A Note on Interpolating Scaling Functions

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Abstract

In this paper, we present a new method to find interpolating refinable functions. The construction can be interpreted as a natural generalization of a well–known univariate approach and applies to scaling matrices A satisfying $|\det A| = 2$. The resulting scaling functions automatically satisfy certain Strang–Fix–conditions.

Key Words: Interpolating scaling functions, Strang–Fix–conditions, expanding scaling matrices.

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1 Introduction

In this note, we present a new approach to construct interpolating scaling functions. In general, a function $\phi \in L_2(\mathbf{R}^d)$ is called a *scaling* function or a *refinable* function if it satisfies a *two–scale–relation*

$$\phi(x) = \sum_{k\in\mathbf{Z}^d} a_k \phi(Ax-k), \qquad \mathbf{a} = \{a_k\}_{k\in\mathbf{Z}^d} \in \ell_2(\mathbf{Z}^d), \tag{1.1}$$

where A is an *expanding* integer scaling matrix, i.e., all its eigenvalues have modulus larger than one. In several practical applications, e.g., in CAGD, it is often convenient

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to work with *interpolating* refinable functions, i.e., in addition to (1.1) one requires that ϕ is at least continuous and satisfies

$$\phi(k) = \delta_{0,k} \qquad k \in \mathbf{Z}^d. \tag{1.2}$$

Furthermore, functions ϕ which are sufficiently smooth and well–located are preferable. In recent studies, several examples of refinable functions satisfying these conditions have been constructed, see, e.g., [2, 3, 4, 5, 6, 7, 12]. In this paper, we present a new approach which yields compactly supported functions and has the advantage that Strang–Fix–conditions of a certain order automatically hold. This is important since the Strang–Fix–conditions always serve as indicators for a certain smoothness, and, moreover, give rise to a certain order of approximation. Our method turns out to be a quite natural generalization of a well–known univariate concept, see Section 2 for details. It applies to scaling matrices A satisfying $|\det A| = 2$ and and can be used in arbitrary spatial dimensions.

This paper is organized as follows. In Section 2, we briefly recall the setting of interpolating scaling functions. In Section 3, we present our new construction and, finally, in Section 4 we discuss some examples to explain the applicability of our approach.

For later use, let us fix some notation. Let $q = |\det A|$. Furthermore, let $R = \{\rho_0, \ldots, \rho_{q-1}\}$, $R^T = \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{q-1}\}$ denote complete sets of representatives of $\mathbf{Z}^d / A\mathbf{Z}^d$ and $\mathbf{Z}^d / B\mathbf{Z}^d$, $B = A^T$, respectively. Without loss of generality, we shall always assume that $\rho_0 = \tilde{\rho}_0 = 0$.

2 The Setting

In the sequel, we shall only consider compactly supported scaling functions. Moreover, we shall always assume that $\operatorname{supp} \mathbf{a} := \{k \in \mathbf{Z}^d \mid a_k \neq 0\}$ is finite. Computing the Fourier transform of both sides of (1.1) yields

$$\hat{\phi}(\omega) = \sum_{k \in \mathbf{Z}^d} \frac{1}{q} a_k e^{-2\pi i \langle k, B^{-1}\omega \rangle} \hat{\phi}(B^{-1}\omega). \tag{2.1}$$

By iterating (2.1) we obtain

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m(B^{-j}\omega), \tag{2.2}$$

where the *symbol* $m(\omega)$ is defined by

$$m(\omega) := \frac{1}{q} \sum_{k \in \mathbf{Z}^d} a_k e^{-2\pi i \langle k, \omega \rangle}. \tag{2.3}$$

Equation (2.2) means that instead of trying to construct a refinable function directly we may also start with a symbol $m(\omega)$. Then the question arises which conditions on m guarantee that $\hat{\phi}$ according to (2.2) is well–defined in $L_2(\mathbf{R}^d)$ and has some additional desirable properties such as sufficient smoothness. Moreover, for our purposes, we have to clarify how the interpolating property (1.2) can be guaranteed. Some sufficient conditions are summarized in the following theorem which goes back to Lemarié [9, 10], see also [2] for a further discussion.

Theorem 2.1 *Let $m(\omega)$ be a trigonometric polynomial which satisfies*

(C1) $m(0) = 1$*;*

(C2) $m(\omega) \geq 0$*;*

(C3) $\sum_{\tilde{\rho} \in R^T} m(\omega + B^{-1}\tilde{\rho}) = 1$*;*

(C4) $m(\omega)$ *satisfies Cohens's condition.*

Then $m(\omega)$ is a symbol of an interpolating refinable function ϕ.

In general, one wants to find scaling functions that have a certain smoothness. To this end, one often requires that the *Strang–Fix–conditions* of order L are satisfied, i.e.,

$$(C5) \qquad \left(\frac{\partial}{\partial \omega}\right)^l m(B^{-1}\tilde{\rho}) = 0 \qquad \text{for all} \quad |l| \leq L \quad \text{and all} \quad \tilde{\rho} \in R^T \backslash \{0\}.$$

In the univariate case, there exist five major approaches to find symbols $m(\omega)$ satisfying (C1)–C(5), see, e.g., [2] for a detailed discussion. There also exist several approaches to generalize some of these concepts to the multivariate case [2, 4]. In this note, we try to find a somewhat natural generalization of the following ansatz which is due to Lemarié and Meyer [10, 11]: Define $m(\omega)$ according to

$$m(\omega) := 1 - c_K \int_0^\omega \sin^{2K-1}(2\pi\omega) d\omega \tag{2.4}$$

and choose c_L such that $m(1/2) = 0$. Then (C5) is clearly satisfied with $L = 2K - 1$.

It turns out that such a generalization can indeed be found, at least for the case $|\det A| = 2$.

3 The Construction

We want to find multivariate versions of (2.4). In a first step, we confine the presentation to the 2D–case. Generalizations to higher–dimensional cases will be discussed later. For notational convenience, we shall always use the abbreviation $\tilde{\rho}_1 = \tilde{\rho}$. (Recall that we always choose $\rho_0 = \tilde{\rho}_0 = 0$).

Observing that in the univariate case $R = R^T = \{0, 1\}$, $B^{-1}\tilde{\rho} = 1/2$, a first guess could be

$$m(\omega_1, \omega_2) = 1 - c_K \int_0^{w_1} \sin^{2K-1}(\pi (B^{-1}\tilde{\rho})_1^{-1} t) dt. \tag{3.1}$$

Using the property

$$\sin(\pi(t+1)) = -\sin(\pi t),$$

it is easily checked that such an approach may work in principle. However, it has the disadvantage that it always leads to some kind of 'separable' symbol. We would clearly prefer a 'non–separable', i.e., truly multivariate symbol. To this end, it is somewhat natural to replace the right–hand side in (3.1) by an expression involving some kind of double integral. As we shall see in Theorem 3.1 stated below, this does not work directly but requires some additional correction terms and further conditions on the integrands. Nevertheless, as explained in Section 4, examples can be constructed in some very natural way.

Theorem 3.1 *Suppose that $m_1(t_1)$, $m_2(t_2)$ are trigonometric polynomials satisfying*

$$m_1((B^{-1}\tilde{\rho})_1 + t) = -m_1(t), \qquad m_2((B^{-1}\tilde{\rho})_2 + t) = m_2(t), \tag{3.2}$$

$$\int_0^{(B^{-1}\tilde{\rho})_2} m_2(t)dt = 0, \tag{3.3}$$

and

$$\left(\frac{d}{dt}\right)^k m_i((B^{-1}\tilde{\rho})_i) = 0 \quad \text{for all} \;\; k \leq L-1, \;\; i = 1,2. \tag{3.4}$$

Furthermore, let the constant c_1 be defined by

$$c_1 := \left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1\right)^{-1} \tag{3.5}$$

and suppose that c_2 and $c_{1,2}$ are related by

$$c_2 = -\frac{c_{1,2}}{2c_1}. \tag{3.6}$$

Then the symbol

$$m(\omega_1, \omega_2) = 1 - c_{1,2}\int_0^{\omega_1}\int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 - c_1\int_0^{\omega_1} m_1(t_1)dt_1 - c_2\int_0^{\omega_2} m_2(t_2)dt_2 \tag{3.7}$$

satisfies (C1), (C3) *and Strang–Fix conditions* (C5) *of order L.*

Proof: Let us start by varifying the Strang–Fix conditions (C5). For $l_1, l_2 > 0$, we obtain by exploiting assumption (3.4)

$$\begin{aligned}\left(\frac{\partial}{\partial\omega}\right)^l (m(B^{-1}\tilde{\rho})) \;\; &= \;\; -c_{1,2}\left(\frac{d}{dt_1}\right)^{l_1-1} m_1((B^{-1}\tilde{\rho})_1)\left(\frac{d}{dt_2}\right)^{l_2-1} m_2((B^{-1}\tilde{\rho})_2) \\ &\quad -c_1\left(\frac{d}{dt_1}\right)^{l_1-1} m_1((B^{-1}\tilde{\rho})_1) - c_2\left(\frac{d}{dt_2}\right)^{l_2-1} m_2((B^{-1}\tilde{\rho})_2) = 0.\end{aligned}$$

The cases $l_1 = 0,\; l_2 > 0$ and $l_2 = 0,\; l_1 > 0$ can be treated analogously. It remains to study the case $l_1 = l_2 = 0$. By using (3.3) and (3.5) we get

$$m(B^{-1}\tilde{\rho}) \;\; = \;\; 1 - c_{1,2}\int_0^{(B^{-1}\tilde{\rho})_1}\int_0^{(B^{-1}\tilde{\rho})_2} m_1(t_1)m_2(t_2)dt_1dt_2 - c_1\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1$$

$$
\begin{aligned}
&-c_2 \int_0^{(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 \\
&= 1 - c_1 \int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 \\
&= 0.
\end{aligned}
$$

The next step is to check the condition (C3). Splitting up the integrals yields

$$
\begin{aligned}
&m(\omega) + m(\omega + B^{-1}\tilde{\rho}) \\
&= 2 - c_{1,2} \int_0^{\omega_1} \int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1 \int_0^{\omega_1} m_1(t_1)dt_1 - c_2 \int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_{1,2} \int_0^{\omega_1+(B^{-1}\tilde{\rho})_1} \int_0^{\omega_2+(B^{-1}\tilde{\rho})_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1 \int_0^{\omega_1+(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 \\
&\quad -c_2 \int_0^{\omega_2+(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 \\
&= 2 - c_{1,2} \int_0^{\omega_1} \int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1 \int_0^{\omega_1} m_1(t_1)dt_1 - c_2 \int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_{1,2} \left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 + \int_{(B^{-1}\tilde{\rho})_1}^{(B^{-1}\tilde{\rho})_1+\omega_1} m_1(t_1)dt_1 \right) \left(\int_0^{(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 + \int_{(B^{-1}\tilde{\rho})_2}^{(B^{-1}\tilde{\rho})_2+\omega_2} m_2(t_2)dt_2 \right) \\
&\quad -c_1 \left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 + \int_{(B^{-1}\tilde{\rho})_1}^{\omega_1+(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 \right) - c_2 \left(\int_0^{(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 + \int_{(B^{-1}\tilde{\rho})_2}^{\omega_2+(B^{-1}\tilde{\rho})_2} m_2(t_2)dt_2 \right).
\end{aligned}
$$

Therefore, by employing the conditions (3.2) and (3.3), we get

$$
\begin{aligned}
&m(\omega) + m(\omega + B^{-1}\tilde{\rho}) \\
&= 2 - c_{1,2} \int_0^{\omega_1} \int_0^{\omega_2} m_1(t_1)m_2(t_2)dt_1dt_2 \quad - c_1 \int_0^{\omega_1} m_1(t_1)dt_1 - c_2 \int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_{1,2} \left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 - \int_0^{\omega_1} m_1(t_1)dt_1 \right) \int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_1 \left(\int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 - \int_{(0}^{\omega_1} m_1(t_1)dt_1 \right) - c_2 \int_0^{\omega_2} m_2(t_2)dt_2 \\
&= 2 - c_2 \int_0^{\omega_2} m_2(t_2)dt_2 - c_{1,2} \int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 \int_0^{\omega_2} m_2(t_2)dt_2 \\
&\quad -c_1 \int_0^{(B^{-1}\tilde{\rho})_1} m_1(t_1)dt_1 - c_2 \int_0^{\omega_2} m_2(t_2)dt_2.
\end{aligned}
$$

By using (3.5), we end up with

$$
m(\omega) + m(\omega + B^{-1}\tilde{\rho}) = 1 + (-2c_2 - c_{1,2}c_1^{-1}) \int_0^{\omega_2} m_2(t_2)dt_2
$$

and (C3) follows from (3.6). It is obvious that the symbol $m(\omega_1, \omega_2)$ satisfies (C1). The theorem is proved. □

Remark 3.1 *The reader should observe that Theorem 3.1 can in fact be used simultaneously for a whole class of matrices satisfying* $|\det A| = 2$*. Assume that a second scaling matrix* M *exists with a representative* $\tilde{\delta}$ *such that* $A^{-T}\tilde{\rho} = M^{-T}\tilde{\delta}$ *holds. Then a symbol* m *constructed according to* (3.7) *for* A *also works for* M*. Nevertheless, from* (2.2) *it is clear that the resulting refinable functions may differ dramatically.*

Theorem 3.1 clearly generalizes to higher dimensional cases, although everything becomes much more complicated from the technical point of view. Therefore we only state one possible 3D–version of our approach. Several other variants are possible.

Theorem 3.2 *Suppose that* $m_1(t_1), m_2(t_2)$ *and* $m_3(t_3)$ *are trigonometric polynomials satisfying* (3.4)*. Let us furthermore assume that* m_2 *and* m_3 *both satisfy* (3.3) *and that*

$$m_1((B^{-1}\tilde{\rho})_1 + t) = -m_1(t), \quad m_2((B^{-1}\tilde{\rho})_2 + t) = m_2(t), \quad m_3((B^{-1}\tilde{\rho})_3 + t) = m_3(t). \tag{3.8}$$

Let c_1 *be defined by* (3.5) *and suppose that* $c_{1,2,3}$ *and* $c_{2,3}$ *are related by*

$$c_{2,3} = -\frac{c_{1,2,3}}{2c_1}. \tag{3.9}$$

Then the symbol

$$\begin{aligned} m(\omega_1,\omega_2,\omega_3) &= 1 - c_{1,2,3}\int_0^{\omega_1}\int_0^{\omega_2}\int_0^{\omega_3} m_1(t_1)m_2(t_2)m_3(t_3)dt_1dt_2dt_3 \\ &\quad -c_{2,3}\int_0^{\omega_2}\int_0^{\omega_3} m_2(t_2)m_3(t_3)dt_2dt_3 - c_1\int_0^{\omega_1} m_1(t_1)dt_1 \end{aligned} \tag{3.10}$$

satisfies (C1), (C3) *and* (C5).

4 Examples

We have applied the construction presented above to the case $d = 2$, $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. In this case, $|\det A| = 2$ as required and we may choose $\tilde{\rho} = \binom{1}{0}$ as the second representative. Quite natural choices for $m_1(t_1)$, $m_2(t_2)$ are given by

$$m_1(t_1) = \sin^{2K-1}(2\pi t_1), \qquad m_2(t_2) = \sin^{2K-1}(4\pi t_2). \tag{4.1}$$

Let us first discuss the case $K = 2$. Then

$$c_1 = \frac{3\pi}{2}, \qquad c_2 = \frac{-c_{1,2}}{3\pi} \tag{4.2}$$

and (3.7) yields

$$\begin{aligned} &m(\omega_1,\omega_2) \\ &= 1 - \frac{c_{1,2}}{72\pi^2}\left(-\cos(2\pi\omega_1)(2+\sin^2(2\pi\omega_1))+2\right)\left(-\cos(4\pi\omega_2)(2+\sin^2(4\pi\omega_2))+2\right) \\ &\quad -\frac{1}{4}\left(-\cos(2\pi\omega_1)(2+\sin^2(2\pi\omega_1))+2\right) + \frac{c_{1,2}}{36\pi^2}\left(-\cos(4\pi\omega_2)(2+\sin^2(4\pi\omega_2))+2\right). \end{aligned} \tag{4.3}$$

The nonvanishing coefficients of the resulting mask can be computed a follows.

$$
\begin{aligned}
a_{(0,0)} &= \frac{1}{2}; \\
a_{(1,2)} &= a_{(1,-2)} = a_{(-1,2)} = a_{(-1,-2)} = -\frac{81c_{1,2}}{4608\pi^2}; \\
a_{(1,6)} &= a_{(1,-6)} = a_{(-1,6)} = a_{(-1,-6)} = a_{(3,2)} = a_{(3,-2)} = a_{(-3,2)} = a_{(-3,-2)} = \frac{9c_{1,2}}{4608\pi^2}; \\
a_{(-3,-6)} &= a_{(-3,6)} = a_{(3,-6)} = a_{(3,6)} = -\frac{c_{1,2}}{4608\pi^2}; \\
a_{(-1,0)} &= a_{(1,0)} = \frac{9c_{1,2}}{288\pi^2} + \frac{9}{32}; \\
a_{(3,0)} &= a_{(-3,0)} = -\frac{c_{1,2}}{288\pi^2} - \frac{1}{32}.
\end{aligned}
\tag{4.4}
$$

A typical symbol obtained by this procedure is displayed in Figure 1.

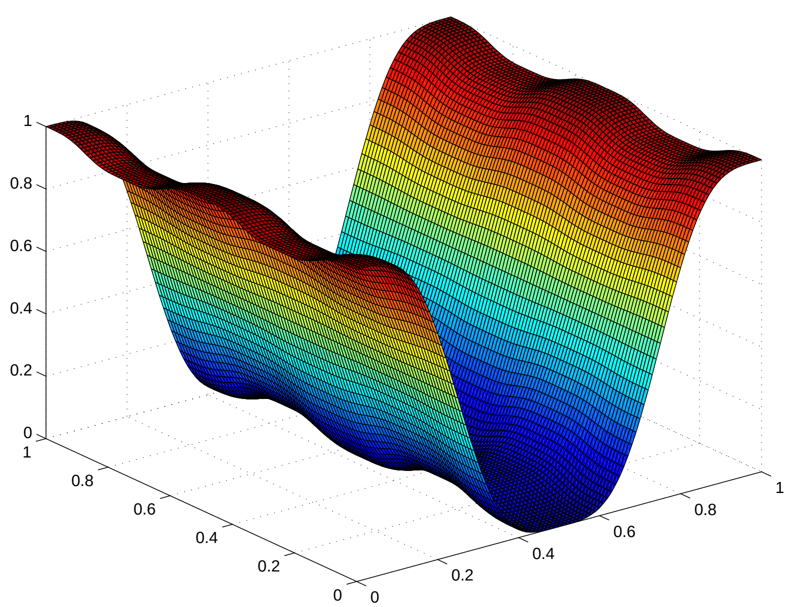

Figure 1: $m(\omega_1, \omega_2)$ for $c_{1,2} = -5$

It remains to estimate the smoothness of the resulting refinable function ϕ, i.e., we want to find

$$\alpha^* := \sup\{\alpha : \ \phi \in C^\alpha\}.$$

It is well–known that $\alpha^* \geq \kappa_{\sup}$, where $\kappa_{\sup}$ is defined by

$$\kappa_{\sup} := \sup\{\kappa : \ \int_{\mathbf{R}^d} (1+|\omega|)^\kappa |\hat{\phi}(\omega)| d\omega < \infty\}. \tag{4.5}$$

The regularity problem, i.e., the problem of estimating κ_{sup} from below, has attracted several people in the last few years, see, e.g., [1, 8, 12, 13]. One typical result in this direction reads as follows.

Theorem 4.1 *For an integer L, let*

$$V_L := \{v \in \ell_0(\mathbf{Z}^d) \ : \ \sum_{k \in \mathbf{Z}^d} p(k) v_k = 0, \quad \text{for all } p \in \Pi_L\},$$

where Π_L denotes the polynomials of total degree L. Assume that A is a dilation matrix with a complete set of orthonormal eigenvectors. If the symbol $m(\omega)$ according to (2.3) *is nonnegative and satisfies Strang–Fix–conditions* (C5) *of order L, then for a suitable choice Ω with supp $\mathbf{a} \subseteq \Omega$, V_L is invariant under the matrix*

$$\mathcal{H} := [q a_{Ak-l}]_{k,l \in \Omega}\,.$$

Let ϱ be the spectral radius of $\mathcal{H}|_{V_L}$. Then the exponent κ_{sup} satisfies

$$\kappa_{\text{sup}} \geq -\frac{\log(\varrho)}{\log(|\lambda_{\max}|)}. \tag{4.6}$$

We used Theorem 4.1 to test several values of $c_{1,2}$. The results are shown in the following table.

$c_{1,2}$	$-\log(\varrho)/\log(\lvert\lambda_{\max}\rvert)$
-50	0.26569
-10	0.55643
-5	0.60106
-3	0.61971
-1	0.63884
-0.5	0.6437
0	0.6486
0.5	0.65352
1	0.65848
3	0.67864
50	0.7298
100	0.0054245

Remark 4.1 *i) We see that the regularity of the resulting interpolating scaling functions decreases significantly for large values of $|c_{1,2}|$. For very large values of $|c_{1,2}|$, one does not even get an L_2–function.*

ii) We also observe that in order to increase the smoothness of the corresponding scaling function it seems to be a good idea to use positive values of $c_{1,2}$. However, then another problem occurs. To use Theorem 4.1, we have to work with a nonnegative symbol, and it can be easily checked that this is only the case for $c_{1,2}$ in a certain interval contained in $(-\infty, 0]$. Therefore the results for positive values of $c_{1,2}$ are not completely justified by Theorem 4.1. But the requirement of a nonnegative symbol in Theorem 4.1 is a sufficient condition which does not need to be necessary in all cases.

As already stressed in Remark 3.1, the symbol computed according to Theorem 3.1 can also be used for other scaling matrices. In our case, it is easy to check that e.g. for the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\tilde{\delta} = \binom{1}{0}$ the conditions of Remark 3.1 are satisfied. It turns out that for this matrix the resulting refinable functions are in fact much smoother as can be seen from the following table.

$c_{1,2}$	$-\log(\varrho)/\log(\vert\lambda_{\max}\vert)$
-10	0.96322
-5	1.2694
-1	1.7589
-0.5	1.8665
0	2
0.2	1.9678
1	1.8562
10	1.2073
50	0.045414

We have also studied the case $K = 3$. In this case, eq. (3.7) yields

$$\begin{aligned} & m(\omega_1, \omega_2) \\ = \; & 1 - \frac{c_{1,2}}{\pi^2}\left(-\frac{5}{16}\cos(2\pi\omega_1) + \frac{5}{96}\cos(6\pi\omega_1) - \frac{1}{160}\cos(10\pi\omega_1) + \frac{4}{15}\right) \\ & \cdot\left(-\frac{5}{32}\cos(4\pi\omega_2) + \frac{5}{192}\cos(12\pi\omega_2) - \frac{1}{320}\cos(20\pi\omega_2) + \frac{2}{15}\right) \\ & -\frac{15}{8}\left(-\frac{5}{16}\cos(2\pi\omega_1) + \frac{5}{96}\cos(6\pi\omega_1) - \frac{1}{160}\cos(10\pi\omega_1) + \frac{4}{15}\right) \\ & +\frac{4c_{1,2}}{15\pi^2}\left(-\frac{5}{32}\cos(4\pi\omega_2) + \frac{5}{192}\cos(12\pi\omega_2) - \frac{1}{320}\cos(20\pi\omega_2) + \frac{2}{15}\right). \end{aligned}$$

The nonvanishing coefficients of the resulting mask are given by

$$\begin{aligned} a_{(0,0)} &= \frac{1}{2}; \qquad (4.7) \\ a_{(1,2)} &= a_{(1,-2)} = a_{(-1,2)} = a_{(-1,-2)} = -\frac{25c_{1,2}}{2048\pi^2}; \\ a_{(1,6)} &= a_{(1,-6)} = a_{(-1,6)} = a_{(-1,-6)} = \frac{25c_{1,2}}{12288\pi^2}; \\ a_{(1,10)} &= a_{(1,-10)} = a_{(-1,10)} = a_{(-1,-10)} = -\frac{5c_{1,2}}{20480\pi^2}; \\ a_{(1,0)} &= a_{(-1,0)} = \frac{75}{256} + \frac{c_{1,2}}{48\pi^2}; \\ a_{(3,2)} &= a_{(3,-2)} = a_{(-3,2)} = a_{(-3,-2)} = \frac{45c_{1,2}}{12288\pi^2}; \\ a_{(-3,-6)} &= a_{(-3,6)} = a_{(3,-6)} = a_{(3,6)} = -\frac{45c_{1,2}}{73728\pi^2}; \end{aligned}$$

$$
\begin{aligned}
a_{(3,10)} &= a_{(3,-10)} = a_{(-3,10)} = a_{(-3,-10)} = \frac{9c_{1,2}}{122880\pi^2};\\
a_{(3,0)} &= a_{(-3,0)} = -\frac{9c_{1,2}}{1440\pi^2} - \frac{75}{1536};\\
a_{(5,2)} &= a_{(5,-2)} = a_{(-5,2)} = a_{(-5,-2)} = -\frac{5c_{1,2}}{20480\pi^2};\\
a_{(5,6)} &= a_{(5,-6)} = a_{(-5,6)} = a_{(-5,-6)} = \frac{5c_{1,2}}{122880\pi^2};\\
a_{(5,10)} &= a_{(5,-10)} = a_{(-5,10)} = a_{(-5,-10)} = -\frac{c_{1,2}}{204800\pi^2};\\
a_{(5,0)} &= a_{(-5,0)} = \frac{15}{2560} + \frac{c_{1,2}}{2400\pi^2}.
\end{aligned}
$$

The regularity of the corresponding scaling functions can again be estimated by using Theroem 4.1.

$c_{1,2}$	$-\log(\varrho)/\log(\lvert\lambda_{\max}\rvert)$
-50	0.42988
-10	0.5938
-3	0.61571
-1	0.62137
-0.5	0.62275
0	0.6241
3	0.63181
10	0.64683
20	0.66002
30	0.625
50	0.4986

Remark 4.2 *A* MATLAB *program to compute the regularity of refinable functions according to Theorem 4.1 can be found on the IGPM–homepage, see* `http://elc2.igpm.rwth-aachen.de/barinka/mattoys/soft.html`.

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