag{36}$$

see, e.g., [St] for details. In this paper, we shall mainly consider the case that ϕ is a box spline. Let us briefly recall the definition and the basic facts. Let $X = \{x^1, \ldots, x^\mu\} \subset \mathbb{Z}^d \backslash \{0\}$, $x^\nu = (x_1^\nu, \ldots, x_d^\nu)^T$ denote a set of not necessarily distinct vectors satisfying $\mu \ge d$ and

$$< X >= \text{span } X = \mathbb{R}^d. \tag{37}$$

Then the *box spline* $B(\cdot|X)$ is defined by requiring that the equation

$$\int\limits_{\mathbb{R}^d} f(x) B(x|X) dx = \int\limits_{[0,1]^\mu} f(Xu) du \tag{38}$$

holds for any continuous function f on $\mathbb{R}^d$. The vectors $x^1, \ldots, x^\mu$ are called the *direction vectors* of $B(\cdot|X)$. Every box spline is a refinable function,

$$B(\cdot|X) = \sum_{k \in \mathbb{Z}^d} a_k B(2 \cdot -k|X), \tag{39}$$

where the mask $\mathbf{a} = \{a_k\}_{k \in \mathbb{Z}^d}$ is given by

$$\sum_{k \in \mathbb{Z}^d} a_k z^k = 2^{d-\mu} \prod_{\nu=1}^{\mu} (1 + z^{x^\nu}). \tag{40}$$

It can be shown that a box spline is a piecewise polynomial. Moreover, it is nonnegative and satisfies

$$B(u|X) = 0, \quad u \notin [X[, \ [X[:= \{t_1 x^1 + \cdots + t_\mu x^\mu : \ 0 \le t_\nu < 1, \ 1 \le \nu \le \mu\}. \tag{41}$$

For further information on box splines, the reader is referred to [DM]. In [BBDK], suitable sets of points and weights for various box splines were constructed. Let us also mention that further examples can be constructed by means of [QM].

5.2 Error analysis

In [BBDK], several quadrature rules of the form (34) have been developed and tested. The performance was quite well in all cases. In this section, we want to go one step further

and derive some rigorous error estimates, at least for some specific box splines. To this end, we apply the general theory as, e.g., outlined in [St]. Let us start by briefly recalling the basic facts.

Let R_2 be a rectangle $a \le x \le b,\ c \le y \le d$. We want to discuss estimates for the error $E_w^n(f)$ in an integration formula

$$\int_{R_2} w(x,y) f(x,y) dx dy = \sum_{i=1}^{n(N)} \lambda_i f(x_i, y_i) + E_w^n(f). \tag{42}$$

We assume that

(i) i, j are nonnegative integers;

(ii) p, q are positive integers;

(iii) $\sigma = p + q \ge 2$.

We define $B_{p,q}(a_0, c_0)$ as the space of all functions $f(x,y)$ with the following properties:

(i) The derivative $f^{(p,q)}(x,y)$ is Riemann integrable on R_2;

(ii) the derivatives $f^{(\sigma-j,j)}(x, c_0),\ j < q$ are Riemann integrable on $a \le x \le b$;

(iii) the derivatives $f^{(i,\sigma-i)}(a_0, y),\ i < p$ are Riemann integrable on $c \le y \le d$;

(iv) the derivatives $f^{(i,j)}(a_0, c_0),\ i + j < \sigma$ exist;

(v) Taylor's formula with respect to the point (a_0, c_0) holds for all (x,y) in R_2.

We shall also need the *kernel functions* $K_{p,q}(u,v)$, $K_{i,j:x}(u)$ and $K_{i,j:y}(v)$ defined by

$$K_{p,q}(u,v) := E_w^n\left(\frac{(x-u)^{p-1}}{(p-1)!}\zeta(a_0,u,x)\frac{(y-v)^{q-1}}{(q-1)!}\zeta(c_0,v,y)\right), \tag{43}$$

$$K_{i,j:x}(u) := E_w^n\left(\frac{(x-u)^{i-1}}{(i-1)!}\zeta(a_0,u,x)\frac{(y-c_0)^{j}}{j!}\right), \quad 0 \le j < q,\ i \le \sigma - j, \tag{44}$$

$$K_{i,j:y}(v) := E_w^n\left(\frac{(x-a_0)^{i}}{i!}\frac{(y-v)^{j-1}}{(j-1)!}\zeta(c_0,v,y)\right), \quad 0 \le i < q,\ j \le \sigma - i, \tag{45}$$

where $\zeta(\iota, \varpi, \varsigma)$ is the step function

$$\zeta(\iota,\varpi,\varsigma) := \begin{cases} 1, & \text{for } \iota \le \varpi < \varsigma, \\ -1, & \text{for } \varsigma \le \varpi < \iota, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

It turns out that $E_w^n(f)$ can be estimated by means of the kernel functions defined above. The following theorem summerizes several results proved in [St].

Theorem 5.1 *Suppose we have an integration formula* (34) *of degree* N *and let* $2 \le \sigma = p+q \le N$. *Let us furthermore assume that* $f \in B_{p,q}(a_0, c_0)$. *Then*

$$|E_w^n(f)| \le \sum_{j<q} e_{\sigma-j,j:x} M_{\sigma-j,j:x} + \sum_{i<p} e_{i,\sigma-i:y} M_{i,\sigma-i:y} + e_{p,q} M_{p,q}, \tag{47}$$

where

$$\begin{aligned} e_{i,j:x} &:= \int_a^b |K_{i,j:x}(u)|du, & M_{i,j:x} &:= \sup_{u\in[a,b]} |f^{(\sigma-j,j)}(u,c_0)|, \\ e_{i,j:y} &:= \int_c^d |K_{i,j:y}(v)|dv, & M_{i,j:y} &:= \sup_{v\in[c,d]} |f^{(i,\sigma-i)}(a_0,v)|, \\ e_{p,q} &:= \int_a^b \int_c^d |K_{p,q}(u,v)|dudv, & M_{p,q} &:= \sup_{(u,v)\in[a,b]\times[c,d]} |f^{(p,q)}(u,v)|. \end{aligned} \tag{48}$$

We see that the error estimate (47) essentially consists of two parts. One part only depends on the function f whereas the other part only depends on the given quadrature rule. Therefore it seems to be worthwhile to compute at least the second part as precisely as possible. Although this might be expensive, the reader should observe that this only has to be done once and for all.

As an example, we have performed such a computation for the box spline $B(\,\cdot\,|\,\begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix})$ which is the famous Courant finite element whose graph looks like a hexagonal pyramid. To determine the quantities $e_{i,j:x}, e_{i,j:y}$ and $e_{p,q}$, we have to compute the functions $K_{p,q}(u,v)$, $K_{i,j:x}(u)$ and $K_{i,j:y}(v)$ with high accuracy on a fine grid. From that, the values of $e_{i,j:x}$, $e_{i,j:y}$ and $e_{p,q}$ can be extracted.

The evaluation of $K_{p,q}$ at a point (u,v) amounts to compute the exact integral and the quadrature rule for the function in brackets on the right–hand side in (43). In our case, the integrals can be computed *exactly*. We evaluate them as functions of u and v on a rectangular grid with meshsize $h = 1/64$. Computed once and for all, these values can be used to determine $K_{p,q}$ for different quadrature rules on the same grid. To compute the error for other quadrature rules, e.g., for different exactness or different algorithms for choosing the knots, respectively, only the application of the quadrature rule to the truncated powers have to be computed. This has to be done on the same grid and is very fast. In the following tables, some of the resulting error terms defined in (48) are displayed.

N | $i!j!e_{i,j:x}(B(\,\cdot\,|\,\begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}))$

2

$i\backslash j$	0	1
1	0.303	0.499
2	0.333	

3

$i\backslash j$	0	1	2
1	0.135	0.0453	0.0462
2	0.0268	0.0420	
3	0.0237		

4

$i\backslash j$	0	1	2	3
1	0.126	0.0259	0.0291	0.0229
2	0.0176	0.0130	0.0130	
3	0.00537	0.0188		
4	0.00625			

5

$i\backslash j$	0	1	2	3	4
1	0.0839	0.0147	0.0147	0.00858	0.00828
2	0.00764	0.00142	0.00284	0.00220	
3	0.00144	0.000336	0.00288		
4	0.000718	0.000241			
5	0.000732				

N | $i!j!e_{i,j:y}(B(\,\cdot\,|\,\begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}))$

2

$i\backslash j$	1	2
0	0.663	0.833
1	0.0833	

3

$i\backslash j$	1	2	3
0	0.164	0.0323	0.0139
1	0.0298	0.0114	
2	0.0475		

4

$i\backslash j$	1	2	3	4
0	0.127	0.0178	0.00505	0.00281
1	0.0195	0.00364	0.00345	
2	0.0294	0.00905		
3	0.0279			

5

$i\backslash j$	1	2	3	4	5
0	0.100	0.0109	0.00238	0.000824	0.000493
1	0.0173	0.00212	0.000577	0.000351	
2	0.0168	0.00212	0.00159		
3	0.00742	0.00294			
4	0.00865				

N	$p!q!e_{p,q}(B(\cdot \mid \begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}))$			
2	$p\backslash q$	1		
	1	0.499		
3	$p\backslash q$	1	2	
	1	0.1016	0.0569	
	2	0.0514		
4	$p\backslash q$	1	2	3
	1	0.0904	0.0382	0.0282
	2	0.0436	0.0170	
	3	0.0322		
5	$p\backslash q$	1	2	3	4
	1	0.0532	0.0188	0.0116	0.00896
	2	0.0197	0.00445	0.00271	
	3	0.0123	0.00338		
	4	0.00918			

In Figure 3, some typical functions $K_{p,q}$ for the box spline $B(\cdot \mid \begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix})$ are depicted.

The knots of the quadrature rules correspond to those lines where the kernel functions show singularities or singularities in a derivative.

The first two pictures correspond to the same values of p and q but different quadrature rules. Since the second one is computed for $N = 4$, more knots and therefore more singularities occur. However, the error is of comparable size because the kernel functions have low regularity and therefore increasing the order of the quadrature rule cannot decrease the error a lot.

The superiority of higher order quadrature rules shows up if the function to be integrated is smooth. Therefore we compare in the second and the third picture the behaviour of $K_{p,q}$ as p and q increase. One observes that the error depicted in the third figure is much smaller than in the second one, because the smoother kernel functions can be integrated more accurately with the (higher order) quadrature rule.

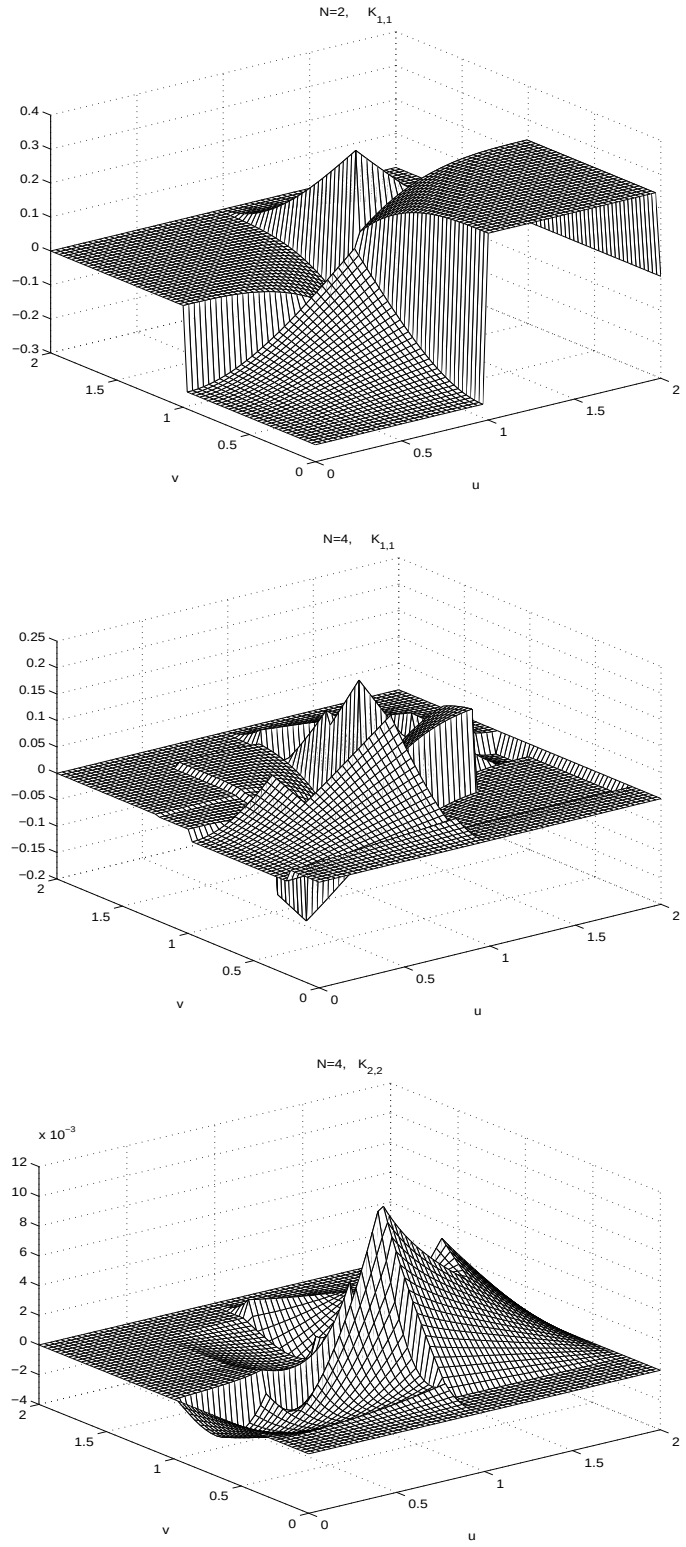

Figure 3: Some typical kernel functions.

We are now in the position to test our error bounds (47) for a typical example. For $f(x,y) := \sin(x+y)$ we want to compute the integral

$$\int_{I\!R^2} f(x,y) B(x,y \,|\, \begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}) dx dy. \tag{49}$$

The exact value of (49) is given by

$$\cos(1) - \cos(3) + \frac{\cos(4) - 1}{2} \;\approx\; 0.703472992036784.$$

Since in our special case

$$|f^{p,q}(x,y)| \le 1, \qquad \text{for all } p, q, x, \text{ and } y,$$

the function f belongs to all spaces $B_{p,q}$ and the evaluation of the error bound (47) is particularly easy: all values $M_{p,q}$, $M_{i,j:x}$ and $M_{i,j:y}$ are bounded by one. Let us denote by $|E^n_{p,q}|$ the error bound given by (47) for some p, q. In the next tabular, we present the integrals as calculated by our quadrature formula together with the true error and some of the corresponding error bounds.

N	n	$I^n_{B(\,\cdot\,\mid \begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix})}(f)$	$\lvert E^n_{B(\,\cdot\,\mid \begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix})}(f)\rvert$	$\lvert E^n_{1,1}(f)\rvert$	$\lvert E^n_{2,1}(f)\rvert$	$\lvert E^n_{2,2}(f)\rvert$	$\lvert E^n_{2,3}(f)\rvert$
2	3	0.2978843607	4.06e-01	1.08e+00			
3	6	0.7072451256	3.77e-03	1.31e−01	3.77e−02		
4	10	0.7053436798	1.87e-03	1.08e−01	2.54e−02	8.32e−03	
5	15	0.7033638796	1.09e-04	6.24e−02	1.16e−02	1.32e−03	5.01e−04

As expected, the error bounds decrease rapidly for larger values of p and q since the function f is clearly analytic. In fact, already for $N = 4$ we obtain reasonably sharp bounds.

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