Convergent Adaptive Wavelet Methods for the Stokes Problem*

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Abstract. We consider wavelet discretizations for the Stokes problem in the mixed and divergence free variational formulation. For both cases, we present convergent adaptive multiscale strategies. Moreover, for adaptive wavelet discretizations of the mixed formulation we provide an easy to implement criterion for enforcing stability.

1 Introduction

Adaptive methods play an important role for the numerical solution of partial differential equations. Adapting the discretization to the structure of the problem and the error quantity one is interested in, allows to resolve complicated problems with local character. Since in many cases no sufficient a priori knowledge on the structure of the solution is available, local refinement or derefinement is based on a posteriori error estimates.

Even though adaptive methods are widely used in industrial codes, there has not been a convergence analysis of these strategies for many years. Quite recently, starting from [9], some rigorous proofs have been given. Whereas in [9] the convergence of an adaptive scheme for the 2d Poisson problem using piecewise linear Finite Elements is given, the subsequent papers [4,5] use the framework of multiscale methods and wavelets. In [5], a convergent adaptive wavelet method for elliptic operators in any spatial dimension has been introduced. This method has been somewhat modified in [4] and it has been proved that this modified strategy is asymptotically optimal efficient, i.e., it has the same rate of convergence as the corresponding best n-term approximation. The overall effort of this method is $\mathcal{O}(N \log N)$, where N denotes the number of unknowns. This latter algorithm has been tested in 1d and 2d and the numerical results presented in [2] are promising.

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In this paper, we consider adaptive wavelet methods for the Stokes problem. Since the mixed formulation gives rise to a saddle point problem, the above mentioned approaches do not cover this case. However, in [6] a convergent wavelet method for general saddle point problems has been given. We apply this strategy to the mixed formulation of the Stokes problem. Moreover, it is known that the Stokes problem can be seen as an elliptic problem reduced to the subspace of divergence free vector fields. We show that using divergence free wavelets [11,13,14] one can in fact apply the theory for symmetric positive definite operators to the divergence free discretization of the Stokes problem.

1.1 The Stokes Problem

The Stokes problem is well-known as a linearized model of the flow of a viscous, incompressible fluid in some domain. A numerical solver can also be used as a kernel for solving the full Navier–Stokes equations, [10]. Let us recall the formulation of the Stokes problem.

Problem 1. *Given the exterior force* $\boldsymbol{f}$*, one has to determine the velocity* $\boldsymbol{u}$ *and the pressure* p *such that*

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega \ , \\ \nabla \cdot \boldsymbol{u} &= 0 \text{ in } \Omega \ , \\ \boldsymbol{u} &= 0 \text{ on } \Gamma := \partial\Omega \ , \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ *is a bounded Lipschitz domain of interest.*

1.2 Multiscale Methods and Wavelets

Let us briefly recall those facts for multiscale methods and wavelets that will be needed in this paper. For extensive surveys, we refer to [3,7]. We call a system of functions $\Psi := \{\psi_\lambda : \lambda \in \nabla\} \subset L^2(\Omega)$ a system of *wavelets*, if they form (besides others) a Riesz basis for $L^2(\Omega)$. Here, ∇ denotes an infinite set of indices. We may think of each index as a couple $\lambda = (j, k)$ where $j =: |\lambda|$ denotes the scale or level of a wavelet whereas k represents its location in space. Moreover, we assume that in particular $\operatorname{diam}(\operatorname{supp}\psi_\lambda) \sim 2^{-|\lambda|}$ and that Ψ characterizes a family of Sobolev spaces $H^s(\Omega)$, $s \in (-\tilde{\gamma}, \gamma)$, in the sense

$$\Big\| \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda \Big\|_{s,\Omega} \sim \left(\sum_{\lambda\in\nabla} 2^{2s|\lambda|} |d_\lambda|^2 \right)^{1/2} .$$

The constants $\tilde{\gamma}$, $\gamma > 0$ are determined by smoothness and approximation properties of Ψ and its biorthogonal (dual) system $\tilde{\Psi}$, [7]. By $A \sim B$, we mean that there exist absolute constants $c, C > 0$ such that $cA \le B \le CA$. The first inequality will be abbreviated by $A \lesssim B$.

2 Adaptive Wavelet Methods for Elliptic Operator Equations

Let A be a linear operator mapping a Hilbert space H into its dual H^*. Then, we consider the operator equation $Au = f$ for a given $f \in H^*$. For the analysis of an adaptive strategy, the following assumptions will be posed.

Assumption 1. *(a) The bilinear form $a(\cdot,\cdot)$ is symmetric and positive definite on H such that $\|v\|_A^2 := a(v,v) \sim \|v\|_H^2$ for $v \in H$.*

(b) The wavelet basis functions Ψ are in H, their duals $\tilde{\Psi}$ in the dual space H^*, each function $v \in H$ has a unique expansion in terms of Ψ

$$v = \boldsymbol{d}^T \Psi := \sum_{\lambda \in \nabla} d_\lambda \, \psi_\lambda, \qquad d_\lambda := \langle v, \tilde{\psi}_\lambda \rangle \ ,$$

such that $\|\boldsymbol{D}^{-1} \boldsymbol{d}\|_{\ell^2(\nabla)} \sim \|\boldsymbol{d}^T \Psi\|_H$, where $\boldsymbol{D}$ is a fixed positive diagonal matrix.

(c) The stiffness matrix $\boldsymbol{A} := \boldsymbol{D}\langle A\Psi, \Psi \rangle^T \boldsymbol{D}$ has the decay property

$$|a_{\lambda,\lambda'}| \ \lesssim \ 2^{-\sigma \left| |\lambda| - |\mu| \right|} \left(1 + d(\lambda,\lambda')\right)^{-\tau},$$

where $d(\lambda,\lambda') := 2^{\min(|\lambda|,|\lambda'|)} \operatorname{dist}\left(\operatorname{supp}\psi_\lambda, \operatorname{supp}\psi_{\lambda'}\right)$ for some $\sigma > \frac{n}{2}$ and $\tau > n$.

For any finite $\Lambda \subset \nabla$ let $u_\Lambda \in \operatorname{span}(\Psi_\Lambda)$ denote the related Galerkin solution. In [4,5], a strategy is described how to enlarge Λ to some $\tilde{\Lambda} \supset \Lambda$ such that the *distance property* holds, i.e., there exists some $0 < \kappa < 1$ such that

$$\|u_\Lambda - u_{\tilde{\Lambda}}\|_A \geq \kappa \, \|u - u_\Lambda\|_A \ .$$

Now, one proceeds using Galerkin orthogonality

$$a(u_\Lambda - u_{\tilde{\Lambda}}, u - u_{\tilde{\Lambda}}) = 0 \tag{2}$$

to conclude $\|u - u_{\tilde{\Lambda}}\|_A^2 = \|u - u_\Lambda\|_A^2 - \|u_\Lambda - u_{\tilde{\Lambda}}\|_A^2 \leq (1-\kappa^2)\|u - u_\Lambda\|_A^2$, which proves the *saturation property*, i.e., a strict error reduction since $0 < 1 - \kappa^2 < 1$.

Obviously, due to the constraint on the divergence, one can not directly apply this result to the Stokes problem.

3 Mixed Discretization

The most common discretizations of the Stokes problem are based on the *mixed formulation*:

Problem 2. *For given* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)^n$*, determine* $\boldsymbol{u} \in \boldsymbol{X} := H_0^1(\Omega)^n$ *and* $p \in M := L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q(x)\,dx = 0\}$ *such that*

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= (\boldsymbol{f},\boldsymbol{v})_{0,\Omega}, & \boldsymbol{v} \in \boldsymbol{X} \ , \\ b(\boldsymbol{u},q) &= 0, & q \in M \ , \end{aligned} \tag{3}$$

where $a(\boldsymbol{u},\boldsymbol{v}) := (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{0,\Omega}$, $b(\boldsymbol{v},q) := (\nabla \cdot \boldsymbol{v}, q)_{0,\Omega}$.

Assuming that the saddle point Problem 2 is well posed, we are interested in a convergent adaptive strategy. Moreover, given adaptive discretizations $\boldsymbol{X}_\Lambda \subset \boldsymbol{X}$ and $M_\Lambda \subset M$, it is well-known that in order to ensure stability, the induced spaces need to fulfill the *Ladyshenskaja–Babuška–Brezzi (LBB)* condition

$$\inf_{q_\lambda \in M_\Lambda} \sup_{\boldsymbol{v}_\lambda \in \boldsymbol{X}_\Lambda} \frac{b(\boldsymbol{v}_\lambda, q_\lambda)}{\|v_\lambda\|_X \, \|q_\lambda\|_M} \geq \beta \tag{4}$$

for some constant $\beta > 0$ independent of Λ.

In [6], we have introduced a convergent adaptive scheme for saddle point problems and we have given explicit criteria (in terms of single basis functions) in order to ensure (4). Let us sketch the main results from [6] for the special case of the Stokes problem.

3.1 Convergent Adaptive Strategy

In order to introduce an adaptive scheme that can be proven to converge, we consider an adaptive version of Uzawa's algorithm [1]. For the *Schur complement* $S := BA^{-1}B'$ (where A and B are induced by the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, respectively, in the usual manner), we assume $2\|S\|^{-1} > \alpha > 0$ and we set $q := \|Id - \alpha S\| < 1$.

Algorithm 1. *Let* $\Lambda_0^M = \emptyset$ *and* $p_{\Lambda_0}^{(0)} = p^{(0)} = 0$*. Then, for* $i = 1, 2, \ldots$ *and chosen* $\varepsilon_i > 0$ *we proceed as follows:*

1. *Determine by an adaptive algorithm a set of indices* Λ_i^X *such that for the Galerkin solution* $\boldsymbol{u}_{\Lambda_i}^{(i)}$ *w.r.t.* Λ_i^X *of*

$$a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) - b(\boldsymbol{v}, p_{\Lambda_{i-1}}^{(i-1)}), \qquad \boldsymbol{v} \in \boldsymbol{X} \ , \tag{5}$$

one has $\|\boldsymbol{u}_{\Lambda_i}^{(i)} - \tilde{\boldsymbol{u}}^{(i)}\|_{1,\Omega} < q^i \varepsilon_i$*, where* $\tilde{\boldsymbol{u}}^{(i)}$ *denotes the exact solution of (5).*

2. *Determine an index set* Λ_i^M *such that the LBB condition holds. Then, define* $p_{\Lambda_i}^{(i)}$ *by*

$$(p_{\Lambda_i}^{(i)}, q_{\Lambda_i}) = (p_{\Lambda_{i-1}}^{(i-1)}, q_{\Lambda_i}) + \alpha b(\boldsymbol{u}_{\Lambda_i}^{(i)}, q_{\Lambda_i}), \qquad q_{\Lambda_i} \in M_{\Lambda_i} \ . \tag{6}$$

The following result has been proven in [6].

Theorem 1. *Under the above assumptions and with $\varepsilon_i > 0$ chosen such that $\sum_{i=1}^{\infty} \varepsilon_i \lesssim 1$ the exact solution of the mixed problem can be approximated with any desired accuracy by Algorithm 1:*

$$\|\boldsymbol{u} - \boldsymbol{u}_{\Lambda_{i+1}}^{(i+1)}\|_{1,\Omega} + \|p - p_{\Lambda_i}^{(i)}\|_{0,\Omega} \lesssim q^i \ .$$

3.2 The LBB Condition

It was already mentioned that (4) is important for the stability of the numerical solution. Moreover, we have seen that it also enters in Algorithm 1. Hence, it is important to have a criterion for (4) at hand that is easy to check. In the above described adaptive framework this means, that we have to be able to construct a space M_Λ for a given $\boldsymbol{X}_\Lambda$ such that (4) holds (see the second step in Algorithm 1). Finally, this construction must be easy to accomplish and to implement.

First, we have to choose the wavelet bases for $\boldsymbol{X}$ and M appropriately. This has been introduced in [8] and may be summarized as follows: choose wavelet bases $\boldsymbol{\Psi} := \{\boldsymbol{\psi}_\lambda : \lambda \in \nabla^X\}$ for $\boldsymbol{X}$ and $\Theta := \{\vartheta_\mu : \mu \in \nabla^M\}$ for M such that the divergence of any vector field $\boldsymbol{\psi}_\lambda$ is a certain linear combination of the *dual* functions $\tilde{\Theta}$, i.e., there exist finite set of indices $\Delta(\lambda) \subset \nabla^M$ such that

$$\nabla \cdot \boldsymbol{\psi}_\lambda = \sum_{\mu \in \Delta(\lambda)} c_{\lambda,\mu} \tilde{\vartheta}_\mu \ . \tag{7}$$

Using this choice, the following fact is a consequence of the general result stated in [6], Theorem 3.2.

Theorem 2. *For subsets $(\Lambda^X, \Lambda^M) \subset (\nabla^X, \nabla^M)$, define the wavelet trial spaces $\boldsymbol{X}_\Lambda := \mathrm{span}(\boldsymbol{\Psi}_{\Lambda^X})$ and $M_\Lambda := \mathrm{span}(\Theta_{\Lambda^M})$. These spaces fulfill the LBB condition (4) provided that*

$$\Lambda^M = \mathcal{B}(\Lambda^X) := \{\mu \in \Delta(\lambda) : \lambda \in \Lambda^X\} \ . \tag{8}$$

Moreover, (8) *ensures the* full equilibrium property*, i.e.,* $(\nabla \cdot \boldsymbol{v}_\Lambda, q_\Lambda) = 0$ *for an* $\boldsymbol{v}_\Lambda \in \boldsymbol{X}_\Lambda$ *and all* $q_\Lambda \in M_\Lambda$ *already implies* $\nabla \cdot \boldsymbol{v}_\Lambda = 0$.

4 Divergence Free Discretization

Another common way to form a variational formulation of (1) is to embed the divergence constraint into the trial and test space. This goes back to Leray [12] in 1934. Here, we will follow [10]. Let us set

$$\boldsymbol{\mathcal{V}} := \{\boldsymbol{\phi} \in C_0^\infty(\Omega)^n : \nabla \cdot \boldsymbol{\phi} = 0\} \ , \qquad \boldsymbol{V} := \mathrm{clos}_{\|\cdot\|_{1,\Omega}}(\boldsymbol{\mathcal{V}}) \ . \tag{9}$$

Then, the divergence free variational formulation of (1) reads:

Problem 3. *Given a vector field* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)^n$, *one has to determine the velocity* $\boldsymbol{u} \in \boldsymbol{V}$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{f})_{0,\Omega}, \qquad \boldsymbol{v} \in \boldsymbol{V} \ . \tag{10}$$

Note that the pressure is eliminated in this formulation and can be obtained by means of a postprocessing, [10]. The advantage of (10) is obvious since we can deal with an elliptic problem in the setting of Section 2. On the other hand, one has to use a basis for $\boldsymbol{V}$ which is problematic for many kinds of discretizations.

In recent years, divergence free wavelet bases have been constructed starting from the pioneering work by Lemarié–Rieusset in [11], who constructed tensor product divergence free wavelets on $\mathbb{R}^n$. Nowadays, there are also constructions on some classes of bounded domains $\Omega \subset \mathbb{R}^n$ available, [13,14]. We will not describe the construction in detail here, but rather summarize those properties that we will need here and refer the reader to [14] for further details. In particular, we will always assume that Ω is chosen in such a way that the subsequent construction actually is possible.

Divergence free wavelets are linear combinations of suitable wavelet functions in the form

$$\psi_{\boldsymbol{\lambda}}^{\mathrm{df}} = \sum_{\boldsymbol{\mu} \in S(\boldsymbol{\lambda})} d_{\boldsymbol{\lambda},\boldsymbol{\mu}} \psi_{\boldsymbol{\mu}} \ , \tag{11}$$

where $S(\boldsymbol{\lambda}) \subset \boldsymbol{\nabla}$ is a finite subset whose cardinality is independent of $\boldsymbol{\lambda}$. Moreover, also the values of $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ do not depend on $\boldsymbol{\lambda}$ in the sense that the following inequality holds independently of $\boldsymbol{\lambda}$

$$\left| \sum_{\boldsymbol{\mu} \in S(\boldsymbol{\lambda})} d_{\boldsymbol{\lambda},\boldsymbol{\mu}} \right| \lesssim 1 \ . \tag{12}$$

Finally, $\boldsymbol{\Psi} = \{\psi_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}$ is a suitable wavelet basis for $H_0^1(\Omega)^n$ fulfilling Assumption 1. Their duals $\tilde{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}$ take the form

$$\tilde{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}} = \sum_{\boldsymbol{\mu} \in \tilde{S}(\boldsymbol{\lambda})} \tilde{d}_{\boldsymbol{\lambda},\boldsymbol{\mu}} \tilde{\psi}_{\boldsymbol{\mu}} \ , \tag{13}$$

where $\tilde{\boldsymbol{\Psi}}$ is a dual basis for $\boldsymbol{\Psi}$. Finally, the following result is in general proven in [14] (see also [11,13]): each vector field $\boldsymbol{\zeta} \in \boldsymbol{V}$ has a unique expansion

$$\boldsymbol{\zeta} = \sum_{\boldsymbol{\lambda} \in \boldsymbol{\nabla}^{\mathrm{df}}} c_{\boldsymbol{\lambda}} \psi_{\boldsymbol{\lambda}}^{\mathrm{df}}, \qquad \boldsymbol{c} \in \ell^2(\boldsymbol{\nabla}^{\mathrm{df}}) \ ,$$

and the following estimate holds for $\boldsymbol{\zeta}$, $\psi_{\boldsymbol{\lambda}}^{\mathrm{df}} \in H^s(\Omega)^n$

$$\|\boldsymbol{\zeta}\|_{s,\Omega}^2 \sim \sum_{\boldsymbol{\lambda} \in \boldsymbol{\nabla}^{\mathrm{df}}} 2^{2s|\boldsymbol{\lambda}|} |c_{\boldsymbol{\lambda}}|^2 \ . \tag{14}$$

Theorem 3. *The divergence free wavelet bases $\boldsymbol{\Psi}^{\mathrm{df}}$ fulfill Assumption 1 for Problem 3.*

Proof. Condition (a) is trivially fulfilled since the bilinear form $a(\cdot,\cdot)$ is elliptic on all of $H_0^1(\Omega)^n$, [10]. Since $\boldsymbol{V}$ is a closed subset of $H_0^1(\Omega)^n$, the norm equivalence (14) for $s=1$ already ensures (b) in Assumption 1. Finally, due to the properties of $S(\boldsymbol{\lambda})$ and $\boldsymbol{\Psi}$ in (11), we obtain

$$a(\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{\mathrm{df}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}^{\mathrm{df}}) = \sum_{\boldsymbol{\mu}\in S(\boldsymbol{\lambda})} \sum_{\boldsymbol{\mu}'\in S(\boldsymbol{\lambda}')} d_{\boldsymbol{\lambda},\boldsymbol{\mu}} d_{\boldsymbol{\lambda}',\boldsymbol{\mu}'} a(\psi_{\boldsymbol{\mu}}, \psi_{\boldsymbol{\mu}'}) \ .$$

In view of (12) and the properties of $S(\boldsymbol{\lambda})$, we obtain that $\boldsymbol{\Psi}^{\mathrm{df}}$ enforces analogous decay properties as $\boldsymbol{\Psi}$ which proves (c). □

References

1. K. Arrow, L. Hurwicz, and H. Uzawa, Studies in Nonlinear Programming, Stanford University Press, Stanford, CA, 1958.
2. A. Barinka, T. Barsch, P. Charton, A. Cohen, S. Dahlke, W. Dahmen, and K. Urban, *Adaptive wavelet schemes for elliptic problems — Implementation and numerical experiments*, RWTH Aachen, IGPM Preprint 173, 1999.
3. A. Cohen, *Wavelet methods in numerical analysis*, in: Handbook of Numerical Analysis, North Holland, Amsterdam, to appear.
4. A. Cohen, W. Dahmen, and R. DeVore, *Adaptive wavelet schemes for elliptic operator equations — Convergence rates*, RWTH Aachen, IGPM Preprint 165, 1998.
5. S. Dahlke, W. Dahmen, R. Hochmuth, and R. Schneider, *Stable multiscale bases and local error estimation for elliptic problems*, Appl. Numer. Math. **23**, No. 1 (1997) 21–48.
6. S. Dahlke, R. Hochmuth, and K. Urban, *Adaptive wavelet methods for saddle point problems*, Preprint 1126, Istituto di Analisi Numerica del C. N. R., 1999.
7. W. Dahmen, *Wavelet and multiscale methods for operator equations*, Acta Numerica **6** (1997) 55–228.
8. W. Dahmen, A. Kunoth, and K. Urban, *A Wavelet–Galerkin method for the Stokes problem*, Computing **56** (1996) 259–302.
9. W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. **33** (1996) 1106–1124.
10. V. Girault and P.-A. Raviart, Finite Element Methods for Navier–Stokes–Equations, Springer–Verlag, Berlin, 2nd edition, 1986.
11. P.G. Lemarié–Rieusset, *Analyses multi-résolutions non orthogonales, Commutation entre Projecteurs et Derivation et Ondelettes Vecteurs à divergence nulle*, Revista Mat. Iberoamericana **8** (1992) 221–236.
12. J. Leray, *Sur le movement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934) 193–248.
13. K. Urban, *On divergence-free wavelets*, Adv. Comput. Math. **4**, No. 1,2 (1995) 51–82.
14. K. Urban, *Wavelet bases in $\boldsymbol{H}(div)$ and $\boldsymbol{H}(\boldsymbol{curl})$*, Preprint 1106, Istituto di Analisi Numerica del C. N. R., 1998.