Optimal Approximation of Elliptic Problems by Linear and Nonlinear Mappings III: Frames

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October 19, 2006

Abstract

We study the optimal approximation of the solution of an operator equation $\mathcal{A}(u) = f$ by certain n-term approximations with respect to specific classes of frames. We consider worst case errors, where f is an element of the unit ball of a Sobolev or Besov space $B^t_q(L_p(\Omega))$ and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain; the error is always measured in the H^s-norm. We study the order of convergence of the corresponding nonlinear frame widths and compare it with several other approximation schemes. Our main result is that the approximation order is the same as for the nonlinear widths associated with Riesz bases, the Gelfand widths, and the manifold widths. This order is better than the order of the linear widths iff $p < 2$. The main advantage of frames compared to Riesz bases, which were studied in our earlier papers, is the fact that we can now handle arbitrary bounded Lipschitz domains—also for the upper bounds.

AMS subject classification: 41A25, 41A46, 41A65, 42C40, 65C99

Key Words: Elliptic operator equation, worst case error, frames, nonlinear approximation methods, best n-term approximation, manifold width, Besov spaces on Lipschitz domains.

*This author acknowledges support through the European Union's Human Potential Programme, under contract HPRN-CT-2002-00285 (HASSIP), and through DFG, Grant Da 360/4-3. He also wants to thank the Friedrich-Schiller-Universität Jena for the hospitality and support.

1 Introduction

We study the optimal approximation of the solution of an operator equation

$$\mathcal{A}(u) = f, \tag{1}$$

where $\mathcal{A}$ is a linear operator

$$\mathcal{A} : H \to G \tag{2}$$

from a Hilbert space H to another Hilbert space G. We always assume that $\mathcal{A}$ is boundedly invertible, hence (1) has a unique solution for any $f \in G$. We have in mind the more specific situation of an elliptic operator equation which is given as follows. Assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and assume that

$$\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega) \tag{3}$$

is an isomorphism, where $s > 0$. For the exact definitions of Lipschitz domains and spaces of distributions defined on such domains we refer to the Appendix, see also [9]. Now we put $H = H_0^s(\Omega)$ and $G = H^{-s}(\Omega)$. Since $\mathcal{A}$ is boundedly invertible, the inverse mapping $S : G \to H$ is well defined. This mapping is sometimes called the solution operator—in particular if we want to compute the solution $u = S(f)$ from the given right-hand side $\mathcal{A}(u) = f$.

We study different mappings S_n for the approximation of the solution $u = \mathcal{A}^{-1}(f)$ for f contained in $F \subset G$. We consider the worst case error

$$e(S_n, F, H) = \sup_{\|f\|_F \leq 1} \|\mathcal{A}^{-1}(f) - S_n(f)\|_H, \tag{4}$$

where F is a normed (or quasi-normed) space, $F \subset G$. In our main results, F will be a Sobolev or Besov space.[1] Hence we use the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{S} & H \\ I \nwarrow & & \nearrow S_F \\ & F. & \end{array}$$

Here $I : F \to G$ denotes the identity and S_F the restriction of S to F. Then one is interested in approximations that have an optimal order of convergence depending

[1] Formally we only deal with Besov spaces. Because of the embeddings $B_1^{-s+t}(L_p(\Omega)) \subset W_p^{-s+t}(\Omega) \subset B_\infty^{-s+t}(L_p(\Omega))$, which hold for $1 \leq p \leq \infty$, $t \geq s$, see [45], our results are valid also for Sobolev spaces.

on n, where n denotes the degrees of freedom. For our purposes, the following approximation schemes are important. Consider the class $\mathcal{L}_n$ of all continuous linear mappings $S_n : F \to H$,

$$S_n(f) = \sum_{i=1}^{n} L_i(f) \cdot \tilde{h}_i$$

with arbitrary $\tilde{h}_i \in H$. The worst case error of optimal linear mappings is given by the *approximation numbers* or *linear widths*

$$e_n^{\mathrm{lin}}(S, F, H) = \inf_{S_n \in \mathcal{L}_n} e(S_n, F, H).$$

We may also use nonlinear approximations with respect to a Riesz basis $\mathcal{R}$ of H, i.e., we consider the class $\mathcal{N}_n(\mathcal{R})$ of all (linear or nonlinear) mappings of the form

$$S_n(f) = \sum_{k=1}^{n} c_k \, h_{i_k},$$

where the c_k and the i_k depend in an arbitrary way on f. Then the *nonlinear widths* $e_{n,C}^{\mathrm{non}}(S, F, H)$ are given by

$$e_{n,C}^{\mathrm{non}}(S, F, H) = \inf_{\mathcal{R} \in \mathcal{R}_C} \inf_{S_n \in \mathcal{N}_n(\mathcal{R})} e(S_n, F, H).$$

Here $\mathcal{R}_C$ denotes a set of Riesz bases for H where C indicates the stability of the basis, i.e., we require $B/A \leq C$ and A, B are the Riesz constants of the basis. The investigation of these widths $e_{n,C}^{\mathrm{non}}$ and its comparison with the linear widths have been the major part of our analysis in [8, 9]. This has continued earlier research on related topics, cf. e.g. [24, 38, 39, 40]. The next type of widths we are interested in has served as a very useful tool in our analysis of the widths $e_{n,C}^{\mathrm{non}}$ in [9]. The *manifold widths* are related to the class $\mathcal{C}_n$ of continuous mappings, given by arbitrary continuous mappings $N_n : F \to \mathbb{R}^n$ and $\varphi_n : \mathbb{R}^n \to H$. Again we define the worst case error of optimal continuous mappings by

$$e_n^{\mathrm{cont}}(S, F, H) = \inf_{S_n \in \mathcal{C}_n} e(S_n, F, H), \tag{5}$$

where $S_n = \varphi_n \circ N_n$. These numbers have been studied in [13, 27] and later in [9, 14, 16, 17]. As mentioned above we have studied the relationships of these widths in [9]. It has turned out that for problems as in (3) with $F = B_q^{-s+t}(L_p(\Omega))$ (with some extra conditions on Ω) one has the following: if $p \geq 2$ and $t > 0$ then

$$\begin{aligned} (6) \quad e_n^{\mathrm{lin}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) &\asymp e_n^{\mathrm{cont}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \\ &\asymp e_{n,C}^{\mathrm{non}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \asymp n^{-t/d}, \end{aligned}$$

whereas in the case $0 < p < 2$

$$e_n^{\mathrm{lin}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \asymp n^{-t/d+1/p-1/2}$$

and

$$e_n^{\mathrm{cont}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \asymp e_{n,C}^{\mathrm{non}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \asymp n^{-t/d} .$$

Hence, if $p < 2$ then there is an essential difference in the behavior, nonlinear approximations can do better than linear ones.

This paper is a continuation of [8, 9]. We are again interested in optimal nonlinear approximation schemes, but this time not related to Riesz bases but to classes of frames. The motivation for this is given by the following observations. In [9], we presented upper and lower bounds for $e_{n,C}^{\mathrm{non}}(S, F, H)$. The proof of the lower bound was quite general and used the fact that $e_{n,C}^{\mathrm{non}}(S, F, H)$ can be estimated from below by the manifold widths $e_n^{\mathrm{cont}}(S, F, H)$ up to some constants. In contrary to this, the proof of the upper bound was based on norm equivalences of Besov norms with weighted sequence norms that are induced by a biorthogonal wavelet basis. However, this restricts the choice of the underlying domain $\Omega \subset \mathbb{R}^d$ since on a general Lipschitz domain the construction of a suitable wavelet basis might be very complicated or even impossible. This problem becomes less serious in the frame setting since a suitable wavelet frame always exists, see Section 5.2 for a detailed discussion. Moreover, in recent years the application of frame methods for the numerical resolution of the solution u in (1) has become a field of increasing importance. Especially, it has been possible to derive adaptive wavelet frame schemes that are guaranteed to converge for a wide range of problems [6, 7, 37]. Therefore it is important to clarify the power that frame schemes can have, in principle.

In this paper, we give a first answer. Our main result states that the nonlinear frame widths show the same asymptotic behavior as the $e_{n,C}^{\mathrm{non}}(S, F, H)$, where we now can allow arbitrary bounded Lipschitz domains.

There is an interesting difference to the Riesz bases case. In the frame setting, we do *not* work with arbitrary n-term approximations, but only with those induced by a frame pair, see Section 2.2 for details. The reason is that, for practical applications, only these canonical representations are used. Actually we prove that if we would allow arbitrary n-term approximations then the associated frame widths would be zero. Moreover, certain conditions related to stability must be satisfied by the admissible frames. Fortunately, these conditions are always satisfied for the known constructions of wavelet frames on Lipschitz domains.

This paper is organized as follows. In Section 2, we describe the basic setting. First of all, we introduce and discuss the frame concept as far as it is needed for our purposes. Then, in Subsection 2.2, we define the nonlinear frame widths and prove some basic properties that are needed in the sequel. Section 3 contains the main results of this paper. In the next section two examples are discussed: the Poisson equation for Lipschitz domains and a Fredholm integral equation of the first kind (the single layer potential). Proofs of our main results are given in Section 5. For general Hilbert spaces H and G we show that similar to the Riesz bases case the nonlinear frame widths can be estimated from below by the manifold widths. Then, for the more specific case of Besov spaces on Lipschitz domains, we also prove an upper estimate which shows that the asymptotic behavior is the same as for the Riesz basis case—but this time for arbitrary bounded Lipschitz domains.

Notation. We write $a \asymp b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}\, a \leq b \leq c\, a\,.$$

One-sided estimates of this type are denoted by $a \lesssim b$. All unimportant constants will be denoted by c, sometimes with additional indices. Identity operators are always denoted by I, also sometimes with additional indices.

2 Frames

In this paper, we will study certain approximations of $u = S(f)$ based on frames. Therefore, in this section we recall the basic properties of frames as far as they are needed for our purposes and introduce the corresponding nonlinear widths. For further information on frames, we refer the reader e.g. to [2, 21]. A sequence $\mathcal{F} = \{h_k\}_{k\in\mathbb{N}}$ in a separable Hilbert space H is a *frame* for H if there exist constants $A, B > 0$ such that

$$A^2 \sum_{k=1}^{\infty} \big|(f, h_k)_H\big|^2 \leq \|f\|_H^2 \leq B^2 \sum_{k=1}^{\infty} \big|(f, h_k)_H\big|^2 \qquad (7)$$

for all $f \in H$. As a consequence of (7), the corresponding operators of analysis and synthesis given by

$$\mathcal{T} : H \to \ell_2(\mathbb{N}), \quad f \mapsto \big((f, h_k)_H\big)_{k\in\mathbb{N}}, \qquad (8)$$

$$\mathcal{T}^* : \ell_2(\mathbb{N}) \to H, \quad \mathbf{c} \mapsto \sum_{k=1}^{\infty} c_k h_k, \qquad (9)$$

are bounded. The composition $\mathcal{T}^*\mathcal{T}$ is a boundedly invertible (positive and self-adjoint) operator called the *frame operator*. Furthermore, $\widetilde{\mathcal{F}} := (\mathcal{T}^*\mathcal{T})^{-1}\mathcal{F}$ is again a frame for H, the *canonical dual frame*. The following formulas hold

$$f = \sum_{k=1}^{\infty} (f, (\mathcal{T}^*\mathcal{T})^{-1} h_k)_H \, h_k = \sum_{k=1}^{\infty} (f, h_k)_H \, (\mathcal{T}^*\mathcal{T})^{-1} h_k \tag{10}$$

for all $f \in H$. This classical concept of a frame is too general, we need an additional stability condition, stronger than (7). Without this additional assumption on the frames, there would not exist lower bounds for corresponding widths as we shall now explain.

Remark 1. *Let H be a separable Hilbert space and let $K \subset H$ be a compact subset. Then for an arbitrary $C > 1$ there exists a frame $\mathcal{F} = \{h_i\}_{i\in\mathbb{N}}$ in H with $B/A < C$ such that the following is true: For all $f \in K$ and for all $\varepsilon > 0$ there exists a $h_i \in \mathcal{F}$ and $c \in \mathbb{R}$ such that*

$$\|f - ch_i\|_H < \varepsilon.$$

Hence the best n-term approximation yields an error 0 already for $n = 1$. To prove this statement, we construct such a frame for a given compact set $K \subset H$. Let $M_1 = \{e_i,\ i \in \mathbb{N}\}$ be a complete orthonormal set of H and let $\{k_i,\ i \in \mathbb{N}\}$ be a dense subset of K. We consider sets of the form

$$M_2^{\delta} = \{\alpha_1 k_1,\, \alpha_2 k_2,\, \ldots\} \subset H$$

with $\alpha_i = \delta^i$, where $0 < \delta < 1$ and put $\mathcal{F}_\delta = M_1 \cup M_2^\delta$. It is not difficult to check that $\mathcal{F}_\delta$ is a frame with all the claimed properties if $\delta = \delta(C)$ is chosen appropriately.

The frames $\mathcal{F}_\delta$ can be considered as "pathological", since the norms of many elements of $\mathcal{F}_\delta$ are extremely small. A first idea would be to request that the norms of the frame elements are uniformly bounded from above and below,

$$0 < c_1 \le \|h_i\|_H \le c_2 < \infty \qquad \text{for all} \quad h_i \in \mathcal{F} = \{h_i\}_{i\in\mathbb{N}}\,,$$

but this does not help: Now we can define $\mathcal{F}_\delta$ as the union of M_1 and multiples of the $e_i \pm \alpha_i k_i$. Then one obtains such a "normed" frame such that: For all $f \in K$ and for all $\varepsilon > 0$ there exist $h_i \in \mathcal{F}$ and $c_i \in \mathbb{R}$ such that

$$\|f - c_1 h_1 - c_2 h_2\|_H < \varepsilon.$$

Therefore we go into a different direction, see Definitions 1 and 2.

2.1 Frame Pairs

As it is well-known, Sobolev spaces defined on domains can be discretized by means of weighted ℓ_2-spaces, see the Appendix for some examples how one can do this. Let $w := (w_k)_{k\in\mathbb{N}}$ be a sequence of nonnegative numbers which we call simply a *weight* in what follows. Then we put

$$\ell_{2,w} := \Big\{ a = (a_k)_{k\in\mathbb{N}} : \quad \| \, a \, \|_{\ell_{2,w}} := \Big(\sum_{k=1}^{\infty} w_k \, |a_k|^2 \Big)^{1/2} < \infty \Big\} .$$

Definition 1. *Let H be a separable Hilbert space with dual space H'. Let $w = (w_k)_k$ be a weight.*
(i) *Two sequences $(\mathcal{F}, \mathcal{G})$, $\mathcal{F} := \{h_k\}_{k\in\mathbb{N}} \subset H', \mathcal{G} := \{g_k\}_{k\in\mathbb{N}} \subset H$, are called a* frame pair *for (H, w), if*

$$f = \sum_{k=1}^{\infty} \langle f, h_k \rangle_{H\times H'} \, g_k , \tag{11}$$

holds for all $f \in H$ and in addition we have the norm equivalence

$$A \, \| \, (\langle f, h_k\rangle_{H\times H'})_{k\in\mathbb{N}} \, \|_{\ell_{2,w}} \le \|f\|_H \le B \, \| \, (\langle f, h_k\rangle_{H\times H'})_{k\in\mathbb{N}} \, \|_{\ell_{2,w}} \tag{12}$$

with some positive constants A, B.
(ii) *Let K be a compact subset of H. A frame pair $(\mathcal{F}, \mathcal{G})$ for (H, w) is called* stable *with respect to K if the inequality*

$$A' \, \|(\langle f, h_n\rangle_{H\times H'})_{n\in\Lambda}\|_{\ell_{2,w}} \le \Big\| \sum_{k\in\Lambda} \langle f, h_n\rangle_{H\times H'} \, g_k \Big\| \tag{13}$$

holds with some $A' > 0$, all finite subsets $\Lambda \subset \mathbb{N}$ and all $f \in K$.
(iii) *Let K be a compact subset of H and let $C \ge 1$ be a given number. By $\mathcal{P}_C(K)$ we denote the set of all stable frame pairs $(\mathcal{G}, \mathcal{F})$ with respect to K such that the constants A, B and A' in* (12) *and* (13) *satisfy $B/\min(A, A') \le C$.*

Remark 2. *To avoid any type of confusion we shall use $(\cdot, \cdot)$ for the scalar product in H and $\langle \cdot, \cdot \rangle$ for duality pairings, in particular for $H \times H'$.*

Some comments are in order.

Remark 3. *(i) A frame pair in the sense of* (11) *and* (12) *is sometimes called an* atomic decomposition, *cf. e.g.* [2, Def. 17.3.1.]. *However, the phrase* atomic decomposition *is used with a different meaning in the theory of function spaces, cf. e.g.* [18, 25, 43, 46]. *For this reason we do not use it here.*

(ii) *Let* $(\mathcal{F},\mathcal{G})$ *be a frame pair for* (H,w)*. As above let* $\mathcal{F}=\{h_k\}_{k\in\mathbb{N}}\subset H'$ *and* $\mathcal{G}:=\{g_k\}_{k\in\mathbb{N}}\subset H$*. By the Riesz representation theorem, for every* h_k *there exists an element* $\widetilde{h}_k\in H$ *such that* $\langle f,h_k\rangle_{H\times H'}=(f,\widetilde{h}_k)_H$*. Consequently,*

$$\|\,(f,\,\sqrt{w_k}\,\widetilde{h}_k)_{k\in\mathbb{N}}\,\|_{\ell_2}=\|\,(\langle f,h_k\rangle_{H\times H'})_{k\in\mathbb{N}}\,\|_{\ell_{2,w}}\qquad \text{for all}\quad f\in H\,.$$

Hence, there is a one-to-one correspondence between $\mathcal{F}$ *and the Hilbert frame* $(\sqrt{w_k}\,\widetilde{h}_k)_k$*. However, note that* $\mathcal{G}$ *need not be related to the canonical dual frame of* $(\sqrt{w_k}\,\widetilde{h}_k)_k$*.*

(iii) *The reader might wonder why we use the concept of frame pairs instead of the classical frame setting as introduced in* (7) *and* (10)*. However, since we are dealing here with Gelfand triples* $(H^s_0(\Omega),L_2(\Omega),H^{-s}(\Omega))$*, this approach would be at least problematic since we are not allowed to indentify the space* $H^s_0(\Omega)$ *with its dual.(Otherwise, it would not be possible to identify* $L_2(\Omega)$ *with its dual at the same time - a strange construction. We refer to* [23] *for further details.)*

(iv) *Our concept is closely related to Banach frames in the sense of* [20, 22]*. A Banach frame for a separable and reflexive Banach space* $\mathcal{B}$ *is a sequence* $\mathcal{F}=\{h_k\}_{k\in\mathbb{N}}$ *in* $\mathcal{B}'$ *with an associated sequence space* $\mathcal{B}_d$ *such that the following properties hold:*

(B1) *norm equivalence: there exist constants* $A,B>0$ *such that*

$$A\left\|\left(\langle f,h_k\rangle_{\mathcal{B}\times\mathcal{B}'}\right)_{k\in\mathbb{N}}\right\|_{\mathcal{B}_d}\le\|f\|_{\mathcal{B}}\le B\left\|\left(\langle f,h_k\rangle_{\mathcal{B}\times\mathcal{B}'}\right)_{k\in\mathbb{N}}\right\|_{\mathcal{B}_d}\tag{14}$$

for all $f\in\mathcal{B}$*;*

(B2) *there exists a bounded operator* R *from* $\mathcal{B}_d$ *onto* $\mathcal{B}$*, a so-called* synthesis *or* reconstruction operator*, such that*

$$R\left(\left(\langle f,h_k\rangle_{\mathcal{B}\times\mathcal{B}'}\right)_{k\in\mathbb{N}}\right)=f.\tag{15}$$

(It is a remarkable fact that for Banach spaces the existence of the reconstruction operator does not follow from the norm equivalence (14) *and has to be explicitly required).*

A frame pair in the sense of Definition 1 (i) *induces a Banach frame* $\mathcal{F}=\{h_k\}_{k\in\mathbb{N}}$ *for the special case* $\mathcal{B}=H$*,* $\mathcal{B}_d=\ell_{2,w}(\mathbb{N})$*. Let* δ_k *denote the elements of the canonical unit vector basis of* $\ell_{2,w}$*. The reconstruction operator is obtained by defining* $R(\delta_k)=g_k$*,* $k\in\mathbb{N}$*, and then taking continuous extension, cf.* [2, Chapt. 17]*. Consequently, in our setting the estimate*

$$\left\|\sum_{k\in\mathbb{N}}c_kg_k\right\|_{\mathcal{B}}\ \lesssim\ \|(c_k)_{k\in\mathbb{N}}\|_{\mathcal{B}_d}\tag{16}$$

always holds.

(v) *We comment on the condition* (13). *Clearly,* (13) *always holds on all of H for a Riesz basis $\{g_k\}_{k\in\mathbb{N}}$ for H. However, there exist frames which are not Riesz bases and for which* (13) *holds on H. E.g., take an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ and define the frame $\mathcal{F} =: \{e_1, 2^{-1/2}e_2, 2^{-1/2}e_2, e_3, e_4...\}$. This is a tight frame,* (12) *holds with $A = B = 1$, so the primal and the canonical dual frame coincide. (We refer again to* [2, Chapt. 5] *for further information). Since $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis, a direct computation shows that* (13) *holds for $A' = 2^{-1/2}$. Nevertheless, requiring* (13) *on all of H would be very restrictive, and most frames would not satisfy it. As an example, consider the frame $\mathcal{F} := \{e_1, 2^{-1/2}e_2, 2^{-1/2}e_2, 3^{-1/2}e_3, 3^{-1/2}e_3, 3^{-1/2}e_3, \dots\}$. This is also a tight frame, but again a direct check shows that* (13) *does not hold. Therefore we require* (13) *only on subsets. Fortunately, such a condition is satisfied in case of the known frame constructions for function spaces on Lipschitz domains.*

(vi) *The example in (v) shows that the two constants A amd A' in Definition* 1 *need not to be related at all. Nevertheless, to avoid unnecessary notational difficulties, we will restrict ourselves to the case $A = A'$ in the sequel. The modifications to the case $A \neq A'$ are straightforward.*

(vii) *For simplicity, we have introduced our basic concepts for frame pairs indexed by the set of natural numbers. Later on, we shall also use frame pairs corresponding to more general countable sets, with the obvious modifications.*

For later use, let us finally state the following simple but useful property: frame pairs are invariant under isomorphic mappings.

Lemma 1. *Let G, H be Hilbert spaces and let $S : G \to H$ be an isomorphism. Let $(\mathcal{F}, \mathcal{G})$ be a frame pair for (G, w) with frame constants A, B. Then the following holds:*
(i) *$(S^{*-1}(\mathcal{F}), S(\mathcal{G}))$ is a frame pair for (H, w) with frame constants $\widetilde{A} = A/\|S^{-1}\|$ and $\widetilde{B} = B\|S\|$.*
(ii) *If $(\mathcal{F}, \mathcal{G})$ is contained in $\mathcal{P}_C(K)$ then $(S^{*-1}(\mathcal{F}), S(\mathcal{G}))$ is contained in $\mathcal{P}_{\widetilde{C}}(S(K))$, where $\widetilde{C} = C\|S\|\|S^{-1}\|$.*

Proof. Step 1. Proof of (i). We start by showing (11). For $f \in H$, we obtain

$$f = S(S^{-1}(f)) = S\Big(\sum_{k\in\mathbb{N}} \langle S^{-1}(f), h_k\rangle_{H\times H'}\, g_k\Big) = \sum_{k\in\mathbb{N}} \langle f, S^{*-1}(h_k)\rangle_{H\times H'}\, S(g_k).$$

It remains to show the norm equivalence (12). We obtain

$$\begin{aligned}\frac{1}{\|S\|}\|f\|_H &= \frac{1}{\|S\|}\|(S\circ S^{-1})(f)\|_H \le \|S^{-1}(f)\|_G \\ &\le B(\langle S^{-1}(f), h_n\rangle_{G\times G'})_{\ell_{2,w}} = B(\langle f, S^{*-1}(h_k)\rangle_{H\times H'})_{\ell_{2,w}} \\ &\le \frac{B}{A}\,\|S^{-1}(f)\|_G \le \frac{B}{A}\,\|S^{-1}\|\|f\|_H\,.\end{aligned}$$

Step 2. Proof of (ii). Clearly, $S(K)$ is a compact subset (since continuous images of compact sets are compact). For $f \in S(K)$, we get

$$\begin{aligned}\|\sum_{k\in\Lambda}\langle f, S^{*-1}(h_k)\rangle_{H\times H'}\, S(g_k)\|_H &\ge \|S^{-1}\|^{-1}\Big\|\sum_{k\in\Lambda}\langle S^{-1}(f), h_k\rangle_{G\times G'}\, g_k\Big\|_G \\ &\ge \|S^{-1}\|^{-1}A\,\|\,(\langle S^{-1}(f), h_k\rangle_{G\times G'})_{k\in\Lambda}\,\|_{\ell_{2,w}} \\ &= \|S^{-1}\|^{-1}A\,\|\,(\langle f, S^{*-1}h_k\rangle_{H\times H'})_{k\in\Lambda}\,\|_{\ell_{2,w}}\,,\end{aligned}$$

and (ii) is proved with $\widetilde{C} = \widetilde{B}/\widetilde{A} = C\|S\|\|S^{-1}\|$. □

2.2 Nonlinear Widths for Frame Pairs

The aim of this paper is to study the asymptotic behavior of specific nonlinear approximation schemes based on frames and to compare them with other well-known widths. Especially, we want to prove frame analogues to the results obtained in [8, 9] for the nonlinear widths associated with classes of Riesz bases.

Let $(\mathcal{F}, \mathcal{G})$ be a frame pair for (H, w) in the sense of Definition 1 and consider specific n-term approximations of the form

$$\sigma_n\Big(u, (\mathcal{F}, \mathcal{G})\Big) := \inf_{|\Lambda|=n}\Big\|\, u - \sum_{k\in\Lambda}\langle u, h_k\rangle_{H\times H'}\, g_k\,\Big\|_H. \tag{17}$$

We do not allow arbitrary expansions in terms of the g_k involving at most n nonvanishing coefficients. The reason is that, for practical applications, only these canonical representations are used. Furthermore, to end up with a reasonable notion of a width we need to restrict us to stable frame pairs.

Definition 2. *Let G and H be separable Hilbert spaces and let $S:\ G \to H$ be an isomorphism. Let F be a subset of G. For a given constant $C \ge 1$ we denote by $\mathcal{K}_C$ the set of all compact subsets $K \subset S(F)$ such that the inequality*

$$e_n^{\text{cont}}(I, S(F), H) \le C e_n^{\text{cont}}(I, K, H) \tag{18}$$

holds for all n. Then, for $n \in \mathbb{N}$, the nonlinear frame width $e_{n,C}^{\text{frame}}(S, F, H)$ *of the operator S is defined by*

$$e_{n,C}^{\text{frame}}(S, F, H) := \inf\Big\{\sup_{\|f\|_F\le 1}\sigma_n\Big(S(f), (\mathcal{F}, \mathcal{G})\Big)\ \Big|\ (\mathcal{F}, \mathcal{G}) \in \mathcal{P}_C(K),\ K \in \mathcal{K}_C\Big\}. \tag{19}$$

Remark 4. *We comment on this definition. To get a reasonable lower bound for* $e^{\text{frame}}_{n,C}(S,F,H)$ *we need to restrict ourselves to frame pairs which are stable with respect to subsets of* K *of* $S(F)$ *which are not too small. "Not too small" is expressed by the inequality* (18).

In the above definition we decided for the manifold widths because they have some nice properties. These widths e_n^{cont} are particular examples of s-numbers in the sense of Pietsch [31], see also [27]. One of the interesting properties consists in the inequality

$$e_n^{\text{cont}}(T_2 \circ T_1 \circ T_0, E_0, F_0) \le \| T_0 \| \, \| T_2 \| \, e_n^{\text{cont}}(T_1, E, F)\,, \tag{20}$$

where $T_0 \in \mathcal{L}(E_0, E)$, $T_1 \in \mathcal{L}(E, F)$, $T_2 \in \mathcal{L}(F, F_0)$ and E_0, E, F, F_0 are arbitrary quasi-Banach spaces. As a consequence one obtains that the asymptotic behavior of the manifold widths remains unchanged under isomorphisms. A similar result is true in case of our nonlinear frame widths. As a consequence we can concentrate on the investigation of identity operators in what follows.

Lemma 2. *Let* G *and* H *be separable Hilbert spaces and let* $S : G \to H$ *be an isomorphism. Let* F *be a quasi-normed subset of* G *and let* $I : F \to G$ *be the identity. For* $C \ge 1$ *and*

$$\widetilde{C} = C\,(\|S^{-1}\|\,\|S\|)^2$$

we obtain

$$e^{\text{frame}}_{n,\widetilde{C}}(S,F,H) \le \|S\|\, e^{\text{frame}}_{n,C}(I,F,G) \tag{21}$$

and

$$e^{\text{frame}}_{n,\widetilde{C}}(I,F,G) \le \|S^{-1}\|\, e^{\text{frame}}_{n,C}(S,F,H). \tag{22}$$

Proof. We shall prove (21), the proof of (22) is very similar. From (19) we can conclude that for any $\varepsilon > 0$ we can find a compact set $K \in \mathcal{K}_C \subset G$ and a frame pair $(\mathcal{F}, \mathcal{G}) \in \mathcal{P}_C(K)$ for (G, w) such that

$$\sup_{\|f\|_F \le 1} \inf_{|\Lambda|=n} \Big\| f - \sum_{k\in\Lambda} \langle f, h_k\rangle_{G\times G'}\, g_k \Big\|_G \le e^{\text{frame}}_{n,C}(I,F,G) + \varepsilon\,.$$

Lemma 1 implies that $(S^{*-1}(\mathcal{F}), S(\mathcal{G}))$ is a frame pair for (H, w) which is contained in $\mathcal{P}_{C_1}(S(K))$, where $C_1 = C\,\|S^{-1}\|\,\|S\|$. We consider the following commutative diagrams:

$$\begin{array}{ccc} S(F) & \xrightarrow{I_1} & H \\ {\scriptstyle S^{-1}}\downarrow & & \uparrow{\scriptstyle S} \\ F & \xrightarrow{I_2} & G \end{array} \qquad\qquad \begin{array}{ccc} K & \xrightarrow{I_2} & G \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle S^{-1}} \\ S(K) & \xrightarrow{I_1} & H\,. \end{array}$$

By means of (20) we derive from these diagrams

$$e_n^{\rm cont}(I_1, S(F), H) \le \|S^{-1}\| \, \|S\| \, e_n^{\rm cont}(I_2, F, G)$$

and

$$e_n^{\rm cont}(I_2, K, G) \le \|S^{-1}\| \, \|S\| \, e_n^{\rm cont}(I_1, S(K), H) \, .$$

Now our assumption $K \in \mathcal{K}_C$ yields

$$\begin{aligned} e_n^{\rm cont}(I_1, S(F), H) &\le \|S^{-1}\| \, \|S\| \, e_n^{\rm cont}(I_2, F, G) \le C \, \|S^{-1}\| \, \|S\| \, e_n^{\rm cont}(I_2, K, G) \\ &\le C \, (\|S^{-1}\| \, \|S\|)^2 \, e_n^{\rm cont}(I_1, S(K), H) \, . \end{aligned}$$

In other words, $S(K)$ belongs to the set $\mathcal{K}_{\widetilde{C}}$. From

$$\Big\| \, S(f) - \sum_{k \in \Lambda} \langle S(f), S^{*-1}(h_k) \rangle_{H \times H'} \, S(g_k) \, \Big\|_H \le \|S\| \, \Big\| \, f - \sum_{k \in \Lambda} \langle f, h_k \rangle_{g \times G'} \, g_k \, \Big\|_G$$

it follows that

$$e_{n,\widetilde{C}}^{\rm frame}(S, F, H) \le \|S\| \, e_{n,C}^{\rm frame}(I, F, G) \, .$$

□

We finish this section by proving two additional properties of nonlinear frame widths that will be used later on in Section 5.3.

Lemma 3. *Let G_1, G_2, H_1, H_2 be Hilbert spaces and let $S_i \in \mathcal{L}(F_i, H_i)$, $i = 1, 2$, be isomorphisms. Let F_1, F_2 be quasi-normed subsets of G_1 and G_2, respectively. Furthermore we suppose $T_1 \in \mathcal{L}(F_1, F_2)$, $T_2 \in \mathcal{L}(H_2, H_1)$ and both are isomorphisms. Finally, we assume that we can decompose $S_1 = T_2 \circ S_2 \circ T_1$. Then,*

$$e_{n,\widetilde{C}}^{\rm frame}(S_1, F_1, H_1) \le \|T_2\| \, \|T_1\| \, e_{n,C}^{\rm frame}(S_2, F_2, H_2) \tag{23}$$

holds with $\widetilde{C} = C \, \|T_2^{-1}\| \, \|T_2\|$.

Proof. Corresponding to our assumptions we have the following commutative diagram:

$$\begin{array}{ccc} F_1 & \xrightarrow[S_1]{} & H_1 \\ {\scriptstyle T_1}\big\downarrow & & \big\uparrow{\scriptstyle T_2} \\ F_2 & \xrightarrow[S_2]{} & H_2 \, . \end{array}$$

By definition, for any $\varepsilon > 0$ we can find a compact set $K \in \mathcal{K}_C \subset G$ and a frame pair $(\mathcal{F}, \mathcal{G}) \in \mathcal{P}_C(K)$ for (H_2, w) such that

$$\sup_{\|f\|_{F_2} \le 1} \inf_{|\Lambda| = n} \Big\| \, S_2 f - \sum_{k \in \Lambda} \langle S_2 f, h_k \rangle_{H_2 \times H_2'} \, g_k \, \Big\|_{H_2} \le e_{n,C}^{\rm frame}(S_2, F_2, H_2) + \varepsilon \, .$$

Lemma 1 implies that $(T_2^{*-1}(\mathcal{F}), T_2(\mathcal{G}))$ is a frame pair for (H_1, w) which is contained in $\mathcal{P}_{\widetilde{C}}(T_2(K))$, where $\widetilde{C} = C\,\|T_2^{-1}\|\,\|T_2\|$. We put

$$u_k := T_2^{*-1}(f_k) \qquad \text{and} \qquad v_k := T_2(g_k)\,.$$

Consequently

$$\begin{aligned} \Big\| S_1 g \; &- \sum_{k\in\Lambda} \langle S_1 g, u_k\rangle_{H_1\times H_1'}\, v_k \Big\|_{H_1} \\ &\le \|T_2\| \, \Big\| S_2(T_1 g) - \sum_{k\in\Lambda} \langle S_2(T_1 g), T_2^* u_k\rangle_{H_2\times H_2'}\, T_2^{-1} v_k \Big\|_{H_2} \\ &\le \|T_2\| \, (e_{n,C}^{\mathrm{frame}}(S_2, F_2, H_2) + \varepsilon)\,, \end{aligned}$$

if $\|\, T_1 g\,\|_{F_2} \le 1$. A homogeneity argument yields

$$\sup_{\|g\|_{F_1}\le 1} \inf_{|\Lambda|=n} \Big\| S_1(g) - \sum_{k\in\Lambda} \langle S_1 g, u_k\rangle_{H_1\times H_1'}\, v_k \Big\|_{H_1} \le \|T_2\|\,\|T_1\|\, e_{n,C}^{\mathrm{frame}}(S_2, F_2, H_2)$$

which proves our claim. □

Lemma 4. *Let U be a closed subspace of the Hilbert space H equipped with the same norm as H. Let G be a Hilbert space and let $S : G \to H$ be an isomorphism. If F is a subset of $S^{-1}(U)$, then*

$$e_{n,C}^{\mathrm{frame}}(S, F, U) \le e_{n,C}^{\mathrm{frame}}(S, F, H)$$

follows.

Proof. The Hilbert space H can be written as the orthogonal sum of U and its orthogonal complement V. By P we denote the orthogonal projection onto U. Let $(\mathcal{F}, \mathcal{G})$ be a frame pair for (H, w). Then the elements $f \in U$ can be written in the form

$$f = \sum_{k=1}^{\infty} \langle f, h_k\rangle\, P g_k\,.$$

The norm equivalences (12) remain unchanged. Hence, $(\mathcal{F}, P(\mathcal{G}))$ is a frame pair for (U, w) with constants $\widetilde{A}, \widetilde{B}$ and $A \le \widetilde{A} \le \widetilde{B} \le B$. Concerning the stability it is enough to notice that only subsets K of $S(F) \subset U$ come into consideration. □

3 Main Results

In this section, we want to state and to prove the main results of this paper. The first theorem is a general result for arbitrary Hilbert spaces H and G that clarifies the relationships of the manifold widths $e_n^{\text{cont}}(S,F,H)$ with the nonlinear frame widths $e_{n,C}^{\text{frame}}(S,F,H)$. The second theorem deals with the more specific situation of function spaces on Lipschitz domains contained in $\mathbb{R}^d$ and provides upper and lower bounds for $e_{n,C}^{\text{frame}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega))$.

Theorem 1. *Let H and G be separable Hilbert spaces. Let $S : G \to H$ be an isomorphism. Suppose that the embedding $F \hookrightarrow G$ is compact. Then for all $C \geq 1$ and all $n \in \mathbb{N}$, we have*

$$e_{4n+1}^{\text{cont}}(S,F,H) \leq 2\,C^2\, e_{n,C}^{\text{frame}}(S,F,H)\,. \tag{24}$$

Theorem 2. *Let Ω be a bounded Lipschitz domain contained in $\mathbb{R}^d$. Let $0 < p, q \leq \infty$, $s > 0$, and $t > d(\frac{1}{p} - \frac{1}{2})_+$. Let $S : H^{-s}(\Omega) \to H_0^s(\Omega)$ be an isomorphism. Then there exists a number C^* such that for any $C \geq C^*$ we have*

$$e_{n,C}^{\text{frame}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \asymp n^{-t/d}\,.$$

Remark 5. *(i) The number C^* depends on Ω. It is known that for any Lipschitz domain there exists an appropriate frame pair as it is needed here. However, optimal estimates about the stability seem to be not known.*

(ii) For exact definitions of the distribution spaces defined on Lipschitz domains we refer to the Appendix and to [9]

(iii) Theorem 2 is a frame analogue to Theorem 4 in [9]*. In* [9]*, it has been shown that if the domain Ω is chosen in such a way that the spaces $B_q^{-s+t}(L_p(\Omega))$ and $H^{-s}(\Omega)$ allow a discretization by one common wavelet system $\tilde{\mathcal{R}}^*$, then also*

$$e_{n,C}^{\text{non}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \asymp n^{-t/d}$$

holds for C sufficiently large. We see that the restrictive condition on the domain that was needed in the Riesz basis case can be dropped in the frame setting.

(iv) Our proof of the upper bounds in Theorem 2 is constructive. One may always use the frame pair constructed in Lemma 5 below.

4 Examples

In this section, we apply the analysis presented above to two classical examples, i.e., the Poisson equation in a Lipschitz domain and the single layer potential equation on the unit circle.

4.1 The Poisson Equation

We consider the Poisson equation in a bounded Lipschitz domain Ω contained in $\mathbb{R}^d$

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{25}$$

As usual, we study (25) in the weak formulation. Then, it can be shown that the operator $\mathcal{A} = \triangle : \; H_0^1 \longrightarrow H^{-1}$ is boundedly invertible, see, e.g., [23] for details. Hence Theorem 2 applies with $s = 1$, so that

$$e_{n,C}^{\mathrm{frame}}(S, B_q^{-1+t}(L_p(\Omega)), H_0^1(\Omega)) \asymp n^{-t/d}$$

if $t > d\,(\frac{1}{p} - \frac{1}{2})_+$.

4.2 The Single Layer Potential

As a second example we shall deal with an integral equation. Let Γ be the unit circle. Then we consider the Fredholm integral equation of the first kind

$$\mathcal{A}f(x) := -\frac{1}{2\pi} \int_\Gamma \log|x-y|\, f(y)\, d\Gamma_y = \varphi(x)\,, \qquad x \in \Gamma\,.$$

The left-hand side is called the *single layer potential.* The following is known, cf. e.g. [5]: the operator $\mathcal{A}$ belongs to $\mathcal{L}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$, where $H^{1/2}(\Gamma)$ is the collection of all functions $g \in L_2(\Gamma)$ such that

$$\int_\Gamma \int_\Gamma \frac{|g(x)-g(y)|^2}{|x-y|^2}\, d\Gamma_x\, d\Gamma_y < \infty$$

and $H^{-1/2}(\Gamma)$ its dual. Furthermore, $\mathcal{A}$ is a bijection of H onto G where

$$G := \{g \in H^{1/2}(\Gamma) : \int_\Gamma g(y)\, d\Gamma_y = 0\} \quad \text{and} \quad H := \{g \in H^{-1/2}(\Gamma) : \langle g, 1\rangle = 0\}\,.$$

The space G can be interpreted as the quotient space $H^{1/2}(\Gamma)/\mathbb{R}$ of $H^{1/2}(\Gamma)$ with $\mathbb{R}$ (the constants) and H can be interpreted as the quotient space $H^{-1/2}(\Gamma)/\mathbb{R}$ of $H^{-1/2}(\Gamma)$ with $\mathbb{R}$. By S we denote $\mathcal{A}^{-1}$, defined on G with values in H. Now we investigate $e_{n,C}^{\mathrm{frame}}(S, F, G)$ where F is chosen to be the quotient space of the Besov space $B_q^{t+1/2}(L_p(\Gamma))$ and the constants, see Subsection 5.3.2 for a definition of $B_q^{t+1/2}(L_p(\Gamma))$. We put

$$Y_q^s(L_p(\Gamma)) := \{g \in B_q^s(L_p(\Gamma)) : \langle g, 1\rangle_\Gamma = 0\}\,.$$

The same principles as above apply. Again we use a commutative diagram

$$
(26)\qquad \begin{array}{ccc} H^{1/2}(\Gamma)/\mathbb{R} & \xrightarrow{S} & H^{-1/2}(\Gamma)/\mathbb{R} \\ I & \nwarrow \quad \nearrow & S_F \\ & F := Y_q^{t+1/2}(L_p(\Gamma)). & \end{array}
$$

Here I denotes the identity and S_F the restriction of S to F. Then the outcome is as follows.

Theorem 3. *Let $0 < p, q \le \infty$ and $t > (\frac{1}{p} - \frac{1}{2})_+$. Then there exists a number C^* such that for any $C \ge C^*$ we have*

$$
e_{n,C}^{\text{frame}}(S, Y_q^{t+1/2}(L_p(\Gamma)), H) \asymp n^{-t}\,.
$$

Remark 6. *There are far-reaching extensions concerning the theory of the mapping properties of the single layer potentials. In particular, much more general curves and surfaces are discussed. We refer to* [44, Sect. 20] *for the discussion of these properties in the framework of d-sets.*

5 Proofs

5.1 Proof of Theorem 1

First we deal with Theorem 1. Here we shall work in the framework of Hilbert frame pairs. Hence we consider sequences $(f_k)_k$ and $(g_k)_k$ in a (separable) Hilbert space H such that

$$
(27)\qquad f = \sum_{k=1}^{\infty} (f, g_k) f_k
$$

for all $f \in H$, compare with Remark 3 (ii). By Remark 3 (iv), we may assume that

$$
(28)\qquad \left\| \sum_{k=1}^{\infty} c_k f_k \right\|^2 \le B^2 \cdot \sum_{k=1}^{\infty} c_k^2
$$

for arbitrary $(c_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})$. Moreover, we assume that the representation (27) is stable on $K \subset H$ in the sense that

$$
(29)\qquad A^2 \sum_{k\in\Lambda} |(f, g_k)|^2 \le \left\| \sum_{k\in\Lambda} (f, g_k) f_k \right\|^2
$$

for arbitrary $f \in K$ and $\Lambda \subset \mathbb{N}$. Moreover we assume that

$$\frac{B}{A} \le C. \tag{30}$$

We consider particular n-term approximations of $f \in K$ by subsums of (27) and their error

$$\sigma_n(f) = \inf_{|\Lambda|=n} \left\| f - \sum_{k\in\Lambda} (f, g_k) f_k \right\|. \tag{31}$$

We define

$$e_{n,C}(K, H) = \inf_{(f_k)_k, (g_k)_k} \sup_{f\in K} \sigma_n(f), \tag{32}$$

with the understanding that (27)-(31) hold true. Moreover, we define

$$e_n^{\mathrm{cont}}(K, H) := \inf_{N_n, \varphi_n} \sup_{u\in K} \|\varphi_n(N_n(u)) - u\|, \tag{33}$$

where the infimum runs over all continuous mappings $\varphi_n : \mathbb{R}^n \to H$ and $N_n : K \to \mathbb{R}^n$. Then the following result is a frame analogue of Proposition 1 from [9].

Proposition 1. *Assume that $K \subset H$ is compact and $C \ge 1$. Then*

$$e_{4n+1}^{\mathrm{cont}}(K, H) \le 2C e_{n,C}(K, H). \tag{34}$$

Proof. Assume that K, n, and $C \ge 1$ are given. Let $\varepsilon > 0$. Then there exist sequences $(f_k)_k$ and $(g_k)_k$ in H such that (27)-(30) as well as

$$\sup_{f\in K} \inf_{|\Lambda|=n} \|f - \sum_{k\in\Lambda} (f, g_k) f_k\| \le e_{n,C}(K, H) + \varepsilon \tag{35}$$

hold. Since we only consider $f \in K$, we can always assume that the index set Λ is a subset of $\{1, 2, \dots, N\}$. We only loose another ε. Here N might be large, but is finite. We write

$$L_N(f) = \sum_{k=1}^{N} (f, g_k) f_k \tag{36}$$

and obtain

$$\sup_{f\in K} \|f - L_N(f)\| \le \varepsilon \tag{37}$$

and

$$\sup_{f\in K} \inf_{|\Lambda|=n} \left\| L_N(f) - \sum_{k\in\Lambda} (f, g_k) f_k \right\| \le e_{n,C}(K, H) + 4\varepsilon. \tag{38}$$

For the n-term approximation in (38) we also write

$$f_n^* = \sum_{k \in \Lambda} a_k f_k, \tag{39}$$

hence $a_k = (f, g_k)$ and $|\Lambda| = n$ for each $f \in K$ and

$$\sup_{f \in K} \|L_N(f) - f_n^*\| \le e_{n,C}(K, H) + 4\varepsilon. \tag{40}$$

For the proof we may assume that $A = 1$. We consider the modification L_N^* of L_N defined by

$$L_N^*(f) = \sum_{k=1}^{N} a_k^* f_k, \tag{41}$$

where $a_k^* = a_k$ if $|a_k| \ge 2\beta$ and $a_k^* = 0$ if $|a_k| \le \beta$. To obtain a continuous dependence of a_k^* from a_k and, hence, a continuous mapping $L_N^* : H \to H$, we define

$$a_k^* = 2\mathrm{sgn} a_k \cdot (|a_k| - \beta)$$

if $|a_k| \in (\beta, 2\beta)$. The number $\beta > 0$ will be defined later.

Assume that for $f \in K$ there are $m > n$ of the a_k with $|a_k| \ge \beta$. Then

$$L_N f - f_n^* = \sum_{k \in \tilde{\Lambda}} a_k f_k,$$

where $\tilde{\Lambda}$ contains at least $m - n$ elements with $|a_k| \ge \beta$. Then we obtain from (29)

$$\|L_N f - f_n^*\| \ge (m - n)^{1/2} \beta$$

and with (40) we get

$$m - n \le \frac{1}{\beta^2} (e_{n,C}(K, H) + 4\varepsilon)^2. \tag{42}$$

Now we consider the sum $\sum_{|a_k| < \beta} a_k^2$ for $f \in K$. We distinguish between those k that are used for f_n^* (there are at most n of those k) and the other indices and obtain

$$\sum_{|a_k| < \beta} a_k^2 \le n\beta^2 + (e_{n,C}(K, H) + 4\varepsilon)^2. \tag{43}$$

Now we are ready to estimate $\|L_N^*(f) - L_N(f)\|$ for $f \in K$. Observe that $|a_k^* - a_k| \le \beta$ for any k. We obtain

$$\|L_N^*(f) - L_N(f)\| \le B(m\beta^2 + n\beta^2 + (e_{n,C}(K, H) + 4\varepsilon)^2)^{1/2}.$$

Using the estimate (42) for m, we obtain

$$\|L_N^*(f) - L_N(f)\| \leq B(2n\beta^2 + 2(e_{n,C}(K,H) + 4\varepsilon)^2)^{1/2}.$$

Now we define β by

$$n\beta^2 = (e_{n,C}(K,H) + 4\varepsilon)^2$$

and obtain the final error estimate (where we replace, for general A, the number B by B/A)

$$\|L_N^*(f) - L_N(f)\| \leq \frac{2B}{A}\,(e_{n,C}(K,H) + 4\varepsilon).$$

In addition we obtain

$$m \leq 2n$$

and therefore L_N^* yields a continuous $2n$-term approximation of $f \in K$ with error at most

$$\sup_{f\in K} \|L_N^*(f) - f\| \leq \frac{2B}{A}\,(e_{n,C}(K,H) + 4\varepsilon) + \varepsilon.$$

The mapping L_N^* is continuous and the image is a complex of dimension $2n$, see, e.g., [14]. Hence we have an upper bound for the so-called Aleksandrov widths, see [14] and [36]. By the famous theorem of Nöbeling, any such mapping can be factorized as $L_N^* = \varphi_{4n+1} \circ N_{4n+1}$ where $N_{4n+1} : K \to \mathbb{R}^{4n+1}$ and $\varphi_{4n+1} : \mathbb{R}^{4n+1} \to H$ are continuous. Hence the result is proved. □

Proof of Theorem 1

First we observe that

$$e_{4n+1}^{\text{cont}}(S,F,H) = e_{4n+1}^{\text{cont}}(I,S(F),H).$$

Condition (18) implies that

$$e_n^{\text{cont}}(I,S(F),H) \leq C\, e_n^{\text{cont}}(I,K,H) = C\, e_n^{\text{cont}}(K,H)$$

so that Proposition 1 yields

$$e_{4n+1}^{\text{cont}}(K,H) \leq 2C\, e_{n,C}(K,H) \leq 2C\, e_{n,C}^{\text{frame}}(I,S(F),H).$$

We also have

$$e_{n,C}^{\text{frame}}(I,S(F),H) = e_{n,C}^{\text{frame}}(S,F,H),$$

hence we finally obtain

$$e_{4n+1}^{\text{cont}}(S,F,H) \leq 2C^2\, e_{n,C}^{\text{frame}}(S,F,H).$$

5.2 Proof of Theorem 2

We want to make a general remark concerning the notation in advance. In what follows we will use the symbol $\langle \cdot, \cdot \rangle$ for different duality pairing. Which one will be always clear from the context. So we avoid indices.

5.2.1 Lower Bounds

The proof of the lower bound follows by combining Theorem 1 with the following proposition proved in [9], see also [13, 14, 16]:

Proposition 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $0 < p, q \le \infty$, $s > 0$, and*

$$t > d\left(\frac{1}{p} - \frac{1}{2}\right)_+ .$$

Then

$$e_n^{\mathrm{cont}}(S, B_q^{-s+t}(L_p(\Omega)), H_0^s(\Omega)) \asymp n^{-t/d} .$$

5.2.2 Upper Bounds

The proof of the upper bound turns out to be a little bit more complicated. However, let us mention that our proof is constructive. As a first step we reduce the proof of Theorem 2 to the proof of the following

Theorem 4. *Let Ω be as above. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and suppose that*

$$t > d\left(\frac{1}{p} - \frac{1}{2}\right)_+$$

holds. Then there exists a number C^* such that for any $C \ge C^*$ we have

$$e_{n,C}^{\mathrm{frame}}(I, B_q^{s+t}(L_p(\Omega)), B_2^s(L_2(\Omega))) \ \lesssim\ n^{-t/d} .$$

Proof of Theorem 2. Since $H^{-s}(\Omega) = B_2^{-s}(L_2(\Omega))$, cf. Remark 10, Theorem 4 yields that

$$e_{n,C}^{\mathrm{frame}}(I, B_q^{-s+t}(L_p(\Omega)), H^{-s}(\Omega)) \ \lesssim\ n^{-t/d}.$$

Since $S : H^{-s}(\Omega) \to H_0^s(\Omega)$ is an isomorphism, Lemma 2 implies the desired result. □

5.2.3 Widths and Discrete Besov Spaces

The proof of Theorem 4 requires several preparations. First of all, let us fix some notation. Let $0 < p,q \le \infty$ and let $s \in \mathbb{R}$. Let $\nabla := (\nabla_j)_{j=-1}^{\infty}$ be a sequence of subsets of finite cardinality of the set $\{1,2,\ldots,2^d-1\} \times \mathbb{Z}^d$. We suppose that there exist $0 < C_1 \le C_2$ and $J \in \mathbb{N}$ such that the cardinality $|\nabla_j|$ of ∇_j satisfies

$$C_1 \le 2^{-jd}\,|\nabla_j| \le C_2 \qquad \text{for all} \quad j \ge J\,. \tag{44}$$

Then $b^s_{p,q}(\nabla)$, where $0 < q < \infty$, denotes the collection of all sequences $a = (a_{j,\lambda})_{j,\lambda}$ of complex numbers such that

$$\|\, a \,\|_{b^s_{p,q}} := \left(\sum_{j=-1}^{\infty} 2^{j(s+d(1/2-1/p))q} \Big(\sum_{\lambda \in \nabla_j} |\, a_{j,\lambda}|^p \Big)^{q/p} \right)^{1/q} < \infty\,. \tag{45}$$

For $q = \infty$, we use the usual modification

$$\|\, a \,\|_{b^s_{p,\infty}} := \sup_{j=-1,0,1,\ldots} 2^{j(s+d(1/2-1/p))} \left(\sum_{\lambda \in \nabla_j} |a_{j,\lambda}|^p \right)^{1/p} < \infty. \tag{46}$$

In our paper [9] we have dealt with several types of widths of embeddings of those discrete Besov spaces. A few of the results we obtained there will be recalled now.

Proposition 3. *Let $0 < p,q \le \infty$ and $s \in \mathbb{R}$. Suppose that*

$$t > d\left(\frac{1}{p} - \frac{1}{2}\right)_+ . \tag{47}$$

It holds

$$e_n^{\mathrm{cont}}(I, b^{s+t}_{p,q}(\nabla), b^s_{2,2}(\nabla)) \asymp e_n^{\mathrm{non}}(I, b^{s+t}_{p,q}(\nabla), b^s_{2,2}(\nabla)) \asymp n^{-t/d}\,.$$

Remark 7. *Of course, the constants in the above inequalities depend on ∇ (and therefore on C_1, C_2 and J) as well as on s,t,p and q. But this will play no role in what follows.*

5.2.4 Frame Pairs for Sobolev Spaces on Domains

Now we turn to the construction of frame pairs for Sobolev spaces with some additional features.

Let $s \in \mathbb{R}$ be fixed and let

(48)

$$\Psi := \Big\{ \varphi_k, \widetilde{\varphi}_k :\ k \in \mathbb{Z}^d \Big\} \cup \Big\{ \psi_{i,j,k}, \widetilde{\psi}_{i,j,k} : \quad i = 1,\ldots 2^d-1,\ j = 0,1,2\ldots, k \in \mathbb{Z}^d \Big\},$$

be a biorthogonal wavelet system such that the parameter r, controlling the smoothness and the moment conditions, satisfies $r > |s|$, see Proposition 4 in the Appendix. Here, as always in this subsection we shall use $H^s(\Omega) = B^s_2(L_2(\Omega))$ in the sense of equivalent norms, see the Appendix. We suppose

$$\operatorname{supp}\varphi\,,\ \operatorname{supp}\psi_i\,,\ \operatorname{supp}\widetilde{\varphi}\,,\ \operatorname{supp}\widetilde{\psi}_i \subset [-N,N]^d\,, \qquad i = 1,\ldots 2^d-1\,.$$

By $B(x^0,R)$ we denote a ball with radius R and center x^0. We may assume $\Omega \subset B(x^0,R)$ for some $R > 0$ and $x^0 \in \Omega$. Ryshkov [33] has proved that in case of a bounded Lipschitz domain there exists a linear and continuous extension operator $\mathcal{E} \in \mathcal{L}(H^s(\Omega) \to H^s(\mathbb{R}^d))$. In addition we may assume that

$$\operatorname{supp}\ \mathcal{E}f\ \subset B(x^0, 2R) \tag{49}$$

holds for all $f \in H^s(\Omega)$. Now we turn to the wavelet decomposition of $\mathcal{E}f$. Defining

$$\Lambda_j := \Big\{ k \in \mathbb{Z}^d: \quad |2^{-j}k_i - x_i^0| \le 2R + 2^{-j}N\,, \quad i = 1,\ldots,d \Big\}\,, \qquad j = 0,1,\ldots\,,$$

we obtain for given $f \in H^s(\Omega)$

$$\mathcal{E}f = \sum_{k\in\Lambda_0} \langle \mathcal{E}f, \widetilde{\varphi}_k\rangle\, \varphi_k + \sum_{i=1}^{2^d-1}\sum_{j=0}^{\infty}\sum_{k\in\Lambda_j} \langle \mathcal{E}f, \widetilde{\psi}_{i,j,k}\rangle\, \psi_{i,j,k} \qquad (\text{convergence in}\quad \mathcal{S}') \tag{50}$$

and

$$\|\,\mathcal{E}f\,|H^s(\mathbb{R}^d)\| \asymp \Big(\sum_{k\in\Lambda_0} |\langle \mathcal{E}f, \widetilde{\varphi}_k\rangle|^2\Big)^{1/2} + \Big(\sum_{i=1}^{2^d-1}\sum_{j=0}^{\infty} 2^{2js}\Big(\sum_{k\in\Lambda_j} |\langle \mathcal{E}f, \widetilde{\psi}_{i,j,k}\rangle|^2\Big)\Big)^{1/2} < \infty\,. \tag{51}$$

This can be rewritten by using

$$\nabla_{-1} := \Lambda_0 \tag{52}$$

$$\nabla_j := \Big\{(i,k): \quad 1 \le i \le 2^d-1\,, \quad k \in \Lambda_j\Big\}\,, \quad j = 0,1,\ldots\,, \tag{53}$$

$\psi_{j,\lambda} := \psi_{i,j,k}$, if $\lambda = (i,k) \in \nabla_j$, $j \in \mathbb{N}_0$, and $\psi_{j,\lambda} := \varphi_k$ if $\lambda = k \in \nabla_{-1}$. Similarly in case of the dual basis. Then (50), (51) read as

$$\mathcal{E}f = \sum_{j=-1}^{\infty}\sum_{\lambda\in\nabla_j} \langle \mathcal{E}f, \widetilde{\psi}_{j,\lambda}\rangle\, \psi_{j,\lambda} \qquad (\text{convergence in}\quad \mathcal{S}') \tag{54}$$

and

$$\|\,f\,|H^s(\Omega)\| \asymp \|\,\mathcal{E}f\,|H^s(\mathbb{R}^d)\| \asymp \|\,(\langle \mathcal{E}f, \widetilde{\psi}_{j,\lambda}\rangle)_{j,\lambda}\,\|_{b^s_{2,2}(\nabla)}\,. \tag{55}$$

Let $\mathcal{X}_\Omega$ denote the characteristic function of Ω. We put

$$g_{j,\lambda} := \mathcal{X}_\Omega \, \psi_{j,\lambda} \, , \qquad j = -1, 0, 1, \dots \, , \qquad \lambda \in \nabla_j \, . \tag{56}$$

For $M \in \mathbb{N}$ we have

$$\sum_{j=-1}^{M} \sum_{\lambda \in \nabla_j} \langle \mathcal{E} f, \widetilde{\psi}_{j,\lambda} \rangle \, g_{j,\lambda} = \Big(\sum_{j=-1}^{M} \sum_{\lambda \in \nabla_j} \langle \mathcal{E} f, \widetilde{\psi}_{j,\lambda} \rangle \, \psi_{j,\lambda} \Big)\Big|_{\Omega}$$

and consequently

$$\lim_{M \to \infty} \sum_{j=-1}^{M} \sum_{\lambda \in \nabla_j} \langle \mathcal{E} f, \widetilde{\psi}_{j,\lambda} \rangle \, g_{j,\lambda} = (\mathcal{E} f)_{|\Omega} = f$$

in $H^s(\Omega)$. Let $\mathcal{E}^*$ denote the adjoint of $\mathcal{E}$. Define

$$h_{j,\lambda} = \mathcal{E}^*(\widetilde{\psi}_{j,\lambda}) \, , \qquad j = -1, 0, 1, \dots \, , \qquad \lambda \in \nabla_j \, . \tag{57}$$

Then, taking into account the norm equivalences (55), it follows that $(\mathcal{F}, \mathcal{G})$ is a frame pair for $(H^s(\Omega), b^s_{2,2}(\nabla))$, where

$$\mathcal{F} \;=\; \{h_{j,\lambda} : \quad j = -1, 0, 1, \dots \, , \quad \lambda \in \nabla_j\} \qquad \text{and} \tag{58}$$

$$\mathcal{G} \;=\; \{g_{j,\lambda} : \quad j = -1, 0, 1, \dots \, , \quad \lambda \in \nabla_j\} \, . \tag{59}$$

Instead of writing (H, w) we used here the notation $(H, \ell_{2,w})$, see Definition 1. We collect our findings in the following lemma.

Lemma 5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let Ψ be a wavelet system, see* (48)*, such that $r > |s|$, see Proposition* 4*. Let $\mathcal{F}$ and $\mathcal{G}$ be defined as in* (56)*-*(59)*. Then $(\mathcal{F}, \mathcal{G})$ is a frame pair for $(H^s(\Omega), b^s_{2,2}(\nabla))$, where $\nabla = \nabla(\Omega)$ is defined in* (52)*,* (53)*.*

5.2.5 Stability of Frame Pairs

Next we need to investigate the stability of this frame pair constructed in the previous subsection. The symbol ∇ will always refer to $\nabla = \nabla(\Omega)$ defined in (52), (53). Let $0 < p, q \le \infty$ and suppose $t > d(\frac{1}{p} - \frac{1}{2})_+$. Furthermore, we require that the parameter r of the wavelet system satisfies

$$r > \max\Big(s + t, d \max(0, \frac{1}{p} - 1) - s, d \max(0, \frac{1}{p} - 1) - (s + t) \Big) \, , \tag{60}$$

see Proposition 4. We choose a rectangular subset $\square$ of Ω such that $\operatorname{dist}(\square, \partial\Omega) > 0$. Then we define

$$\nabla_j^* := \Big\{ (i, k) \in \Lambda_j : \quad \operatorname{supp} \psi_{j,\lambda} \subset \square \Big\} \, , \quad j = 0, 1, \dots \, , \tag{61}$$

Of course, it may happen that $\nabla_j^* = \emptyset$ if j is small. Let $J \in \mathbb{N}$ be a number such that $\nabla_j^* \neq \emptyset$ for all $j \geq J$. Then we put

(62)

$$K := \Big\{ f \in \mathcal{D}'(\Omega) : \quad \text{there exists } (a_{j,\lambda})_{j,\lambda} \in b^{s+t}_{p,q}(\nabla^*) \text{ s.t.} \quad f = \sum_{j=J}^{\infty} \sum_{\lambda \in \nabla_j^*} a_{j,\lambda}\, \psi_{j,\lambda} \Big\} .$$

Because of dist $(\square, \partial\Omega) > 0$ we can extend f by zero outside of Ω and obtain from Proposition 4 that $K \subset B^{s+t}_q(L_p(\Omega))$. Again making use of Proposition 4 we find that

$$\Big\| \sum_{(j,\lambda)\in\Lambda} a_{j,\lambda}\, \psi_{j,\lambda} \Big\|_{H^s(\Omega)} \asymp \Big\| \sum_{(j,\lambda)\in\Lambda} a_{j,\lambda}\, \psi_{j,\lambda} \Big\|_{H^s(\mathbb{R}^d)} \asymp \| \, (a_{j,\lambda})_{(j,\lambda)\in\Lambda} \, \|_{b^s_{2,2}(\nabla^*)} ,$$

if $\Lambda \subset \bigcup_{j=J}^{\infty} \nabla_j^*$. Here the constants do not depend on Λ.

Finally we have to show that K is sufficiently large or more exactly, that $K \in \mathcal{K}_C$ for some sufficiently large C. By definition of K the mapping

$$T : f \mapsto (\langle f, \widetilde{\psi}_{j,\lambda} \rangle)_{(j,\lambda)\in\nabla_j^*}$$

belongs to $\mathcal{L}(K, b^{s+t}_{p,q}(\nabla^*))$. Moreover, it is invertible and $T^{-1} \in \mathcal{L}(b^{s+t}_{p,q}(\nabla^*), K)$. Once again we shall use the extension operator $\mathcal{E}$. In addition we apply the fact that $\mathcal{E}$ may be chosen such that $\mathcal{E} \in \mathcal{L}(B^{s+t}_q(L_p(\Omega)), B^{s+t}_q(L_p(\mathbb{R}^d)))$, cf. Ryshkov [33]. Now we extend T by defining

$$T : f \mapsto (\langle \mathcal{E} f, \widetilde{\psi}_{j,\lambda} \rangle)_{(j,\lambda)\in\nabla_j} .$$

This extension is again bounded, cf. Proposition 4. Let us have a look at the commutative diagram

$$\begin{array}{ccc} b^{s+t}_{p,q}(\nabla^*) & \xrightarrow{\;I_1\;} & b^s_{2,2}(\nabla) \\ {\scriptstyle T^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle T} \\ K & \xrightarrow{\;I_2\;} & B^s_2(L_2(\Omega)) . \end{array}$$

Because of $\nabla_j^* \subset \nabla_j$, $j \geq J$, there is a natural embedding operator between these sequence spaces, here denoted by I_1. Since $T \in \mathcal{L}(B^s_2(L_2(\Omega)), b^s_{2,2}(\nabla))$ we can apply (20) and conclude

$$e_n^{\text{cont}}(I_1, b^{s+t}_{p,q}(\nabla^*), b^s_{2,2}(\nabla)) \leq \| \, T^{-1} \, \| \, \| \, T \, \| \, e_n^{\text{cont}}(I_2, K, B^s_2(L_2(\Omega))) . \tag{63}$$

Furthermore

$$e_n^{\text{cont}}(I_1, b^{s+t}_{p,q}(\nabla^*), b^s_{2,2}(\nabla^*)) = e_n^{\text{cont}}(I_1, b^{s+t}_{p,q}(\nabla^*), b^s_{2,2}(\nabla)) .$$

To explain this we split $b^s_{2,2}(\nabla))$ into $b^s_{2,2}(\nabla^*)$ and its orthogonal complement U. Then the claimed identity follows from the observation that optimal approximations $S_n = \varphi_n \circ N_n$, see (5), of elements of $b^{s+t}_{p,q}(\nabla^*))$ are obtained with $\varphi_n : \mathbb{R}^n \to b^s_{2,2}(\nabla^*)$. The behavior of the left-hand side in (63) is known, see Proposition 3. As a consequence we obtain

$$\begin{aligned} c_1\, n^{-t/d} & \le & e_n^{\mathrm{cont}}(I_1, b^{s+t}_{p,q}(\nabla^*), b^s_{2,2}(\nabla^*)) = e_n^{\mathrm{cont}}(I_1, b^{s+t}_{p,q}(\nabla^*), b^s_{2,2}(\nabla)) \\ & \le & c_2\, e_n^{\mathrm{cont}}(I_2, K, B^s_2(L_2(\Omega))) \end{aligned} \tag{64}$$

with some positive c_1, c_2. Summarizing we have proved that the frame pair $(\mathcal{F}, \mathcal{G})$ from Lemma 5 is admissible in the sense of Definition 2 for C sufficiently large.

Lemma 6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\Box$ be a rectangular subset of Ω such that* $\operatorname{dist}(\Box, \partial\Omega) > 0$. *Let $s \in \mathbb{R}$, $0 < p, q \le \infty$ and $t > d(\frac{1}{p} - \frac{1}{2})_+$. Let Ψ be a wavelet system, see* (48)*, such that r satisfies* (60)*, see Proposition* 4*. Let $\mathcal{F}$ and $\mathcal{G}$ be defined as in* (56)-(59)*. Then the frame pair $(\mathcal{F}, \mathcal{G})$ is stable with respect to the set K defined in* (62)*, i.e. it belongs to $\mathcal{P}_C(K)$, and it also belongs to $\mathcal{K}_C \subset B^{s+t}_q(L_p(\Omega))$ if C is sufficiently large.*

5.2.6 Proof of Theorem 4

To prove Theorem 4 we shall use the frame pair from Lemmata 5 and 6.
Let $\Lambda \subset \nabla$ be a set of cardinality n. Then

$$\begin{aligned} \sigma_n(f, (\mathcal{F}, \mathcal{G}))_{B^s_2(L_2(\Omega))} & \le \Big\| \sum_{(j,\lambda)\notin\Lambda} \langle f, \mathcal{E}^* \tilde{\psi}_{j,\lambda}\rangle\, g_{j,\lambda} \Big\|_{B^s_2(L_2(\Omega))} \\ & \le c_1 \, \| \, (\langle f, \mathcal{E}^* \tilde{\psi}_{j,\lambda}\rangle)_{(j,\lambda)\notin\Lambda} \, \|_{b^s_{2,2}} , \end{aligned}$$

where we have once again used (16). By $\mathcal{O}$ we denote the canonical orthonormal basis of $b^0_{2,2}(\nabla)$ and by $e_{j,\lambda}$ its elements, respectively. For $a \in b^s_{2,2}(\nabla)$ we put

$$\sigma_n\big(a, \mathcal{O}\big)_{b^s_{2,2}} := \inf_{|\Lambda|=n} \Big\| \sum_{(j,\lambda)\notin\Lambda} a_{j,\lambda}\, e_{j,\lambda} \Big\|_{b^s_{2,2}(\nabla)} .$$

If Λ contains the n largest terms $2^{js}\, |\langle f, \mathcal{E}^* \tilde{\psi}_{j,\lambda}\rangle|$ then

$$\sigma_n(f, (\mathcal{F}, \mathcal{G}))_{B^s_2(L_2(\Omega))} \le c_1\, \sigma_n\Big((\langle f, \mathcal{E}^* \tilde{\psi}_{j,\lambda}\rangle)_{(j,\lambda)\in\nabla}, \mathcal{O}\Big)_{b^s_{2,2}}$$

follows. Next we shall use the following abbreviations: let $F_1 = B^{s+t}_q(L_p(\Omega))$ and $F_2 = b^{s+t}_{p,q}(\nabla)$. Using Proposition 3 with respect to ∇ and a simple homogeneity argument we find

$$\sup_{\|f\|_{F_1} \le 1} \sigma_n(f, (\mathcal{F}, \mathcal{G}))_{B^s_2(L_2(\Omega))} \le c_2 \sup_{\|a\|_{F_2} \le 1} \sigma_n(a, \mathcal{O})_{b^s_{2,2}} \le c_3\, n^{-t/d} ,$$

since

$$\| \, (\langle f, \mathcal{E}^* \tilde{\psi}_{j,\lambda} \rangle)_{j,\lambda \in \nabla} \|_{b^{s+t}_{p,q}} \asymp \| \, f \, \|_{B^{s+t}_q(L_p(\Omega))} \, .$$

This completes the proof of Theorem 4. □

Remark 8. *The advantage of our frame construction consists in the fact that it is universal for all bounded Lipschitz domains. The disadvantage of our frame construction lies in the use of the operator $\mathcal{E}^*$. This limits its value in case of concrete calculations. There are other frame constructions in the literature. Let us mention here the constructions given in* [4], [47] *and* [6]*. We add a few comments to these frames:*

- *The frame pairs constructed in* [4] *allow a discretization of Besov spaces on domains Ω under certain restrictions, both with respect to the domains and with respect to the parameters of the Besov space. In particular, only the case $1 \le p \le \infty$, $0 < q \le \infty$ and $s > 0$ is considered. With $(\mathcal{F}, \mathcal{G})$ denoting the frame pairs constructed in the aforementioned paper we obtain*

$$\sup_{\|f\|_{F_1} \le 1} \sigma_n(f, (\mathcal{F}, \mathcal{G}))_{H^{-s}(\Omega)} \asymp n^{-t/d}$$

  where

$$F_1 := B^{-s+t}_q(L_p(\Omega)) \, , \qquad t - s > 0 \, , \quad 1 \le p, q \le \infty \, .$$

  Generalization to the case $0 < q, p < 1$ have been given in [15].

- *The frames constructed in* [47] *allow a discretization of Besov spaces on Lipschitz domains Ω under the restrictions $0 < p, q \le \infty$ and $s < 0$. The frame pairs consist of either wavelets originating from a wavelet basis on $\mathbb{R}^d$ or dilated and shifted versions of the associated scaling function. They all have the property that their support is contained in Ω. Furthermore, these dilated and shifted copies of the scaling functions show up only near the boundary. Inside a box contained in Ω and with some distance to the boundary the frame pair reduces to a biorthogonal wavelet subsystem. The same construction can be made to discretize the Besov spaces $\widetilde{B}^s_q(L_p(\Omega))$ if $s > d \max(0, 1/p - 1)$, see the Appendix for a definition. Hence, with $(\mathcal{F}, \mathcal{G})$ denoting the frame pair of* [47] *we obtain*

$$\sup_{\|f\|_{F_1} \le 1} \sigma_n(f, (\mathcal{F}, \mathcal{G}))_{H^{-s}(\Omega)} \asymp n^{-t/d},$$

  where

$$F_1 := \begin{cases} B^{-s+t}_q(L_p(\Omega)) & \text{if} \quad t - s < 0 \\ \widetilde{B}^{-s+t}_q(L_p(\Omega)) & \text{if} \quad t - s > \, d \max(0, \frac{1}{p} - 1). \end{cases}$$

- *The frame pairs constructed in* [6] *allow a discretization of* $H^s(\Omega)$*-spaces with* $s > 0$*. This construction works for domains with piecewise analytic boundary and is based on an* overlapping *partition of the domain by means of sufficiently smooth parametric images of the unit cube. On the reference cube, a tensor product biorthogonal wavelet basis employing the boundary adapted wavelets on the interval from* [10] *is constructed. Under certain conditions, the union of all the parametric images of these bases gives rise to frame pair for* $H^s(\Omega),\ s > 0$.

- *Of course, all the examples of biorthogonal wavelet bases on polyhedral domains also fit into our setting. One natural way as, e.g., outlined in* [1] *and* [11]*, is to decompose the domain into a* disjoint *union of parametric images of reference cubes. Then one constructs wavelet bases on the reference cubes and glues everything together in a judicious fashion. However, due to the glueing procedure, only Sobolev spaces* H^s *with smoothness* $s < 3/2$ *can be characterized. This bottleneck can be circumvented by the approach in* [12]*. There, a much more tricky domain decomposition method involving certain projection and extension operators is used. By proceeding in this way, norm equivalences for all spaces* $B^t_q(L_p(\Omega))$ *can be derived, at least for the case* $p > 1$*, see* [12, Theorem 3.4.3]*. However, the authors also mention that their results can be generalized to the case* $p < 1$*, see* [12, Remark 3.1.2].

5.3 Proof of Theorem 3

Periodic Besov spaces have analoguous properties than the Besov spaces defined on smooth domains or on $\mathbb{R}^d$. Our general reference for these classes is [34]. A definition of periodic Besov spaces is given in the Appendix.

5.3.1 Widths of Periodic Besov Spaces

As a preparation of the proof of Theorem 3 we shall investigate the widths of embeddings of periodic Besov spaces, a topic which is also of self-contained interest. In [9] we reduced the corresponding problem for the nonperiodic Besov spaces on a Lipschitz domain to that one for the discrete Besov spaces. It would be of interest to construct an isomorphism between these periodic spaces $B^s_q(L_p(\mathbb{T}))$ and $b^s_{p,q}$ as well, see Subsection 5.2.3. Periodic wavelet constructions exist in the literature. However, up to our knowledge, those characterizations of periodic Besov spaces are established only with additional restrictions for the parameters. So we employ a different strategy here.

Theorem 5. *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and suppose that*

$$t > \Big(\frac{1}{p} - \frac{1}{2}\Big)_+$$

holds. Then there exists a constant C^* such that for any $C \ge C^*$ we have

$$e_{n,C}^{\text{frame}}(I, B_q^{s+t}(L_p(\mathbb{T})), B_2^s(L_2(\mathbb{T}))) \asymp n^{-t}\,.$$

Proof. *Step 1.* Preparations. For the estimate from above we shall use a connection between periodic and weighted spaces. Let $\varrho_\kappa(x) := (1+|x|^2)^{-\kappa/2}$, $x \in \mathbb{R}$, $\kappa > 0$. We define

$$B_q^s(L_p(\mathbb{R}, \varrho_\kappa)) := \Big\{ f \in \mathcal{S}'(\mathbb{R}) : \quad f\, \varrho_\kappa \in B_q^s(L_p(\mathbb{R})) \Big\}\,, \tag{65}$$

endowed with the natural quasi-norm

$$\|\, f\, | B_q^s(L_p(\mathbb{R}, \varrho_\kappa))\| := \|\, f\, \varrho_\kappa\, | B_q^s(L_p(\mathbb{R}))\|\,.$$

Here $\mathcal{S}'(\mathbb{R})$ denotes the collection of the tempered distributions on $\mathbb{R}$. As a combination of Franke's characterization of weighted spaces, see Theorem 5.1.3 in [34], and a result of Triebel [41] we find that $f \in B_q^s(L_p(\mathbb{T}))$ if and only if f is a 2π-periodic distribution in $\mathcal{S}'(\mathbb{R})$ which belongs to $B_q^s(L_p(\mathbb{R}, \varrho_\kappa))$ with $\kappa > (1/p)$. Moreover, there exist positive constants c_1, c_2 such that

$$c_1 \,\|\, f\, | B_q^s(L_p(\mathbb{R}, \varrho_\kappa))\| \le \|\, f\, | B_q^s(L_p(\mathbb{T})\| \le c_2 \,\|\, f\, | B_q^s(L_p(\mathbb{R}, \varrho_\kappa))\|$$

holds for all such f.

Step 2. Let $\psi \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function such that $\psi(x) = 1$ if $|x| \le \pi$ and $\psi(x) = 0$ if $|x| \ge 2\pi$. We shall study the mapping $T : f \mapsto \psi \cdot f$. Let $J = [-3\pi, 3\pi]$. Obviously

$$\begin{aligned} \|\, f\,\psi\, | B_q^s(L_p(J))\| &\le \|\, f\,\psi\, | B_q^s(L_p(\mathbb{R}))\| = \|\, f\,\psi\,\varrho_\kappa\,(1/\varrho_\kappa)\ \psi(\cdot/2)\, | B_q^s(L_p(\mathbb{R}))\| \\ &\le c_3 \,\|\, (1/\varrho_\kappa)\ \psi(\cdot/2)\, | C^\mu(\mathbb{R})\| \,\|\, f\,\psi\,\varrho_\kappa\, | B_q^s(L_p(\mathbb{R}))\| \\ &\le c_4 \,\|\, f\,\psi\, | B_q^s(L_p(\mathbb{R}, \varrho_\kappa))\|\,. \end{aligned}$$

where μ has to be chosen sufficiently large, cf. e.g. [42, 2.8] or [32, 4.7]. Since ψ is a pointwise multiplier for these weighted Besov spaces as well we end up with $T \in \mathcal{L}(B_q^s(L_p(\mathbb{T})), B_q^s(L_p(J)))$. Moreover, T is a bijection onto a closed subspace of $B_q^s(L_p(J))$, denoted by $T_q^s(L_p(J))$, simultenuously for all parameters. Now we consider the commutative diagram:

$$\begin{array}{ccc} B_q^{s+t}(L_p(\mathbb{T})) & \xrightarrow{\ I_1\ } & B_2^{s+t}(L_2(\mathbb{T})) \\ {\scriptstyle T}\big\downarrow & & \big\uparrow{\scriptstyle T^{-1}} \\ T_q^{s+t}(L_p(J)) & \xrightarrow{\ I_2\ } & T_2^s(L_2(J))\,. \end{array}$$

Lemma 3 yields

$$e^{\text{frame}}_{n,\widetilde{C}}(I_1, B^{s+t}_q(L_p(\mathbb{T})), B^s_2(L_2(\mathbb{T}))) \le \|T\|\,\|T^{-1}\|\, e^{\text{frame}}_{n,C}(I_2, T^{s+t}_q(L_p(J)), T^s_2(L_2(J)))$$

with $\widetilde{C} = C\,\|T^{-1}\|\,\|T\|$. Now we employ Lemma 4 and obtain

$$e^{\text{frame}}_{n,C}(I_2, T^{s+t}_q(L_p(J)), T^s_2(L_2(J))) \le e^{\text{frame}}_{n,C}(I_2, T^{s+t}_q(L_p(J)), B^s_2(L_2(J)))\,.$$

This, together with a monotonicity arguments leads to

$$e^{\text{frame}}_{n,\widetilde{C}}(I_1, B^{s+t}_q(L_p(\mathbb{T})), B^s_2(L_2(\mathbb{T}))) \le \|T\|\,\|T^{-1}\|\, e^{\text{frame}}_{n,C}(I_2, B^{s+t}_q(L_p(J)), B^s_2(L_2(J)))\,.$$

The estimate from above is finished by using Theorem 4 with $\Omega = J$ and $d = 1$.
Step 3. Let $J = (-1/2, 1/2)$. Then there exists a linear extension operator $\mathcal{E}$: $B^s_q(L_p(J)) \to B^s_q(L_p(\mathbb{R}))$, see [33]. Let ψ be as above. We define

$$Tf(x) := \begin{cases} \mathcal{E}f(x)\,\psi(6x) & \text{if} \quad -\pi \le x \le \pi\,, \\ 2\pi\text{-periodic extension} & \text{otherwise}\,. \end{cases}$$

We claim that $T \in \mathcal{L}(B^s_q(L_p(J)), B^s_q(L_p(\mathbb{T})))$ for all parameter constellations. To see that we first construct an appropriate decomoposition of unity. We put

$$\varphi(x) := \frac{\psi(x)}{\sum_{k=-\infty}^{\infty} \psi(x - 2\pi k)}\,, \quad x \in \mathbb{R}\,.$$

It follows that

$$1 = \sum_{m=-\infty}^{\infty} \varphi(x - 2\pi m) \qquad \text{for all} \quad x \in \mathbb{R}$$

and $\operatorname{supp}\varphi \subset \{x \in \mathbb{R} : \psi(x/2) = 1\}$. Hence, with $t = \min(1, p, q)$ and $\kappa > 1/t \ge 1/p$, we obtain

$$\begin{aligned}
&\|\,Tf\,|B^s_q(L_p(\mathbb{T}))\|^t \le c_2^t\,\|\,(Tf)\,\varrho_\kappa\,|B^s_q(L_p(\mathbb{R}))\|^t \\
&= \; c_2^t\,\|\sum_{m=-\infty}^{\infty} \varphi(\cdot - 2\pi m)\,(Tf)\,\varrho_\kappa\,|B^s_q(L_p(\mathbb{R}))\|^t \\
&\le \; c_2^t \sum_{m=-\infty}^{\infty} \|\,\varphi(\cdot - 2\pi m)\,(Tf)\,\varrho_\kappa\,|B^s_q(L_p(\mathbb{R}))\|^t \\
&= \; c_2^t \sum_{m=-\infty}^{\infty} \|\,\varphi(\cdot - 2\pi m)\,\psi\Big(\frac{\cdot - 2\pi m}{2}\Big)\,(Tf)\,\varrho_\kappa\,|B^s_q(L_p(\mathbb{R}))\|^t \\
&\le \; c_3 \sum_{m=-\infty}^{\infty} \|\,\varphi(\cdot - 2\pi m)\,\varrho_\kappa\,|C^\mu(\mathbb{R})\|^t\,\|\,\psi\Big(\frac{\cdot - 2\pi m}{2}\Big)\,(Tf)\,|B^s_q(L_p(\mathbb{R}))\|^t\,,
\end{aligned}$$

where we used again assertions on pointwise multipliers, see, e.g., [42, 2.8] or [32, 4.7]. The shift-invariance of $\| \cdot \, |B^s_q(L_p(\mathbb{R}))\|$ and the periodicity of Tf imply

$$\| \, \psi\Big(\frac{\cdot - 2\pi m}{2}\Big) \, (Tf) \, |B^s_q(L_p(\mathbb{R}))\| = \| \, \psi(\, \cdot \, /2) \, (Tf) \, |B^s_q(L_p(\mathbb{R}))\|$$

for all $m \in \mathbb{Z}$. Furthermore, elementary calculations yield

$$\| \, \varphi(\cdot - 2\pi m) \, \varrho_\kappa \, |C^\mu(\mathbb{R})\| \le c_4 \, \varrho_\kappa(2\pi m)$$

with c_4 independent of m. Altogether this proves

$$\begin{aligned} \| \, Tf \, |B^s_q(L_p(\mathbb{T}))\| & \le c_5 \, \| \, \psi(\, \cdot \, /2) \, (Tf) \, |B^s_q(L_p(\mathbb{R}))\| \Big(\sum_{m=-\infty}^{\infty} \varrho(2\pi m)^t \Big)^{1/t} \\ & \le c_6 \, \| \, \psi(\, \cdot \, /2) \, (Tf) \, |B^s_q(L_p(\mathbb{R}))\| \, . \end{aligned}$$

Taking into account the identity

$$\psi(x/2) \, Tf(x) = \psi(x/2) \, \Big(\sum_{m=-2}^{2} \mathcal{E}f(x - 2\pi m) \, \psi(6(x - 2\pi m)) \Big)$$

we have

$$\begin{aligned} \| \, \psi(\, \cdot \, /2) \, (Tf) \, |B^s_q(L_p(\mathbb{R}))\| & \le c_7 \sum_{m=-2}^{2} \| \, \psi(\, \cdot \, /2) \, \mathcal{E}f(x - 2\pi m) \, \psi(6(x - 2\pi m)) \, |B^s_q(L_p(\mathbb{R}))\| \\ & \le c_8 \sum_{m=-2}^{2} \| \, \mathcal{E}f(x - 2\pi m) \, \psi(6(x - 2\pi m)) \, |B^s_q(L_p(\mathbb{R}))\| \\ & \le c_9 \| \, \mathcal{E}f \, \psi(6(\, \cdot \,)) \, |B^s_q(L_p(\mathbb{R}))\| \\ & \le c_{10} \| \, \mathcal{E}f \, |B^s_q(L_p(\mathbb{R}))\| \\ & \le c_{10} \, \|\mathcal{E}\| \, \| \, f \, |B^s_q(L_p(J))\| \, , \end{aligned}$$

which proves the claim. Moreover, T is a bijection onto a closed subspace of $B^s_q(L_p(\mathbb{T}))$. This subspace will be denoted by $T^s_q(L_p(\mathbb{T}))$. Now we can argue as in Step 2. The commutative diagram

$$\begin{array}{ccc} B^{s+t}_q(L_p(J)) & \xrightarrow{I_1} & B^{s+t}_2(L_2(J)) \\ {\scriptstyle T}\big\downarrow & & \big\uparrow{\scriptstyle T^{-1}} \\ T^{s+t}_q(L_p(\mathbb{T})) & \xrightarrow{I_2} & T^s_2(L_2(\mathbb{T})) \end{array}$$

implies

$$e^{\rm frame}_{n,\widetilde{C}}(I_1, B^{s+t}_q(L_p(J)), B^s_2(L_2(J))) \le \|T\| \, \|T^{-1}\| \, e^{\rm frame}_{n,C}(I_2, B^{s+t}_q(L_p(\mathbb{T})), B^s_2(L_2(\mathbb{T}))) \, .$$

with $\widetilde{C} = C \, \|T^{-1}\| \, \|T\|$. The estimate from below is finished by using Theorem 4 with $\Omega = J$ and $d = 1$. □

Now we consider some subspaces of $B^s_q(L_p(\mathbb{T}))$. Let

$$Z^s_q(L_p(\mathbb{T})) := \Big\{ f \in B^s_q(L_p(\mathbb{T})) : \quad \langle f, 1\rangle_{\mathbb{T}} = 0 \Big\} . \tag{66}$$

Observe that the function $g(x) = 1$ belongs to $D(\mathbb{T})$, the collection of all complex-valued, 2π-periodic and infinitely differentiable function. Since

$$D(\mathbb{T}) \hookrightarrow B^s_q(L_p(\mathbb{T})) \hookrightarrow D'(\mathbb{T})$$

the scalar product $\langle f, 1\rangle_{\mathbb{T}}$ is well-defined for all $f \in B^s_q(L_p(\mathbb{T}))$, cf. [34, 3.5.1].

Corollary 1. *Let* $0 < p, q \le \infty$, $s \in \mathbb{R}$ *and suppose that*

$$t > \Big(\frac{1}{p} - \frac{1}{2}\Big)_+$$

holds. Then there exists a constant C^* *such that for any* $C \ge C^*$ *we have*

$$e^{\text{frame}}_{n,C}(I, Z^{s+t}_q(L_p(\mathbb{T})), Z^s_2(L_2(\mathbb{T}))) \asymp n^{-t} .$$

Proof. The upper estimate can be established as above. For the estimate from below we start with $f \in B^s_q(L_p(J))$ and $J = [-1/2, -1/4]$. The operator T has to be replaced by

$$\widetilde{T} f(x) := \begin{cases} \mathcal{E} f(x)\, \psi(14(x+1/2)) - \mathcal{E} f(-x)\, \psi(14(-x+1/2)) & \text{if} \quad -\pi \le x \le \pi , \\ 2\pi\text{-periodic extension} & \text{otherwise} . \end{cases}$$

Hence $\langle \widetilde{T} f, 1\rangle_{\mathbb{T}} = 0$ which is clear for $f \in D(\mathbb{T})$. Since $D(\mathbb{T})$ is dense in $D'(\mathbb{T})$ it follows in general. □

5.3.2 Besov Spaces on the Unit Circle

There is a simple transformation of the interval $[0, 2\pi)$ onto the unit circle given by

$$t \mapsto (\cos t, \sin t) , \qquad 0 \le t < 2\pi .$$

For a given distribution $f \in D'(\Gamma)$ we define

$$h(t) := f(\cos t, \sin t) , \qquad t \in \mathbb{R} . \tag{67}$$

Observe that $\varphi \in D(\Gamma)$ implies $\varphi(\cos t, \sin t) \in D(\mathbb{T})$. Hence, if $f \in D'(\Gamma)$ then $h \in D'(\mathbb{T})$.

Definition 3. *Let* $s \in \mathbb{R}$ *and* $0 < p, q \le \infty$. *Then* $B^s_q(L_p(\Gamma))$ *is the collection of all distributions* $f \in D'(\Gamma)$ *such that the corresponding distribution* h *is contained in* $B^s_q(L_p(\mathbb{T}))$. *We put*

$$\| f \,|B^s_q(L_p(\Gamma))\| := \| h \,|B^s_q(L_p(\mathbb{T}))\| .$$

Lemma 7. *In the sense of equivalent norms we have* $H^{1/2}(\Gamma) = B_2^{1/2}(L_2(\Gamma))$ *as well as* $H^{-1/2}(\Gamma) = B_2^{-1/2}(L_2(\Gamma))$.

Proof. It holds

$$B_2^{1/2}(L_2(\mathbb{T})) = \Big\{ h \in L_2(\mathbb{T}) : \quad \int_0^{2\pi}\int_0^{2\pi} \frac{|h(x)-h(y)|^2}{|x-y|^2}\, dx\, dy < \infty \Big\}\,,$$

see e.g. [34, 3.5.4]. Furthermore, the norms $\| h \,|B^s_q(L_p(\mathbb{T}))\|$ and

$$\| h \,|L_2(\mathbb{T})\| + \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|h(x)-h(y)|^2}{|x-y|^2}\, dx\, dy\Big)^{1/2}$$

are equivalent. Now it remains to observe that

$$\begin{aligned} \| f \,|L_2(\Gamma)\| \quad &+ \quad \Big(\int_\Gamma\int_\Gamma \frac{|f(x)-f(y)|^2}{|x-y|^2}\, d\Gamma_x\, d\Gamma_y\Big)^{1/2} \\ &\asymp \quad \| h \,|L_2(\mathbb{T})\| + \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|h(x)-h(y)|^2}{|x-y|^2}\, dx\, dy\Big)^{1/2} \end{aligned}$$

since there exist positive constants c_1, c_2 such that

$$c_1\, |x-y|^2 \le (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \le c_2\, |x-y|^2$$

for all $x, y \in [0, 2\pi]$. This proves $H^{1/2}(\Gamma) = B_2^{1/2}(L_2(\Gamma))$ in the sense of equivalent norms. The second assertion follows from $(H^{1/2}(\Gamma))' = H^{-1/2}(\Gamma)$ (just by definition) and the duality relation $(B_2^{1/2}(L_2(\mathbb{T})))' = B_2^{-1/2}(L_2(\mathbb{T}))$, see [34, 3.5.6]. □

5.3.3 Proof of Theorem 3

We consider the commutative diagram

$$\begin{array}{ccc} Y_q^{t+1/2}(L_p(\Gamma)) & \xrightarrow{\ I_1\ } & H^{1/2}(\Gamma) \\ {\scriptstyle T}\downarrow & & \uparrow{\scriptstyle T^{-1}} \\ Z_q^{t+1/2}(L_p(\mathbb{T})) & \xrightarrow{\ I_2\ } & Z_2^{1/2}(L_2(\mathbb{T})) \end{array}$$

Here the operator T is chosen to be the mapping $f \mapsto h$. Since T is a bijection considered as a mapping defined on $D'(\Gamma)$ with values in $D'(\mathbb{T})$ we obtain that T is an isomorphism belonging to $\mathcal{L}(B_q^{t+1/2}(L_p(\Gamma)), B_q^{t+1/2}(L_p(\mathbb{T})))$. Consequently, $T : Y_q^{t+1/2}(L_p(\Gamma)) \to Z_q^{t+1/2}(L_p(\mathbb{T}))$ is an isomorphism as well. Lemma 3 yields

(68)
$$e^{\rm frame}_{n,\widetilde{C}}(I_1, Y_q^{t+1/2}(L_p(\Gamma)), H^{1/2}(\Gamma)) \le \|T\|\, \|T^{-1}\|\, e^{\rm frame}_{n,C}(I_2, Z_q^{t+1/2}(L_p(\mathbb{T})), Z_2^{1/2}(L_2(\mathbb{T})))$$

with $\widetilde{C} = C\,\|T^{-1}\|\,\|T\|$. As a consequence of the commutative diagram

$$\begin{array}{ccc} Z_q^{t+1/2}(L_p(\mathbb{T})) & \xrightarrow{I_1} & Z_2^{1/2}(L_2(\mathbb{T})) \\ {\scriptstyle T^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle T} \\ Y_q^{t+1/2}(L_p(\Gamma)) & \xrightarrow{I_2} & H^{1/2}(\Gamma) \end{array}$$

Lemma 3, and inequality (68) we conclude

$$e_{n,\widetilde{C}}^{\mathrm{frame}}(I_1, Y_q^{t+1/2}(L_p(\Gamma)), H^{1/2}(\Gamma)) \asymp e_{n,C}^{\mathrm{frame}}(I_2, Z_q^{t+1/2}(L_p(\mathbb{T})), Z_2^{1/2}(L_2(\mathbb{T}))) \,.$$

¿From Corollary 1 we derive

$$e_{n,C}^{\mathrm{frame}}(I_1, Y_q^{t+1/2}(L_p(\Gamma)), H^{1/2}(\Gamma)) \asymp n^{-t}$$

for C sufficiently large. Now the assertion follows from the commutative diagram (26) and Lemma 2.

6 Appendix – Besov Spaces

Here we collect some properties of Besov spaces which have been used in the text before. For general information on Besov spaces we refer to the monographs [28, 29, 30, 32, 42, 43, 46]. A collection of results for Besov as well as Sobolev spaces on domains can be found in [9]. There detailed references are given.

6.1 Wavelet Characterizations

For the construction of biorthogonal wavelet bases as considered below we refer to the recent monograph of Cohen [3, Chapt. 2]. Let φ be a compactly supported scaling function of sufficiently high regularity and let ψ_i, $i = 1, \ldots 2^d - 1$ be corresponding wavelets. More exactly, we suppose for some $N > 0$ and $r \in \mathbb{N}$

$$\begin{aligned} &\mathrm{supp}\,\varphi\,,\ \mathrm{supp}\,\psi_i \quad \subset \quad [-N,N]^d\,, \qquad i = 1,\ldots,2^d-1\,, \\ &\varphi, \psi_i \in C^r(\mathbb{R}^d)\,, \qquad i = 1,\ldots,2^d-1\,, \\ &\int x^\alpha\,\psi_i(x)\,dx = 0 \qquad \text{for all} \quad |\alpha| \le r\,, \qquad i = 1,\ldots,2^d-1\,, \end{aligned}$$

and

$$\varphi(x-k),\, 2^{jd/2}\,\psi_i(2^jx-k)\,, \qquad j \in \mathbb{N}_0\,, \quad k \in \mathbb{Z}^d\,,$$

is a Riesz basis in $L_2(\mathbb{R}^d)$. We shall use the standard abbreviations

$$\psi_{i,j,k}(x) = 2^{jd/2}\,\psi_i(2^jx-k) \qquad \text{and} \qquad \varphi_k(x) = \varphi(x-k)\,.$$

Further, the dual Riesz basis should fulfill the same requirements, i.e., there exist functions $\widetilde{\varphi}$ and $\widetilde{\psi}_i$, $i = 1, \ldots, 2^d - 1$, such that

$$\begin{aligned}
\langle \widetilde{\varphi}_k, \psi_{i,j,k} \rangle &= \langle \widetilde{\psi}_{i,j,k}, \varphi_k \rangle = 0\,, \\
\langle \widetilde{\varphi}_k, \varphi_\ell \rangle &= \delta_{k,\ell} \qquad \text{(Kronecker symbol)}\,, \\
\langle \widetilde{\psi}_{i,j,k}, \psi_{u,v,\ell} \rangle &= \delta_{i,u}\, \delta_{j,v}\, \delta_{k,\ell}\,, \\
\operatorname{supp} \widetilde{\varphi}\,, &\qquad \operatorname{supp} \widetilde{\psi}_i \quad \subset \quad [-N,N]^d\,, \qquad i = 1, \ldots, 2^d - 1\,, \\
\widetilde{\varphi}, \widetilde{\psi}_i &\in C^r(\mathbb{R}^d)\,, \qquad i = 1, \ldots, 2^d - 1\,, \\
\int x^\alpha\, \widetilde{\psi}_i(x)\, dx &= 0 \qquad \text{for all} \quad |\alpha| \le r\,, \qquad i = 1, \ldots, 2^d - 1\,.
\end{aligned}$$

For $f \in \mathcal{S}'(\mathbb{R}^d)$ we put

$$\langle f, \psi_{i,j,k} \rangle = f(\overline{\psi_{i,j,k}}) \qquad \text{and} \qquad \langle f, \varphi_k \rangle = f(\overline{\varphi_k})\,, \tag{69}$$

whenever this makes sense.

Proposition 4. *Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$. Suppose*

$$r > \max\Big(s, d\, \max(0, \frac{1}{p} - 1) - s\Big)\,. \tag{70}$$

Then $B^s_q(L_p)$ is the collection of all tempered distributions f such that f is representable as

$$f = \sum_{k \in \mathbb{Z}^d} a_k\, \varphi_k + \sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} a_{i,j,k}\, \psi_{i,j,k} \qquad (\textit{convergence in} \quad \mathcal{S}')$$

with

$$\|\, f\, |B^s_q(L_p)\|^* := \Big(\sum_{k \in \mathbb{Z}^d} |a_k|^p\Big)^{1/p} + \Big(\sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \Big(\sum_{k \in \mathbb{Z}^d} |a_{i,j,k}|^p\Big)^{q/p}\Big)^{1/q} < \infty\,,$$

if $q < \infty$ and

$$\|\, f\, |B^s_\infty(L_p)\|^* := \Big(\sum_{k \in \mathbb{Z}^d} |a_k|^p\Big)^{1/p} + \sup_{i=1,\ldots,2^d-1}\ \sup_{j=0,\ldots}\ 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))} \Big(\sum_{k \in \mathbb{Z}^d} |a_{i,j,k}|^p\Big)^{1/p} < \infty\,.$$

The representation is unique and

$$a_{i,j,k} = \langle f, \widetilde{\psi}_{i,j,k} \rangle \qquad \textit{and} \qquad a_k = \langle f, \widetilde{\varphi}_k \rangle$$

hold. Further $I : f \mapsto \{\langle f, \widetilde{\varphi}_k \rangle, \langle f, \widetilde{\psi}_{i,j,k} \rangle\}$ is an isomorphic map of $B^s_q(L_p(\mathbb{R}^d))$ onto the sequence space (equipped with the quasi-norm $\|\cdot|B^s_q(L_p)\|^*$), i.e. $\|\cdot|B^s_q(L_p)\|^*$ may serve as an equivalent quasi-norm on $B^s_q(L_p)$.

A proof of Proposition 4 has been given in [47], see also [25] for a homogeneous version. A different proof, but restricted to $s > d(\frac{1}{p} - 1)_+$, is given in [3, Thm. 3.7.7]. However, there are many forerunners with some restrictions on s, p and q.

6.2 Besov Spaces on Domains

Let $\Omega \subset \mathbb{R}^d$ be an bounded open nonempty set. Then we define $B^s_q(L_p(\Omega))$ to be the collection of all distributions $f \in \mathcal{D}'(\Omega)$ such that there exists a tempered distribution $g \in B^s_q(L_p(\mathbb{R}^d))$ satisfying

$$f(\varphi) = g(\varphi) \qquad \text{for all} \quad \varphi \in \mathcal{D}(\Omega)\,,$$

i.e. $g|_\Omega = f$ in $\mathcal{D}'(\Omega)$. We put

$$\|\, f\,|B^s_q(L_p(\Omega))\| := \inf \,\|\, g\,|B^s_q(L_p(\mathbb{R}^d))\|\,,$$

where the infimum is taken with respect to all distributions g as above.

6.3 Sobolev Spaces on Domains

Let Ω be a bounded Lipschitz domain. Let $m \in \mathbb{N}$. As usual $H^m(\Omega)$ denotes the collection of all functions f such that the distributional derivatives $D^\alpha f$ of order $|\alpha| \le m$ belong to $L_2(\Omega)$. The norm is defined as

$$\|\, f\,|H^m(\Omega)\| := \sum_{|\alpha|\le m} \|\, D^\alpha f\,|L_2(\Omega)\|\,.$$

It is well-known that $H^m(\mathbb{R}^d) = B^m_2(L_2(\mathbb{R}^d))$ in the sense of equivalent norms, cf. e.g. [42]. As a consequence of the existence of a bounded linear extension operator for Sobolev spaces on bounded Lipschitz domains, cf. [35, p. 181], it follows

$$H^m(\Omega) = B^m_2(L_2(\Omega)) \qquad \text{(equivalent norms)}\,,$$

for such domains. For fractional $s > 0$ we introduce the classes by complex interpolation. Let $0 < s < m$, $s \notin \mathbb{N}$. Then, following [26, 9.1], we define

$$H^s(\Omega) := \Big[H^m(\Omega), L_2(\Omega)\Big]_\Theta\,, \qquad \Theta = 1 - \frac{s}{m}\,.$$

This definition does not depend on m in the sense of equivalent norms, cf. [45]. The outcome $H^s(\Omega)$ coincides with $B^s_2(L_2(\Omega))$, cf. [9] for further details.

6.4 Spaces on Domains and Boundary Conditions

We concentrate on homogeneous boundary conditions. Here it makes sense to introduce two further scales of function spaces (distribution spaces).

Definition 4. *Let $\Omega \subset \mathbb{R}^d$ be an open nontrivial set. Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$.*
(i) *Then $\mathring{B}^s_q(L_p(\Omega))$ denotes the closure of $\mathcal{D}(\Omega)$ in $B^s_q(L_p(\Omega))$, equipped with the quasi-norm of $B^s_q(L_p(\Omega))$.*
(ii) *Let $s \ge 0$. Then $H^s_0(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$, equipped with the norm of $H^s(\Omega)$.*
(iii) *By $\widetilde{B}^s_q(L_p(\Omega))$ we denote the collection of all $f \in \mathcal{D}'(\Omega)$ such that there is a $g \in B^s_q(L_p(\mathbb{R}^d))$ with*

$$g_{|\Omega} = f \qquad \text{and} \qquad \operatorname{supp} g \subset \overline{\Omega}\,, \tag{71}$$

equipped with the quasi-norm

$$\| f \,|\widetilde{B}^s_q(L_p(\Omega))\| = \inf \| g \,|B^s_q(L_p(\mathbb{R}^d))\| \,,$$

where the infimum is taken over all such distributions g as in (71).

Remark 9. *For a bounded Lipschitz domain it holds $\mathring{B}^s_q(L_p(\Omega)) = \widetilde{B}^s_q(L_p(\Omega)) = B^s_q(L_p(\Omega))$ if*

$$0 < p, q < \infty\,, \quad \max\Big(\frac{1}{p} - 1,\, d\Big(\frac{1}{p} - 1\Big)\Big) < s < \frac{1}{p}\,,$$

cf. [19, Cor. 1.4.4.5] *and* [45]. *Hence,*

$$H^s_0(\Omega) = \mathring{B}^s_2(L_2(\Omega)) = \widetilde{B}^s_2(L_2(\Omega)) = B^s_2(L_2(\Omega)) = H^s(\Omega)$$

if $0 \le s < 1/2$.

6.5 Sobolev Spaces with Negative Smoothness

Definition 5. *For $s > 0$ we define*

$$H^{-s}(\Omega) := \begin{cases} \Big(H^s_0(\Omega)\Big)' & \text{if} \quad s - \frac{1}{2} \ne \textit{integer}\,, \\[2ex] \Big(\widetilde{B}^s_2(L_2(\Omega))\Big)' & \textit{otherwise}\,. \end{cases}$$

Remark 10. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then*

$$H^s_0(\Omega) = \widetilde{B}^s_2(L_2(\Omega))\,, \qquad s > 0\,, \quad s - \frac{1}{2} \ne \textit{integer}\,,$$

Furthermore

$$H^{-s}(\Omega) = B^{-s}_2(L_2(\Omega))\,, \qquad s > 0\,, \tag{72}$$

to be understood in the sense of equivalent norms. Again we refer to [9] *for detailed references.*

6.6 Besov Spaces on the Torus

Here our general reference is [34, Chapt. 3]. Since we are using also spaces with negative smoothness $s < 0$ and/or $p, q < 1$ we shall give a definition, which relies on Fourier analysis.

Let $D(T)$ denote the collection of all complex-valued infinitely differentiable functions on $\mathbb{T}$ (i.e. 2π-periodic). By $D'(T)$ we denote its dual. Any $f \in D'(T)$ can be identified with its Fourier series $\sum_{k=-\infty}^{\infty} c_k(f)\, e^{ikx}$ where $c_k(f) = (2\pi)^{-1}\, f(e^{-ikx})$. Next we need a smooth dyadic decompositions of unity. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi(x) = 1$ if $|x| \le 1$ and $\varphi(x) = 0$ if $|x| \ge 2$. Then we put

$$\varphi_0(x) := \varphi(x), \qquad \varphi_j(x) := \varphi(2^{-j}x) - \varphi(2^{-j+1}x)\,, \quad j \in \mathbb{N}\,. \tag{73}$$

It follows

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1\,, \qquad x \in \mathbb{R}\,,$$

and

$$\text{supp}\ \varphi_j \subset \Big\{ x \in \mathbb{R}^d : \quad 2^{j-2} \le |x| \le 2^{j+1} \Big\}\,, \qquad j = 1, 2, \ldots\,.$$

By means of these functions we define the Besov classes.

Definition 6. *Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$. Then $B^s_q(L_p(\mathbb{T}))$ is the collection of all periodic tempered distributions f such that*

$$\|\, f\, |B^s_q(L_p(\mathbb{T}))\| = \Big(\sum_{j=0}^{\infty} 2^{sjq}\, \| \sum_{k=-\infty}^{\infty} \varphi_j(k)\, c_k(f)\, e^{ikx}\, |L_p(\mathbb{T})\|^q \Big)^{1/q} < \infty$$

if $q < \infty$ and

$$\|\, f|B^s_\infty(L_p(\mathbb{T}))\| = \sup_{j=0,1,\ldots} 2^{sj}\, \| \sum_{k=-\infty}^{\infty} \varphi_j(k)\, c_k(f)\, e^{ikx}\, |L_p(\mathbb{T})\, |L_p(\mathbb{R}^d)\| < \infty$$

if $q = \infty$.

Remark 11. *i) These classes are quasi-Banach spaces. They do not depend on the chosen function φ (up to equivalent quasi-norms).*

(ii) There is a number of different characterizations of periodic Besov spaces, cf. e.g. [34, Chapt. 3]. *In particular we wish to refer to the characterization by differences* [34, 3.5.4].

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