Weighted Coorbit Spaces and Banach Frames on Homogeneous Spaces

Stephan Dahlke*
Universität Marburg
Fachbereich Mathematik
und Informatik
Hans–Meerwein–Str., Lahnberge
35032 Marburg
Germany

Gabriele Steidl†
Universität Mannheim
Fakultät für Mathematik
und Informatik
D7, 27
68131 Mannheim
Germany

Gerd Teschke‡
Fachbereich 3
Universität Bremen
Postfach 33 04 40
28334 Bremen
Germany

July 29, 2003

Abstract

This paper is concerned with frame constructions on domains and manifolds. The starting point is a unitary group representation which is square integrable modulo a suitable subgroup and therefore gives rise to a generalized continuous wavelet transform. Then generalized coorbit spaces can be defined by collecting all functions for which this wavelet transform is contained in a weighted L_p–space. Moreover, we show that a judicious discretization of the representation leads to an atomic decomposition and to Banach frames for these coorbit spaces.

*The work of this author has been partially supported by Deutsche Forschungsgemeinschaft, Grant Da 360/4–1, and by the EU Research Training Network *Harmonic Analysis and Statistics for Image Processing.*

†The work of this author has been partially supported by Deutsche Forschungsgemeinschaft, Grant We 2602/2–1

‡The work of this author has been partially supported by Deutsche Forschungsgemeinschaft, Grant Ma 1657/6–1, and by the EU Research Training Network *Harmonic Analysis and Statistics for Image Processing.*

Key Words: Square integrable group representations, time–frequency analysis, atomic decompositions, frames, homogeneous spaces, weighted coorbit spaces.

AMS Subject classification: 57S25, 42C15, 42C40, 46E15, 46E35.

1 Introduction

One of the classical tasks in applied analysis is the efficient representation/analysis of a given signal. Usually, the first step is the decomposition of the signal into suitable building blocks. Starting with Fourier analysis around 1820, many more or less successful approaches have been suggested. Current interest especially centers around multiscale representations of wavelet type. Wavelet bases have several remarkable advantages. Among others, they give rise to characterizations of function spaces such as Besov spaces and provide powerful approximation schemes, see, e.g., [4, 5]. However, in recent studies, it has turned out that the use of Riesz bases may have some serious drawbacks. One important problem is the lack of flexibility which is in some sense a consequence of the uniqueness of the representation. Therefore, one natural way out suggests itself: why not using a slightly weaker concept and allowing some redundancies, i.e., why not working with frames? In general, given a Hilbert space $\mathcal{H}$, a collection of elements $\{e_i\}_{i\in\mathbb{Z}}$ is called a *frame* if there exit constants $0 < A_1 \leq A_2 < \infty$ such that

$$A_1\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in\mathbb{Z}} |\langle f, e_i\rangle_{\mathcal{H}}|^2 \leq A_2\|f\|_{\mathcal{H}}^2. \tag{1.1}$$

The frame concept has been introduced by Duffin and Schäfer [6] in 1952. However, the starting point of the modern frame theory was the fundamental Feichtinger/Gröchenig theory which has been developed since 1986 in a series of papers [8, 9, 10, 11, 12]. This very aestetic and subtle theory is essentially based on group theory. Given a space $\mathcal{N}$, the first step is to find a suitable group $\mathcal{G}$ that admits a (square) integrable representation in $L_2(\mathcal{N})$ and therefore gives rise to a generalized (continuous) wavelet transform. Then, so–called *coorbit spaces* can be defined by collecting all functions for which this wavelet transform is contained in some (weighted) L_p–space. Finally, a judicious discretization of the representation produces the desired frames for the coorbit spaces. This approach works fine for the whole Euclidean plane and produces a general framework that covers, e.g., the classical wavelet and Weyl–Heisenberg frames. However, when it comes to practical applications, also the case of bounded domains and manifolds is important. Then, very often the problem arises that the group acting on the manifold is too 'large', i.e., its representation is not square–integrable. One natural remedy as suggested, e.g., by Ali et al. [1] and Torresani [16], is the concept of square–integrability modulo quotients. In this case, one has to find a certain subgroup $\mathcal{P}$ such that, after restricting the representation to the induced quotient space $\mathcal{G}/\mathcal{P}$ by fixing a Borel section $\sigma : \mathcal{G}/\mathcal{P} \longrightarrow \mathcal{G}$, one is again in a square–integrable setting. However, by this passage to quotients the very convenient group structure gets lost, so that many of the building blocks used in the Feichtinger/Gröchenig theory such as convolutions are no longer available. Nevertheless, in the previous paper [3], we have shown that a quite natural generalization of the Feichtinger/Gröchenig theory to quotient spaces is indeed possible. The major tool was a generalized reproducing kernel. The application of the corresponding integral operator in some sense replaces the usual convolution. Then, under certain integrability conditions on this kernel it has turned out that all the basic steps of the Feichtinger/Gröchenig approach can still be performed. By employing the concept of square integrability modulo quotients, generalized coorbit spaces may be defined. Moreover, one can define an approximation operator which produces atomic decompositions for these coobit spaces. Furthermore, a reconstruction operator can be introduced in a similar fashion and the frame bounds can be established.

To keep the technical difficulties at a reasonable level, in [3] only the 'simplest' class of coorbit spaces was considered. However, the coorbit approach allows the definition of whole scales of smoothness spaces by collecting all functions for which the generalized wavelet transform has certain decay properties, i.e., by considering *weighted* spaces. To fill this gap is the major aim of the present work.

This paper is organized as follows. In Section 2, we collect all the facts on group theory that are needed for our purposes. Then, in Section 3, we introduce and analyze our generalized weighted coorbit spaces. Section 4 contains the main results of this paper. In Subsection 4.1 we explain the setting and state all the conditions that are needed to establish atomic decompositions and Banach frames for the generalized weighted coorbit spaces. Subsection 4.2 is devoted to the definition and the analysis of the underlying approximation operators. Finally, in Subsection 4.3 we establish the frame bounds. This part of our analysis is essentially based on a version of the Riesz–Thorin interpolation theorem. Since this specific version was not found in the literature, we have included a proof based on complex interpolation in the appendix. There, we also state and prove a version of the generalized Young inequality for weighted L_p–spaces.

2 Group Theoretical Background

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{G}$ be a separable Lie group with (right) Haar measure ν. A *continuous representation* of $\mathcal{G}$ in $\mathcal{H}$ is defined as a mapping

$$U : \mathcal{G} \longrightarrow \mathcal{L}(\mathcal{H})$$

of $\mathcal{G}$ into the space $\mathcal{L}(\mathcal{H})$ of unitary operators on $\mathcal{H}$, such that $U(gg') = U(g)U(g')$ for all $g, g' \in \mathcal{G}$, $U(e) = Id$ and for any $\phi, \psi \in \mathcal{H}$, the function $g \in \mathcal{G} \to \langle \phi, U(g)\psi\rangle_{\mathcal{H}}$ is continuous. The representation U is said to be *square–integrable* if it is irreducible and there exists a nonzero $\psi \in \mathcal{H}$ such that

$$\int_{\mathcal{G}} |\langle \psi, U(g)\psi\rangle_{\mathcal{H}}|^2 \, d\nu(g) < \infty \, .$$

Such a function ψ is called *admissible*. In the sequel, we shall always be concerned with the case that the Hilbert space $\mathcal{H}$ is given as some L_2–space on a manifold $\mathcal{N}$, i.e.

$\mathcal{H} = L_2(\mathcal{N})$. Unfortunately, there are many cases of practical interest where no square integrable representation exists. Very often, these cases can be handled by restricting U to a convenient quotient $\mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is a closed subgroup of $\mathcal{G}$. Unless otherwise stated, we shall always consider right coset spaces, i.e.,

$$g_1 \sim g_2 \qquad \text{if and only if} \qquad g_1 = h \circ g_2 \qquad \text{for some} \qquad h \in \mathcal{P}.$$

Because U is not directly defined on $\mathcal{G}/\mathcal{P}$, it is necessary to embed $\mathcal{G}/\mathcal{P}$ in $\mathcal{G}$. This can be realized by using the canonical fiber bundle structure of $\mathcal{G}$ with projection $\Pi : \mathcal{G} \to \mathcal{G}/\mathcal{P}$. Let $\sigma : \mathcal{G}/\mathcal{P} \to \mathcal{G}$ be a Borel section of this fiber bundle, i.e., $\Pi \circ \sigma(h) = h$ for all $h \in \mathcal{G}/\mathcal{P}$. We introduce $U \circ \sigma$ and suppose that $\mathcal{G}/\mathcal{P}$ carries a $\mathcal{G}$–invariant measure μ. An attractive notation of square integrability on a homogeneous space appears in [1]. An irreducible representation U is *square integrable* mod $(\mathcal{P}, \sigma)$, if there exists a nonzero function $\psi \in L_2(\mathcal{N})$, called *admissible* (with respect to σ), such that

$$\int\limits_{\mathcal{G}/\mathcal{P}} |\langle f, U(\sigma(h)^{-1})\psi\rangle|^2 \, d\mu(h) < \infty \quad \text{for all } f \in L_2(\mathcal{N}),$$

i.e., the operator V_ψ given by

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi\rangle \tag{2.1}$$

maps $L_2(\mathcal{N})$ into $L_2(\mathcal{G}/\mathcal{P})$. Unless otherwise stated, in this paper $\langle \cdot, \cdot \rangle$ always denotes the L_2–inner product with respect to $L_2(\mathcal{G}/\mathcal{P})$ or $L_2(\mathcal{N})$, e.g.,

$$\langle F, G\rangle = \int_{\mathcal{N}} F(x)\overline{G(x)}dx$$

whenever the integral is defined. The admissibility condition can be rewritten as

$$0 < \int\limits_{\mathcal{G}/\mathcal{P}} |\langle f, U(\sigma(h)^{-1})\psi\rangle|^2 \, d\mu(h) = \langle f, A_\sigma f\rangle < \infty \quad \text{for all } f \in L_2(\mathcal{N}),$$

where A_σ is a positive, bounded, and invertible operator. If $A_\sigma = \lambda \mathcal{I}$ for some $\lambda > 0$, then U is called *strictly square integrable* mod $(\mathcal{P}, \sigma)$ and ψ *strictly admissible*. Moreover, we say that (ψ, σ) is a *strictly admissible pair* [16]. In order to keep the notation simple we focus our attention to strictly square integrable representations, where we normalize ψ so that $\lambda = 1$. Then $V_\psi : L_2(\mathcal{N}) \to L_2(\mathcal{G}/\mathcal{P})$ in (2.1) is an isometry.

Assume now that (ψ, σ) is a strictly admissible pair for our setting. Then the following facts are well–known [1]:

- The set $S_\sigma := \{U(\sigma(h)^{-1})\psi : h \in \mathcal{G}/\mathcal{P}\}$ is total in $L_2(\mathcal{N})$, i.e., $(S_\sigma)^\perp = \{0\}$.
- The map V_ψ is an isometry from $L_2(\mathcal{N})$ onto the reproducing kernel Hilbert space

$$\mathcal{M}_2 := \{F \in L_2(\mathcal{G}/\mathcal{P}) : \ \langle F(\cdot), R(h, \cdot)\rangle = F(h)\}$$

with reproducing kernel

$$R(h,l) \;=\; R_\psi(h,l) \;:=\; \langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1})\psi\rangle \tag{2.2}$$
$$= \langle \psi, U(\sigma(h)\sigma(l)^{-1})\psi\rangle$$
$$= V_\psi(U(\sigma(h)^{-1})\psi)(l). \tag{2.3}$$

In other words, the spaces $L_2(\mathcal{N})$ and $\mathcal{M}_2$ are isometrically isomorphic. In particular, $||f||_{L_2(\mathcal{N})} = ||V_\psi f||_{L_2(\mathcal{G}/\mathcal{P})}$. Note that $R(h,l) = \overline{R(l,h)}$. Further, we see by (2.3) that $R(h,\cdot) \in L_2(\mathcal{G}/\mathcal{P})$ for any fixed $h \in \mathcal{G}/\mathcal{P}$ and by applying Schwarz's inequality in (2.2) that $R \in L_\infty(\mathcal{G}/\mathcal{P} \times \mathcal{G}/\mathcal{P})$.

- The map V_ψ can be inverted on its image by its adjoint V_ψ^*, which is obviously given by

$$V_\psi^* F(s) := \int_{\mathcal{G}/\mathcal{P}} F(h) U(\sigma(h)^{-1})\psi(s)\, d\mu(h).$$

This provides us with the reconstruction formula

$$f = V_\psi^* V_\psi f = \int_{\mathcal{G}/\mathcal{P}} \langle f, U(\sigma(h)^{-1})\psi\rangle U(\sigma(h)^{-1})\psi \, d\mu(h) \tag{2.4}$$

for $f \in L_2(\mathcal{N})$.

3 Weighted Coorbit Spaces on Homogeneous Spaces

In this section we extend our considerations of functions belonging to coorbit spaces on manifolds, cf. [3], to the concept of weighted coorbit spaces. By this extention we are able to characterize a wide range of function spaces on manifolds, e.g., in dependence on the underlying group we may obtain general modulation and Besov spaces, respectively, or some mixed function spaces. In order to keep comparisons as simple as possible, we adapt the notations given in [3, 8, 9, 10, 11, 12].

Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{P},\sigma)$ with a strictly admissible function ψ. Furthermore, we introduce a positive, continuous weight function w on $\mathcal{G}$ which is in addition submultiplicative, i.e., $w(g\,\tilde{g}) \le w(g)\,w(\tilde{g})$ for all $g, \tilde{g} \in \mathcal{G}$, and uniformly bounded from below, i.e., $\inf_{g\in\mathcal{G}} w(g) \ge C_w > 0$. Associated with w we are concerned with the weighted L_p–spaces on $\mathcal{G}/\mathcal{P}$ defined for $1 \le p < \infty$ by

$$L_{p,w}(\mathcal{G}/\mathcal{P}) := \{f \text{ measurable on } \mathcal{G}/\mathcal{P} : \|f\|_{L_{p,w}(\mathcal{G}/\mathcal{P})} := \left(\int_{\mathcal{G}/\mathcal{P}} |f(h)|^p w(\sigma(h))^p d\mu(h)\right)^{1/p} < \infty\},$$

and for $p = \infty$ by

$$L_{\infty,w}(\mathcal{G}/\mathcal{P}) := \{f \text{ measurable on } \mathcal{G}/\mathcal{P} : \|f\|_{L_{\infty,w}(\mathcal{G}/\mathcal{P})} := \operatorname{ess}\sup_{h\in\mathcal{G}/\mathcal{P}} |f(h)| w(\sigma(h)) < \infty\}.$$

In the following we suppose the fundamental condition

$$\int_{\mathcal{G}/\mathcal{P}} |R(h,l)| w(\sigma(l))\, d\mu(l) \le C \tag{3.1}$$

with a constant C independent of h. This condition is equivalent with the assumption that the functions $V_\psi(U(\sigma(h)^{-1})\psi)$ are in $L_{1,w}$ with norm bounded independently of h. In addition to the kernel R we define a non–symmetric kernel $\tilde{R}$ by

$$\tilde{R}(h,l) := R(h,l)\frac{w(\sigma(h))}{w(\sigma(l))}. \tag{3.2}$$

Of course, (3.1) together with the lower boundedness of our weight function w implies that

$$\int_{\mathcal{G}/\mathcal{P}} |\tilde{R}(h,l)|\, d\mu(h) \le \frac{C}{C_w} \le C_\psi. \tag{3.3}$$

Moreover, we assume conversely that

$$\int_{\mathcal{G}/\mathcal{P}} |\tilde{R}(h,l)|\, d\mu(l) \le C_\psi \tag{3.4}$$

and finally that

$$\sup_{h,l\in\mathcal{G}} |\tilde{R}(h,l)| \le C_\psi. \tag{3.5}$$

These requirements replace the usual integrability conditions in the group case. In our setting, the general problem occurs that a group structure does no longer exist and therefore we need a substitute for the usual convolution operation. It seems to us that a powerful approach is to use the weighted Young inequality as presented in the appendix, Theorem 5.1. However, the application of this inequality requires exactly integrability conditions of the form (3.3) and (3.4).

The first problem is to provide a suitable large set that may serve as a reservoir of selection for the objects of our coorbit spaces. By $H'_{1,w}$ we denote the space of all continuous linear functionals on

$$H_{1,w} := \{f \in L_2(\mathcal{N}) :\ V_\psi f \in L_{1,w}(\mathcal{G}/\mathcal{P})\}.$$

As usual, the norm $\|\cdot\|_{H_{1,w}}$ on $H_{1,w}$ is defined as

$$\|f\|_{H_{1,w}} := \|V_\psi f\|_{L_{1,w}(\mathcal{G}/\mathcal{P})}.$$

By (3.1) we observe that the elements $U(\sigma(h)^{-1})\psi$ of our $L_2(\mathcal{N})$ total set are in $H_{1,w}$. Further, for $f \in H_{1,w}$, we have by the Schwarz inequality and since w is uniformly bounded from below that

$$\begin{aligned} \|f\|^2_{L_2(\mathcal{N})} &= ||V_\psi f||^2_{L_2(\mathcal{G}/\mathcal{P})} = \int\limits_{\mathcal{G}/\mathcal{P}} |\langle f, U(\sigma(h)^{-1})\psi\rangle| \, |V_\psi f(h)| \, d\mu(h) \\ &\leq ||f||_{L_2(\mathcal{N})} \, \|\psi\|_{L_2(\mathcal{N})} \int\limits_{\mathcal{G}/\mathcal{P}} |V_\psi f(h)| \, d\mu(h) \\ &\leq ||f||_{L_2(\mathcal{N})} \, \|\psi\|_{L_2(\mathcal{N})} \, ||V_\psi f||_{L_{1,w}(\mathcal{G}/\mathcal{P})} C_w^{-1} \end{aligned}$$

which implies the following dense continuous embeddings

$$H_{1,w} \hookrightarrow L_2(\mathcal{N}) \hookrightarrow H'_{1,w}. \tag{3.6}$$

Since $U(\sigma(h)^{-1})\psi \in H_{1,w}$ for every $h \in \mathcal{G}/\mathcal{P}$, the following generalization of the operator V_ψ in (2.1) on $H'_{1,w}$ is well–defined

$$V_\psi f := \langle f, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w} \times H_{1,w}} \tag{3.7}$$

where $f \in H'_{1,w}$. For any $f \in H'_{1,w}$, we obtain by (3.3) that

$$\begin{aligned} \|V_\psi f\|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} &= \|\langle f, U(\sigma(h)^{-1})\psi\rangle\|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \\ &\leq \|f\|_{H'_{1,w}} \operatorname*{ess\,sup}_{h\in\mathcal{G}/\mathcal{P}} \frac{1}{w(\sigma(h))} \|U(\sigma(h)^{-1})\psi\|_{H_{1,w}} \\ &\leq \|f\|_{H'_{1,w}} \operatorname*{ess\,sup}_{h\in\mathcal{G}/\mathcal{P}} \frac{1}{w(\sigma(h))} \, C_\psi \, w(\sigma(h)) \\ &\leq C_\psi \, \|f\|_{H'_{1,w}} \ . \end{aligned} \tag{3.8}$$

Thus, $V_\psi : H'_{1,w} \to L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$ is a bounded operator. For $F \in L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$, we define $\tilde{V}_\psi F$ by

$$\begin{aligned} \langle \tilde{V}_\psi F, g\rangle_{H'_{1,w}\times H_{1,w}} := \langle F, V_\psi g\rangle &= \int\limits_{\mathcal{G}/\mathcal{P}} F(l) \overline{V_\psi g(l)} \, d\mu(l) \\ &= \int\limits_{\mathcal{G}/\mathcal{P}} F(l) \overline{\langle g, U(\sigma(l)^{-1})\psi\rangle} \, d\mu(l) \end{aligned}$$

for all $g \in H_{1,w}$. It is easy to check that $\tilde{V}_\psi : L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P}) \to H'_{1,w}$ is also a bounded operator. Now we obtain for $F \in L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$ that

$$\begin{aligned} V_\psi \tilde{V}_\psi F &= \langle \tilde{V}_\psi F, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}} = \langle F, V_\psi(U(\sigma(h)^{-1})\psi)\rangle \\ &= \langle F, R(h,\cdot)\rangle \ . \end{aligned} \tag{3.9}$$

Similar to the definition of coorbit spaces in [3] we define *weighted coorbit spaces* by

$$M_{p,w} := \{f \in H'_{1,w} : \ V_\psi f \in L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})\},$$

with $1 \le p \le \infty$ and norm

$$\|f\|_{M_{p,w}} := \|V_\psi f\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}.$$

As we shall see in the following , the choice of $L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$ is natural. It is straightforward that $\|\cdot\|_{M_{p,w}}$ defines a seminorm. The property that $\|f\|_{M_{p,w}} = 0, i.e., V_\psi f = 0$, implies $f = 0$ follows similarly as in [9] since $\{U(\sigma(h)^{-1})\psi : h \in \mathcal{G}/\mathcal{P}\}$ is a dense subset of $H_{1,w}$ and since w is positive. The basic step for the investigations outlined below is a correspondence principle between these weighted coorbit spaces and certain subspaces on the quotient group $\mathcal{G}/\mathcal{P}$ which are defined by means of the reproducing kernel R. To this end, we consider the subspaces

$$\mathcal{M}_{p,w} := \{F \in L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P}) : \ \langle F, R(h,\cdot)\rangle = F(h)\}$$

of $L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$ with $1 \le p \le \infty$. Then the desired correspondence principle can be formulated as follows:

Theorem 3.1 *Let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{P},\sigma)$ and ψ a strictly admissible function. Let V_ψ be defined by (3.7) and let R in (2.2) fulfill (3.3) and (3.5).*

i) For every $f \in M_{p,w}$, the following equation is satisfied

$$\langle V_\psi f, R(h,\cdot)\rangle = V_\psi f(h)\,,$$

i.e., $V_\psi f \in \mathcal{M}_{p,w}$.

ii) For every $F \in \mathcal{M}_{p,w}$, $1 \le p \le \infty$, there exists a uniquely determined functional $f \in M_{p,w}$ such that $F = V_\psi f$.

Consequently, the spaces $M_{p,w}$ and $\mathcal{M}_{p,w}$, $1 \le p \le \infty$, are isometrically isomorphic.

Proof: *i)* Since $U(\sigma(h)^{-1})\psi \in L_2(\mathcal{N})$ we have by (2.4) that

$$\begin{aligned} V_\psi f(h) &= \langle f, U(\sigma(h)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}} \\ &= \langle f, \int\limits_{\mathcal{G}/\mathcal{P}} R(h,l)U(\sigma(l)^{-1})\psi\, d\mu(l)\rangle_{H'_{1,w}\times H_{1,w}}. \end{aligned}$$

By (3.1), the Fubini theorem and (3.6) we can change the order of integration and get

$$\begin{aligned} V_\psi f(h) &= \int\limits_{\mathcal{G}/\mathcal{P}} \overline{R(h,l)}\langle f, U(\sigma(l)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}}\, d\mu(l) \\ &= \langle V_\psi f, R(h,\cdot)\rangle\ . \end{aligned}$$

ii). For $F \in \mathcal{M}_{p,w}$, $1 \le p \le \infty$, we have that

$$\begin{aligned}
\|F\|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} &= \| \int\limits_{\mathcal{G}/\mathcal{P}} F(l)\overline{R(h,l)}\, d\mu(l) \|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \\
&= \operatorname*{ess\ sup}_{h\in\mathcal{G}/\mathcal{P}} | \int\limits_{\mathcal{G}/\mathcal{P}} F(l)\overline{R(h,l)}\, d\mu(l) | \frac{1}{w(\sigma(h))},
\end{aligned}$$

and further, by applying Hölder's inequality with $1/p + 1/q = 1$, the assumptions (3.3) and (3.5),

$$\begin{aligned}
| \int\limits_{\mathcal{G}/\mathcal{P}} F(l)\overline{R(h,l)}\, d\mu(l) | &\le \int\limits_{\mathcal{G}/\mathcal{P}} |F(l)| \frac{1}{w(\sigma(l))} (|R(h,l)| w(\sigma(l)))^{1/p+1/q}\, d\mu(l) \\
&\le \left(\int\limits_{\mathcal{G}/\mathcal{P}} |F(l)|^p \frac{1}{w^p(\sigma(l))} |R(h,l)| w(\sigma(l)) d\mu(l) \right)^{1/p} \times \\
&\quad \left(\int\limits_{\mathcal{G}/\mathcal{P}} |R(h,l)| w(\sigma(l)) d\mu(l) \right)^{1/q} \\
&\le C_\psi \ \|F\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} w(\sigma(h)) \ .
\end{aligned}$$

Consequently, we have that

$$\|F\|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \le C_\psi \|F\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \ .$$

Thus, $F \in L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$ and by (3.9) we obtain that $F = V_\psi(\tilde{V}_\psi F)$, where $\tilde{V}_\psi F \in H'_{1,w}$ and since $F \in L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$ also $\tilde{V}_\psi F \in M_{p,w}$. The uniqueness condition follows by definition of $M_{p,w}$. ■

Applying Theorem 3.1 *i)* and (3.9) we get for $f \in H'_{1,w}$ that

$$V_\psi \tilde{V}_\psi (V_\psi f) = \langle V_\psi f, R(h,\cdot) \rangle = V_\psi f \ .$$

Hence, $\tilde{V}_\psi V_\psi$ is the identity in $H'_{1,w}$ and we have the reconstruction formula

$$f = \tilde{V}_\psi V_\psi f = \int\limits_{\mathcal{G}/\mathcal{P}} \langle f, U(\sigma(h)^{-1})\psi \rangle U(\sigma(h)^{-1})\psi\, d\mu(h) \ .$$

We finish this section by establishing the relationship

$$M_{\infty,w} = H'_{1,w}.$$

This can be seen as follows: For $f \in H'_{1,w}$ we have by (3.8) that $\|V_\psi f\|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \leq C_\psi ||f||_{H'_{1,w}}$. Conversely, we have for $f \in M_{\infty,w}$

$$
\begin{aligned}
\|f\|_{H'_{1,w}} &= \sup_{\|g\|_{H_{1,w}}=1} |\langle f, g\rangle_{H'_{1,w}\times H_{1,w}}| = \sup_{\|g\|_{H_{1,w}}=1} |\langle \tilde{V}_\psi V_\psi f, g\rangle_{H'_{1,w}\times H_{1,w}}| \\
&= \sup_{\|g\|_{H_{1,w}}=1} |\langle V_\psi f, V_\psi g\rangle| \leq \|V_\psi f\|_{L_{\infty,\frac{1}{w}}} .
\end{aligned}
$$

4 Atomic Decompositions and Banach Frames for Weighted Coorbit Spaces

Once our generalized coorbit spaces are established, the next step is to derive some atomic decompositions for these spaces and to construct suitable Banach frames. This program is performed in several steps. In the next subsection, we present some preparations and state our main results. The remaining two subsections are devoted to the building blocks which are necessary to prove these results. The major step is the construction of suitable approximation operators which are defined and analyzed in Subsection 4.2.

The results in this section are again inspired by the pioneering work of Feichtinger and Gröchenig, [9, 10, 11, 12]. Furthermore, they are a generalization of [3].

4.1 Setting and Main Results

Before we can state and prove our main results, some preparations are necessary. Given some neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$, a family $X = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called $\mathcal{U}$*–dense* if $\bigcup_{i\in\mathcal{I}} \mathcal{U}x_i = \mathcal{G}$. A family $X = (x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ is called *relatively separated,* if for any compact set $\mathcal{Q} \subseteq \mathcal{G}$ there exists a finite partition of the index set $\mathcal{I}$, i.e., $\mathcal{I} = \bigcup_{r=1}^{r_0} \mathcal{I}_r$, such that $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for all $i, j \in \mathcal{I}_r$ with $i \neq j$. Note that these technical conditions can be easily fulfilled by some families X in all the settings we are interested in.

Let $\mathcal{U}$ be an arbitrary compact neighborhood of the identity in $\mathcal{G}$. By [7], there exists a bounded uniform partition of unity (of size $\mathcal{U}$), i.e., a family of continuous functions $(\varphi_i)_{i\in\mathcal{I}}$ on $\mathcal{G}$ such that

- $0 \leq \varphi_i(g) \leq 1$ for all $g \in \mathcal{G}$;
- there is a $\mathcal{U}$–dense, relatively separated family $(x_i)_{i\in\mathcal{I}}$ in $\mathcal{G}$ such that supp $\varphi_i \subseteq \mathcal{U}x_i$;
- $\sum_{i\in\mathcal{I}} \varphi_i(g) \equiv 1$ for all $g \in \mathcal{G}$.

Furthermore, we define the (left and right) $\mathcal{U}$*–oscillation* with respect to the analyzing wavelet ψ as

$$
\begin{aligned}
\mathrm{osc}^l_{\mathcal{U}}(l, h) &:= \sup_{u\in\mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u^{-1}\sigma(l)\sigma(h)^{-1})\psi\rangle|, \\
\mathrm{osc}^r_{\mathcal{U}}(l, h) &:= \sup_{u\in\mathcal{U}} |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(\sigma(l)\sigma(h)^{-1}u)\psi\rangle|.
\end{aligned}
$$

In analogy to (3.2) we need for applying the weighted Young inequality the *w–modified $\mathcal{U}$–oscillations*

$$\begin{aligned} \mathrm{osc}^l_{\mathcal{U},w}(l,h) &:= \mathrm{osc}^l_{\mathcal{U}}(l,h)\frac{w(\sigma(l))}{w(\sigma(h))}\ , \\ \mathrm{osc}^r_{\mathcal{U},w}(l,h) &:= \mathrm{osc}^r_{\mathcal{U}}(l,h)\frac{w(\sigma(l))}{w(\sigma(h))}\ . \end{aligned}$$

In the sequel, we shall always assume that $(x_i)_{i\in\mathcal{I}}$ can be chosen such that $\sigma(\mathcal{G}/\mathcal{P})\cap\mathcal{U}x_i \neq \emptyset$ implies $x_i \in \sigma(\mathcal{G}/\mathcal{P})$. Let

$$\mathcal{I}_\sigma := \{i \in \mathcal{I} :\ \sigma(\mathcal{G}/\mathcal{P}) \cap \mathcal{U}x_i \neq \emptyset\}\ .$$

Then there exist $h_i \in \mathcal{G}/\mathcal{P}$ such that $x_i = \sigma(h_i)$, where $i \in \mathcal{I}_\sigma$. Note that

$$\sum_{i\in\mathcal{I}_\sigma} \varphi_i(\sigma(h)) = 1\ ,$$

where $h \in \mathcal{G}/\mathcal{P}$.

In this setting, we can formulate our main theorems which we shall prove in the following subsections. The first one is a decomposition theorem which says that discretizing the representation $U(\sigma(\cdot)^{-1})$ by means of a $\mathcal{U}$–dense set indeed produces an atomic decomposition of $M_{p,w}$.

Theorem 4.1 *Let $\mathcal{G}$ be a separable Lie group with closed subgroup $\mathcal{P}$, w a weight function and let μ be an invariant measure on $\mathcal{G}/\mathcal{P}$. Further, let U be a strictly square integrable representation of $\mathcal{G}$ mod $(\mathcal{P},\sigma)$ in $L_2(\mathcal{N})$ with strictly admissible function ψ. Let a compact neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ be chosen such that*

$$\int_{\mathcal{G}/\mathcal{P}} \mathrm{osc}^l_{\mathcal{U},w}(l,h)d\mu(l) \leq \gamma \quad \textit{and} \quad \int_{\mathcal{G}/\mathcal{P}} \mathrm{osc}^l_{\mathcal{U},w}(l,h)d\mu(h) \leq \gamma\,, \tag{4.1}$$

where $\gamma < 1$. Let $X = (x_i)_{i\in\mathcal{I}}$ be a $\mathcal{U}$–dense and relatively separated family. Furthermore, suppose that for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$

$$\mu\{h \in \mathcal{G}/\mathcal{P} : \sigma(h) \in \mathcal{Q}\sigma(h_i)\} \geq C_\mathcal{Q} > 0 \tag{4.2}$$

holds for all $i \in \mathcal{I}_\sigma$. Finally, let us assume that for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$ our analyzing function ψ fulfills the following inequality

$$\int_{\mathcal{G}/\mathcal{P}} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \frac{w(q^{-1}\sigma(l))}{w(\sigma(h))}\, d\mu(l) \leq \tilde{C}_\mathcal{Q} \tag{4.3}$$

with a constant $\tilde{C}_\mathcal{Q} < \infty$ independent of $h \in \mathcal{G}/\mathcal{P}$. Then $M_{p,w}$, $1 \leq p \leq \infty$, has the following atomic decomposition: if $f \in M_{p,w}$, $1 \leq p \leq \infty$, then f can be represented as

$$f = \sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi,$$

where the sequence of coefficients $(c_i)_{i\in I_\sigma} = (c_i(f))_{i\in I_\sigma} \in \ell_{p,\frac{1}{w}}$ *depends linearly on* f *and satisfies*

$$||(c_i)_{i\in I_\sigma}||_{\ell_{p,\frac{1}{w}}} \leq A||f||_{M_{p,w}}. \tag{4.4}$$

If $(c_i)_{i\in I_\sigma} \in \ell_{p,\frac{1}{w}}$*, then* $f = \sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi$ *is contained in* $M_{p,w}$ *and*

$$||f||_{M_{p,w}} \leq B||(c_i)_{i\in I_\sigma}||_{\ell_{p,\frac{1}{w}}}. \tag{4.5}$$

Here we use $w = (w(x_i))_{i\in I_\sigma}$ as discretized weight sequence and

$$\ell_{p,w} := \{c = (c_i)_{i\in I_\sigma} : \|c\|_{\ell_{p,w}} := \|c\,w\|_{\ell_p} < \infty\}$$

for $1 \leq p \leq \infty$.

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by the moments or coefficients $\langle f, U(\sigma(h_i)^{-1})\psi\rangle_{H'_{1,w}\times H_{1,w}}$ and how f can be reconstructed from these coefficients. This question is answered by the following theorem which shows that our generalized coherent states indeed give rise to Banach frames.

Theorem 4.2 *Impose the same assumptions as in Theorem 4.1 with*

$$\int_{\mathcal{G}/\mathcal{P}} \mathrm{osc}^r_{\mathcal{U},w}(l,h)d\mu(l) \leq \frac{\tilde{\gamma}}{C_\psi} \quad \text{and} \quad \int_{\mathcal{G}/\mathcal{P}} \mathrm{osc}^r_{\mathcal{U},w}(l,h)d\mu(h) \leq \frac{\tilde{\gamma}}{C_\psi}, \tag{4.6}$$

where $\tilde{\gamma} < 1$*, instead of* (4.1) *and with*

$$\int_{\mathcal{G}/\mathcal{P}} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \frac{w(\sigma(h))}{w(q^{-1}\sigma(l))}\, d\mu(l) \leq \tilde{C}_\mathcal{Q} \tag{4.7}$$

where $\tilde{C}_\mathcal{Q} < \infty$ *is a constant independent of* $h \in \mathcal{G}/\mathcal{P}$*, instead of* (4.3)*. Then the set*

$$\{\psi_i := U(\sigma(h_i)^{-1})\psi : i \in \mathcal{I}_\sigma\}$$

is a Banach frame for $M_{p,w}$*. This means that*

(i) $f \in M_{p,w}$ *if and only if* $(\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma} \in \ell_{p,\frac{1}{w}}$*;*

(ii) there exist two constants $0 < A' \leq B' < \infty$ *such that*

$$A'\,\|f\|_{M_{p,w}} \leq \|(\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} \leq B'\,\|f\|_{M_{p,w}}; \tag{4.8}$$

(iii) there exists a bounded, linear reconstruction operator $\mathcal{S}$ *from* $\ell_{p,\frac{1}{w}}$ *to* $M_{p,w}$ *such that* $\mathcal{S}\left((\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}})_{i\in\mathcal{I}_\sigma}\right) = f$.

For further information concerning Banach frames see [14].

4.2 Approximation Operators

In this section, we examine two different approximation operators on $\mathcal{M}_{p,w}$. We use the results to construct expansions for the spaces $\mathcal{M}_{p,w}$, which then, by the correspondence principle in Theorem 3.1, lead to expansions for the coorbit spaces $M_{p,w}$.

We consider the following approximation operators on $\mathcal{M}_{p,w}$:

$$\begin{aligned} T_\varphi F(h) &:= \sum_{i\in\mathcal{I}_\sigma} \langle F, \varphi_i \circ \sigma \rangle \, R(h_i, h) \\ &= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{P}} F(l)\varphi_i(\sigma(l))\, d\mu(l) R(h_i, h), \\ S_\varphi F(h) &:= \sum_{i\in\mathcal{I}_\sigma} F(h_i) \langle \varphi_i \circ \sigma, R(h, \cdot)\rangle \\ &= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{P}} F(h_i)\varphi_i(\sigma(l)) R(l,h)\, d\mu(l). \end{aligned}$$

So far, it is not clear a prior whether these formal expressions make sense at all and on which spaces they are bounded operators. This will be clarified in Theorem 4.3 below. Another remark is required on the meaning of the sum over I_σ. We order the finite subsets of I_σ by inclusion, then $\sum_{i\in\mathcal{I}_\sigma} \ldots$ will be understood as the limit of the partial sums over finite subsets of I_σ.

The first step is to establish the invertibility of the operators T_φ and S_φ.

Theorem 4.3 *i) If the conditions* (4.1) *are fulfilled, then the operator* $T_\varphi : \mathcal{M}_{p,w} \to \mathcal{M}_{p,w}$ *is bounded with bounded inverse.*
ii) If the conditions (4.6) *are fulfilled, then the operator* $S_\varphi : \mathcal{M}_{p,w} \to \mathcal{M}_{p,w}$ *is bounded with bounded inverse.*

Proof: By definition of $\mathcal{M}_{p,w}$, we have for $F \in \mathcal{M}_{p,w}$ that

$$\begin{aligned} F(h) &= \langle F, R(h,\cdot)\rangle = \int_{\mathcal{G}/\mathcal{P}} F(l)\overline{R(h,l)}\, d\mu(l) \\ &= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{P}} F(l)\varphi_i(\sigma(l)) R(l,h)\, d\mu(l) \end{aligned}$$

and consequently

$$\begin{aligned} F(h) - T_\varphi F(h) &= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{P}} F(l)\varphi_i(\sigma(l)) \left[R(l,h) - R(h_i,h)\right] \, d\mu(l), \\ F(h) - S_\varphi F(h) &= \sum_{i\in\mathcal{I}_\sigma} \int_{\mathcal{G}/\mathcal{P}} [F(l) - F(h_i)]\varphi_i(\sigma(l)) R(l,h)\, d\mu(l). \end{aligned} \tag{4.9}$$

Let us first consider $F - T_\varphi F$. By the definition of R we obtain

$$\begin{aligned}
|F(h) - T_\varphi F(h)| &\leq \sum_{i\in\mathcal{I}_\sigma} \int\limits_{\mathcal{G}/\mathcal{P}} |F(l)|\varphi_i(\sigma(l))|R(l,h) - R(h_i,h)|\, d\mu(l) \\
&= \sum_{i\in\mathcal{I}_\sigma} \int\limits_{\mathcal{G}/\mathcal{P}} |F(l)|\varphi_i(\sigma(l)) \times \\
&\qquad |\langle \psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(\sigma(h_i)\sigma(h)^{-1})\psi\rangle|\, d\mu(l).
\end{aligned}$$

Now $\sigma(l) \in \mathcal{U}x_i$ implies that there exists $u \in \mathcal{U}$ such that $\sigma(l) = u\, x_i = u\, \sigma(h_i)$. Thus $\sigma(h_i) = u^{-1}\sigma(l)$ and we get

$$|F(h) - T_\varphi F(h)| \leq \sum_{i\in\mathcal{I}_\sigma} \int\limits_{\mathcal{G}/\mathcal{P}} |F(l)|\varphi_i(\sigma(l))\, \mathrm{osc}^l_{\mathcal{U}}(l,h)\, d\mu(l) = \int\limits_{\mathcal{G}/\mathcal{P}} |F(l)|\, \mathrm{osc}^l_{\mathcal{U}}(l,h)\, d\mu(l).$$

By recalling the assumptions (4.1) and applying the weighted Young inequality (see appendix), we obtain

$$\|F - T_\varphi F\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} = \|(I - T_\varphi)F\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \leq \gamma\, \|F\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}.$$

Consequently $|||I - T_\varphi||| < 1$, i.e., $I - T_\varphi$ is a contraction on $\mathcal{M}_{p,w}$. Thus, regarding that $|||T_\varphi||| \leq |||T_\varphi - I||| + |||I|||$, we see that T_φ is a bounded operator with bounded inverse.

Next we consider $F - S_\varphi F$. Since $F \in \mathcal{M}_{p,w}$ and by the definition of R we obtain

$$\begin{aligned}
|F(l) - F(h_i)| &\leq \int\limits_{\mathcal{G}/\mathcal{P}} |F(g)||R(g,l) - R(g,h_i)|\, d\mu(g) \\
&= \int\limits_{\mathcal{G}/\mathcal{P}} |F(g)||\langle \psi, U(\sigma(g)\sigma(l)^{-1})\psi - U(\sigma(g)\sigma(h_i)^{-1})\psi\rangle|\, d\mu(g).
\end{aligned}$$

By (4.9) we are only interested in $l \in \mathcal{G}/\mathcal{P}$ with $\sigma(l) \in \mathcal{U}x_i$, i.e., $\sigma(l) = u\,\sigma(h_i)$ for some $u \in \mathcal{U}$ and $\sigma(h_i)^{-1} = \sigma(l)^{-1}u$. Thus

$$|F(l) - F(h_i)| \leq \int\limits_{\mathcal{G}/\mathcal{P}} |F(g)|\, \mathrm{osc}^r_{\mathcal{U}}(g,l)\, d\mu(g)$$

and since (φ_i) is a partition of unity

$$\sum_{i\in\mathcal{I}_\sigma} |F(l) - F(h_i)|\varphi_i(\sigma(l)) \leq \int\limits_{\mathcal{G}/\mathcal{P}} |F(g)|\, \mathrm{osc}^r_{\mathcal{U}}(g,l)\, d\mu(g).$$

By the weighted Young inequality and (4.6) this implies

$$||\sum_{i\in\mathcal{I}_\sigma}|F(l)-F(h_i)|\varphi_i(\sigma(l))||_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}\leq\frac{\tilde{\gamma}}{C_\psi}\,||F||_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}.$$

It is easy to check that the order of summation and integration in (4.9) can be changed. Then we obtain by (4.9), (3.3) – (3.4) and the weighted Young inequality

$$\|F-S_\varphi F\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}\leq C_\psi\,||\sum_{i\in\mathcal{I}_\sigma}|F(l)-F(h_i)|\varphi_i(\sigma(l))||_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}\leq\tilde{\gamma}||F||_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}.$$

Consequently, $I-S_\varphi$ is a contraction on $\mathcal{M}_{p,w}$ and S_φ is a bounded operator with bounded inverse on $\mathcal{M}_{p,w}$. ■

Using the correspondence principle we can derive the following representation of functions from our coorbit spaces.

Corollary 4.1 *Any function $f\in M_{p,w}$ can be decomposed as*

$$f=\sum_{i\in\mathcal{I}_\sigma}c_i\,U(\sigma(h_i)^{-1})\psi\ , \tag{4.10}$$

where

$$c_i=c_i(f):=\langle T_\varphi^{-1}F,\varphi_i\circ\sigma\rangle$$

and $F:=V_\psi f$.

Proof: By Theorem 3.1 *i)* and Theorem 4.3 *i)* we have that

$$V_\psi f(h)=F(h)=T_\varphi T_\varphi^{-1}F(h)=\sum_{i\in\mathcal{I}_\sigma}\langle T_\varphi^{-1}F,\varphi_i\circ\sigma\rangle R(h_i,h).$$

Since $\tilde{V}_\psi V_\psi$ is the identity on $H'_{1,w}$ and $\tilde{V}_\psi$ is bounded on $L_{\infty,\frac{1}{w}}$, we obtain

$$f=\tilde{V}_\psi V_\psi f=\sum_{i\in\mathcal{I}_\sigma}\langle T_\varphi^{-1}F,\varphi_i\circ\sigma\rangle\tilde{V}_\psi(R(h_i,\cdot)). \tag{4.11}$$

Now, for any $g\in H_{1,w}$,

$$\begin{aligned}\langle\tilde{V}_\psi(R(h_i,\cdot)),g(\cdot)\rangle_{H'_{1,w}\times H_{1,w}} &= \langle R(h_i,\cdot),V_\psi g(\cdot)\rangle\ =\ \overline{V_\psi g(h_i)}\\ &= \langle U(\sigma(h_i)^{-1})\psi,g\rangle_{H'_{1,w}\times H_{1,w}}\end{aligned}$$

so that $\tilde{V}_\psi(R(h_i,\cdot))=U(\sigma(h_i)^{-1})\psi$. Together with (4.11) this yields the assertion. ■

Moreover, the operator S_φ induces the reconstruction operator as stated in Theorem 4.2 iii).

Corollary 4.2 *Any function $f \in M_{p,w}$ can be reconstructed as*

$$f = \sum_{i \in \mathcal{I}_\sigma} \langle f, U(\sigma(h_i)^{-1}\psi\rangle_{H'_{1,w} \times H_{1,w}} e_i,$$

where

$$e_i = \tilde{V}_\psi(E_i), \quad E_i := S_\varphi^{-1}(\langle \varphi_i \circ \sigma, R(h, \cdot)\rangle).$$

Proof: Since S_φ has a continuous inverse, we obtain for $F := V_\psi f \in \mathcal{M}_{p,w}$ that

$$\begin{aligned} F(h) &= S_\varphi^{-1} S_\varphi F(h) \\ &= \sum_{i \in \mathcal{I}_\sigma} F(h_i) S_\varphi^{-1} \langle \varphi_i \circ \sigma, R(h, \cdot)\rangle = \sum_{i \in \mathcal{I}_\sigma} F(h_i) E_i. \end{aligned}$$

Now the correspondence principle and the continuity of $\tilde{V}_\psi$ on $L_{\infty, \frac{1}{w}}$ implies

$$\begin{aligned} f &= \tilde{V}_\psi V_\psi f = \tilde{V}_\psi \left(\sum_{i \in \mathcal{I}_\sigma} V_\psi(f)(h_i) E_i \right) \\ &= \sum_{i \in \mathcal{I}_\sigma} \langle f, U(\sigma(h_i)^{-1}\psi\rangle_{H'_{1,w} \times H_{1,w}} \tilde{V}_\psi(E_i) = \sum_{i \in \mathcal{I}_\sigma} \langle f, U(\sigma(h_i)^{-1}\psi\rangle_{H'_{1,w} \times H_{1,w}} e_i. \qquad \blacksquare \end{aligned}$$

4.3 Frame Bounds

In this section, we want to prove the norm equivalences in Theorem 4.1 and 4.2. For the verification that the infinite sums appearing in the following lemmatas converge (unconditionally) in $\mathcal{M}_{p,w}$, respectively $M_{p,w}$, it suffices to obtain for $p < \infty$ the estimates for finite sequences. Then all the estimates can be extended in the usual way, see again [9, 10, 11] for details. Only the case $p = \infty$ requires some additional effort. The necessary modifications are left to the reader.

In the following, 'C' always denotes a generic constant which is independent of all the other parameters under consideration, but whose concrete value may be different in each particular estimate.

We start with Theorem 4.1, relation (4.4).

Lemma 4.1 *Suppose that the conditions in Theorem 4.1 are satisfied. For any $f \in M_{p,w}$ let*

$$(c_i)_{i \in \mathcal{I}_\sigma} := (\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle)_{i \in \mathcal{I}_\sigma}.$$

Then there exists a constant $A < \infty$ such that the following inequality holds:

$$\|(c_i)_{i \in \mathcal{I}_\sigma}\|_{\ell_{p, \frac{1}{w}}} \leq A \|f\|_{M_{p,w}} .$$

In particular, we have that $(c_i)_{i \in \mathcal{I}_\sigma} \in \ell_{p, \frac{1}{w}}$.

Proof: 1. First we show that for any sequence $(\eta_i)_{i\in\mathcal{I}_\sigma}$ the inequality

$$\|(\eta_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} \leq C\,\|\sum_{i\in\mathcal{I}_\sigma}|\eta_i|1_{\mathcal{U}x_i}\circ\sigma\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \tag{4.12}$$

holds, where again $x_i=\sigma(h_i)$ and where $1_{\mathcal{U}x_i}$ denotes the characteristic function of $\mathcal{U}x_i$.

Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, there exists a splitting $\mathcal{I}=\bigcup_{r=1}^{r_0}\mathcal{I}_r$ such that $\mathcal{U}x_i\cap\mathcal{U}x_j=\emptyset$ for $i,j\in\mathcal{I}_r$ and $i\neq j$. This results in a decomposition $\mathcal{I}_\sigma=\bigcup_{r=1}^{r_0}\mathcal{I}_{\sigma r}$, where

$$\mathcal{I}_{\sigma r}=\{i\in\mathcal{I}_r:\ \mathcal{U}x_i\cap\sigma(\mathcal{G}/\mathcal{P})\neq\emptyset\}\ .$$

Then we obtain

$$\begin{aligned}
\|\sum_{i\in\mathcal{I}_\sigma}|\eta_i|1_{\mathcal{U}x_i}\circ\sigma\|^p_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} &= \int_{\mathcal{G}/\mathcal{P}}\left(\sum_{r=1}^{r_0}\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|1_{\mathcal{U}x_i}(\sigma(h))\frac{1}{w(\sigma(h))}\right)^p d\mu(h)\\
&\geq \sum_{r=1}^{r_0}\int_{\mathcal{G}/\mathcal{P}}\left(\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|1_{\mathcal{U}x_i}(\sigma(h))\frac{1}{w(\sigma(h))}\right)^p d\mu(h)\\
&= \sum_{r=1}^{r_0}\int_{\mathcal{G}/\mathcal{P}}\sum_{i\in\mathcal{I}_{\sigma r}}|\eta_i|^p1_{\mathcal{U}x_i}(\sigma(h))\frac{1}{w^p(\sigma(h))}d\mu(h).
\end{aligned}$$

Moreover, since $w(\sigma(h))\leq w(u)w(x_i)$ for $\sigma(h)\in\mathcal{U}x_i$, we can conlude from (4.2) that

$$\|\sum_{i\in\mathcal{I}_\sigma}|\eta_i|1_{\mathcal{U}x_i}\circ\sigma\|^p_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}\ \geq\ (\max_{u\in\mathcal{U}}w(u))^{-1}\,C_\mathcal{U}\sum_{i\in\mathcal{I}_\sigma}\frac{|\eta_i|^p}{w^p(x_i)}$$

which implies (4.12) by continuity of w and since $\mathcal{U}$ is compact.
2. Let $F\in L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})$. Then the application of (4.12) yields

$$\begin{aligned}
\|(\langle F,\varphi_i\circ\sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} &\leq \|(\langle|F|,\varphi_i\circ\sigma\rangle)_{i\in\mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}}\\
&\leq C\,\|\sum_{i\in\mathcal{I}_\sigma}\langle|F|,\varphi_i\circ\sigma\rangle 1_{\mathcal{U}x_i}\circ\sigma\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}\ .
\end{aligned}$$

Further, we see for an arbitrary fixed $h\in\mathcal{G}/\mathcal{P}$ that

$$\sum_{i\in\mathcal{I}_\sigma}\langle|F|,\varphi_i\circ\sigma\rangle 1_{\mathcal{U}x_i}(\sigma(h))=\sum_{i\in\mathcal{I}_h}\langle|F|,\varphi_i\circ\sigma\rangle\ ,$$

where $\mathcal{I}_h:=\{i\in\mathcal{I}_\sigma:\ x_i\in\mathcal{U}^{-1}\sigma(h)\}$, and

$$\sum_{i\in\mathcal{I}_h}\langle|F|,\varphi_i\circ\sigma\rangle\ =\ \sum_{i\in\mathcal{I}_h}\langle|F|,\varphi_i(\sigma(\cdot))\rangle\leq\langle|F|,1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(\cdot)\sigma(h)^{-1})\rangle\ .$$

Now $\sigma(l)\sigma(h)^{-1} \in \mathcal{U}\mathcal{U}^{-1}$ means that there exist some $u_1, u_2 \in \mathcal{U}$ depending on h, l such that $\sigma(l)\sigma(h)^{-1} = u_1 u_2^{-1}$. Then the submultiplicativity of our weight function implies that

$$w(\sigma(l)) = w(u_1 u_2^{-1}\sigma(h)) \leq w(u_1 u_2^{-1})\, w(\sigma(h)),$$

and since $\mathcal{U}\mathcal{U}^{-1}$ is compact and w continuous

$$\frac{w(\sigma(l))}{w(\sigma(h))} \leq C$$

with a constant independent of h and l. Consequently, since

$$\int_{\mathcal{G}/\mathcal{P}} 1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(l)\sigma(h)^{-1}) \frac{w(\sigma(l))}{w(\sigma(h))}\, d\mu(l) \;\leq\; C \int_{\mathcal{G}/\mathcal{P}} 1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(l)\sigma(h)^{-1})\, d\mu(l) \;\leq\; C, \quad (4.13)$$

for all $h \in \mathcal{G}/\mathcal{P}$ and similarly for the integration with respect to $d\mu(h)$ for all $l \in \mathcal{G}/\mathcal{P}$, we obtain by the weighted Young inequality, compare again with the appendix, Theorem 5.1, where $K(l,h) := 1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(l)\sigma(h)^{-1})$, that

$$\begin{aligned} \|(\langle F, \varphi_i \circ \sigma\rangle)_{i \in I_\sigma}\|_{\ell_{p,\frac{1}{w}}} &\leq C \|\langle |F|, 1_{\mathcal{U}\mathcal{U}^{-1}}(\sigma(\cdot)\sigma(h)^{-1})\rangle\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \\ &\leq C \|F\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \,. \end{aligned}$$

3. Finally, we conclude by the correspondence principle and by using $F = T_\varphi^{-1} V_\psi f \in \mathcal{M}_{p,w}$ in the above inequality that

$$\begin{aligned} \|(\langle T_\varphi^{-1} V_\psi f, \varphi_i \circ \sigma\rangle)_{i \in \mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} &\leq C\, \|T_\varphi^{-1} V_\psi f\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \\ &\leq C\, |||T_\varphi^{-1}|||\; \|V_\psi f\|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \\ &\leq C\, |||T_\varphi^{-1}|||\; \|f\|_{M_{p,w}} \,. \end{aligned}$$

■

The next step is to establish (4.5).

Lemma 4.2 *Suppose that the conditions in Theorem 4.1 are satisfied. Then there exists a constant $B < \infty$ such that for any sequence $(c_i)_{i \in \mathcal{I}_\sigma} \in \ell_{p,\frac{1}{w}}$, $1 \leq p \leq \infty$, the following inequality holds:*

$$\| \sum_{i \in \mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi \|_{M_{p,w}} \leq B\, \|(c_i)_{i \in \mathcal{I}_\sigma}\|_{\ell_{p,\frac{1}{w}}} \,.$$

Proof: 1. First we prove that

$$\| \sum_{i \in \mathcal{I}_\sigma} c_i R(h_i, h) \|_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \leq B \|(c_i)_{i \in I_\sigma}\|_{\ell_{p,\frac{1}{w}}} .$$

To this end, we want to use the Riesz–Thorin Interpolation Theorem as outlined in the appendix. That is, we show that

$$\mathcal{T} : (c_i)_{i\in I_\sigma} \longrightarrow \sum_{i\in I_\sigma} c_i R(h_i,\cdot)$$

is a bounded operator from $\ell_{1,\frac{1}{w}}$ to $L_{1,\frac{1}{w}}$ and from $\ell_{\infty,\frac{1}{w}}$ to $L_{\infty,\frac{1}{w}}$. Then the weighted Riesz–Thorin Theorem implies that $\mathcal{T}$ is also a bounded operator from $\ell_{p,\frac{1}{w}}$ to $L_{p,\frac{1}{w}}$ for all $1 \le p \le \infty$.

For $p = 1$, we obtain by (3.4) that

$$\begin{aligned}
\|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,\cdot)\|_{L_{1,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} &\le \int\limits_{\mathcal{G}/\mathcal{P}} \sum_{i\in\mathcal{I}_\sigma} |c_i|\ |R(h_i,h)| \frac{1}{w(\sigma(h))}\, d\mu(h) \\
&\le \sum_{i\in\mathcal{I}_\sigma} |c_i| \frac{1}{w(\sigma(h_i))} \sup_{i\in\mathcal{I}_\sigma} \int\limits_{\mathcal{G}/\mathcal{P}} |R(h_i,h)| \frac{w(\sigma(h_i))}{w(\sigma(h))}\, d\mu(h) \\
&\le C_\psi \|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{1,\frac{1}{w}}} .
\end{aligned}$$

For $p = \infty$ it follows that

$$\begin{aligned}
\|\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)\|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} &= \sup_{h\in\mathcal{G}/\mathcal{P}} |\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h)\, \frac{1}{w(\sigma(h))}| \\
&\le \sup_{i\in\mathcal{I}_\sigma} \frac{|c_i|}{w(\sigma(h_i))} \sup_{h\in\mathcal{G}/\mathcal{P}} \sum_{i\in\mathcal{I}_\sigma} |R(h_i,h)| \frac{w(\sigma(h_i))}{w(\sigma(h))} \\
&= \|(c_i)_{i\in I_\sigma}\|_{\ell_{\infty,\frac{1}{w}}} \sup_{h\in\mathcal{G}/\mathcal{P}} \sum_{i\in\mathcal{I}_\sigma} |\tilde{R}(h_i,h)|. \qquad (4.14)
\end{aligned}$$

Since $(x_i)_{i\in\mathcal{I}}$ is a relatively separated family, we have for any compact neighborhood $\mathcal{Q}$ of the identity in $\mathcal{G}$ that $\mathcal{I}_\sigma = \bigcup_{r=1}^{r_0} \mathcal{I}_{\sigma r}$ and $\mathcal{Q}x_i \cap \mathcal{Q}x_j = \emptyset$ for $i,j \in \mathcal{I}_{\sigma r}$ and $i \neq j$. Hence we obtain

$$\sum_{i\in\mathcal{I}_\sigma} |\tilde{R}(h_i,h)| = \sum_{r=1}^{r_0} \sum_{i\in\mathcal{I}_{\sigma r}} |\tilde{R}(h_i,h)| .$$

For all $l \in \mathcal{G}/\mathcal{P}$ with the property that $\sigma(l) \in \mathcal{Q}\sigma(h_i)$, we have that $\sigma(h_i)^{-1} \in \sigma(l)^{-1}\mathcal{Q}$ and hence

$$\begin{aligned}
&\sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \frac{w(q^{-1}\sigma(l))}{w(\sigma(h))} \ge \\
&\ge |\langle U(\sigma(h)^{-1})\psi, U(\sigma(h_i)^{-1})\psi\rangle| \frac{w(\sigma(h_i))}{w(\sigma(h))} \\
&= |R(h_i,h)| \frac{w(\sigma(h_i))}{w(\sigma(h))} = |\tilde{R}(h_i,h)| .
\end{aligned}$$

Let $\mathcal{B}_i := \{l \in \mathcal{G}/\mathcal{P} : \sigma(l) \in \mathcal{Q}\sigma(h_i)\}$. Then the above inequality implies

$$\int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \, \frac{w(q^{-1}\sigma(l))}{w(\sigma(h))} \, d\mu(l) \geq |\tilde{R}(h_i,h)|\mu(\mathcal{B}_i) \ .$$

Now we have that for $i,j \in \mathcal{I}_{\sigma r}$ and $i \neq j$ the sets $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint. Consequently, we obtain by (4.2)

$$\begin{aligned}
&\int_{\mathcal{G}/\mathcal{P}} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \frac{w(q^{-1}\sigma(l))}{w(\sigma(h))} \, d\mu(l) \ \geq \\
&\geq \ \sum_{i\in\mathcal{I}_{\sigma r}} \int_{\mathcal{B}_i} \sup_{q\in\mathcal{Q}} |\langle U(\sigma(h)^{-1})\psi, U(\sigma(l)^{-1}q)\psi\rangle| \frac{w(q^{-1}\sigma(l))}{w(\sigma(h))} \, d\mu(l) \\
&\geq \ \sum_{i\in\mathcal{I}_{\sigma r}} |\tilde{R}(h_i,h)|\mu(\mathcal{B}_i) \\
&\geq \ C_{\mathcal{Q}} \sum_{i\in\mathcal{I}_{\sigma r}} |\tilde{R}(h_i,h)|
\end{aligned}$$

and further by (4.3) for all $h \in \mathcal{G}/\mathcal{P}$

$$\sum_{i\in\mathcal{I}_{\sigma r}} |\tilde{R}(h_i,h)| \leq \frac{\tilde{C}_{\mathcal{Q}}}{C_{\mathcal{Q}}} \quad , \quad \sum_{i\in\mathcal{I}_{\sigma}} |\tilde{R}(h_i,h)| \leq \frac{r_0\tilde{C}_{\mathcal{Q}}}{C_{\mathcal{Q}}} \ . \tag{4.15}$$

Together with (4.14) this yields

$$\| \sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h) \|_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \leq \|(c_i)_{i\in\mathcal{I}_\sigma}\|_{\ell_{\infty,\frac{1}{w}}} \frac{r_0\tilde{C}_{\mathcal{Q}}}{C_{\mathcal{Q}}} \ .$$

2. Now it is easy to check that $\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h) \in \mathcal{M}_{p,\frac{1}{w}}$. Since $V_\psi \tilde{V}_\psi$ is the identity on $L_{\infty,\frac{1}{w}}$ and $\tilde{V}_\psi V_\psi$ on $H'_{1,w}$, we obtain

$$\begin{aligned}
\sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h) &= V_\psi \tilde{V}_\psi \left(\sum_{i\in\mathcal{I}_\sigma} c_i V_\psi(U(\sigma(h_i)^{-1})\psi) \right) \\
&= V_\psi \left(\sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi \right) .
\end{aligned}$$

Thus,

$$\| \sum_{i\in\mathcal{I}_\sigma} c_i U(\sigma(h_i)^{-1})\psi \|_{M_{p,w}} = \| \sum_{i\in\mathcal{I}_\sigma} c_i R(h_i,h) \|_{L_{p,\frac{1}{w}}}$$

and we are done. ■

Next let us turn to the estimates (4.8) in Theorem 4.2.

Lemma 4.3 *Suppose that the conditions in Theorem 4.2 are satisfied. For $i \in I_\sigma$, let $\psi_i := U(\sigma(h_i)^{-1})\psi$. Then, for $f \in M_{p,w}$, there exists a constant $B' < \infty$ such that*

$$|| \left(\langle f, \psi_i \rangle_{H'_{1,w} \times H_{1,w}} \right)_{i \in I_\sigma} ||_{\ell_{p,\frac{1}{w}}} \leq B' ||f||_{M_{p,w}}.$$

Proof: Let $F := V_\psi f$. By the correspondence principle the assertion is equivalent to

$$|| \left(F(h_i)\right)_{i \in I_\sigma} ||_{\ell_{p,\frac{1}{w}}} \leq B' ||F||_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}. \tag{4.16}$$

We prove (4.16) for $p = 1$ and $p = \infty$ and apply again the weighted Riesz–Thorin Interpolation Theorem to obtain the inequality for all $1 \leq p \leq \infty$.

For $p = 1$, we conclude as follows

$$\begin{aligned}
\sum_{i \in I_\sigma} |F(h_i)| \frac{1}{w(\sigma(h_i))} &= \sum_{i \in I_\sigma} |\langle F, R(h_i, \cdot) \rangle| \frac{1}{w(\sigma(h_i))} \\
&\leq \sum_{i \in I_\sigma} \int_{\mathcal{G}/\mathcal{P}} |F(l)| |R(h_i, l)| \frac{1}{w(\sigma(h_i))} \, d\mu(l) \\
&= \int_{\mathcal{G}/\mathcal{P}} |F(l)| \frac{1}{w(\sigma(l))} \sum_{i \in I_\sigma} |R(h_i, l)| \frac{w(\sigma(l))}{w(\sigma(h_i))} \, d\mu(l) \\
&\leq ||F||_{L_{1,\frac{1}{w}}} \sup_{l \in \mathcal{G}/\mathcal{P}} \sum_{i \in I_\sigma} |\tilde{R}(l, h_i)|.
\end{aligned}$$

Using (4.7) we obtain as in (4.15) that $\sum_{i \in I_\sigma} |\tilde{R}(l, h_i)| \leq r_0 \tilde{C}_\mathcal{Q} / C_\mathcal{Q}$ and consequently

$$\sum_{i \in I_\sigma} |F(h_i)| \frac{1}{w(\sigma(h_i))} \leq \frac{r_0 \tilde{C}_\mathcal{Q}}{C_\mathcal{Q}} ||F||_{L_1(\mathcal{G}/\mathcal{P})}.$$

For $p = \infty$, we get

$$\begin{aligned}
\sup_{i \in I_\sigma} |F(h_i)| \frac{1}{w(\sigma(h_i))} &= \sup_{i \in I_\sigma} |\langle F, R(h_i, \cdot) \rangle| \frac{1}{w(\sigma(h_i))} \\
&\leq \sup_{i \in I_\sigma} \int_{\mathcal{G}/\mathcal{P}} |F(l)| |R(h_i, l)| \frac{1}{w(\sigma(h_i))} \, d\mu(l) \\
&\leq \sup_{l \in \mathcal{G}/\mathcal{P}} |F(l)| \frac{1}{w(\sigma(l))} \sup_{i \in I_\sigma} \int_{\mathcal{G}/\mathcal{P}} |R(h_i, l)| \frac{w(\sigma(l))}{w(\sigma(h_i))} \, d\mu(l) \\
&= \sup_{l \in \mathcal{G}/\mathcal{P}} |F(l)| \frac{1}{w(\sigma(l))} \sup_{i \in I_\sigma} \int_{\mathcal{G}/\mathcal{P}} |\tilde{R}(l, h_i)| \, d\mu(l) \\
&\leq C_\psi \, ||F||_{L_{\infty,\frac{1}{w}}(\mathcal{G}/\mathcal{P})},
\end{aligned}$$

where we have used (3.3) for the last estimate. This finishes the proof. ■

Lemma 4.4 *Suppose that the conditions in Theorem 4.2 are satisfied. For $i \in I_\sigma$, let $\psi_i := U(\sigma(h_i)^{-1})\psi$. Then, for $\Big(\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}}\Big)_{i\in I_\sigma} \in \ell_{p,\frac{1}{w}}$, there exists a constant $A' > 0$ such that*

$$||f||_{M_{p,w}} \leq \frac{1}{A'}\, ||\, \Big(\langle f, \psi_i\rangle_{H'_{1,w}\times H_{1,w}}\Big)_{i\in I_\sigma}\, ||_{\ell_{p,\frac{1}{w}}}.$$

Proof: 1. First we show that

$$\tilde{\mathcal{T}} : (c_i)_{i\in I_\sigma} \mapsto \langle \sum_{i\in I_\sigma} c_i\varphi_i \circ \sigma, R(h,\cdot)\rangle$$

is a bounded operator from $\ell_{p,\frac{1}{w}}$ to $\mathcal{M}_{p,w}$ Again by the Riesz–Thorin Theorem, if suffices to show the boundedness for $p = 1$ and $p = \infty$.

For $p = 1$, we get by (3.3), (3.4) and the weighted Young inequality

$$\begin{aligned}
&||\langle \sum_{i\in I_\sigma} c_i\,\varphi_i \circ \sigma, R(h,\cdot)\rangle||_{L_{1,\frac{1}{w}}} \;\leq\; C_\psi\, ||\sum_{i\in I_\sigma} c_i\,\varphi_i\circ\sigma||_{L_{1,\frac{1}{w}}(\mathcal{G}/\mathcal{P})}\\
&\leq C_\psi \int\limits_{\mathcal{G}/\mathcal{P}} \sum_{i\in I_\sigma} \frac{|c_i|}{w(\sigma(h_i))}\; |\varphi_i\circ\sigma|\; \frac{w(\sigma(h_i))}{w(\sigma(h))}\, d\mu(h)\\
&\leq C_\psi\, ||(c_i)_{i\in I_\sigma}||_{\ell_{1,\frac{1}{w}}}\; \sup_{i\in I_\sigma} \int\limits_{\mathcal{G}/\mathcal{P}} |\varphi_i(\sigma(h))|\; \frac{w(\sigma(h_i))}{w(\sigma(h))}\, d\mu(h).
\end{aligned} \tag{4.17}$$

By $\operatorname{supp}\varphi_i \subseteq \mathcal{U}\sigma(h_i)$ we consider $h \in \mathcal{G}/\mathcal{P}$ with $\sigma(h) = u\sigma(h_i)$. Then, by using similar arguments as in the proof of Lemma 4.1, we obtain

$$\frac{w(\sigma(h_i))}{w(\sigma(h))} \leq C \tag{4.18}$$

with a constant C independent of h_i and h. Hence we can estimate (4.17) by

$$||\langle \sum_{i\in I_\sigma} c_i\,\varphi_i \circ \sigma, R(h,\cdot)\rangle||_{L_{1,\frac{1}{w}}} \leq C_\psi\, C\, ||(c_i)_{i\in I_\sigma}||_{\ell_{1,\frac{1}{w}}}.$$

For $p = \infty$, we obtain in a similar way by using the weighted Young inequality

$$\begin{aligned}
||\langle \sum_{i\in I_\sigma} c_i\,\varphi_i \circ \sigma, R(h,\cdot)\rangle||_{L_{\infty,\frac{1}{w}}} \quad &\leq \quad C_\psi \sup_{h\in\mathcal{G}/\mathcal{P}} |\sum_{i\in I_\sigma} c_i\,\varphi_i(\sigma(h))|\, \frac{1}{w(\sigma(h))}\\
&\leq \quad C_\psi \sup_{i\in I_\sigma} \frac{|c_i|}{w(\sigma(h_i))}\; \sup_{h\in\mathcal{G}/\mathcal{P}} \sum_{i\in I_\sigma} \varphi_i(\sigma(h))\, \frac{w(\sigma(h_i))}{w(\sigma(h))},
\end{aligned}$$

and further by (4.18) and since $\{\varphi_i\}$ is a partition of unity that

$$||\langle\sum_{i\in I_\sigma} c_i, \varphi_i \circ \sigma, R(h,\cdot)\rangle||_{L_{\infty,\frac{1}{w}}} \leq C_\psi\, C\, ||(c_i)_{i\in I_\sigma}||_{\ell_{\infty,\frac{1}{w}}}.$$

2. Next it is easy to check that

$$\langle\sum_{i\in I_\sigma} c_i\varphi_i \circ \sigma, R(h,\cdot)\rangle = \sum_{i\in I_\sigma} c_i\langle\varphi_i \circ \sigma, R(h,\cdot)\rangle.$$

Since S_φ^{-1} is a bounded operator on $\mathcal{M}_{p,\frac{1}{w}}$, we conclude that

$$(c_i)_{i\in I_\sigma} \mapsto S_\varphi^{-1}\left(\sum_{i\in I_\sigma} c_i\langle\varphi_i \circ \sigma, R(h,\cdot)\rangle\right) = \sum_{i\in I_\sigma} c_i S_\varphi^{-1}\left(\langle\varphi_i \circ \sigma, R(h,\cdot)\rangle\right)$$

is also bounded from $\ell_{p,\frac{1}{w}}$ to $\mathcal{M}_{p,w}$.

3. Finally, we apply part 1 and 2 of the proof to the special sequence $\left(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma} = (F(h_i))_{i\in I_\sigma}$, where $F := V_\psi f$, and obtain

$$||\sum_{i\in I_\sigma}\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}} S_\varphi^{-1}\left(\langle\varphi_i \circ \sigma, R(h,\cdot)\rangle\right)||_{L_{p,\frac{1}{w}}(\mathcal{G}/\mathcal{P})} \leq C\, ||\left(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma}||_{\ell_{p,\frac{1}{w}}}$$

and together with Corollary 4.2 and the correspondence principle

$$||f||_{M_{p,w}} \leq C\, ||\left(\langle f,\psi_i\rangle_{H'_{1,w}\times H_{1,w}}\right)_{i\in I_\sigma}||_{\ell_{p,\frac{1}{w}}}.$$

■

5 Appendix

In this section, we want to collect some basic facts that were needed before. Let us start with extending the classical Young inequality, see, e.g., [13], p. 185, Theorem 6.18, to weighted L_p–spaces.

Theorem 5.1 (**Weighted Young Inequality**) *Let $(X,\mathcal{A},\eta)$ and $(Y,\mathcal{B},\zeta)$ be σ–finite measure spaces, let K be an $\mathcal{A}\otimes\mathcal{B}$–measurable function on $X\times Y$, and let w be a positive weight function. Suppose that K satisfies the following conditions*

$$\int_X |K(x,y)|\frac{w(y)}{w(x)}\, d\eta(x) \leq C_K$$

for a.e. $y\in Y$ and

$$\int_Y |K(x,y)|\frac{w(y)}{w(x)}\, d\zeta(y) \leq C_K$$

for a.e. $x \in X$. *If* $f \in L_{p,\frac{1}{w}}$, $1 \le p \le \infty$, *then the integral*

$$Tf(x) = \int_Y K(x,y) f(y)\, d\zeta(y)$$

converges absolutely for a.e. $x \in X$, *the function* Tf *thus defined is in* $L_{p,\frac{1}{w}}$ *and*

$$\|Tf\|_{L_{p,\frac{1}{w}}} \le C_K \|f\|_{L_{p,\frac{1}{w}}}.$$

Proof: To show that the operator T is bounded we apply the assumptions of Theorem 5.1 and the Hölder inequality with $1/p + 1/q = 1$ as follows:

$$\begin{aligned}
\|Tf\|^p_{L_{p,\frac{1}{w}}} &= \int \Big| \int K(x,y) f(y) d\zeta(y) \Big|^p \frac{1}{w^p(x)}\, d\eta(x) \\
&\le \int \left(\int (|K(x,y)| w(y))^{1/p+1/q} \frac{|f(y)|}{w(y)}\, d\zeta(y) \right)^p \frac{1}{w^p(x)}\, d\eta(x) \\
&\le \int \left(\int |K(x,y)| w(y) \frac{|f(y)|^p}{w^p(y)}\, d\zeta(y) \right)^{p/p} \left(\int |K(x,y)| w(y) d\zeta(y) \right)^{p/q} \\
&\quad \times \frac{1}{w^p(x)}\, d\eta(x) \\
&\le C_K^{p/q} \int \int |K(x,y)| w(y) \frac{|f(y)|^p}{w^p(y)}\, d\zeta(y) w(x)^{p/q-p}\, d\eta(x) \\
&= C_K^{p/q} \int w(y) \frac{|f(y)|^p}{w^p(y)} \int \frac{|K(x,y)|}{w(x)}\, d\eta(x)\, d\zeta(y) \\
&\le C_K^p \|f\|^p_{L_{p,\frac{1}{w}}}.
\end{aligned}$$

■

In order to establish the frame bounds, we need a variant of the Riesz–Thorin interpolation theorem for the case of weighted L_p–spaces. For $p_0, p_1 < \infty$, the desired result is essentially a special case of the Stein–Weiss interpolation theorem, see, e.g., [2], Corollary 5.5.4, for details. However, for our approach we definitely need the corresponding result for $p_0 = 1, p_1 = \infty$. The resulting theorem is stated and proved below. It might be already known to the specialists, however, in this special form, it was not found in the literature.

The proof is based on complex interpolation. Therefore we start by briefly recalling the basic setting. For further information concerning real and complex interpolation, the reader is, e.g., refered to [2] and [15]. Let A_0 and A_1 be two complex Banach spaces. Then (A_0, A_1) is called an *interpolation couple* if there exists a linear complex Hausdorff space such that both A_0 and A_1 are linearly and continuously embedded in this space. Then $A_0 \cap A_1$ with norm $\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A1}\}$ and $A := A_0 + A_1$ with norm $\|a\|_{A_0+A_1} = \inf_{a=a_0+a_1} \{\|a\|_{A_0}, \|a\|_{A1}\}$ are also complex Banach spaces. Let

$$\mathcal{S} := \{z \in \mathbb{C} : \quad 0 < \Re z < 1\}$$

be a strip in the complex plane. The collection $\mathcal{F}$ of all functions $f(z)$ defined on $\overline{\mathcal{S}}$ with values in A with the two properties

i) $f(z)$ is continuous in $\overline{\mathcal{S}}$ and analytic in $\mathcal{S}$ with

$$\sup_{z\in\overline{\mathcal{S}}} \|f(z)\|_A < \infty,$$

ii) $f(it) \in A_0$ and $f(1+it) \in A_1$, with $t \in \mathbb{R}$, are continuous in the respective Banach spaces and

$$\|f\|_{\mathcal{F}} := \max\{\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1+it)\|_{A_1}\} < \infty$$

is again a Banach space.

For a given interpolation couple (A_0, A_1) and $\theta \in (0,1)$, the space $(A_0, A_1)_{[\theta]}$ is defined as

$$(A_0, A_1)_{[\theta]} := \{a \in A : \text{ there exists } f(z) \in \mathcal{F} \text{ with } f(\theta) = a\}.$$

Equipped with the norm

$$\|a\|_{[\theta]} := \inf\{\|f\|_{\mathcal{F}} : \ f(\theta) = a\},$$

$(A_0, A_1)_{[\theta]}$ becomes a Banach space which has the following *interpolation property*:

Theorem 5.2 *Let (A_0, A_1) and (B_0, B_1) be two interpolation couples and let T be a linear operator from $A_0 + A_1$ into $B_0 + B_1$ such that its restriction to A_j is a bounded linear operator from A_j into B_j, with norm $\leq M_j, j = 0, 1$. Then for any $\theta \in (0,1)$, the restriction of T to (A_0, A_1) is a bounded linear operator from $(A_0, A_1)_{[\theta]}$ into $(B_0, B_1)_{[\theta]}$ with norm $\leq M_0^{1-\theta} M_1^{\theta}$.*

Theorem 5.2 is the main ingredient for the proof of Theorem 5.3. For technical reasons, we shall also need the so–called three line theorem, see [2], page 4 for details.

Lemma 5.1 *(The three line theorem) Assume that $F(z)$ is analytic on S and bounded and continuous on $\overline{S}$. If*

$$|F(it)| \leq N_0, \qquad |F(1+it)| \leq N_1, \qquad -\infty < t < \infty,$$

then we have for $\theta \in [0,1]$ that

$$|F(\theta + it)| \leq N_0^{1-\theta} N_1^{\theta}, \qquad -\infty < t < \infty.$$

Now we are ready to establish the desired interpolation result with respect to $L_{1,w}$ and $L_{\infty,w}$.

Theorem 5.3 *Let T be a bounded linear operator from $L_{1,w}$ into $\ell_{1,w}$ with norm M_1 and from $L_{\infty,w}$ into $\ell_{\infty,w}$ with norm M_∞. Then, for any $1 < p < \infty$, the operator T is also a bounded from $L_{p,w}$ into $\ell_{p,w}$ with norm $M_1^{1/p} M_\infty^{(p-1)/p}$.*

Proof: According to Theorem 5.2, it remains to show that

$$(L_{1,w}, L_{\infty,w})_{[\theta]} = L_{p,w} \quad \text{and} \quad (\ell_{1,w}, \ell_{\infty,w})_{[\theta]} = \ell_{p,w} \ , \tag{5.1}$$

where

$$\frac{1}{p} = 1 - \theta.$$

We only prove the first statement in (5.1), the second one follows analogously. We have to show that

$$\|a\|_{[\theta]} = \|a\|_{(L_{1,w}, L_{\infty,w})_{[\theta]}} = \|a\|_{L_{p,w}} \ .$$

We start with the proof of $\|a\|_{[\theta]} \leq \|a\|_{L_{p,w}}$. Without loss of generality we may assume that $\|a\|_{L_{p,w}} = 1$. For our purposes, it is convenient to define f as follows

$$f(z) := w(x)^{p(1-z)-1} \exp(\varepsilon(z^2 - \theta^2)) |a(x)|^{p(1-z)} \frac{a(x)}{|a(x)|} \ .$$

We observe that f is an analytic function on the strip $\mathcal{S}$ with $f(\theta) = a$. In order to compute $\|a\|_{[\theta]}$ we note that

$$\|f\|_{\mathcal{F}} = \max\{\sup_t \|f(it)\|_{L_{1,w}}, \sup_t \|f(1+it)\|_{L_{\infty,w}}\} \ . \tag{5.2}$$

For $\|f(it)\|_{L_{1,w}}$, we obtain

$$\begin{aligned}
\|f(it)\|_{L_{1,w}} &= \int w(x) |w(x)^{p(1-it)-1} \exp(\varepsilon(-t^2 - \theta^2)) |a(x)|^{p(1-it)} \frac{a(x)}{|a(x)|}| dx \\
&= \exp(\varepsilon(-t^2 - \theta^2)) \int |a(x)|^p w(x)^p \, dx \\
&= \exp(\varepsilon(-t^2 - \theta^2) \|a\|_{L_{p,w}}^p = \exp(\varepsilon(-t^2 - \theta^2).
\end{aligned}$$

Consequently, for some suitable ε,

$$\sup_t \|f(it)\|_{L_{1,w}} = \exp(-\varepsilon\theta^2) \leq 1 \ . \tag{5.3}$$

The $L_{\infty,w}$–norm of $f(1+it)$ can be estimated as

$$\begin{aligned}
\|f(1+it)\|_{L_{\infty,w}} &= \sup_x w(x) |w(x)^{p(1-(1+it))-1} \exp(\varepsilon((1+it)^2 - \theta^2)) |a(x)|^{p(1-(1+it))} \frac{a(x)}{|a(x)|}| \\
&= \exp(\varepsilon(1 - t^2 - \theta^2)) \leq \exp(\varepsilon) \ . \qquad (5.4)
\end{aligned}$$

Combining (5.3) and (5.4) we obtain by (5.2)

$$\|f\|_{\mathcal{F}} \leq \exp(\varepsilon) \to 1 \ \text{ for } \ \varepsilon \to 0 \ ,$$

and taking the infimum yields

$$\|a\|_{[\theta]} \leq \|a\|_{L_{p,w}} \ , \text{i.e.,} \ \ L_{p,w} \subset (L_{1,w}, L_{\infty,w})_{[\theta]} \ .$$

The next step is to show $\|a\|_{L_{p,w}} \leq \|a\|_{[\theta]}$. Without loss of generality we may again assume that $\|a\|_{[\theta]} = 1$. Then we have

$$\|a\|_{L_{p,w}} = \sup\{|\langle a, b\rangle_w| : \ \|b\|_{L'_{p,w}} = 1\},$$

where, for $1 \leq p < \infty$, the dual pairing can be written as

$$\langle a, b\rangle_w := \int a(x)b(x)w(x)^p \, dx \, .$$

We define

$$F(z) := \langle f(z), g(z)\rangle_w$$

for some $f \in \mathcal{F}$ satisfying $f(\theta) = a$ and g given by

$$g(z) := w(x)^{1-p(1-z)} \exp(\varepsilon(z^2 - \theta^2))|b(x)|^{pz/(p-1)} \frac{b(x)}{|b(x)|}$$

for some $b \in L'_{p,w}$ with $\|b\|_{L'_{p,w}} = 1$. We want to estimate $F(z)$ by means of Lemma 5.1. Since $\|a\|_{[\theta]} = 1$ we can find $f \in \mathcal{F}$ with $f(\theta)) = a$ such that $\|f(it)\|_{L_{1,w}} \leq 1 + \varepsilon$ and $\|f(1+it)\|_{L_{\infty,w}} \leq 1 + \varepsilon$ for all $\varepsilon > 0$. Any such function f provides us with suitable bounds for $|F(it)|$ and $|F(1+it)|$. Indeed,

$$\begin{aligned}
|F(it)| &= |\int f(it)g(it)w(x)^p \, dx| \\
&\leq \int |f(it)||w(x)^{1-p(1-it)}|w(x)^p \, dx \exp(\varepsilon(-t^2 - \theta^2)) \\
&\leq \int |f(it)|w(x) \, dx \exp(\varepsilon(-t^2 - \theta^2)) \\
&\leq \|f(it)\|_{L_{1,w}} \exp(\varepsilon(-t^2 - \theta^2)) \\
&\leq (1+\varepsilon)\exp(-\varepsilon\theta^2) \leq \exp(\varepsilon) =: N_0
\end{aligned}$$

and

$$\begin{aligned}
|F(1+it)| &= |\int g(1+it)f(1+it)w(x)^p \, dx| \\
&\leq \|f(1+it)\|_{L_{\infty,w}} \int |b(x)|^{p(1+it)/(p-1)} w(x)^{p(1+it)} \, dx \exp(\varepsilon(1 - t^2 - \theta^2)) \\
&\leq (1+\varepsilon) \int |b(x)|^{p/(p-1)} w(x)^p \, dx \exp(\varepsilon) \exp(\varepsilon(-t^2 - \theta^2)) \\
&\leq \exp(2\varepsilon) =: N_1 \, .
\end{aligned}$$

Hence, by using Lemma 5.1,

$$|F(\theta + it)| \leq \exp(2\varepsilon) \quad \text{for all } 0 \leq \theta \leq 1 \, .$$

Consequently,

$$|\langle a, b\rangle_w| \leq |F(\theta)| \leq \exp(2\varepsilon) \, ,$$

that is, $\|a\|_{L_{p,w}} \leq 1$ and therefore $(L_{1,w}, L_{\infty,w})_{[\theta]} \subset L_{p,w}$. ■

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