Quarkonial frames with compression properties

Stephan Dahlke*, Philipp Keding, and Thorsten Raasch

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Abstract

In the spirit of subatomic or quarkonial decomposition of function spaces [26], we construct compactly supported, piecewise polynomial functions whose properly weighted dilates and translates generate frames for Sobolev spaces on the real line. All frame elements except for those on the coarsest level have vanishing moment properties. As a consequence, the matrix representation of certain elliptic operators in frame coordinates is compressible, i.e., well-approximable by sparse submatrices.

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1 Introduction

The theory of function spaces and their decompositions is an intensively studied field of research with many potential applications. Stable decompositions of function spaces by means of atoms or molecules usually give rise to equivalences of smoothness norms and weighted sequence norms of expansion coefficients. This property enables one to switch from a continuous to a discrete setting which is essential for practical applications. Prominent examples are atomic decompositions by means of wavelets. In this case, the atoms are designed by dilating, translating and scaling of a finite set of functions. These wavelet atoms give rise to stable decompositions for many important function spaces such as Besov and Triebel-Lizorkin spaces, see [16, 18, 28]. However, in the last years, many more decomposition techniques have been designed. In particular, the concept of subatomic or quarkonial decompositions seems to be a quite powerful approach. These decompositions are based on a partition of unity (PUM) whose elements are not only translated and dilated as in the wavelet case, but also multiplied by polynomials up to a certain order. By proceeding this way, the collection of atoms is highly enriched and therefore allows for much more flexible decomposition strategies. However, on the other

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hand, the representation of a given function then gets highly redundant. Therefore, we do not end up with a basis, but with a frame. Nevertheless, these frames are again stable in the sense that they give rise to norm equivalences for certain function spaces, see, e.g, [25–27].

Stable decompositions also have successful applications in numerical analysis. In recent years, in particular the design of adaptive numerical algorithms for operator equations has become more and more the center of attraction. In general, an adaptive algorithm is an updating strategy. Based on a local a posteriori error estimator, a partition of a domain (or more general a finite-dimensional subspace of a function space) is refined (enriched) only in regions where the approximation is still far away from the exact solution. We refer, e.g., to the monograph [29] for an overview. Once again, the wavelet setting stands out since the strong analytical properties of wavelets can be used to design refinement strategies that are guaranteed to converge for a large set of problems, including operators of negative order [5–7]. Moreover, the order of convergence is optimal in the sense that the convergence order of best N-term wavelet approximation is asymptotically realized. In the meantime, these algorithms have also been generalized to the case of (wavelet) frames [9, 10, 23].

Once we know how adaptive numerical algorithms based on wavelet frames can be designed, it is clearly an interesting and challenging task to design adaptive numerical schemes based on quarkonial decompositions. The motivation can be explained as follows: Standard adaptive wavelet methods are essentially based on space refinements (h-method). In the case of quarks, also the p-enrichment induced by the polynomials comes into play, so that the resulting algorithm would resemble an hp-method. It is well-known that hp-methods for operator equations usually converge quite fast, however, rigorous proofs are often missing. So there is some hope that in the long run, by combining the knowledge on the design of adaptive wavelet methods with the concept of quarkonial decompositions, it might be possible to derive very powerful schemes with a provable order of convergence. However, to achieve this goal, it is a long way to go, and this paper can be viewed as one first step in this direction.

The idea to use subatomic decompositions for numerical purposes has some history. Variants of the (PUM)-method have already been employed in numerical analysis under various names (meshless particle methods, generalized finite element methods, hp-clouds, etc.). Formally, this method has been introduced by Babuška and Melenk [1, 2]. There also exist combinations with multiscale methods (MPUM), see [22].

From the viewpoint of wavelet methods, the design of numerical solvers based on subatomic decompositions may be structured into the following two major steps. First of all, one should establish the approximation and stability properties of the desired ansatz system. When using subatomic decompositions, we can expect that the set of basis functions will form at least a frame for the solution space which typically is a Sobolev space on a domain or on a closed manifold. The second step entails the choice of a suitable convergent refinement strategy. Here, at least two alternative refinement strategies are available from the wavelet context. On the one hand, one may consider Galerkin-type methods, where a finite subset of active frame elements is iteratively refined, e.g., by chasing the large residual coefficients of the associated Galerkin projections. This choice is closely related with prominent refinement strategies of adaptive finite element methods. However, due to the redundancy of the quarkonial frame, the uniform well-posedness of the finite-dimensional Galerkin subproblems would not be guaranteed without spending additional effort. On the other hand, one may reinterpret the given operator equation as an equivalent, biinfinite system of equations for the expansion coefficients of the unknown solution in the quarkonial frame. In order to realize well-known iterative schemes from numerical linear algebra within this infinite-dimensional setting, it turns out that the infinite-dimensional stiffness matrices should have certain compression properties which are enforced, e.g., by vanishing moment properties of the individual frame elements, see again [6] for details.

The investigations in this paper exactly follow these observations. Based on a given biorthogonal wavelet basis, we construct a quarkonial frame system that indeed possesses the same order of vanishing moments as the underlying wavelet basis. Moreover, these frame systems are stable in the Sobolev spaces H^s for a certain range of parameters $0 < s < \gamma$, where γ depends on the properties of the wavelet basis. The techniques are based on smoothness (Bernstein) and approximation (Jackson) estimates combined with abstract axiomatic principles to design multi-scale frames of *hp*-type.

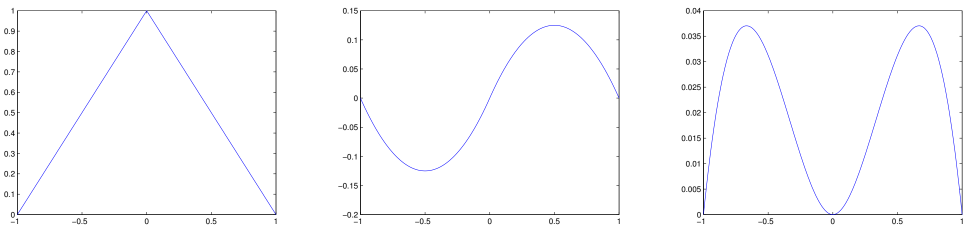
In a certain sense, this paper supplements the investigations in [11]. In loc. cit., also stable quarkonial systems for function spaces have been designed. However, the basis functions used there usually do not possess vanishing moments. On the other hand, the analysis in [11] is more general in the sense that whole scales of Besov spaces are considered, whereas in this paper we confine the discussion to the case of L_2 Sobolev spaces.

This paper is organized as follows. In Section 2, we introduce the basic quarkonial setting and fixe some notation. In Section 3 we derive Jackson- and Berstein estimates related to our specific quarkonial decompositions. Then, in Section 4 we show that by switching to generalized wavelets associated with the underlying PUM, we end up with a frame for L_2. In Section 5, we show that by combining the investigations in the Sections 3 and 4 with an abstract approach to design multiscale *hp*-frames we also obtain stable frames in $H^s, 0 < s < \gamma$. Finally, in Section 6, we prove first compression results for stiffness matrices induced by classical elliptic differential operators backed up by some numerical experiments.

2 Preliminaries and Notation

For $\gamma > 0$, let $\varphi \in H^\gamma(\mathbb{R}) \cap L_\infty(\mathbb{R})$ have compact support in $[-M, M]$, $M \in \mathbb{N}$, and suppose that it holds the partition of unity property

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = 1, \quad \text{for all } x \in \mathbb{R}. \tag{1}$$



(a) $p=0$ (b) $p=1$ (c) $p=2$

Figure 1: Some B-spline quarks φ_p of order $m=2$

We assume that the integer translates of φ are ℓ_2-stable and therefore a Riesz basis for their closed span

$$V := \operatorname{clos}_{L_2(\mathbb{R})} \operatorname{span} \bigl\{\varphi(\cdot - k) : k \in \mathbb{Z}\bigr\}. \tag{2}$$

In particular, there exist stability constants c_φ, $C_\varphi > 0$, such that

$$c_\varphi \|\mathbf{c}\|_{\ell_2(\mathbb{Z})} \le \Bigl\| \sum_{k\in\mathbb{Z}} c_k \varphi(\cdot - k) \Bigr\|_{L_2(\mathbb{R})} \le C_\varphi \|\mathbf{c}\|_{\ell_2(\mathbb{Z})}, \quad \text{for all } \mathbf{c} = (c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}). \tag{3}$$

As a typical example in which these requirements are satisfied, we think of φ being a symmetrized cardinal B-spline of order $m > \gamma + \frac{1}{2}$, i.e., $\varphi = N_m(\cdot + \lfloor \frac{m}{2} \rfloor)$ with $\operatorname{supp}\varphi = [-\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil]$ and $M = \lceil \frac{m}{2} \rceil$.

Our aim is to analyze systems of dilates and integer translates of the *quarks*

$$\varphi_p(x) := \left(\tfrac{x}{\lceil m/2 \rceil}\right)^p \varphi(x), \quad \text{for all } p \ge 0, x \in \mathbb{R}, \tag{4}$$

and their stability properties in relevant function spaces, see also Figure 1. In particular, let us define

$$\varphi_{p,j,k}(x) := 2^{j/2} \varphi_p(2^j x - k), \quad \text{for all } p, j \ge 0, k \in \mathbb{Z}, x \in \mathbb{R}. \tag{5}$$

For given $j, p \ge 0$, we shall consider the closed subspaces

$$V_{j,p} := \operatorname{clos}_{L_2(\mathbb{R})} \operatorname{span}\{\varphi_{i,j,k} : 0 \le i \le p, k \in \mathbb{Z}\}. \tag{6}$$

In case that $\varphi = N_m(\cdot + \lfloor \frac{m}{2} \rfloor)$, the spaces $V_{j,p} = \{f(2^j \cdot) : f \in V_{0,p}\}$ are closely related to certain polynomial spline spaces. In fact, it obviously holds that $V_{0,p} \subset \operatorname{clos}_{L_2(\mathbb{R})} S^{m-2}_{m+p}$, where S^r_n is the polynomial spline space of order n and regularity r with respect to integer nodes of multiplicity $n-r-1$,

$$S^r_n := \bigl\{f \in L_2(\mathbb{R}) : f|_{[k,k+1)} \in \mathbb{P}_{n-1}, k \in \mathbb{Z}\bigr\} \cap C^r(\mathbb{R}).$$

However, for $m \ge 2$, S^{m-2}_{m+p} is strictly larger than $\operatorname{span}\{\varphi_i(\cdot - k) : 0 \le i \le p, k \in \mathbb{Z}\}$. A simple counterexample for $m = 2$ is the quadratic B-spline with respect to double integer knots, $s(x) = \max\{0, 1 - |1 - x|\}^2$. $s \in S_3^0$ does not have a finite linear expansion with respect to the integer translates of $N_2(x+1)$ and $xN_2(x+1)$.

In the sequel, it is our aim to verify the following three properties. First, we will prove a *Jackson estimate* of the form

$$\|f\|_{L_2(\mathbb{R})}^2 + \sum_{j=0}^{\infty} 2^{2js} E_{j,p}(f)^2 \le A_s \|f\|_{H^s(\mathbb{R})}^2, \quad \text{for all } f \in H^s(\mathbb{R}), 0 < s < \gamma, p \ge 0, \tag{7}$$

where $E_{j,p}(f) := \inf_{v \in V_{j,p}} \|f - v\|_{L_2(\mathbb{R})}$ is the error of the best $L_2(\mathbb{R})$ approximation from $V_{j,p}$. Second, we will establish a *Bernstein estimate*,

$$\|g\|_{H^s(\mathbb{R})} \le B_{s,p} 2^{js} \|g\|_{L_2(\mathbb{R})}, \quad \text{for all } g \in V_{j,p}, 0 < s < \gamma, p \ge 0. \tag{8}$$

Finally, we will prove that the system $\{\varphi_{i,j,k} : 0 \le i \le p, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$ frame for its closed span $V_{j,p}$, i.e.

$$(C_p)^{-1} \|g_{j,p}\|_{L_2(\mathbb{R})}^2 \le \inf_{g_{j,p} = \sum_{i=0}^{p} \sum_{k \in \mathbb{Z}} c_{i,k} \varphi_{i,j,k}} \sum_{i=0}^{p} \sum_{k \in \mathbb{Z}} |c_{i,k}|^2 \le D_p \|g_{j,p}\|_{L_2(\mathbb{R})}^2, \quad \text{for all } g_{j,p} \in V_{j,p}, \tag{9}$$

where the frame constants C_p, D_p only depend on the current maximal polynomial degree $p \ge 0$. Under the conditions (7), (8) and (9), the stability of a properly weighted system of dilates and translates of φ_p, $p \ge 0$, follows from general principles via the theory of stable subspace splittings.

3 Direct and Inverse Estimates

3.1 Direct Estimates

We shall first derive direct estimates for the approximation spaces $V_{j,p}$ from (6). They are closely related to known results from spline theory, however we do not yet need that φ is a B-spline.

Theorem 3.1. *Assume that* (1) *holds. There exists $A_m > 0$, such that*

$$(p+1)^{2s} \sum_{j=0}^{\infty} 2^{2js} E_{j,p}(f)^2 \le A_m \|f\|_{H^s(\mathbb{R})}^2, \quad \textit{for all } f \in H^s(\mathbb{R}), 0 < s \le m. \tag{10}$$

In particular, it holds that

$$E_{j,p}(f) \le A_m^{1/2} (p+1)^{-s} 2^{-js} \|f\|_{H^s(\mathbb{R})}, \quad \textit{for all } f \in H^s(\mathbb{R}), 0 < s \le m. \tag{11}$$

Proof. Let $j, p \ge 0$ and $f \in L_2(\mathbb{R})$ be fixed. In view of (4), (5) and (6), $V_{j,p}$ contains at least all $v \in L_2(\mathbb{R})$ of the form

$$v(x) = \sum_{k \in \mathbb{Z}} p_k(x) \varphi(2^j x - k), \tag{12}$$

where $p_k \in \mathbb{P}_p$ are polynomials of degree at most p, for all $k \in \mathbb{Z}$, and the sum converges in $L_2(\mathbb{R})$. From the partition property (1) and from (12), we can deduce that

$$f(x) - v(x) = \sum_{k\in\mathbb{Z}} \big(f(x) - p_k(x)\big)\varphi(2^j x - k), \quad \text{for almost every } x \in \mathbb{R}.$$

Define $I_{j,l} := 2^{-j}[l, l+1]$ and $S_{j,k} := \operatorname{supp}\varphi(2^j \cdot -k)$, for all $l, k \in \mathbb{Z}$. By the compact support of φ, $\#\{k \in \mathbb{Z} : S_{j,k} \cap I_{j,l} \neq \emptyset\}$ is uniformly bounded in $l \in \mathbb{Z}$ and $j \geq 0$. For any $f \in L_2(\mathbb{R})$, we can therefore estimate

$$\begin{aligned}
\|f - v\|^2_{L_2(\mathbb{R})} &= \sum_{l\in\mathbb{Z}} \int_{I_{j,l}} \Big(\sum_{k\in\mathbb{Z}} \big(f(x) - p_k(x)\big)\varphi(2^j x - k)\Big)^2 \,\mathrm{d}x \\
&\leq C_1 \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \int_{I_{j,l}} \big(f(x) - p_k(x)\big)^2 \varphi(2^j x - k)^2 \,\mathrm{d}x \\
&= C_1 \sum_{k\in\mathbb{Z}} \int_{S_{j,k}} \big(f(x) - p_k(x)\big)^2 \varphi(2^j x - k)^2 \,\mathrm{d}x \\
&\leq C_1 \|\varphi\|^2_{L_\infty(\mathbb{R})} \sum_{k\in\mathbb{Z}} \big\|f - p_k\big\|^2_{L_2(S_{j,k})}.
\end{aligned}$$

with $C_1 = C_1(\varphi)$.

Now let $f \in H^m(\mathbb{R})$ and let $p_k \in \mathbb{P}_p$ be the orthogonal projection of $f|_{S_{j,k}}$ onto $\mathbb{P}_p$ in $L_2(S_{j,k})$. It follows that $\|p_k\|_{L_2(S_{j,k})} \leq \|f\|_{L_2(S_{j,k})}$ and due to $\varphi \in L_\infty(\mathbb{R})$, the sum (12) really converges in $L_2(\mathbb{R})$, so that this particular v is contained in $V_{j,p}$. Moreover, standard results from polynomial approximation tell us that

$$\|f - p_k\|_{L_2(S_{j,k})} \leq C_2(p+1)^{-m} 2^{-jm} |f|_{H^m(S_{j,k})},$$

where $C_2 = C_2(m, \varphi) > 0$ is independent of j, k and p, see [21, Cor. 3.12]. We deduce that with $C_3 = C_3(m, \varphi) > 0$,

$$E_{j,p}(f) \leq \|f - v\|_{L_2(\mathbb{R})} \leq C_3(p+1)^{-m} 2^{-jm} |f|_{H^m(\mathbb{R})}. \tag{13}$$

For arbitrary $f \in L_2(\mathbb{R})$, using the triangle inequality and (13), we see that for each $g \in H^m(\mathbb{R})$, we have

$$E_{j,p}(f) \leq \|f - g\|_{L_2(\mathbb{R})} + E_{j,p}(g) \leq \|f - g\|_{L_2(\mathbb{R})} + C_4(p+1)^{-m} 2^{-jm} |g|_{H^m(\mathbb{R})}.$$

By consequence, taking the infimum over $g \in H^m(\mathbb{R})$, $E_{j,p}(f)$ can be estimated by values of the K functional $K(f, t) := \inf_{g\in H^m(\mathbb{R})} \|f - g\|_{L_2(\mathbb{R})} + t|g|_{H^m(\mathbb{R})}$,

$$E_{j,p}(f) \leq C_4 K\big(f, (p+1)^{-m} 2^{-jm}\big). \tag{14}$$

We will now use the fact that for $0 < s \leq m$, an equivalent norm on $H^s(\mathbb{R})$ is given by

$$\|f\|_{[L_2(\mathbb{R}), H^m(\mathbb{R})]_{s/m,2}} = \|f\|_{L_2(\mathbb{R})} + \Big(\int_0^\infty \big(t^{-s/m} K(f, t)\big)^2 \frac{\mathrm{d}t}{t}\Big)^{1/2},$$

with constants in the norm equivalence only depending on m, see [3] for details. Similar to [3, Lemma 3.1.3], we can replace the latter integral by a discrete sum, losing constants that only depend on m. In fact, for $(p+1)^{-m}2^{-jm} \leq t \leq (p+1)^{-m}2^{-(j-1)m}$, it follows from the monotonicity property $K(f,as) \leq \max\{1,a\}K(f,s)$ of the K functional that

$$2^{-s}(p+1)^s 2^{js} K\big(f,(p+1)^{-m}2^{-jm}\big) \leq t^{-s/m}K(f,t) \leq 2^m(p+1)^s 2^{js} K\big(f,(p+1)^{-m}2^{-jm}\big).$$

We can therefore estimate

$$\begin{aligned}\int_0^\infty \big(t^{-s/m}K(f,t)\big)^2\frac{\mathrm{d}t}{t} &= \sum_{j\in\mathbb{Z}}\int_{(p+1)^{-m}2^{-jm}}^{(p+1)^{-m}2^{-(j-1)m}} \big(t^{-s/m}K(f,t)\big)^2\frac{\mathrm{d}t}{t}\\ &\left\{\begin{array}{l}\leq 2^{2m}(\log 2^m)(p+1)^{2s}\sum_{j\in\mathbb{Z}}2^{2js}K\big(f,(p+1)^{-m}2^{-jm}\big)^2\\ \geq 2^{-2s}(\log 2^m)(p+1)^{2s}\sum_{j\in\mathbb{Z}}2^{2js}K\big(f,(p+1)^{-m}2^{-jm}\big)^2\end{array}\right.,\end{aligned}$$

so that the claim follows from (14) and summation over $j \geq 0$. □

3.2 Norm estimates

We will now establish sharp bounds for the L_q norms of single B-spline quarks, as $p \to \infty$. In view of $\varphi \in L_\infty(\mathbb{R})$, (4) and the identity

$$\big\|\big(\tfrac{\cdot}{\lceil m/2\rceil}\big)^p\big\|_{L_q(-\lfloor m/2\rfloor,\lceil m/2\rceil)} = \tfrac{1}{\lceil m/2\rceil^p}\left(\tfrac{\lceil m/2\rceil^{pq+1}+\lfloor m/2\rfloor^{pq+1}}{pq+1}\right)^{1/q}, \quad \text{for all } 0<q<\infty,$$

we obtain the simple estimate

$$\|\varphi_p\|_{L_q(\mathbb{R})} \leq \big(\tfrac{2\lceil m/2\rceil}{pq+1}\big)^{1/q}\|\varphi\|_{L_\infty(\mathbb{R})}, \quad \text{for all } p\geq 0, 0<q<\infty. \tag{15}$$

These asymptotics in p are already sharp, e.g., if φ is the step function $\chi_{[0,1)}$, with

$$\|\varphi_p\|_{L_q(\mathbb{R})} = (pq+1)^{-1/q}, \quad \text{for all } p\geq 0, 0<q<\infty. \tag{16}$$

In case that φ has higher regularity in L_∞, the L_q norms of φ_p decay even faster with p. As the most important example, let us establish sharp bounds for the L_q norms of monomial B-spline quarks φ_p, as $p \to \infty$. We start with an auxiliary result on the location of the extrema of φ_p.

Lemma 3.2. *Let $2 \leq m \in \mathbb{N}$, $\varphi = N_m(\cdot + \lfloor \frac{m}{2}\rfloor)$ and φ_p be given by (4). Then*

$$\|\varphi_p\|_{L_\infty(\mathbb{R})} = \big|\varphi_p(\hat{x})\big|, \quad \hat{x} := \begin{cases}\frac{pm}{2(p+m-1)}, & \text{if } m \text{ is even, for all } p \geq (\frac{m}{2}-1)(m-1),\\ \frac{p(m+1)}{2(p+m-1)}, & \text{if } m \text{ is odd, for all } p \geq \frac{(m-1)^2}{2}.\end{cases} \tag{17}$$

Proof. Let $2 \leq m \in \mathbb{N}$ be fixed. Consider first the case that m is even, and let $p \in \mathbb{N}$ with $p \geq (\frac{m}{2}-1)(m-1)$ be fixed. It is sufficient to determine the extrema of φ in $[0,\infty)$, because $\varphi_p(-x) = (-1)^p\varphi_p(x)$ for all $x \in \mathbb{R}$. We will prove that φ_p is nondecreasing on $[0, \frac{m}{2} - 1]$. For $m = 2$, there is nothing to prove. For $m \geq 4$, φ_p is continuously differentiable and for all $x \in [0, \frac{m}{2} - 1]$, we have $\min\{x + \frac{m}{2}, m - x - \frac{m}{2}\} \geq 1$ and thus

$$
\begin{aligned}
(\tfrac{m}{2})^p \varphi_p'(x) &= x^{p-1}\big(pN_m(x + \tfrac{m}{2}) + xN_m'(x + \tfrac{m}{2})\big) \\
&\geq x^{p-1}\Big(pN_m(x + \tfrac{m}{2}) - (\tfrac{m}{2} - 1)\big|N_m'(x + \tfrac{m}{2})\big|\Big) \\
&= x^{p-1}\Big(pN_m(x + \tfrac{m}{2}) - (\tfrac{m}{2} - 1)\big|N_{m-1}(x + \tfrac{m}{2}) - N_{m-1}(x + \tfrac{m}{2} - 1)\big|\Big) \\
&\geq x^{p-1}\Big(pN_m(x + \tfrac{m}{2}) - (\tfrac{m}{2} - 1)\big(N_{m-1}(x + \tfrac{m}{2}) + N_{m-1}(x + \tfrac{m}{2} - 1)\big)\Big) \\
&\geq x^{p-1}\Big(pN_m(x + \tfrac{m}{2}) - (\tfrac{m}{2} - 1)\big((x + \tfrac{m}{2})N_{m-1}(x + \tfrac{m}{2}) \\
&\qquad\qquad\qquad + (m - x - \tfrac{m}{2})N_{m-1}(x + \tfrac{m}{2} - 1)\big)\Big) \\
&= x^{p-1}\big(p - (\tfrac{m}{2} - 1)(m - 1)\big)N_m(x + \tfrac{m}{2}),
\end{aligned}
$$

which is nonnegative because $p \geq (\frac{m}{2} - 1)(m - 1)$. Therefore, all local maxima of φ_p are located in $I := [\frac{m}{2} - 1, \frac{m}{2}]$, whenever $p \geq (\frac{m}{2} - 1)(m - 1)$. On I, we have $N_m(x + \frac{m}{2}) = \frac{1}{(m-1)!}(\frac{m}{2} - x)^{m-1}$, so that from

$$
\begin{aligned}
(\tfrac{m}{2})^p (m-1)! \varphi_p'(x) &= px^{p-1}(\tfrac{m}{2} - x)^{m-1} - (m-1)x^p(\tfrac{m}{2} - x)^{m-2} \\
&= x^{p-1}(\tfrac{m}{2} - x)^{m-2}\big(\tfrac{pm}{2} - (p + m - 1)x\big),
\end{aligned}
$$

we obtain the critical points $\frac{m}{2} \in I$ and $\hat{x} := \frac{pm}{2(p+m-1)}$. Using that $p \geq (\frac{m}{2} - 1)(m - 1)$ we observe that indeed $\hat{x} \in I$, since

$$
\tfrac{m}{2} - 1 = \tfrac{m}{2} - \tfrac{m(m-1)/2}{(m/2-1)(m-1)+m-1} \leq \tfrac{m}{2} - \tfrac{m(m-1)/2}{p+m-1} = \hat{x} \leq \tfrac{m}{2}.
$$

Due to $\varphi_p(\frac{m}{2}) = 0$ and by the symmetry of φ_p, the global maximum of φ_p is attained in I, so that the unique local maximum $\hat{x}$ is also global.

In case that $m \geq 3$ is odd, let $p \in \mathbb{N}$ with $p \geq \frac{(m-1)^2}{2}$ be fixed. Using the symmetry of N_m, we derive for $x \in [-\frac{m-1}{2}, \frac{1}{2}]$ that

$$
\begin{aligned}
\big|\varphi_p(x)\big| &= |\tfrac{2x}{m+1}|^p N_m(x + \tfrac{m-1}{2}) \\
&= |\tfrac{2x}{m+1}|^p N_m\big(m - (x + \tfrac{m-1}{2})\big) \\
&= |\tfrac{2x}{m+1}|^p N_m(-x + 1 + \tfrac{m-1}{2}) \\
&\leq |\tfrac{2(-x+1)}{m+1}|^p N_m(-x + 1 + \tfrac{m-1}{2}) \\
&= \big|\varphi_p(-x + 1)\big|.
\end{aligned}
$$

By consequence, the global maximum of $|\varphi_p|$ is located in $[\frac{1}{2}, \frac{m+1}{2}]$. On this very interval, because φ_p is continuously differentiable for $m \geq 3$, and analogously to the case of even

m, we obtain for $x \in [\frac{1}{2}, \frac{m-1}{2}]$ and hence $\min\{x + \frac{m-1}{2}, m - x - \frac{m-1}{2}\} \geq 1$ that

$$\begin{aligned}
(\tfrac{m+1}{2})^p \varphi_p'(x) &= x^{p-1}\big(pN_m(x + \tfrac{m-1}{2}) + xN_m'(x + \tfrac{m-1}{2})\big) \\
&\geq x^{p-1}\Big(pN_m(x + \tfrac{m-1}{2}) - \tfrac{m-1}{2}\big|N_m'(x + \tfrac{m-1}{2})\big|\Big) \\
&= x^{p-1}\Big(pN_m(x + \tfrac{m-1}{2}) - \tfrac{m-1}{2}\big|N_{m-1}(x + \tfrac{m-1}{2}) - N_{m-1}(x + \tfrac{m-1}{2} - 1)\big|\Big) \\
&\geq x^{p-1}\Big(pN_m(x + \tfrac{m-1}{2}) - \tfrac{m-1}{2}\big(N_{m-1}(x + \tfrac{m-1}{2}) + N_{m-1}(x + \tfrac{m-1}{2} - 1)\big)\Big) \\
&\geq x^{p-1}\Big(pN_m(x + \tfrac{m-1}{2}) - \tfrac{m-1}{2}\big((x + \tfrac{m-1}{2})N_{m-1}(x + \tfrac{m-1}{2}) \\
&\qquad\qquad + (m - x - \tfrac{m-1}{2})N_{m-1}(x + \tfrac{m-1}{2} - 1)\big)\Big) \\
&= x^{p-1}\big(p - \tfrac{(m-1)^2}{2}\big)N_m(x + \tfrac{m-1}{2}),
\end{aligned}$$

which is nonnegative because $p \geq \frac{(m-1)^2}{2}$. Therefore, all local maxima of φ_p are located in $J := [\frac{m-1}{2}, \frac{m+1}{2}]$. On J, we have $N_m(x + \frac{m-1}{2}) = \frac{1}{(m-1)!}(\frac{m+1}{2} - x)^{m-1}$, so that from

$$\begin{aligned}
(\tfrac{m+1}{2})^p(m-1)!\varphi_p'(x) &= px^{p-1}(\tfrac{m+1}{2} - x)^{m-1} - (m-1)x^p(\tfrac{m+1}{2} - x)^{m-2} \\
&= x^{p-1}(\tfrac{m+1}{2} - x)^{m-2}\big(\tfrac{p(m+1)}{2} - (p+m-1)x\big)
\end{aligned}$$

we obtain the critical points $\frac{m+1}{2} \in J$ and $\hat{x} := \frac{p(m+1)}{2(p+m-1)}$. Since

$$\tfrac{m-1}{2} = \tfrac{m+1}{2} - \tfrac{(m^2-1)/2}{(m-1)^2/2+m-1} \leq \tfrac{m+1}{2} - \tfrac{(m^2-1)/2}{p+m-1} = \hat{x} \leq \tfrac{m+1}{2},$$

indeed $\hat{x} \in J$. Due to $\varphi(\frac{m+1}{2}) = 0$, the local maximum $\hat{x} \in J$ is also global, and the claim is proved. $\square$

Proposition 3.3. *Let $m \in \mathbb{N}$, $\varphi = N_m(\cdot + \lfloor \frac{m}{2} \rfloor)$ and φ_p be given by (4). For each $1 \leq q \leq \infty$, there exist $c = c(m,q), C = C(m,q) > 0$ such that*

$$c(p+1)^{-(m-1+1/q)} \leq \|\varphi_p\|_{L_q(\mathbb{R})} \leq C(p+1)^{-(m-1+1/q)}, \quad \text{for all } p \geq (m-1)^2. \tag{18}$$

Proof. The special case $m = 1$ is already covered by (16), so we can assume that $m \geq 2$, without loss of generality, and hence $p \geq (m-1)^2 \geq 1$.

In order to show the upper bound in (18), we study the extremal values $q \in \{1, \infty\}$ and conclude by real interpolation. For $q = 1$, we exploit that for any $g \in C^m[0,m]$,

$$\int_0^m g^{(m)}(x)N_m(x)\,\mathrm{d}x = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}g(k).$$

In the case that m and p are even, we can use $g(x) := \frac{1}{(p+m)\cdots(p+1)}(x - \frac{m}{2})^{p+m}$ and the nonnegativity of φ_p to infer that with $C_1(m) := \sum_{k=0}^m \binom{m}{k}|k-\frac{m}{2}|^m$,

$$
\begin{aligned}
\|\varphi_p\|_{L_1(\mathbb{R})} &= \left(\frac{2}{m}\right)^p \int_{-m/2}^{m/2} x^p N_m(x+\tfrac{m}{2})\,\mathrm{d}x \\
&= \left(\frac{2}{m}\right)^p \int_0^m (y-\tfrac{m}{2})^p N_m(y)\,\mathrm{d}y \\
&= \left(\frac{2}{m}\right)^p \sum_{k=0}^m \frac{(-1)^{m-k}\binom{m}{k}(k-\frac{m}{2})^{p+m}}{(p+m)\cdots(p+1)} \\
&\le C_1(p+1)^{-m}. \qquad (19)
\end{aligned}
$$

If m is odd and p is even, we obtain by analogous arguments that

$$
\begin{aligned}
\|\varphi_p\|_{L_1(\mathbb{R})} &= \left(\frac{2}{m+1}\right)^p \int_{-(m-1)/2}^{(m+1)/2} x^p N_m(x+\tfrac{m-1}{2})\,\mathrm{d}x \\
&= \left(\frac{2}{m+1}\right)^p \int_0^m (y-\tfrac{m-1}{2})^p N_m(y)\,\mathrm{d}y \\
&= \left(\frac{2}{m+1}\right)^p \sum_{k=0}^m \frac{(-1)^{m-k}\binom{m}{k}(k-\frac{m-1}{2})^{p+m}}{(p+m)\cdots(p+1)} \\
&\le C_1^*(p+1)^{-m}, \qquad (20)
\end{aligned}
$$

where $C_1^*(m) := \sum_{k=0}^m \binom{m}{k}|k-\frac{m-1}{2}|^m$. Finally, if $p \ge 1$ is odd and m is arbitrary, the estimate $|x| \le \lceil \frac{m}{2} \rceil$ for all $x \in \operatorname{supp}\varphi_p$ yields

$$
\|\varphi_p\|_{L_1(\mathbb{R})} \le \|\varphi_{p-1}\|_{L_1(\mathbb{R})} \le C_1^* p^{-m} \le C_1^* 2^m (p+1)^{-m}. \qquad (21)
$$

For $q=\infty$ and m even, Lemma 3.2 tells us that for all $p \ge (\frac{m}{2}-1)(m-1)$ and $\hat{x} := \frac{pm}{2(p+m-1)} \in [\frac{m}{2}-1, \frac{m}{2}]$,

$$
\begin{aligned}
\|\varphi_p\|_{L_\infty(\mathbb{R})} &= (\tfrac{2\hat{x}}{m})^p N_m(\hat{x}+\tfrac{m}{2}) \\
&= \tfrac{1}{(m-1)!}(\tfrac{2\hat{x}}{m})^p(\tfrac{m}{2}-\hat{x})^{m-1} \\
&= \tfrac{1}{(m-1)!}(\tfrac{p}{p+m-1})^p(\tfrac{m(m-1)}{2(p+m-1)})^{m-1}.
\end{aligned}
$$

Analogously, if m is odd, Lemma 3.2 tells us that for all $p \ge (m-1)^2$ and $\hat{x} := \frac{p(m+1)}{2(p+m-1)} \in [\frac{m+1}{2}-1, \frac{m+1}{2}]$,

$$
\begin{aligned}
\|\varphi_p\|_{L_\infty(\mathbb{R})} &= (\tfrac{2\hat{x}}{m+1})^p N_m(\hat{x}+\tfrac{m-1}{2}) \\
&= \tfrac{1}{(m-1)!}(\tfrac{2\hat{x}}{m+1})^p(\tfrac{m+1}{2}-\hat{x})^{m-1} \\
&= \tfrac{1}{(m-1)!}(\tfrac{p}{p+m-1})^p(\tfrac{(m+1)(m-1)}{2(p+m-1)})^{m-1}.
\end{aligned}
$$

Combining both cases, we obtain that for each m and $p \ge (m-1)^2$ that

$$
c_2(p+1)^{-(m-1)} \le \|\varphi_p\|_{L_\infty(\mathbb{R})} \le C_2(p+1)^{-(m-1)}, \quad \text{for all } p \ge (m-1)^2, \qquad (22)
$$

with $c_2 = c_2(m), C_2 = C_2(m) > 0$ independent of p, thereby already showing the lower estimate in (18) for $q = \infty$.

Finally, let $1 < q < \infty$ and $p \geq (m-1)^2$. By real interpolation between $L_1(\mathbb{R})$ and $L_\infty(\mathbb{R})$ and due to $\varphi_p \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$, we obtain from (19), (20), (21) and (22) the upper estimate in (18)

$$\|\varphi_p\|_{L_q(\mathbb{R})} \leq C_3(q) \|\varphi_p\|_{L_1(\mathbb{R})}^{1/q} \|\varphi_p\|_{L_\infty(\mathbb{R})}^{1-1/q} \leq C_4(m,q)(p+1)^{-(m-1+1/q)}.$$

It remains to show the lower estimate in (18) for $1 \leq q < \infty$. Let us consider first the case $q \in \mathbb{N}$. For $m \geq 2$ even, we can estimate

$$\begin{aligned}
\|\varphi_p\|_{L_q(\mathbb{R})}^q \geq \int_{\frac{m}{2}-1}^{\frac{m}{2}} \varphi_p(x)^q \,\mathrm{d}x &= \frac{1}{\big((m-1)!\big)^q} \int_{\frac{m}{2}-1}^{\frac{m}{2}} (\tfrac{2x}{m})^{pq} (\tfrac{m}{2} - x)^{(m-1)q} \,\mathrm{d}x \\
&= \frac{1}{\big((m-1)!\big)^q} \int_0^1 (1 - \tfrac{2y}{m})^{pq} y^{(m-1)q} \,\mathrm{d}y \\
&\geq \frac{1}{\big((m-1)!\big)^q} \int_0^1 (1-y)^{pq} y^{(m-1)q} \,\mathrm{d}y.
\end{aligned}$$

For $m \geq 1$ odd, analogous steps lead to the same estimate. Due to $q \in \mathbb{N}$, the latter integral can be computed explicitly, by means of $(m-1)q$ times partial integration,

$$\begin{aligned}
\int_0^1 (1-y)^{pq} y^{(m-1)q} \,\mathrm{d}y &= \frac{\big((m-1)q\big)!}{(pq+1)\cdots\big(pq+(m-1)q\big)} \int_0^1 (1-y)^{(p+m-1)q} \,\mathrm{d}y \\
&= \frac{\big((m-1)q\big)!}{(pq+1)\cdots\big(pq+(m-1)q+1\big)} \\
&\geq C_5(m)(p+1)^{-(m-1)q-1},
\end{aligned}$$

from which the lower estimate in (18) immediately follows. Finally, let $1 \leq q < \infty$ be arbitrary. If $1 \leq q \leq 2$, real interpolation between $L_q(\mathbb{R})$ and $L_\infty(\mathbb{R})$ yields that

$$\|\varphi_p\|_{L_2(\mathbb{R})} \leq C_6 \|\varphi_p\|_{L_q(\mathbb{R})}^{q/2} \|\varphi_p\|_{L_\infty(\mathbb{R})}^{1-q/2},$$

where $C_6 > 0$ does not depend on q. Isolating $\|\varphi_p\|_{L_q(\mathbb{R})}$ and an application of (18) for the L_2 and L_∞ case yields

$$\begin{aligned}
\|\varphi_p\|_{L_q(\mathbb{R})} &\geq C_6^{2/q} \|\varphi_p\|_{L_2(\mathbb{R})}^{2/q} \|\varphi_p\|_{L_\infty(\mathbb{R})}^{1-2/q} \\
&\geq C_7(m,q)(p+1)^{-2(m-1/2)/q}(p+1)^{-(m-1)(1-2/q)} \\
&= C_7(m,q)(p+1)^{-(m-1+1/q)}.
\end{aligned}$$

Analogously, if $2 \leq q < \infty$, real interpolation between $L_1(\mathbb{R})$ and $L_q(\mathbb{R})$ yields

$$\|\varphi_p\|_{L_2(\mathbb{R})} \leq C_8 \|\varphi_p\|_{L_1(\mathbb{R})}^{1-1/(2(1-1/q))} \|\varphi_p\|_{L_q(\mathbb{R})}^{1/(2(1-1/q))},$$

so that isolation of $\|\varphi_p\|_{L_q(\mathbb{R})}$ and an application of (18) for L_1 and L_2 prove the claim,

$$\begin{aligned}\|\varphi_p\|_{L_q(\mathbb{R})} &\geq C_8^{2(1-1/q)}\|\varphi_p\|_{L_2(\mathbb{R})}^{2-2/q}\|\varphi_p\|_{L_1(\mathbb{R})}^{2/q-1}\\ &\geq C_9(m,q)(p+1)^{-(m-1/2)(2-2/q)}(p+1)^{-m(2/q-1)}\\ &= C_9(m,q)(p+1)^{-(m-1+1/q)}.\end{aligned}$$

□

3.3 Inverse Estimates

Theorem 3.4. *Let $m \in \mathbb{N}$, $\varphi = N_m$ and let $V_{j,p}$ be given by* (6). *Then for $1 \leq q \leq \infty$, there exists $C = C(m) > 0$, such that for all $f \in V_{j,p}$,*

$$\omega_m(f,t)_{L_q(\mathbb{R})} \leq C \min\left\{1,(p+1)^2 2^j t\right\}^{m-1+1/q}\|f\|_{L_q(\mathbb{R})}. \tag{23}$$

Proof. Let $f = \sum_{0\leq i\leq p}\sum_{k\in\mathbb{Z}} c_{i,k}\varphi_{i,j,k} \in V_{j,p}$. If $t \geq (p+1)^{-2}2^{-j}$, we simply use

$$\omega_m(f,t)_{L_q(\mathbb{R})} \leq 2^m\|f\|_{L_q(\mathbb{R})} = 2^m\min\{1,(p+1)^2 2^j t\}^{m-1+1/q}\|f\|_{L_q(\mathbb{R})}.$$

Now let $t < (p+1)^{-2}2^{-j}$. By using $V_{j,p} \subset W^{m-1}(L_q(\mathbb{R}))$ and standard arithmetics for the moduli of smoothness, see [14, Ch. 2 §7], we see that $\omega_m(f,t)_{L_q(\mathbb{R})} \leq t^{m-1}\omega_1(f^{(m-1)},t)_{L_q(\mathbb{R})}$. But $f^{(m-1)}$ is piecewise polynomial of degree p without continuity assumptions at the nodes $x_l := 2^{-j}l$, $l \in \mathbb{Z}$. We compute for $0 < h \leq t \leq 2^{-j}$ and $q < \infty$ that

$$\begin{aligned}&\left\|f^{(m-1)}(\cdot+h) - f^{(m-1)}\right\|_{L_q(\mathbb{R})}^q = \sum_{l\in\mathbb{Z}}\left\|f^{(m-1)}(\cdot+h) - f^{(m-1)}\right\|_{L_q(x_l,x_{l+1})}^q\\ &= \sum_{l\in\mathbb{Z}}\left(\left\|f^{(m-1)}(\cdot+h) - f^{(m-1)}\right\|_{L_q(x_l,x_{l+1}-h)}^q + \left\|f^{(m-1)}(\cdot+h) - f^{(m-1)}\right\|_{L_q(x_{l+1}-h,x_{l+1})}^q\right)\\ &\leq 2^{1-1/q}\sum_{l\in\mathbb{Z}}\left(h^q\|f^{(m)}\|_{L_q(x_l,x_{l+1})}^q + \|f^{(m-1)}\|_{L_q(x_{l+1}-h,x_{l+1})}^q + \|f^{(m-1)}\|_{L_q(x_l,x_l+h)}^q\right).\end{aligned}$$

An application of standard estimates for polynomials yields

$$\|f^{(m-1)}\|_{L_q(x_{l+1}-h,x_{l+1})}^q + \|f^{(m-1)}\|_{L_q(x_l,x_l+h)}^q \leq C_1 h 2^j p^2\|f^{(m-1)}\|_{L_q(x_l,x_{l+1})}^q$$

with $C_1 > 0$ independent of m, p and q. Using the L_q Markov inequality for algebraic polynomials P of degree i on an interval I,

$$\|P'\|_{L_q(I)} \leq C_2\frac{i^2}{|I|}\|P\|_{L_q(I)},$$

with $C_2 = C_2(q)$ independent of i, we end up with

$$\begin{aligned}&\omega_m(f,t)_{L_q(\mathbb{R})}^q \leq t^{(m-1)q}\sup_{|h|\leq t}\left\|f^{(m-1)}(\cdot+h) - f^{(m-1)}\right\|_{L_q(\mathbb{R})}^q\\ &\leq C_3(m,q)t^{(m-1)q}\big(h^q(p+1)^{2mq}2^{jmq} + h2^{j(1+(m-1)q)}p^{2+2(m-1)q}\big)\sum_{l\in\mathbb{Z}}\|f\|_{L_q(x_l,x_{l+1})}^q\\ &\leq C_4(m,q)t^{(m-1)q+1}(p+1)^{2(m-1)q+2}2^{j(1+(m-1)q)}\|f\|_{L_q(\mathbb{R})}^q.\end{aligned}$$

The case $q = \infty$ is completely analogous. □

Corollary 3.5. *Let $m \in \mathbb{N}$, $\varphi = N_m$ and let $V_{j,p}$ be given by* (6)*. Then for $1 \le q < \infty$, there exists $C = C(m,q) > 0$, such that for all $f \in V_{j,p}$*

$$\|f^{(k)}\|_{L_q(\mathbb{R})} \le C(p+1)^{2k} 2^{jk} \|f\|_{L_q(\mathbb{R})}, \quad \textit{for all } 0 \le k \le m-1. \tag{24}$$

Proof. Without loss of generality, let $m \ge 2$ and $1 \le k \le m-1$. Note that $V_{j,p} \subset W^{m-1}(L_q(\mathbb{R}))$, so that $f^{(k)}$ is well-defined for each $f \in V_{j,p}$. Let us first consider the case $k = 1$. We can use that for all $f \in W^1(L_q(\mathbb{R}))$, $1 \le q < \infty$,

$$\|f'\|_{L_q(\mathbb{R})} = \lim_{t \to 0} \frac{\omega_1(f,t)_{L_q(\mathbb{R})}}{t},$$

see [17, Prop. 2.4]. Using a Marchaud-type inequality

$$\omega_1(f,t)_{L_q(\mathbb{R})} \le C_1 t \int_t^\infty \frac{\omega_m(f,s)_{L_q(\mathbb{R})}}{s^2} \, \mathrm{d}s,$$

with $C_1 = C_1(m)$, confer [14, Ch. 2 §8] for details, we derive from (23) that

$$\begin{aligned}
\|f'\|_{L_q(\mathbb{R})} &\le C_1 \limsup_{t \to 0} \int_t^\infty \frac{\omega_m(f,s)_{L_q(\mathbb{R})}}{s^2} \, \mathrm{d}s \\
&= C_1 \Big(\limsup_{t \to 0} \int_t^{(p+1)^{-2}2^{-j}} \frac{\omega_m(f,s)_{L_q(\mathbb{R})}}{s^2} \, \mathrm{d}s + \int_{(p+1)^{-2}2^{-j}}^\infty \frac{\omega_m(f,s)_{L_q(\mathbb{R})}}{s^2} \, \mathrm{d}s \Big) \\
&\le C_2 \Big(\big((p+1)^2 2^j\big)^{m-1+1/q} \int_0^{(p+1)^{-2}2^{-j}} s^{m-3+1/q} \, \mathrm{d}s + \int_{(p+1)^{-2}2^{-j}}^\infty s^{-2} \, \mathrm{d}s \Big) \|f\|_{L_q(\mathbb{R})} \\
&= C_2 \Big(\frac{1}{m-2+\frac{1}{q}} + 1 \Big) (p+1)^2 2^j \|f\|_{L_q(\mathbb{R})},
\end{aligned}$$

with $C_2 = C_2(m,q)$. The case of general $2 \le k \le m-1$ can be treated by induction over k, repeating the previous Marchaud-type estimate k times. $\square$

Corollary 3.6. *Let $m \in \mathbb{N}$, $\varphi = N_m$ and let $V_{j,p}$ be given by* (6)*. For each $0 \le s < m - \frac{1}{2}$, there exists $C = C(m,s) > 0$, such that*

$$|f|_{H^s(\mathbb{R})} \le C(p+1)^{2s} 2^{js} \|f\|_{L_2(\mathbb{R})}, \quad \textit{for all } p, j \in \mathbb{N}_0, f \in V_{j,p}. \tag{25}$$

Proof. Let $s > 0$, without loss of generality. In view of the norm estimate

$$|f|_{H^s(\mathbb{R})} \le C_1 \Big(\int_0^\infty \big(t^{-s} \omega_m(f,t)_{L_2(\mathbb{R})}\big)^2 \frac{\mathrm{d}t}{t} \Big)^{1/2}, \quad \text{for all } 0 < s < m,$$

where $C_1 = C_1(s) > 0$, we can compute that by (23),

$$\begin{aligned}
|f|^2_{H^s(\mathbb{R})} &\le \Big(C_2 (p+1)^{2(2m-1)} 2^{j(2m-1)} \int_0^{(p+1)^{-2}2^{-j}} t^{-2s+2m-2} \, \mathrm{d}t \\
&\qquad + C_2 \int_{(p+1)^{-2}2^{-j}}^\infty t^{-(2s+1)} \, \mathrm{d}t \Big) \|f\|^2_{L_2(\mathbb{R})} \\
&\le \Big(\frac{C_2}{2m-1-2s} + \frac{C_2}{2s} \Big) (p+1)^{4s} 2^{2js} \|f\|^2_{L_2(\mathbb{R})}, \quad \text{for all } 0 < s < m - \frac{1}{2},
\end{aligned}$$

with $C_2 = C_2(m, s)$. Due to the fact that (25) trivially holds for $s = 0$ with $C = 1$, an interpolation argument shows that C does in fact only depend on s as $s \to m - \frac{1}{2}$. $\square$

We note, however, that (25) is not sharp for single quarks. If $\varphi = N_2(\cdot + 1)$ is the symmetrized hat function and $s = 1$, one can explicitly compute that

$$\begin{aligned}
\|\varphi'\|_{L_2(\mathbb{R})}^2 &= 2\int_0^1 x^{2p-2}\big(p - (p+1)x\big)^2 \,\mathrm{d}x \\
&= 2\Big(p^2\int_0^1 x^{2p-2}\,\mathrm{d}x - 2p(p+1)\int_0^1 x^{2p-1}\,\mathrm{d}x + (p+1)^2\int_0^1 x^{2p}\,\mathrm{d}x\Big) \\
&= 2\frac{p^2(2p+1) - (p+1)(2p-1)(2p+1) + (2p-1)(p+1)^2}{(2p-1)(2p+1)} \\
&= \frac{2p}{4p^2-1},
\end{aligned}$$

i.e., $\|\varphi_p'\|_{L_2(\mathbb{R})} \eqsim (p+1)^{-1/2} \eqsim (p+1)\|\varphi\|_{L_2(\mathbb{R})}$, as $p \to \infty$, while (25) only yields $|\varphi_p|_{H^1(\mathbb{R})} \leq C(p+1)^2\|\varphi_p\|_{L_2(\mathbb{R})}$, as $p \to \infty$.

4 Quarklet frames for $L_2(\mathbb{R})$

We have seen that systems of dilated and translated quarks $\varphi_{p,j,k}$ alone can only be stable in $H^s(\mathbb{R})$ for $s > 0$ when being properly rescaled. Stability of quarkonial systems in $L_2(\mathbb{R})$ and Sobolev spaces of negative order requires further conditions on the frame elements. We will show now that certain moment conditions by means of a wavelet-type modification of the quark system are sufficient to ensure stability in $L_2(\mathbb{R})$.

In the sequel, we restrict the discussion to the case of symmetrized cardinal B-splines $\varphi = N_m(\cdot + \lfloor \frac{m}{2} \rfloor)$ of order $m \in \mathbb{N}$. As shown in [8], for a given $\tilde{m} \in \mathbb{N}$ such that $\tilde{m} \geq m$ and $m + \tilde{m}$ is even, there exists a compactly supported wavelet ψ with

$$\psi(x) = \sum_{k\in\mathbb{Z}} b_k \varphi(2x - k), \quad \text{for all } x \in \mathbb{R}, \tag{26}$$

and $\tilde{m}$ vanishing moments, $\langle \psi, P \rangle = 0$ for $\deg P < \tilde{m}$. Moreover, the collection

$$\Psi_R := \big\{\varphi(\cdot - k), 2^{j/2}\psi(2^j \cdot - k) : j \in \mathbb{N}_0, k \in \mathbb{Z}\big\} \tag{27}$$

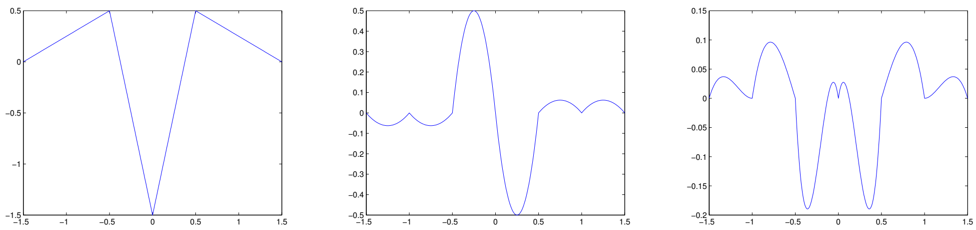
is a Riesz basis for $L_2(\mathbb{R})$.

In complete analogy to the wavelet ψ, let us consider the following *quarklets* ψ_p,

$$\psi_p(x) := \sum_{k\in\mathbb{Z}} b_k \varphi_p(2x - k), \quad \text{for all } p \in \mathbb{N}_0, x \in \mathbb{R}. \tag{28}$$

We refer to Figure 2 for an illustrative example. By assumption, $\psi_0 = \psi$ has $\tilde{m}$ vanishing moments. The following lemma shows that the other ψ_p have the same property.

Lemma 4.1. *For each $p \geq 0$, the quarklet ψ_p has $\tilde{m}$ vanishing moments.*



(a) $p = 0$ (b) $p = 1$ (c) $p = 2$

Figure 2: Some B-spline quarklets ψ_p of order $m = 2$ with $\tilde{m} = 2$ vanishing moments, where $\{b_k\}$ are the Cohen/Daubechies/Feauveau wavelet coefficients $b_{-2} = b_2 = \frac{1}{4}$, $b_{-1} = b_1 = \frac{1}{2}$, $b_0 = -\frac{3}{2}$, $b_k = 0$ otherwise.

Proof. Let us first prove the auxiliary result that the coefficient sequence $\{b_k\}_{k\in\mathbb{Z}}$ has $\tilde{m}$ discrete moments,

$$\sum_{k\in\mathbb{Z}} k^q b_k = 0, \quad \text{for all } 0 \le q < \tilde{m}. \tag{29}$$

We proceed by induction over q. For $q = 0$, $\mu := \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x \neq 0$, the compact support of φ and (26) imply that

$$\sum_{k\in\mathbb{Z}} b_k = \frac{1}{\mu}\sum_{k\in\mathbb{Z}} b_k \int_{\mathbb{R}} \varphi(x-k)\,\mathrm{d}x = \frac{2}{\mu}\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0.$$

Now assume that (29) holds for all $0 \le r \le q-1$, where $0 \le q < \tilde{m}$. By the vanishing moment property of ψ, we compute that

$$0 = \int_{\mathbb{R}} x^q \psi(x)\,\mathrm{d}x = \sum_{k\in\mathbb{Z}} b_k \int_{\mathbb{R}} x^q \varphi(2x-k)\,\mathrm{d}x = \frac{1}{2^{q+1}}\sum_{k\in\mathbb{Z}} b_k \int_{\mathbb{R}} (y+k)^q \varphi(y)\,\mathrm{d}y,$$

so that the induction hypothesis yields (29),

$$0 = \sum_{k\in\mathbb{Z}} b_k \int_{\mathbb{R}} \sum_{r=0}^{q} \binom{q}{r} k^r y^{q-r} \varphi(y)\,\mathrm{d}y = \sum_{r=0}^{q} \binom{q}{r} \int_{\mathbb{R}} y^{q-r}\varphi(y)\,\mathrm{d}y \sum_{k\in\mathbb{Z}} k^r b_k = \mu \sum_{k\in\mathbb{Z}} k^q b_k.$$

In view of (29), the vanishing moment property of ψ_p easily follows from

$$\begin{aligned}
\int_{\mathbb{R}} x^q \psi_p(x)\,\mathrm{d}x &= \sum_{k\in\mathbb{Z}} b_k \int_{\mathbb{R}} x^q \varphi_p(2x-k)\,\mathrm{d}x \\
&= \frac{1}{2^{q+1}} \sum_{k\in\mathbb{Z}} b_k \int_{\mathbb{R}} (y+k)^q \varphi_p(y)\,\mathrm{d}y \\
&= \frac{1}{2^{q+1}} \sum_{l=0}^{q} \binom{q}{l} \int_{\mathbb{R}} y^{q-l}\varphi_p(y)\,\mathrm{d}y \sum_{k\in\mathbb{Z}} k^l b_k = 0, \quad \text{for all } 0 \le q < \tilde{m}.
\end{aligned}$$

□

In the sequel, we consider the usual dyadic dilates and translates of the quarklets,

$$\psi_{p,j,k}(x) := 2^{j/2}\psi_p(2^j x - k), \quad \text{for all } x \in \mathbb{R}, p \geq 0, j \geq 0, k \in \mathbb{Z}. \tag{30}$$

Based on the vanishing moment properties of the quarklets ψ_p, we immediately get the following cancellation estimates for inner products of the $\psi_{p,j,k}$ with smooth functions, using standard techniques from wavelet analysis.

Lemma 4.2. *There exists $C = C(m, \psi)$, such that for all $f \in W^r(L_\infty(\mathbb{R}))$, $r \leq \tilde{m} - 1$,*

$$\left|\langle f, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\right| \leq C(p+1)^{-m} 2^{-j(r+1/2)} |f|_{W^r(L_\infty(\operatorname{supp}\psi_{p,j,k}))}, \quad \textit{for all } p \geq 0, j \geq 0, k \in \mathbb{Z}. \tag{31}$$

Proof. By Lemma 4.1, each quarklet ψ_p and hence $\psi_{p,j,k}$ has $\tilde{m}$ vanishing moments. Therefore, given some $f \in L_2(\mathbb{R})$, an application of the Hölder inequality implies that

$$\left|\langle f, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\right| = \inf_{P \in \mathbb{P}_r} \left|\langle f - P, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\right| \leq \inf_{P \in \mathbb{P}_r} \|f - P\|_{L_\infty(\operatorname{supp}\psi_{p,j,k})} \|\psi_{p,j,k}\|_{L_1(\mathbb{R})}.$$

A Whitney-type estimate on $\operatorname{supp}\psi_{p,j,k}$, (28) and (18) immediately yield (31),

$$\begin{aligned}\left|\langle f, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\right| &\leq C_1(\psi) 2^{-j(r+1/2)} |f|_{W^r(L_\infty(\operatorname{supp}\psi_{p,j,k}))} \|\varphi_p\|_{L_1(\mathbb{R})} \\ &\leq C_2(m, \psi)(p+1)^{-m} 2^{-j(r+1/2)} |f|_{W^r(L_\infty(\operatorname{supp}\psi_{p,j,k}))}.\end{aligned}$$

□

We shall now study the stability properties of the full quarklet system. In particular, we will investigate under which conditions on the weights $w_p \geq 0$, the weighted system

$$\Psi_{Q,w} := \left\{w_p \varphi_p(\cdot - k), w_p 2^{j/2} \psi_p(2^j \cdot - k) : p, j \in \mathbb{N}_0, k \in \mathbb{Z}\right\}. \tag{32}$$

is a frame for $L_2(\mathbb{R})$. Setting $w_0 := 1$, $\Psi_{Q,w}$ contains the L_2 Riesz basis Ψ_R, so that we are left with proving the Bessel property of $\Psi_{Q,w}$. We have to show that the synthesis operator $T : \ell_2(\mathbb{N}_0 \times \mathbb{Z}) \oplus \ell_2(\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{Z}) \to L_2(\mathbb{R})$,

$$T(\mathbf{c}, \mathbf{d}) := \sum_{p=0}^{\infty} \sum_{k \in \mathbb{Z}} c_{p,k} w_p \varphi_p(\cdot - k) + \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} d_{p,j,k} w_p \psi_{p,j,k} \tag{33}$$

is bounded. We will exploit the following proposition.

Proposition 4.3. *Let $m \geq 2$. There exists $C = C(m, \psi) > 0$, such that the Gramian matrices*

$$\mathbf{G}_p := \left(\langle \varphi_p(\cdot - k), \varphi_p(\cdot - k')\rangle_{L_2(\mathbb{R})}\right)_{k,k' \in \mathbb{Z}}, \quad \mathbf{H}_p := \left(\langle \psi_{p,j,k}, \psi_{p,j',k'}\rangle_{L_2(\mathbb{R})}\right)_{(j,k),(j',k') \in \mathbb{N}_0 \times \mathbb{Z}} \tag{34}$$

are bounded operators on $\ell_2(\mathbb{Z})$ and $\ell_2(\mathbb{N}_0 \times \mathbb{Z})$, respectively, with

$$\|\mathbf{G}_p\|_{L(\ell_2(\mathbb{Z}))} \leq C(p+1)^{-(2m-1)}, \quad \|\mathbf{H}_p\|_{L(\ell_2(\mathbb{N}_0 \times \mathbb{Z}))} \leq C(p+1)^{-1}, \quad \textit{for all } p \in \mathbb{N}_0. \tag{35}$$

Proof. Let us discuss $\mathbf{G}_p$ first. By the Schur lemma, it is sufficient to prove that $\mathbf{G}_p$ is bounded on $\ell_1(\mathbb{Z})$ and $\ell_\infty(\mathbb{Z})$, where the operator norms are independent of p. Due to the symmetry of $\mathbf{G}_p$, a norm bound in $\ell_\infty(\mathbb{Z})$ is sufficient. Let $k' \in \mathbb{Z}$ and $\mathbf{c} = (c_k)_{k\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z})$. Then

$$\big|(\mathbf{G}_p\mathbf{c})_{k'}\big| = \Big|\sum_{k\in\mathbb{Z}} c_k \big\langle \varphi_p(\cdot - k'), \varphi_p(\cdot - k)\big\rangle_{L_2(\mathbb{R})}\Big| \le \|\mathbf{c}\|_{\ell_\infty(\mathbb{Z})} \sum_{k\in\mathbb{Z}} \Big|\big\langle \varphi_p(\cdot - k'), \varphi_p(\cdot - k)\big\rangle_{L_2(\mathbb{R})}\Big|.$$

In view of the compact support of φ, the latter sum is finite. Therefore, using the Cauchy-Schwarz inequality, it can be bounded independently of k' by a constant multiple of $\|\varphi_p\|^2_{L_2(\mathbb{R})}$, where the constant only depends on m. An application of (18) yields

$$\|\mathbf{G}_p\|_{L(\ell_2(\mathbb{Z}))} \le \|\mathbf{G}_p\|_{L(\ell_\infty(\mathbb{Z}))} \le C_1(m)(p+1)^{-(2m-1)},$$

showing the estimate (35) for $\mathbf{G}_p$.

Concerning the boundedness of $\mathbf{H}_p$, we shall exploit the compression property (31). Let $j' \in \mathbb{N}_0$, $k' \in \mathbb{Z}$, and $\mathbf{d} = (d_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{Z}} \in \ell_\infty(\mathbb{N}_0 \times \mathbb{Z})$. We start estimating with

$$\begin{aligned}
\big|(\mathbf{H}_p\mathbf{d})_{(j',k')}\big| &= \Big|\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} d_{j,k}\langle \psi_{p,j',k'}, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\Big| \\
&\le \|\mathbf{d}\|_{\ell_\infty(\mathbb{N}_0\times\mathbb{Z})}\Big(\sum_{j=0}^{j'-1}\sum_{k\in\mathbb{Z}} \big|\langle \psi_{p,j',k'}, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\big| + \sum_{j=j'}^{\infty}\sum_{k\in\mathbb{Z}} \big|\langle \psi_{p,j',k'}, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\big|\Big).
\end{aligned}$$

In the first sum over k, where $j < j'$, we can estimate the nonzero inner products between quarklets by an application of (31), the Markov inequality for the piecewise polynomial function $\psi_p^{(m-1)}$, and (22)

$$\begin{aligned}
\big|\langle \psi_{p,j',k'}, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\big| &\le C_2(m,\psi)(p+1)^{-m}2^{-j'(m-1/2)}|\psi_{p,j,k}|_{W^{m-1}(L_\infty(\mathbb{R}))} \\
&\le C_3(m,\psi)(p+1)^{-m}2^{-(j'-j)(m-1/2)}\|\psi_p^{(m-1)}\|_{L_\infty(\mathbb{R})} \\
&\le C_4(m,\psi)(p+1)^{m-2}2^{-(j'-j)(m-1/2)}\|\psi_p\|_{L_\infty(\mathbb{R})} \\
&\le C_5(m,\psi)(p+1)^{-1}2^{-(j'-j)(m-1/2)}.
\end{aligned}$$

The number of nonzero inner products per j in the first sum is bounded by a constant independent of j and j',

$$\sum_{j=0}^{j'-1}\sum_{k\in\mathbb{Z}} \big|\langle \psi_{p,j',k'}, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\big| \le C_6(m,\psi)(p+1)^{-1}\sum_{j=0}^{j'-1} 2^{-(j'-j)(m-1/2)}.$$

In a completely analogous way, using that the number of nonzero inner products per j in the second sum is bounded by a constant multiple of $2^{j-j'}$, the second sum can be estimated by

$$\sum_{j=j'}^{\infty}\sum_{k\in\mathbb{Z}} \big|\langle \psi_{p,j',k'}, \psi_{p,j,k}\rangle_{L_2(\mathbb{R})}\big| \le C_7(m,\psi)(p+1)^{-1}\sum_{j=j'}^{\infty} 2^{-(j-j')(m-3/2)}.$$

Therefore, due to $m \geq 2$, we obtain (35),

$$\big|(\mathbf{H}_p\mathbf{d})_{(j',k')}\big| \leq C_8(m,\psi)\|\mathbf{d}\|_{\ell_\infty(\mathbb{N}_0\times\mathbb{Z})}(p+1)^{-1}.$$

□

In case that the weights w_p decay sufficiently fast, we finally obtain the boundedness of T and hence the L_2 frame property.

Theorem 4.4. *Let $w_p \geq 0$ be chosen such that $w_0 = 1$ and $w_p(p+1)^{-1/2}$ is summable. Then $\Psi_{Q,w}$ is a frame for $L_2(\mathbb{R})$.*

Proof. For $(\mathbf{c},\mathbf{d}) \in \ell_2(\mathbb{N}_0\times\mathbb{Z}) \oplus \ell_2(\mathbb{N}_0\times\mathbb{N}_0\times\mathbb{Z})$, we compute by using Proposition 4.3 and the Cauchy-Schwarz inequality that

$$\begin{aligned}\big\|T(\mathbf{c},\mathbf{d})\big\|_{L_2(\mathbb{R})} &\leq \sum_{p=0}^{\infty} w_p\Big\|\sum_{k\in\mathbb{Z}} c_{p,k}\varphi_p(\cdot-k)\Big\|_{L_2(\mathbb{R})} + \sum_{p=0}^{\infty} w_p\Big\|\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} d_{p,j,k}\psi_{p,j,k}\Big\|_{L_2(\mathbb{R})}\\ &\leq C_1(m,\psi)\sum_{p=0}^{\infty} w_p(p+1)^{-1/2}\Big(\Big(\sum_{k\in\mathbb{Z}}|c_{p,k}|^2\Big)^{1/2} + \Big(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}|d_{p,j,k}|^2\Big)^{1/2}\Big)\\ &= C_1(m,\psi)\sum_{p=0}^{\infty} w_p(p+1)^{-1/2}\big\|(\mathbf{c},\mathbf{d})\big\|_{\ell_2(\mathbb{N}_0\times\mathbb{Z})\oplus\ell_2(\mathbb{N}_0\times\mathbb{N}_0\times\mathbb{Z})}.\end{aligned}$$

□

5 The Frame Property in $H^s(\mathbb{R})$, $s > 0$

In the preceding sections, we have derived all the necessary building blocks that are needed to construct stable quarklet frames not only for $L_2(\mathbb{R})$, but also for scales of L_2-Sobolev spaces $H^s(\mathbb{R})$, $0 < s < \gamma$. We will follow an abstract axiomatic approach to build multi-scale hp–frames (frames build by dyadic dilation, translation and p–enrichment) from families of multi-scale h-frames (built by dyadic dilation and translation). We refer to the manuscript [19] and the references therein for further details.

Suppose we have a family of MRAs $\mathcal{U}_p = \{U_{j,p}\}_{j\geq 0}, p \geq p_0$ satisfying the monotonicity constraints

$$U_{j-1,p} \subset U_{j,p} \subset U_{j,p+1} \subset H^\gamma(\mathbb{R}), \quad j \geq 1, p \geq p_0, \tag{36}$$

where $\gamma > 0$ is fixed. Then, we make the following assumptions:

Assumption A. For the smallest $p = p_0$, assume that one has proved a Jackson theorem such that with certain constants A_s the bound

$$\|f\|_{L_2}^2 + \sum_{j=0}^{\infty} 2^{2js}E_{j,p}(f)_{L_2}^2 \leq A_s\|f\|_{H^s}^2 \tag{37}$$

holds for arbitrary $f \in H^s$ and $0 < s < \gamma$.

Assumption B. Suppose for $p \geq p_0, j \geq 0$, and $0 < s < \gamma$ a Bernstein estimate holds:

$$\|g\|_{H^s(\mathbb{R})} \leq B_{s,p} 2^{js} \|g\|_{L_2(\mathbb{R})}, \quad \text{for all } g \in U_{j,p}. \tag{38}$$

Assumption C. For every j, the ladder

$$\tilde{\mathcal{U}}_j \ : \ U_{j,p_0} \subset \ldots \subset U_{j,p} \subset U_{j,p+1} \subset \ldots$$

possesses a hierarchical frame system $\Psi_j = \cup_{p \geq p_0} \Psi_{j,p}$ where $\Psi_{j,p} = \{\psi_{p,j,k}\}_{k \in \mathbb{Z}} \subset U_{j,p}$. More precisely, we assume that the sections

$$\tilde{\Psi}_{j,p} = \cup_{q=p_0}^{p} \Psi_{j,q}$$

of Ψ_j form frames in $U_{j,p}$ (considered as subspaces of L_2), with frame bounds independent of j, but dependent on p:

$$(C_p)^{-1} \|g_{j,p}\|^2_{L_2(\mathbb{R})} \leq \inf_{g_{j,p} = \sum_{i=0}^{p} \sum_{k \in \mathbb{Z}} c_{i,k} \psi_{i,j,k}} \sum_{i=0}^{p} \sum_{k \in \mathbb{Z}} |c_{i,k}|^2 \leq D_p \|g_{j,p}\|^2_{L_2(\mathbb{R})}, \quad \text{for all } g_{j,p} \in U_{j,p}. \tag{39}$$

These three assumptions together with standard Sobolev spaces properties allow us to conclude the frame property in $H^s, 0 < s < \gamma$, see again [19] for details.

Theorem 5.1. *Under the assumptions A, B, C, there exist weights $w_{p,j,s} > 0$ such that*

$$\tilde{\Psi} := \cup_{j \geq 0} \cup_{p \geq p_0} w_{p,j,s} \Psi_{j,p}$$

has the frame property in $H^s, 0 < s < \gamma$.

Now we want to apply this abstract machinery and in addition pinpoint exact weights to our special case.

Theorem 5.2. *For a given $\gamma > 0$, let $\varphi = N_m(\cdot + \lfloor \frac{m}{2} \rfloor), m > \gamma + 1/2$. Then, for the scaling factors $w_{p,j,s} := 2^{-js}(p+1)^{-2s-\delta}$, with $\delta > 1$, the system*

$$\Psi_{Q,w,s} = \{w_{p,0,s} \varphi_p(\cdot - k), w_{p,j,s} 2^{j/2} \psi_p(2^j \cdot -k) \ : \ p, j \in \mathbb{N}_0, k \in \mathbb{Z}\}$$

has the frame property in $H^s, 0 < s < \gamma$.

Proof. We define

$$\Psi_{j,p} := \{w_p \varphi_p(\cdot - k), w_p 2^{l/2} \psi_p(2^l \cdot -k) : k \in \mathbb{Z}, l < j\}, \ U_{j,p} := \text{clos}_{L_2(\mathbb{R})} \text{span}\{\cup_{q=0}^{p} \Psi_{j,q}\}, \tag{40}$$

cf. (32). Then the sequences $U_{j,p}$ is obviously nested in the sense of (36). For the existence of weights $w_{p,j,s} > 0$ so that $\Psi_{Q,\omega,s}$ constitutes a frame in H^s, it remains to check conditions A,B, and C.

Assumptions A. For $p = 0$, the corresponding elements of $\Psi_{j,0}$ coincide with the primal wavelets of a biorthogonal B-spline wavelet basis. In this case, it is well-known that for $m > \gamma + 1/2$, a Jackson-type estimate of the form (37) holds.

Assumption B. Due to the two-scale-equation (28) the function $\psi_{q,l,k}, 0 \le q \le p$ is contained in the space $V_{l+1,p}$ as defined in (6). As we will show in the appendix, the sequence $\{V_{j,p}\}_{j\ge 0}$ is nested. Consequently, it follows that

$$U_{j,p} \subset V_{j,p}.$$

Therefore, Corollary 3.6 implies (38) with $B_{s,p} = C(p+1)^{2s}$.

Assumption C. In Section 4 we have already shown that the system $\Psi_{Q,w}$ is a frame for $L_2(\mathbb{R})$. The collections $\tilde{\Psi}_{j,p}$ are subsets of $\Psi_{Q,w}$, and it is well-known that a subset of a given frame constitutes a frame for its span, see, e.g., [4], Example 5.1.4. Furthermore let us denote that due to this fact the frame bounds in (39) are not only independent of j but also on p, which will be important in the following part of the proof.

By Theorem 5.1 it follows the existence of weights $\tilde{w}_{j,p,s} > 0$ such that

$$\Psi_{Q,w,s} = \cup_{j\ge 0} \cup_{p\ge 0} \tilde{w}_{j,p,s} \Psi_{j,p}$$

has the frame property in H^s, $0 < s < \gamma$. Now we want to determine the weights $w_{j,p,s} = \tilde{w}_{j,p,s}\, w_p$. In Theorem 4.4 we have already shown, that the weights w_p only need to be nonnegative and $w_p(p+1)^{-1/2}$ has to be summable, such that $w_p = (p+1)^{-\delta_1}$, with $\delta_1 > \frac{1}{2}$ would do the job. To choose the weights $\tilde{w}_{j,p,s}$ we look at the proof of Proposition 1 in [19]. There it is shown in a first step, that for fixed $p \ge 0$, $0 < s < \gamma$ the system

$$\tilde{\Psi}_p := \cup_{j\ge 0} \cup_{q=0}^{p} 2^{-js} \Psi_{j,q}$$

constitutes a frame in H^s with frame bounds $c = c(p, s, \varepsilon) > 0$, $0 < s - \varepsilon < s + \varepsilon < \gamma$ and $C = C(p, s) > 0$. The first constant is of the form $c = (C'_{s,\varepsilon}\, B_{s+\varepsilon,p}\, B_{s-\varepsilon,p})^{-1} = C_{s,\varepsilon}(p+1)^{-4s}$, with $C'_{s,\varepsilon}, C_{s,\varepsilon} > 0$ independently of p. The independence of p relies on the aforementioned fact, that in our case the frame bounds in (39) are independent of p. We are choosing now $\tilde{w}_{j,p,s} = 2^{-js}(p+1)^{-2s-\delta_2}$ with $\delta_2 > \frac{1}{2}$ and show that $\Psi_{Q,w,s}$ is a frame in H^s. With $f \in H^s(\mathbb{R})$ and $\phi_{p,j,k} \in \{w_p\, \varphi_{p,0,k}, w_p\, \psi_{p,j,k}\}$ it holds

$$\begin{aligned}
\inf_{f=\sum\limits_{p\ge 0}\sum\limits_{j\ge 0}\sum\limits_{k\in\mathbb{Z}} c_{p,j,k}\tilde{w}_{p,j,s}\phi_{p,j,k}} \sum_{p\ge 0}\sum_{j\ge 0}\sum_{k\in\mathbb{Z}} |c_{p,j,k}|^2 &\le \inf_{f=\sum\limits_{j\ge 0}\sum\limits_{k\in\mathbb{Z}} c_{0,j,k}2^{-js}\phi_{0,j,k}} \sum_{j\ge 0}\sum_{k\in\mathbb{Z}} |c_{0,j,k}|^2 \\
&\le C(0,s)\, \|f\|^2_{H^s(\mathbb{R})}\,,
\end{aligned}$$

showing the lower bound estimate. To show the upper bound estimate we assume $\sum_{p\geq 0}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} c_{p,j,k}\,\tilde{w}_{p,j,s}\,\phi_{p,j,k}$ to be a fixed decomposition of f. Furthermore for $p,j\in\mathbb{N}_0$, $k\in\mathbb{Z}$ we define $f_p := \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} c_{p,j,k}\,\tilde{w}_{p,j,s}\,\phi_{p,j,k}$ and $\tilde{c}_{p,j,k} := c_{p,j,k}(p+1)^{-2s-\delta_2}$. By using the frame property of $\tilde{\Psi}_p$ with the lower frame bound $c = C_{s,\varepsilon}(p+1)^{-4s}$ we compute

$$\begin{aligned}
\sum_{p\geq 0}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |c_{p,j,k}|^2 &= \sum_{p\geq 0}(p+1)^{4s+2\,\delta_2}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}}|\tilde{c}_{p,j,k}|^2 \\
&\geq \sum_{p\geq 0}(p+1)^{4s+2\,\delta_2}\inf_{f_p=\sum\limits_{j\geq 0}\sum\limits_{k\in\mathbb{Z}} d_{p,j,k}2^{-js}\phi_{p,j,k}}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}}|d_{p,j,k}|^2 \\
&\geq \sum_{p\geq 0}(p+1)^{4s+2\,\delta_2}\inf_{f_p=\sum\limits_{q=0}^{p}\sum\limits_{j\geq 0}\sum\limits_{k\in\mathbb{Z}} d'_{q,j,k}2^{-js}\phi_{q,j,k}}\sum_{q=0}^{p}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}}|d'_{q,j,k}|^2 \\
&\geq C_{s,\varepsilon}\sum_{p\geq 0}(p+1)^{2\,\delta_2}\,\|f_p\|^2_{H^s(\mathbb{R})}\,.
\end{aligned}$$

Exploiting the Cauchy-Schwarz-inequality leads to

$$\begin{aligned}
\sum_{p\geq 0}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |c_{p,j,k}|^2 &\geq C_{s,\varepsilon,\delta_2}\sum_{p'\geq 0}(p'+1)^{-2\,\delta_2}\sum_{p\geq 0}(p+1)^{2\,\delta_2}\,\|f_p\|^2_{H^s(\mathbb{R})} \\
&\geq C_{s,\varepsilon,\delta_2}\left(\sum_{p\geq 0}\|f_p\|_{H^s(\mathbb{R})}\right)^2 \\
&\geq C_{s,\varepsilon,\delta_2}\left\|\sum_{p\geq 0} f_p\right\|^2_{H^s(\mathbb{R})} \\
&= C_{s,\varepsilon,\delta_2}\,\|f\|^2_{H^s(\mathbb{R})}\,,
\end{aligned}$$

where $C_{s,\varepsilon,\delta_2} = C_{s,\varepsilon}\left(\sum_{p'\geq 0}(p'+1)^{-2\,\delta_2}\right)^{-1}$. Taking the infimum finally shows the upper bound estimate and so the claim is proofed.

□

6 Compressibility of differential operators in quarklet coordinates

As already mentioned in the introduction, the stability of weighted quarkonial frames in Sobolev spaces and the compression properties of the individual quarklets can be used to derive adaptive discretization schemes for linear elliptic operator equations in a quite systematic way, see [7, 9, 10, 23] for the general reasoning.

In order to briefly illustrate the main ideas of such schemes, let us consider a linear elliptic variational problem of the form

$$a(u, v) = F(v), \quad \text{for all } v \in H, \tag{41}$$

where H is the solution Hilbert space and $a : H \times H \to \mathbb{R}$ a symmetric, bounded and coercive bilinear form and $F : H \to \mathbb{R}$ a continuous functional. Given a frame $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ for H with countable index set Λ, it is well-known [7, 9, 23] that (41) is equivalent to the linear system of equations

$$\mathbf{A}\mathbf{u} = \mathbf{F}, \tag{42}$$

where $\mathbf{A} := (a(\psi_\mu, \psi_\lambda))_{\mu,\lambda\in\Lambda} \in \mathcal{L}(\ell_2(\Lambda))$ is the biinfinite stiffness matrix, $\mathbf{u} := (u_\lambda)_{\lambda\in\Lambda}$ is a coefficient array of the unknown solution $u = \sum_{\lambda\in\Lambda} u_\lambda \psi_\lambda$ with respect to the frame Ψ, and $\mathbf{F} := (F(\psi_\lambda)_{\lambda\in\Lambda})$ contains the values of the right-hand side F at individual frame elements. Due to the redundancy of the frame Ψ, the system matrix $\mathbf{A}$ has a non-trivial kernel, so that (42) is not uniquely solvable. Straightforward Galerkin-type approaches might hence run into stability problems.

Nonetheless, classical iterative schemes like the damped Richardson iteration

$$\mathbf{u}^{(j+1)} := \mathbf{u}^{(j)} + \omega(\mathbf{F} - \mathbf{A}\mathbf{u}^{(j)}), \quad 0 < \omega < \frac{2}{\|\mathbf{A}\|_{\mathcal{L}(\ell_2(\Lambda))}}, \quad j = 0, 1, \ldots \tag{43}$$

or variations thereof, like steepest descent or conjugate gradient iterations, can still be applied in a numerically stable way, and the associated expansions $u^{(j)} := \sum_{\lambda\in\Lambda} u_\lambda^{(j)} \psi_\lambda \in H$ will converge to the solution u under quite general assumptions. By judiciously choosing the respective tolerances, convergence can even be preserved under perturbation of the exact iterations when, e.g., each evaluation of the infinite-dimensional right-hand side $\mathbf{F}$ and each matrix-vector product $\mathbf{A}\mathbf{v}$ are replaced by suitable numerical approximations [7, 9, 10, 12, 23].

Inexact matrix-vector multiplications play a key role within adaptive wavelet methods. In order to realize them in a computationally efficient way, it is essential to exploit that the system matrix $\mathbf{A}$ is not arbitrarily structured but features certain compressibility properties. By this we mean that $\mathbf{A}$ can be approximated well by sparse matrices with a finite number of entries per row and column. Such approximations can be constructed in a quite generic way, see [6, 23, 24] if the entries of $\mathbf{A}$ have a sufficiently fast off-diagonal decay.

In the sequel, we will show that similar to wavelet systems, also quarklet frames can induce compressible stiffness matrices in the aforementioned sense. As a concrete example, let us consider the variational formulation of the one-dimensional Poisson equation with periodic boundary conditions and a right-hand side $f \in L_2(0, 1)$. Let

$$H^1_{\mathrm{per}}(0, 1) := \big\{v \in L_2(0, 1) : v' \in L_2(0, 1), v(0) = v(1)\big\}$$

be the periodic first-order Sobolev space on the unit interval, with norm

$$\|v\|_{H^1_{\mathrm{per}}(0,1)} := \big(\|v\|^2_{L_2(0,1)} + \|v'\|^2_{L_2(0,1)}\big)^{1/2}, \quad \text{for all } v \in H^1_{\mathrm{per}}(0, 1).$$

We are looking for

$$u \in H := \Big\{ v \in H^1_{\mathrm{per}}(0,1) : \int_0^1 v(x)\,\mathrm{d}x = 0 \Big\}$$

which solves (41) on H, where we set $a(u,v) := \int_0^1 u'v'\,\mathrm{d}x$ and $F(v) := \int_0^1 fv\,\mathrm{d}x$. This variational problem is well-posed because of Wirtinger's inequality, see [15], for weakly differentiable, moment-free periodic functions.

Our aim to discretize the periodic Poisson equation with a periodic quarkonial frame Ψ^H for H. To this end, assume that for suitable weights $w_{p,j} > 0$,

$$\Psi = \Big\{ w_{p,0}\varphi_p(\cdot - k), w_{p,j}\psi_{p,j,k}p(2^j \cdot - k) : p, j \in \mathbb{N}_0, k \in \mathbb{Z} \Big\}$$

is a B-spline quarklet frame for $H^1(\mathbb{R})$ with at least one vanishing moment of the quarklets $\psi_{p,j,k}$, as constructed in the previous section. Similar to the case of periodic wavelet bases on the unit inverval, see [13], we consider the 1-periodized quarks

$$\varphi^{\mathrm{per}}_{p,j_0,k} := \sum_{l \in \mathbb{Z}} \varphi_{p,j_0,k}(\cdot - l), \quad \text{for all } p \geq 0,\ 0 \leq k \leq 2^{j_0} - 1,$$

and the 1-periodized quarklets

$$\psi^{\mathrm{per}}_{p,j,k} := \sum_{l \in \mathbb{Z}} \psi_{p,j,k}(\cdot + l), \quad \text{for all } p \geq 0,\ j \geq j_0,\ 0 \leq k \leq 2^j - 1,$$

with $j_0 \in \mathbb{N}_0$. To avoid some overlap we choose j_0 as the smallest integer so that the support length of the nonperiodized quarklets on the coarsest level is lower or equal to one. Then the system

$$\Psi^{\mathrm{per}} := \Big\{ w_{p,j_0}\varphi^{\mathrm{per}}_{p,j_0,l}, w_{p,j}\psi^{\mathrm{per}}_{p,j,k} : p \in \mathbb{N}_0, j \geq j_0, 0 \leq l \leq 2^{j_0} - 1,\ 0 \leq k \leq 2^j - 1 \Big\}$$

is readily shown to be a frame for $H^1_{\mathrm{per}}(0,1)$. Because of their vanishing moment property, the periodized quarklets $\psi^{\mathrm{per}}_{p,j,k}$ are contained in H, while the periodized quarks $\varphi^{\mathrm{per}}_{p,j_0,k}$ are not. By using that each $v \in H^1_{\mathrm{per}}(0,1)$ can be written as $v = v_1 + v_2$ with $v_2 := \int_0^1 v\,\mathrm{d}x$, $v_1 = v - v_2 \in H$ and $\|v\|_{H^1_{\mathrm{per}}(0,1)} \eqsim \|v_1\|_{H^1_{\mathrm{per}}(0,1)} + \|v_2\|_{H^1_{\mathrm{per}}(0,1)}$, we simply project the periodized quarks onto H via

$$\check{\varphi}^{\mathrm{per}}_{p,j_0,k} := \varphi^{\mathrm{per}}_{p,j_0,k} - \int_0^1 \varphi^{\mathrm{per}}_{p,j_0,k}\,\mathrm{d}x, \quad \text{for all } p \geq 0,\ 0 \leq k \leq 2^{j_0} - 1,$$

and we obtain the desired frame

$$\Psi^H := \Big\{ w_{p,j_0}\check{\varphi}^{\mathrm{per}}_{p,j_0,l}, w_{p,j}\psi^{\mathrm{per}}_{p,j,k} : p \in \mathbb{N}_0, j \geq j_0, 0 \leq l \leq 2^{j_0} - 1,\ 0 \leq k \leq 2^j - 1 \Big\}$$

for H.

By using similar ideas as in Proposition 4.3, one can prove the following compression estimate, which is important to get compression results for the biinfinite stiffness matrix of the one-dimensional Laplacian in quarklet coordinates. For the readers convenience and to keep it simple the following results are stated for the non-periodized quarks and quarklets on the real line. One can easily check that these results can be transfered to the case of periodic quarks and quarklets on the interval $[0, 1]$.

Proposition 6.1. *Let $m \geq 3$, $\varphi = N_m(\cdot + \lfloor m/2 \rfloor)$, $e \in \{0, 1\}$, $\phi^{(0)} = \varphi$ and $\phi^{(1)} = \psi$. There exists $C = C(m, \psi)$, such that*

$$2^{-(j+j')} \big| \langle \phi^{(e)'}_{p,j,k}, \phi^{(e)'}_{p',j',k'} \rangle_{L_2(\mathbb{R})} \big| \leq C(p+1)^{m-1}(p'+1)^{m-1} 2^{-|j-j'|(m-3/2)}. \tag{44}$$

Proof. Note first that by $m \geq 3$, φ and hence ψ and ψ_p have $m-1$ weak derivatives in $L_q(\mathbb{R})$, $1 \leq q \leq \infty$. We first consider the case were both functions are quarklets ψ. For $j' \geq j$ we use the compact support of ψ and the fact that ψ' has $\tilde{m}+1$ vanishing moments to compute that for each $0 \leq r \leq \tilde{m}$,

$$\begin{aligned}\big| \langle \psi'_{p,j,k}, \psi'_{p',j',k'} \rangle_{L_2(\mathbb{R})} \big| &= \inf_{P \in \mathbb{P}_r} \big| \langle \psi'_{p,j,k} - P, \psi'_{p',j',k'} \rangle_{L_2(\mathbb{R})} \big| \\ &\leq \inf_{P \in \mathbb{P}_r} \| \psi'_{p,j,k} - P \|_{L_\infty(\operatorname{supp} \psi_{p',j',k'})} \| \psi'_{p',j',k'} \|_{L_1(\mathbb{R})}.\end{aligned}$$

On the one hand, from (30), (28), (24) and (18), we obtain that with $C_1 = C_1(\psi) > 0$ and $C_2 = C_2(m, \psi) > 0$

$$\| \psi'_{p',j',k'} \|_{L_1(\mathbb{R})} = 2^{j'/2} \| \psi'_{p'} \|_{L_1(\mathbb{R})} \leq C_1 2^{j'/2} \| \varphi'_{p'} \|_{L_1(\mathbb{R})} \leq C_2 (p'+1)^{-(m-2)} 2^{j'/2}.$$

On the other hand, by Whitney's theorem, for each choice of p, j, k, p', j', k', there exists $Q \in \mathbb{P}_r$ such that with $C_3 = C_3(r) > 0$ and $C_4 = C_4(r) > 0$,

$$\| \psi'_{p,j,k} - Q \|_{L_\infty(\operatorname{supp} \psi_{p',j',k'})} = C_3 \omega_{r+1}(\psi'_{p,j,k}, 2^{-j'})_{L_\infty(\mathbb{R})} \leq C_4 2^{-j'(r+1)} |\psi'_{p,j,k}|_{W^{r+1}(L_\infty(\mathbb{R}))}.$$

Due to $\psi' \in W^{m-2}(L_\infty(\mathbb{R}))$, the latter norm is finite for all $0 \leq r \leq m-3$. Picking $r = m-3 \geq 0$, an application of (30), (28), (24) and (18) shows that with $C_5 = C_5(m) > 0$, $C_6 = C_6(m, \psi) > 0$ and $C_7 = C_7(m, \psi) > 0$

$$\begin{aligned}\inf_{P \in \mathbb{P}_{m-3}} \| \psi'_{p,j,k} - P \|_{L_\infty(\operatorname{supp} \psi_{p',j',k'})} &\leq C_5 2^{-j'(m-2)} |\psi'_{p,j,k}|_{W^{m-2}(L_\infty(\mathbb{R}))} \\ &= C_5 2^{-j'(m-2)} 2^{j(m-1/2)} \| \psi_p^{(m-1)} \|_{L_\infty(\mathbb{R})} \\ &\leq C_6 2^{-j'(m-2)} 2^{j(m-1/2)} \| \varphi_p^{(m-1)} \|_{L_\infty(\mathbb{R})} \\ &\leq C_7 (p+1)^{m-1} 2^{-j'(m-2)} 2^{j(m-1/2)}.\end{aligned}$$

Combining the previous estimates, we obtain that with $C_8 = C_8(m, \psi) > 0$

$$\big| \langle \psi'_{p,j,k}, \psi'_{p',j',k'} \rangle_{L_2(\mathbb{R})} \big| \leq C_8 (p+1)^{m-1} (p'+1)^{-(m-2)} 2^{j+j'} 2^{-(j'-j)(m-3/2)}.$$

If $j' \le j$, we obtain in a completely analogous way that

$$\left|\langle \psi'_{p,j,k}, \psi'_{p',j',k'} \rangle_{L_2(\mathbb{R})}\right| \le C_8 (p'+1)^{m-1}(p+1)^{-(m-2)} 2^{j+j'} 2^{-(j-j')(m-3/2)}.$$

In the case that both functions are quarks φ, we estimate

$$\left|\langle \varphi'_{p,0,k}, \varphi'_{p',0,k'} \rangle_{L_2(\mathbb{R})}\right| \le \left\|\varphi'_p\right\|_{L_\infty(\mathbb{R})} \left\|\varphi'_{p'}\right\|_{L_1(\mathbb{R})}.$$

Previously in the proof we have already seen that $||\varphi'_{p'}||_{L_1(\mathbb{R})} \le C_9(p'+1)^{-(m-2)}$ with $C_9 = C_9(m) > 0$ and from (18) and (24) we obtain $||\varphi'_p||_{L_\infty(\mathbb{R})} \le C_{10}(p'+1)^{-(m-3)}$ with $C_{10} = C_{10}(m) > 0$. With $C_{11} = C_{11}(m) > 0$, this leads to

$$\left|\langle \varphi'_{p,0,k}, \varphi'_{p',0,k'} \rangle_{L_2(\mathbb{R})}\right| \le C_{11}(p+1)^{-(m-2)}(p'+1)^{-(m-3)}.$$

If both φ and ψ are involved with $C_{12} = C_{12}(\psi, m) > 0$, we obtain similar to the first case

$$\begin{aligned}
\left|\langle \varphi'_{p,0,k}, \psi'_{p',j',k'} \rangle_{L_2(\mathbb{R})}\right| &= \inf_{P\in\mathbb{P}_r} \left|\langle \varphi'_{p,0,k} - P, \psi'_{p',j',k'} \rangle_{L_2(\mathbb{R})}\right| \\
&\le \inf_{P\in\mathbb{P}_r} \|\varphi'_{p,0,k} - P\|_{L_\infty(\operatorname{supp}\psi_{p',j',k'})} \|\psi'_{p',j',k'}\|_{L_1(\mathbb{R})} \\
&\le C_{12}(p+1)^{m-1}(p'+1)^{-(m-2)} 2^{-j'(m-3/2)} 2^{j'},
\end{aligned}$$

and therefore (44). $\square$

To discretize the one-dimensional Laplacian we will exploit the $H^1(\mathbb{R})$-frame $\Psi_{Q,w,1}$ with the weights $w_{p,j,1} = 2^{-j}(p+1)^{-2-\delta}$. The entries of the corresponding biinfinite stiffness matrix $\mathbf{A}_{H^1(\mathbb{R})}$ are

$$a_{\lambda,\lambda'} = 2^{-(j+j')}(p+1)^{-2-\delta}(p'+1)^{-2-\delta} \langle \phi^{(e)\prime}_{p,j,k}, \phi^{(e)\prime}_{p',j',k'} \rangle_{L_2(\mathbb{R})}, \tag{45}$$

with $\lambda, \lambda' \in \mathcal{I} := \Big\{(p,j,e,k) : p, j \in \mathbb{N}_0, e \in \{0,1\}, k \in \mathbb{Z}\Big\}$, $\phi^{(0)} = \varphi$ and $\phi^{(1)} = \psi$. Combining (44) and (45) leads to

$$|a_{\lambda,\lambda'}| \le C(p+1)^{m-3-\delta}(p'+1)^{m-3-\delta} 2^{-|j-j'|(m-3/2)}.$$

If $\tau := \delta + 3 - m > 0$, it holds

$$\left(\frac{1}{(p+1)(p'+1)}\right)^\tau \le \left(\frac{1}{1+|p-p'|}\right)^\tau,$$

so in this case we get the crucial compression result

$$|a_{\lambda,\lambda'}| \le C(1+|p-p'|)^{-\tau} 2^{-|j-j'|(m-3/2)}. \tag{46}$$

With this result at hand we can proof the compressibility of the stiffness matrix $\mathbf{A}_{H^1(\mathbb{R})}$.

Theorem 6.2. *Let $m \geq 3$. For $J \in \mathbb{N}_0$, we define the biinfinite matrix $\mathbf{A}_J$ by dropping the entries $a_{\lambda,\lambda'}$ from $\mathbf{A}_{H^1(\mathbb{R})}$ when*

$$a \log_2(1 + |p - p'|) + b\,|j' - j| > J, \tag{47}$$

with $a > 1$, $b \geq \frac{a}{a-1}$ and $-\tau + \frac{a(m-2)}{b} < -1$. Then the number of non-zero entries in each row and colum of $\mathbf{A}_J$ is of order 2^J, and

$$||\mathbf{A}_{H^1(\mathbb{R})} - \mathbf{A}_J||_{\mathcal{L}(\ell_2(\mathcal{I}))} \lesssim 2^{-J(m-2)/b}. \tag{48}$$

Proof. For fixed indices $\lambda \in \mathcal{I}$, $p' \in \mathbb{N}_0$ the number of indices $\lambda' \in \mathcal{I}$ for which $\mathrm{vol}(\operatorname{supp}\phi_\lambda \cap \operatorname{supp}\phi_{\lambda'}) > 0$ is of order $\max(1, 2^{j'-j})$, since the support of the quarklets is local. In the case $j' \geq j$ we use the definition of $\mathbf{A}_J$ and get the estimation

$$\begin{aligned}
\sum_{p':|p'-p|\leq 2^{J/a}-1} \; \sum_{j'=j}^{\left\lfloor \frac{J-a\log_2(1+|p'-p|)}{b}\right\rfloor + j} 2^{j'-j} &\lesssim \sum_{p'':|p''|\leq 2^{J/a}-1} \; \sum_{j''=0}^{\left\lfloor \frac{J-a\log_2(1+|p''|)}{b}\right\rfloor} 2^{j''} \\
&\lesssim \sum_{p''=0}^{\lfloor 2^{J/a}\rfloor -1} 2^{J/b} p''^{-a/b} \\
&\lesssim 2^{J/b} 2^{J/a} \\
&= 2^{J(1/a+1/b)}
\end{aligned}$$

for the number of non-zero entries in each row and column of $\mathbf{A}_J$. For $j' < j$ it holds

$$\begin{aligned}
\sum_{p':|p'-p|\leq 2^{J/a}-1} \; \sum_{j'=j-\left\lceil \frac{J-a\log_2(1+|p'-p|)}{b}\right\rceil}^{j-1} 1 &\lesssim \sum_{p''=0}^{\lfloor 2^{J/a}\rfloor -1} \left\lceil \frac{J - a\log_2(1+p'')}{b} \right\rceil \\
&\lesssim 2^{J/a}\frac{J}{b}.
\end{aligned}$$

This shows that for $a > 1$ and $b \geq \frac{a}{a-1}$ the number of non-zero entries in each row and column of $\mathbf{A}_J$ is of order 2^J. Now we want to estimate the operator norm in the space $\ell_2(\mathcal{I})$ of $\mathbf{A}_{H^1(\mathbb{R})} - \mathbf{A}_J$. For that purpose we use the Schur-lemma which states that for a matrix B, an index set $\mathcal{J}$ and weights $\omega_\lambda > 0$ it suffices to show

$$\sup_{\lambda\in\mathcal{J}} \omega_\lambda^{-1} \sum_{\lambda'\in\mathcal{J}} \left|b_{\lambda,\lambda'}\right| \omega'_\lambda \leq C, \qquad \sup_{\lambda'\in\mathcal{J}} \omega_\lambda'^{-1} \sum_{\lambda\in\mathcal{J}} \left|b_{\lambda,\lambda'}\right| \omega_\lambda \leq C, \tag{49}$$

to proof $\|B\|_{\mathcal{L}(\ell_2(\mathcal{J}))} \leq C$. We define $\alpha := \sup_{\lambda\in\mathcal{I}} \omega_\lambda^{-1} \sum_{\lambda'\in\mathcal{I}} \left|a_{\lambda,\lambda'} - \tilde{a}_{\lambda,\lambda'}\right| \omega'_\lambda$ and get

$$\alpha \lesssim \sup_{\lambda\in\mathcal{I}} \omega_\lambda^{-1} \sum_{p'\geq 0} \sum_{j'\geq 0} \max(1, 2^{j'-j}) \left|a_{\lambda,\lambda'} - \tilde{a}_{\lambda,\lambda'}\right| \omega'_\lambda.$$

Using the definition of $\mathbf{A}_J$ and (46) we find

$$\begin{aligned}
\alpha &\lesssim \sup_{\lambda\in\mathcal{I}} \omega_\lambda^{-1} \sum_{p',j'\geq 0|(*)>J} \max(1, 2^{j'-j}) \left|a_{\lambda,\lambda'}\right| \omega'_\lambda \\
&\lesssim \sup_{\lambda\in\mathcal{I}} \omega_\lambda^{-1} \sum_{p',j'\geq 0|(*)>J} \max(1, 2^{j'-j})(1+\left|p-p'\right|)^{-\tau} 2^{-|j'-j|(m-3/2)} \omega'_\lambda,
\end{aligned}$$

with $(*) := a\,\log_2(1+|p-p'|) + b\,|j'-j|$. By choosing the weights $\omega_\lambda = 2^{-j/2}$, for the case $j' \geq j$ we obtain

$$\begin{aligned}
\alpha &\lesssim \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau} \sum_{j'\geq 0|(*)>J} 2^{j/2} 2^{-(j'-j)(m-5/2)} 2^{-j'/2} \\
&\leq \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau} \sum_{j'=j+\beta}^{\infty} 2^{-(m-2)(j'-j)},
\end{aligned}$$

with $\beta := \left\lceil \frac{J-a\,\log_2(1+|p-p'|)}{b} \right\rceil$. Substitution leads to

$$\begin{aligned}
\alpha &\lesssim \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau} \sum_{k=\beta}^{\infty} 2^{-(m-2)k} \\
&\lesssim \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau+a(m-2)/b}\, 2^{-J(m-2)/b}.
\end{aligned}$$

If $-\tau + a(m-2)/b < -1$, the sum in the last expression converges and we reach

$$\alpha \lesssim 2^{-J(m-2)/b}.$$

For $j' < j$ we do similar conversions and for $-\tau + a(m-2)/b < -1$ we obtain again

$$\begin{aligned}
\alpha &\lesssim \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau} \sum_{j'\geq 0|(*)>J} 2^{j/2} 2^{-(m-3/2)(j-j')} 2^{-j'/2} \\
&\leq \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau} \sum_{j'=0}^{j-\beta} 2^{-(m-2)(j-j')} \\
&= \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau} \sum_{k=\beta}^{j} 2^{-(m-2)k} \\
&< \sup_{\lambda\in\mathcal{I}} \sum_{p'\geq 0} (1+\left|p-p'\right|)^{-\tau} \sum_{k=\beta}^{\infty} 2^{-(m-2)k} \\
&\lesssim 2^{-J(m-2)/b},
\end{aligned}$$

what finally claimes the proof. $\square$

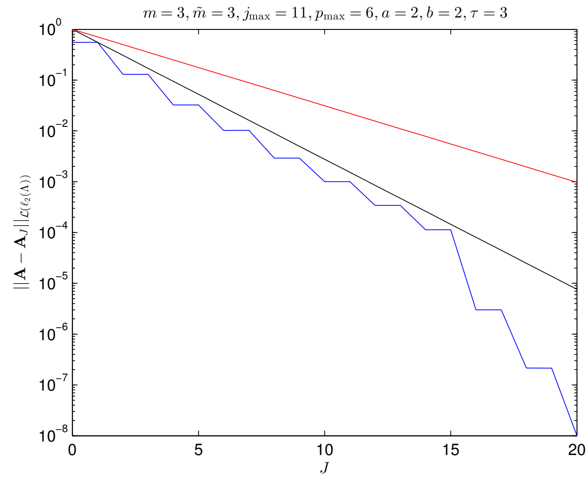


Figure 3: Numerical results in logarithmic scale. The red and black line are the functions $2^{-0.5J}$ respectively $2^{-0.8J}$, whereas the blue line shows the actual scaled results.

In numerical experiments we were able to verify the result of the last theorem for the case of the Poisson-equation with periodic boundary conditions. Actually they seem to be even better than expected. This indicates that it might be possible to improve the result of Proposition 6.1, e.g., by using second compression ideas as outlined in [20] or [24]. In the numerical experiments we computed the operatornorm of $\mathbf{A} - \mathbf{A}_J$, where $\mathbf{A}$ is the stiffness matrix of the Poisson-equation in the periodical setting and $\mathbf{A}_J$ are the corresponding sparse matrices. As parameters we chose $m = \tilde{m} = \tau = 3$ and $a = b = 2$ so that after Theorem 6.2 we could expect

$$||\mathbf{A} - \mathbf{A}_J||_{\mathcal{L}(\ell_2(\Lambda))} \lesssim 2^{-J/2}.$$

The maximal refinement level and the maximal polynomial order are $j_{\max} = 11$ and $p_{\max} = 6$. In Figure 3 the red line shows this expected results whereas the blue line stands for the actual scaled results. We would like to emphasize that the constants appearing in the norm-estimation seem to be of moderate size. The black line shows a possible better estimation. The stronger decay of the blue line for big J occurs due to the inevitable cap of the involved parameters p and j.

A Appendix

In this section, we show a refinement property of the functions $\varphi_0, \ldots, \varphi_p$. Although each individual φ_q is usually not a refinable function, the whole collection $(\varphi_0, \ldots \varphi_p)$ forms a refinable function vector. Consequently, the sequence $V_{j,p}$ as defined in (6) is nested.

Proposition A.1. *For any $p \geq 0$, the vector $(\varphi_0, \ldots, \varphi_p)$ is refinable, i.e., there exist $(p+1) \times (p+1)$-matrices $\mathbf{C}_k$ such that*

$$\begin{pmatrix} \varphi_0(x) \\ \vdots \\ \varphi_p(x) \end{pmatrix} = \sum_{k \in \mathbb{Z}} \mathbf{C}_k \begin{pmatrix} \varphi_0(2x-k) \\ \vdots \\ \varphi_p(2x-k) \end{pmatrix}.$$

Proof. By using the definition of φ_p and the refinability of $\varphi = \varphi_0$ we obtain:

$$\begin{aligned}
\varphi_q(x) &= x^q \varphi(x) \\
&= \frac{1}{2^q}(2x)^q \sum_{k \in \mathbb{Z}} a_k \varphi(2x-k) \\
&= \frac{1}{2^q} \sum_{k \in \mathbb{Z}} a_k (2x-k+k)^q \varphi(2x-k) \\
&= \frac{1}{2^q} \sum_{k \in \mathbb{Z}} a_k \sum_{l=0}^{q} (2x-k)^l \binom{q}{l} k^{q-l} \varphi(2x-k) \\
&= \sum_{k \in \mathbb{Z}} \frac{1}{2^q} a_k \sum_{l=0}^{q} \binom{q}{l} k^{q-l} \varphi_l(2x-k).
\end{aligned}$$

Now setting

$$(\mathbf{C}_k)_{q,l} := \frac{1}{2^q} a_k^q \binom{q}{l} k^{q-l}$$

yields the result. □

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Stephan Dahlke, Philipp Keding
Philipps-Universität Marburg
FB12 Mathematik und Informatik
Hans-Meerwein Straße, Lahnberge
35032 Marburg
Germany
e–mail: `dahlke/keding@mathematik.uni-marburg.de`
WWW: `http://www.mathematik.uni-marburg.de/~dahlke/(~keding/)`

Thorsten Raasch
Johannes Gutenberg-Universität
Institut für Mathematik
Arbeitsgruppe Numerische Mathematik
Staudingerweg 9
55099 Mainz
Germany
e–mail: `raasch@uni-mainz.de`
WWW: `http://www.mathematik.uni-mainz.de/Members/raasch`