On the Stability of Multiscale Wavelet Methods with Applications to Navier–Stokes Equations

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Abstract

Quite recently, wavelet Galerkin methods have been very successfully applied to linear elliptic operator equations, especially to selfadjoint and saddle point problems. In fact, (adaptive) numerical wavelet schemes have been derived which were guaranteed to converge for a large class of problems including saddle point problems such as the Stokes problem. In this paper, we introduce a first numerical scheme to treat also more general problems such as general elliptic and the Navier–Stokes equations. We show that applying the general convergence theory as outlined in [5, 6, 28] to the wavelet setting produces a stable discretization method for a large class of problems including general elliptic and Navier–Stokes equations.

Keywords: Multiscale methods, wavelets, saddle point problems, Stokes problem, elliptic and Navier–Stokes equations, stability, convergence of Galerkin schemes, bordered systems.

AMS subject classification: Primary 42C40, 65N12, 65P30, secondary 65N55.

1 Introduction

In recent years, wavelet analysis has become a field of increasing importance. The first applications of wavelet methods were in image and signal processing. During the last years, they have also shown to offer some potential for the numerical treatment of partial differential and integral equations, see, e.g., [1, 8, 9, 10, 11, 14]. The advantages of wavelet methods can be described as follows. It turns out that a simple diagonal scaling applied to stiffness matrices relative to wavelet bases suffices to produce uniformly bounded condition numbers. Moreover, for a wide class of integral or pseudo–differential operators the stiffness matrix relative to wavelet bases can be shown to be sufficiently close to sparse matrices so that efficient sparse solvers can be applied. These are consequences of the following facts:

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- Weighted norms for sequences of wavelet expansion coefficients are equivalent in a certain range to Sobolev norms;
- for a wide class of operators their representation in the wavelet basis is nearly diagonal;
- the vanishing moments of wavelets remove the smooth part of a function.

So far all these potential advantages of wavelet methods have been exploited in many settings and yield powerful stable and convergent Galerkin schemes. The most far–reaching results were obtained for selfadjoint and saddle point problems. For these problems, it has even been possible to derive optimal convergent adaptive wavelet schemes [8, 9, 10].

This paper is, for different reasons, concentrated around the stability of discrete linear operators. First of all, the wavelet schemes we study here are in the general sense conforming variational methods. So convergence of these methods for linear problems is an immediate consequence of Ceas Lemma: if the solution u is well approximated, here $\inf_{v_n \in S_n} \|u - v_n\|_{L_2(\Omega)} \rightarrow 0$, and the stability is guaranteed, we obtain 'optimal' convergence. These convergence results can be extended to nonlinear problems in several steps, [2, 3, 4]. We assume for simplicity the Hilbert space setting still to be applicable to the nonlinear situation, e.g., for the Navier–Stokes operator. Extensions to more general Banach space settings are studied in [5, 6, 25], see [28] as well. [26] shows, under some technical conditions, that stability for the nonlinear problem is guaranteed if it is correct for the linearized operator. The additionally necessary consistency is granted for continuous operators and their discretizations if the evaluation of all parts of the nonlinear operators is possible in the Hilbert space setting. For the Navier–Stokes operator and finite element methods this has been carefully studied in, e.g. [27]. For wavelet methods this is still a problem for future research.

In this paper, we are mainly interested in the question how stability of a given Galerkin scheme is preserved under perturbations. The setting can be described as follows. Suppose that the operator $\mathcal{A}$ under consideration is of the form $\mathcal{A} = \mathcal{B} + \mathcal{C}$, where we assume that a stable Galerkin scheme for $\mathcal{B}$ exists and $\mathcal{C}$ denotes a compact perturbation. We show that the given stable discretization of $\mathcal{B}$ produces a stable discretization for $\mathcal{A}$ provided that $\mathcal{A}$ is boundedly invertible. This result can be applied to many problems, e.g., the linearized Navier–Stokes equations fall into this category since they can be interpreted as compact perturbations of the Stokes problem, at least for moderate Reynolds numbers.

Finally, to numerically study bifurcation, center and inertial manifolds we use the standard bordered systems with a possibly noninvertable linearized operator. So the well–known stability is violated and has to be replaced by a 'bordered stability'. Again this will be proved via a compact perturbation argument.

Numerical tests for selfadjoint and saddle point problems, e.g., the Stokes problem, with wavelet methods are documented in [1, 10]. To extend these results to general elliptic and saddle point problems, e.g., the Navier–Stokes problem, is a project for future research.

This paper is organized as follows. In Section 2, we briefly discuss the scope of problems we shall be concerned with. Especially, we introduce the setting of elliptic and saddle point problems and discuss typical examples, i.e., the Stokes and the Navier–Stokes equations. Section 3 is concerned with the definition of wavelets and their basic properties. In Section 4, we explain how suitable Galerkin schemes based on wavelets can be constructed and discuss their stability properties. Especially, we show that under certain conditions stability is preserved under compact perturbations. This result is obtained by combining the investigations developed in [28] and [5, 6] with the specific properties of wavelets. In Section 5, we show how these concepts can be applied to the (linearized) Navier–Stokes equations. Finally, we indicate in Section 6 how the stability for discrete bordered systems can be proved via a compact perturbation argument.

2 The Scope of Problems

In this section, we shall briefly explain the scope of problems we shall be concerned with. The goal is to derive a stable numerical scheme for problems that can be interpreted as boundedly invertible compact perturbations of an operator equation with stable discretization. Especially, we are interested in problems related with well–known saddle point problems. As we shall see later, the famous (linearized) Navier–Stokes equations fall into this category. Therefore we first recall the general setting of operator equations and discuss a typical saddle point problem, i.e., the Stokes problem. Furthermore, we introduce and discuss the Navier–Stokes system.

Suppose that $\mathcal{H}$ is a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$ induced by the inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{A} : \mathcal{H} \longrightarrow \mathcal{H}'$ denote a linear operator into the normed dual $\mathcal{H}'$ of $\mathcal{H}$. We shall mainly discuss the case that $\mathcal{A}$ can be written as

$$\mathcal{A} = \mathcal{B} + \mathcal{C}, \tag{1}$$

where $\mathcal{B}$ is a bounded operator, $\mathcal{B} \in \mathcal{L}(\mathcal{H},\mathcal{H}')$, and $\mathcal{C}$ is compact, $\mathcal{C} \in C(\mathcal{H},\mathcal{H}')$. Typical examples for $\mathcal{B}$ are given by general elliptic partial differential equations. E.g., the Poisson equation

$$\begin{aligned} -\triangle u &= f, \quad \text{in } \Omega \subset I\!R^d \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

would play the role of $\mathcal{B}$. This $\mathcal{B} = -\triangle$ is a boundedly invertible mapping of $H_0^1(\Omega)$ onto its dual $H^{-1}(\Omega)$, i.e. $\|\mathcal{B}u\|_{\mathcal{H}'} \sim \|u\|_{\mathcal{H}}$. Here '$a \sim b$' means that both quantities can be uniformly bounded by some constant multiple of each other. Likewise, ' $\lesssim$ ' indicates inequality up to constant factors.

We shall mainly be concerned with the case that $\mathcal{B}$ is induced by a saddle point problem, however we start with a short presentation of general elliptic equations. It is well–known that, given a Hilbert space X, they induce a continuous bilinear form

$$a: \; X \times X \to I\!R.$$

For a general elliptic operator $\mathcal{A}$ this $a(\cdot,\cdot)$ is coercive, hence

$$a(v,v) \geq \alpha\|v\|_X^2 - \beta\|v\|_{H_X}^2, \;\alpha\; > 0, \text{ for a Hilbert space } X \hookrightarrow H_X \hookrightarrow X'. \tag{3}$$

Furthermore, this $\mathcal{A} = \mathcal{B}+\mathcal{C}$ is a compact perturbation of a $\mathcal{B}$, inducing an elliptic bilinear form $c(\cdot,\cdot)$, i.e.,

$$c(v,v) \geq \alpha \|v\|_X^2, \qquad \alpha > 0, \qquad \text{for all } v \in X. \tag{4}$$

Then it is known that, for an invertible $\mathcal{A}$, the equation

$$a(u,v) = \langle f, v\rangle_{X'\times X} \qquad \text{for all } v \in X, \tag{5}$$

is uniquely solvable; here $\langle\cdot,\cdot\rangle_{X'\times X}$ denotes the dual pairing. In the sequel, we will only now and then give some hints for general elliptic operators and their coercive bilinear forms. Mainly we shall concentrate ourselves on the case that $\mathcal{B}$ is induced by a saddle point problem. Then we are given *two* Hilbert spaces X and M, two continuous bilinear forms

$$a: \; X \times X \to I\!R, \qquad b: \; X \times M \to I\!R$$

and $f \in X'$ as well as $g \in M'$. Moreover, we assume $X \subseteq H_X$, $M \subseteq H_M$, where H_X, H_M are Hilbert spaces such that

$$X \hookrightarrow H_X \hookrightarrow X', \qquad M \hookrightarrow H_M \hookrightarrow M'. \tag{6}$$

Then, one has to determine a pair $(u,p) \in X \times M$ such that

$$\begin{array}{lcll} a(u,v) + b(v,p) & = & \langle f, v\rangle_{X'\times X} & \text{for all } v \in X, \\ b(u,q) & = & \langle g, q\rangle_{M'\times M} & \text{for all } q \in M. \end{array} \tag{7}$$

In general, we assume the bilinear form $a(\cdot,\cdot)$ to be elliptic on the subspace

$$V := \{v \in X : \, b(v,q) = 0 \text{ for all } q \in M\} \subset X,$$

i.e., there exists a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|_X^2 \tag{8}$$

holds for all $v \in V$, compare (4). To ensure that the problem (7) is uniquely solvable, we also have to assume that X and M fulfill the *inf–sup condition*:

$$\inf_{q\in M} \sup_{v \in X} \frac{b(v,q)}{\|v\|_X \, \|q\|_M} \geq \beta \tag{9}$$

for some constant $\beta > 0$. For details, we refer e.g. to [22]. The following equivalent formulation will be very useful in the sequel. Defining the operators

$$\begin{array}{lllll} A : X \to X', & \langle Au, v\rangle_{X'\times X} & := & a(u,v), & v \in X, \\ B : M \to X', & \langle Bp, v\rangle_{X'\times X} & := & b(v,p), & v \in X, \\ B' : X \to M', & \langle B'u, q\rangle_{M'\times M} & := & b(u,q), & q \in M, \end{array}$$

the problem (7) is equivalent to find $(u,p) \in X \times M =: \mathcal{H}$ such that

$$\begin{array}{lcll} Au + Bp & = & f & \text{in } X', \\ B'u & = & g & \text{in } M'. \end{array} \tag{10}$$

If (7) is well–posed, the operator

$$\mathcal{B} := \begin{pmatrix} A & B \\ B' & 0 \end{pmatrix} \tag{11}$$

is boundedly invertible with respect to the usual graph norm, i.e., there exist constants $c_{\mathcal{B}}, C_{\mathcal{B}}$ such that

$$c_{\mathcal{B}} \|\mathcal{B}(u,p)\|_{\mathcal{H}'} \le \|(u,p)\|_{\mathcal{H}} \le C_{\mathcal{B}} \|\mathcal{B}(u,p)\|_{\mathcal{H}'}, \tag{12}$$

where $\|(u,p)\|_{\mathcal{H}}^2 := \|u\|_X^2 + \|p\|_M^2$, see again [22] for details.

We shall be concerned with an important special case, i.e., with the *Stokes problem.* Let Ω be a bounded, simply connected domain in $I\!R^d$. Then, given a vector field $f \in H^{-1}(\Omega)^d$ and a function $g \in L_{2,0}(\Omega) := \{q \in L_2(\Omega) \;:\; \int_\Omega q(x)dx = 0\}$, one has to determine the velocity $u \in H_0^1(\Omega)^d$ and the pressure $p \in L_{2,0}(\Omega)$ such that

$$\begin{aligned} -\triangle u + \nabla p &= f \quad \text{in } \Omega, \\ -\nabla \cdot u &= g \quad \text{in } \Omega. \end{aligned} \tag{13}$$

In the *mixed formulation*, the problem reads as follows: find a pair $(u,p) \in H_0^1(\Omega)^d \times L_{2,0}(\Omega)$ such that

$$\begin{aligned} a(u,v) \;+\; b(v,p) &= \langle f, v\rangle \qquad \text{for all } v \in H_0^1(\Omega)^d, \\ b(u,q) \qquad\qquad &= \langle g, q\rangle \qquad \text{for all } q \in L_{2,0}(\Omega), \end{aligned} \tag{14}$$

where

$$\begin{aligned} a(u,v) &:= \langle \nabla u, \nabla v\rangle = \sum_{i,j=1}^d \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x)dx, \\ b(v,q) &:= -\langle \nabla \cdot v, q\rangle = -\sum_{i=1}^d \int_\Omega q(x) \frac{\partial}{\partial x_i} v_i(x)dx. \end{aligned}$$

For further information concerning the theory and the numerical treatment of the Stokes equations, the reader is referred, e.g., to [21], [27].

One of our goals is to present a stable numerical scheme for the stationary Navier-Stokes equation which has the form

$$\begin{aligned} G(u,p) : &= \begin{pmatrix} -\nu\Delta u + \sum_{i=1}^d u_i D_i u + \text{grad } p \\ \text{div } u \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad \text{in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \qquad \int_\Omega p dx = 0. \end{aligned} \tag{15}$$

After multiplying with test functions in the usual way, this problem fits into our setting as follows, see (13), (14): find a pair $(u,p) \in H_0^1(\Omega)^d \times L_{2,0}(\Omega)$ such that

$$\begin{aligned} \nu a(u,v) \;+\; d(u,u,v) + b(v,p) &= \langle f, v\rangle \qquad \text{for all } v \in H_0^1(\Omega)^d, \\ b(u,q) \qquad\qquad\qquad &= \langle g, q\rangle \qquad \text{for all } q \in L_{2,0}(\Omega), \end{aligned} \tag{16}$$

where

$$d(u,v,w) := \sum_{i,j=1}^{d} \int_\Omega u_i(D_i v_j) w_j dx. \tag{17}$$

For bounded Ω, and $d \le 4$, see [27], Lemma 1.2, Ch. II, U 1,

$$d(u,v,w) \text{ is a bounded trilinear form on } H_0^1(\Omega)^d \times H_0^1(\Omega)^d \times H_0^1(\Omega)^d.$$

To treat (16) numerically, we employ its linearized form. We consider for fixed u, v and small w

$$d(u+w, u+w, v) - d(u,u,v) = d(u,w,v) + d(w,u,v) + o(w).$$

So we obtain

$$G'(u,p)(w,r) = \begin{pmatrix} -\nu\Delta w + \sum_{i=1}^{n}(w_i D_i u + u_i D_i w) + \text{grad } r \\ \text{div } w \end{pmatrix} \tag{18}$$

and, with the $a(\cdot,\cdot), b(\cdot,\cdot), d(\cdot,\cdot)$ in (16),

$$(G'(u,p)(w,r), v, q)_2 = \begin{pmatrix} \nu a(w,v) + d(u,w,v) + d(w,u,v) + b(v,r) \\ b(w,q) \end{pmatrix}. \tag{19}$$

In Section 5, we shall derive a stable numerical scheme for the treatment of (18).

3 Multiresolution

Our goal is to develop Galerkin methods for the approximate solution of $\mathcal{A}u = f$ for an operator $\mathcal{A}$ as in (1). However, in contrast to conventional finite element discretizations we will work with trial spaces that do not only exhibit the usual approximation properties and good localization but in addition lead to *expansions* of any element in the underlying Hilbert spaces in terms of *multiscale* or *wavelet* bases with certain stability properties. To correspond to the above range of applications we formulate the relevant facts for the following general framework. These results are essentially known (cf. [13, 17]) but for the convenience of the reader we include a brief summary of the relevant facts.

Let again $\mathcal{H}$ be a Hilbert space (of functions defined on Ω, say) with inner product $\langle\cdot,\cdot\rangle$. Throughout this section orthogonality will always be understood relative to this inner product. Again typical examples are $\mathcal{H} = L_2(\Omega)$, $\mathcal{H} = H^s(\Omega)$ or products of such spaces. Let $\mathcal{S} = \{S_j\}_{j=0}^\infty$ be a sequence of closed nested subspaces of $\mathcal{H}$ whose union is dense in $\mathcal{H}$. In all cases of practical relevance the spaces S_j are spanned by *single scale bases* $\Phi_j = \{\phi_{j,k} : k \in I_j\}$ which are *uniformly stable*, i.e.,

$$\|\mathbf{c}\|_{\ell_2(I_j)} \sim \left\| \sum_{k \in I_j} c_k \phi_{j,k} \right\|_{\mathcal{H}} \tag{20}$$

uniformly in $j \in \mathbb{N}_0$. Here we denote as usual $\|\mathbf{c}\|^2_{\ell_2(I_j)} = \sum_{k\in I_j} |c_k|^2$.

Successively updating a current approximation in S_{j-1} to a better one in S_j can be facilitated if stable bases

$$\Psi_j = \{\psi_{j,k} : k \in J_j\}$$

for some complement W_j of S_{j-1} in S_j are available. Defining for convenience $\Psi_0 := \Phi_0$, $W_0 := S_0$, any $v_n = \sum_{k\in I_n} c_k \phi_{n,k} \in S_n$ has then an alternative *multiscale* representation

$$v_n = \sum_{j=0}^{n} \sum_{k\in J_j} d_{j,k}\psi_{j,k}$$

which corresponds to the direct sum decomposition

$$S_n = \bigoplus_{j=0}^{n} W_j.$$

Of course, there is a continuum of possible choices of such complements. Orthogonal decompositions would correspond to the classical wavelet setting. However, orthogonality often interferes with locality and the actual computation of orthonormal bases might be too expensive. Moreover, in certain applications orthogonal decompositions are actually not best possible [17]. The essential constraint on the choice of W_j is that

$$\Psi = \bigcup_{j\in\mathbb{N}_0} \Psi_j$$

forms a *Riesz-basis* of $\mathcal{H}$, i.e., every $v \in \mathcal{H}$ has a unique expansion

$$v = \sum_{j=0}^{\infty} \sum_{k\in J_j} \langle v, \tilde{\psi}_{j,k}\rangle \psi_{j,k} \tag{21}$$

such that

$$\|v\|_{\mathcal{H}} \sim \left(\sum_{j=0}^{\infty} \sum_{k\in J_j} |\langle v, \tilde{\psi}_{j,k}\rangle|^2 \right)^{\frac{1}{2}}, \quad v \in \mathcal{H}, \tag{22}$$

where $\tilde{\Psi} = \{\tilde{\psi}_{j,k} : k \in J_j, j \in \mathbb{N}_0\}$ forms a biorthogonal system

$$\langle \psi_{j,k}, \tilde{\psi}_{j',k'}\rangle = \delta_{j,j'}\delta_{k,k'}, \quad j, j' \in \mathbb{N}_0, \quad k \in J_j, \quad k' \in J_{j'} \tag{23}$$

and is in fact also a Riesz-basis for $\mathcal{H}$ (cf. [13]).

We explain one aspect why this is important. Let $\mathbf{T}_n$ denote the transformation that takes the coefficients $d_{j,k}$ in the multiscale representation of v_n into the coefficients c_k of the single scale representation. It corresponds to the synthesis part of the fast wavelet transform. In fact, it is known that the Riesz basis property of Ψ is equivalent to $\mathbf{T}_n$ being well conditioned, i.e.,

$$\|\mathbf{T}_n\|, \|\mathbf{T}_n^{-1}\| = \mathcal{O}(1), \quad n \to \infty, \tag{24}$$

where $\|\cdot\|$ denotes the spectral norm [12, 13].

With such a pair of biorthogonal bases Ψ and $\tilde{\Psi}$ one can associate canonical truncation projectors

$$Q_n v := \sum_{j=0}^{n} \sum_{k \in J_j} \langle v, \tilde{\psi}_{j,k} \rangle \psi_{j,k}, \quad Q'_n v := \sum_{j=0}^{n} \sum_{k \in J_j} \langle v, \psi_{j,k} \rangle \tilde{\psi}_{j,k} \tag{25}$$

which are obviously adjoints of each other. Of course, when Ψ is a Riesz-basis then the Q_n and hence their adjoints Q'_n are uniformly bounded in $\mathcal{H}$. Denoting by $\tilde{S}_n$ the range of Q'_n we have therefore *two* sequences $\mathcal{S}$ and $\tilde{\mathcal{S}}$ of nested closed subspaces S_j and $\tilde{S}_j$, respectively, whose union is easily seen to be dense in $\mathcal{H}$ [12].

While the Riesz-basis property of Ψ implies the existence of a biorthogonal Riesz-basis $\tilde{\Psi}$ as well as the uniform boundedness of the projectors Q_n and Q'_n, the converse is known not to be true in general [13]. Additional conditions that do ensure the Riesz-basis property for a general Hilbert space setting have been established in [13]. Here we are only interested in their specialization to the particular case $\mathcal{H} = L_2(\Omega)$. What turns out to matter is that both $\mathcal{S}$ and $\tilde{\mathcal{S}}$ should have some approximation and regularity properties which can be stated in terms of the following pair of estimates. There exists some $\gamma > 0$ such that the inverse estimate

$$\|v_n\|_{H^s(\Omega)} \lesssim 2^{ns} \|v_n\|_{L_2(\Omega)}, \quad v_n \in S_n, \tag{26}$$

holds for $s < \gamma$. Moreover, there exists some $m \geq \gamma$ such that the direct estimate

$$\inf_{v_n \in S_n} \|v - v_n\|_{L_2(\Omega)} \lesssim 2^{-sn} \|v\|_{H^s(\Omega)}, \quad v \in H^s(\Omega), \tag{27}$$

holds for $s \leq m$. Such estimates are known to hold for every finite element or spline space. For instance, for piecewise linear finite elements one has $\gamma = 3/2, m = 2$.

It will be convenient to introduce the following notation. Let

$$J := \{\lambda = (j,k) : k \in J_j, j \in \mathbb{N}_0\} = \bigcup_{j=0}^{\infty} (\{j\} \times J_j).$$

and define

$$|\lambda| := j \quad \text{if} \quad \lambda \in J_j.$$

Then the following result holds [13].

Theorem 3.1 *Suppose that $\Psi = \{\psi_\lambda : \lambda \in J\}$ and $\tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in J\}$ are biorthogonal collections in $L_2(\Omega)$ and that the associated sequence of projectors $\{Q_j\}_{j=0}^{\infty}$ is uniformly bounded. If both $\mathcal{S}$ and $\tilde{\mathcal{S}}$ satisfy* (26) *and* (27) *relative to some $\gamma, \gamma' > 0$, $\gamma \leq m$, $\gamma' \leq m'$, then*

$$\begin{aligned} \|v\|_{H^s(\Omega)} &\sim \left(\sum_{\lambda \in J} 2^{2|\lambda| s} |\langle v, \tilde{\psi}_\lambda \rangle|^2 \right)^{\frac{1}{2}}, \quad s \in (-\gamma', \gamma), \\ &\sim \left(\sum_{\lambda \in J} 2^{2|\lambda| s} |\langle v, \psi_\lambda \rangle|^2 \right)^{\frac{1}{2}}, \quad s \in (-\gamma, \gamma'), \quad v \in H^s(\Omega). \end{aligned} \tag{28}$$

Moreover, the projectors Q_j and Q_j' are uniformly bounded in $H^s(\Omega)$, $s \in (-\gamma', \gamma)$ and $s \in (-\gamma, \gamma')$, respectively.

For more information about the construction of multiscale bases $\Psi, \tilde{\Psi}$ with the above properties the reader is referred to [7, 16, 19].

We are now prepared to employ these bases in a Galerkin scheme. However, before we discuss this topic in the next section, let us finish with some remarks concerning the special setting of saddle point problems. As saddle point problems are defined on product spaces of the form $X \times M$, we need *two* biorthogonal wavelet bases $\Psi = \{\psi_\lambda \;:\; \lambda \in J^X\}$ and $\Theta = \{\vartheta_\mu \;: \mu \in J^M\}$ that form Riesz–bases for H_X and H_M, respectively. The second pair of biorthogonal basis Θ and $\tilde{\Theta}$ also induces a pair of projectors in the sense of (25):

$$P_n q := \sum_{j=0}^{n} \sum_{k \in J_j^M} \langle q, \tilde{\vartheta}_{j,k} \rangle \vartheta_{j,k}, \quad P_n' q := \sum_{j=0}^{n} \sum_{k \in J_j^M} \langle q, \vartheta_{j,k} \rangle \tilde{\vartheta}_{j,k}. \tag{29}$$

In our applications, X and M are mainly Hilbertian Sobolev spaces on suitable domains or manifolds $\Omega_1 \subset I\!R^d$, $\Omega_2 \subset I\!R^{d'}$, i.e.,

$$X = H^t(\Omega_1), \qquad M = H^s(\Omega_2). \tag{30}$$

Then, we assume that the norm equivalences of the form (28) hold for both spaces,

$$\|v\|_{H^\tau(\Omega_1)} \sim \left(\sum_{\lambda \in J^X} 2^{2|\lambda|\tau} |\langle v, \tilde{\psi}_\lambda \rangle|^2 \right)^{1/2}, \qquad \tau \in [-t, t], \tag{31}$$

$$\|q\|_{H^\zeta(\Omega_2)} \sim \left(\sum_{\lambda \in J^X} 2^{2|\mu|\zeta} |\langle q, \tilde{\vartheta}_\mu \rangle|^2 \right)^{1/2}, \qquad \zeta \in [-s, s]. \tag{32}$$

Throughout the remainder of this paper we will assume that the underlying wavelet bases satisfy either the conditions (28) or (31) and (32).

4 Stable Discretizations

Our goal is to develop a suitable Galerkin scheme to approximate the solution of $\mathcal{A}u = f$ for an $\mathcal{A}$ as in (1) which is based on a wavelet basis as introduced in Section 3. Therefore we consider subspaces of the form

$$S_\Lambda := \{\psi_\lambda : \; \lambda \in \Lambda\}, \qquad \Lambda \subset J, \tag{33}$$

and project our problem onto these spaces, i.e., the Galerkin approximation u_Λ is defined by

$$\langle \mathcal{A} u_\Lambda, v \rangle = \langle f, v \rangle \qquad \text{for all} \quad v \in S_\Lambda. \tag{34}$$

In this paper, we shall mainly consider the case that S_Λ consists of the spaces of the underlying multiresolution analysis, i.e., $S_\Lambda = S_j$. Such a method corresponds to *uniform* *mesh refinement.* The general case will be studied in a forthcoming paper. In terms of the projectors Q_Λ and Q'_Λ,

$$Q_\Lambda v := \sum_{\lambda\in\Lambda} \langle v, \tilde{\psi}_\lambda\rangle \psi_\lambda, \qquad Q'_\Lambda v = \sum_{\lambda\in\Lambda} \langle v, \psi_\lambda\rangle \tilde{\psi}_\lambda, \tag{35}$$

the Galerkin scheme (34) may be very conveniently be written as

$$Q'_\Lambda \mathcal{A} Q_\Lambda u_\Lambda = Q'_\Lambda f. \tag{36}$$

In any case, to obtain an applicable numerical algorithm, it is essential that the Galerkin scheme has some basic *stability* properties. By using again the projectors Q_Λ, Q'_Λ this requirement can be formulated as

$$\|Q'_\Lambda \mathcal{A} u_\Lambda\|_{\mathcal{H}'} \sim \|u_\Lambda\|_{\mathcal{H}}, \qquad u_\Lambda \in S_\Lambda. \tag{37}$$

When $\mathcal{A}$ is positive definite and selfadjoint, this is the case for any trial space. Moreover, in the framework of pseudo–differential operators, sufficient conditions have been derived by [17]. It turns out that injectivity of $\mathcal{A}$ and coercivity of the real part of the principal part also imply stability. More precisely, it turns out that if

- $\mathcal{A}$ is in the class $S^n_{1,0}$ which is the subclass of Hörmander's class with the property that
$$\mid D_x^\beta D_\xi^\alpha \sigma(x,\xi) \mid \leq c_{\alpha,\beta}(1+|\xi|)^{(n-|\alpha|)}, \tag{38}$$
- $\mathcal{A}$ is *strongly elliptic*, i.e., a Garding inequality
$$\Re\sigma_0(x,\xi) \geq c|\xi|^n, \qquad \xi \in I\!R^d, \tag{39}$$
holds, where σ_0 denotes the principle part representing an operator of order n,
- $\mathcal{A}$ is injective,
$$\mathrm{Ker}\ \mathcal{A} = \{0\}, \tag{40}$$

then the resulting uniform Galerkin scheme based on the projectors Q_j, Q'_j will be stable and convergent.

For saddle point problems of the form (10), the trial spaces $(X_\Lambda, M_\Lambda) \subset (X, M)$ are defined by a pair of index sets

$$\Lambda := (\Lambda_X, \Lambda_M) \subset (J^X, J^M). \tag{41}$$

It is well–known that stability of the discretization is ensured if the *Ladyshenskaja–Babuska–Brezzi (LBB)* condition

$$\inf_{q_\lambda\in M_\Lambda} \sup_{v_\lambda\in X_\Lambda} \frac{b(v_\lambda, q_\lambda)}{\|v_\lambda\|_X\,\|q_\lambda\|_M} \geq \beta \tag{42}$$

is satisfied. Quite recently, explicit conditions to check (42) in the wavelet context have been derived in [11], see also [15]. Before we can state the result, some preparations are necessary. The basic idea was to use the following well–known condition of Fortin [20].

Proposition 4.1 ([20]) *The LBB condition holds if and only if there exists an operator $Q_\Lambda \in \mathcal{L}(X, X_\Lambda)$ satisfying*

$$\begin{aligned} b(v - Q_\Lambda v, q_\Lambda) &= 0 \quad \text{for all } v \in X,\ q_\Lambda \in M_\Lambda, \text{ and} && (43)\\ \|Q_\Lambda\|_{\mathcal{L}(X,X)} &\lesssim 1, && (44)\end{aligned}$$

independent of Λ.

For any subset $\bar{X} \subseteq X$ we will use the notations

$$\bar{X}^{\perp_b} := \{q \in M : b(v,q) = 0 \quad \text{for all } v \in \bar{X}\}, \tag{45}$$

and similar for $\bar{M} \subseteq M$

$$\bar{M}^{\perp_b} := \{v \in X : b(v,q) = 0 \quad \text{for all } q \in \bar{M}\}. \tag{46}$$

In terms of these sets, the fundamental result from [11] reads as follows.

Theorem 4.2 *The multiscale spaces X_Λ, M_Λ defined above fulfill the LBB condition* (42) *provided that one of the following equivalent conditions holds:*

(a) $M_\Lambda \subseteq (X \ominus X_\Lambda)^{\perp_b}$,

(b) $B(M_\Lambda) \subseteq \tilde{X}_\Lambda$,

(c) $B'(X \ominus X_\Lambda) \subseteq M' \ominus \tilde{M}_\Lambda$.

In summary, it is by now possible to construct stable wavelet Galerkin schemes for a large class of problems. One of the aims of this paper is to investigate to what extent these stability properties are preserved under perturbations. These relationships are clarified in the following theorem which is the main result of this paper. This is essentially a special case of a result in [6, 25]. For our Hilbert space setting it is given in [28] as well.

Theorem 4.3 *Let $\mathcal{B} \subset L(\mathcal{H}, \mathcal{H}')$ and suppose that the biorthogonal wavelet Galerkin scheme $\mathcal{B}_j := Q_j'\mathcal{B}Q_j$ is stable. Let $\mathcal{A} := \mathcal{B} + \mathcal{C}$ with $\mathcal{C} \in C(\mathcal{H}, \mathcal{H}')$, the set of compact operators from $\mathcal{H} \to \mathcal{H}'$. Then*

$$\mathcal{A}^{-1} \in L(\mathcal{H}', \mathcal{H}) \implies \mathcal{A}_j := Q_j'\mathcal{A}Q_j \quad \text{is stable.}$$

Proof. See [6, 25]. We determine for an arbitrary $u \in \mathcal{H}$ and $v' := \mathcal{C}u$ the unique exact and discrete solutions, $\hat{u}$ and $\hat{u}_j$, of the equations $\mathcal{B}\hat{u} = v'$ and $\mathcal{B}_j\hat{u}_j = Q_j'v' = Q_j'\mathcal{B}\hat{u}$. We introduce the notations $T = \mathcal{B}^{-1}$, $T_j = \mathcal{B}_j^{-1}Q_j'$. Since $\mathcal{B}_j$ is assumed to be stable, the corresponding Galerkin scheme converges, hence for any $u \in \mathcal{H}$ we obtain

$$\lim_{j\to\infty} \|\mathcal{B}_j^{-1}Q_j'\mathcal{C}u - \mathcal{B}^{-1}\mathcal{C}u\|_{\mathcal{H}} = \lim_{j\to\infty} \|(T - T_j)\mathcal{C}u\|_{\mathcal{H}} = 0.$$

$\mathcal{C}$ is compact, so we get

$$\lim_{j\to\infty} \|(T - T_j)\mathcal{C}\| = 0. \tag{47}$$

Now let $u_j \in S_j$. Because $\mathcal{A}$ is boundedly invertible, we can estimate

$$\begin{aligned} \|u_j\|_{\mathcal{H}} &\leq \|\mathcal{A}^{-1}\mathcal{A}u_j\|_{\mathcal{H}} \leq \|\mathcal{A}^{-1}\|\|\mathcal{A}u_j\|_{\mathcal{H}'} \leq \|\mathcal{A}^{-1}\|\|\mathcal{B}(I+\mathcal{B}^{-1}\mathcal{C})u_j\|_{\mathcal{H}'} \\ &\leq \|\mathcal{A}^{-1}\|\|\mathcal{B}\|\|(I+T\mathcal{C})u_j\|_{\mathcal{H}}. \end{aligned} \tag{48}$$

Hence, we obtain

$$\begin{aligned} \|\mathcal{A}_j u_j\|_{\mathcal{H}'} &= \|Q_j'(\mathcal{B}+\mathcal{C})Q_j u_j\|_{\mathcal{H}'} \\ &= \|\mathcal{B}_j(I+\mathcal{B}_j^{-1}Q_j'\mathcal{C}Q_j)u_j\|_{\mathcal{H}'} \\ &\geq 1/\|\mathcal{B}_j\|^{-1}\|(I+\mathcal{B}_j^{-1}Q_j'\mathcal{C}Q_j)u_j\|_{\mathcal{H}} \\ &= 1/\|\mathcal{B}_j\|^{-1}\|(I+T_j\mathcal{C}Q_j)u_j\|_{\mathcal{H}} \\ &\geq 1/\|\mathcal{B}_j\|^{-1}(\|(I+T\mathcal{C})Q_j u_j\|_{\mathcal{H}} - \|(T-T_j)\mathcal{C}Q_j u_j\|_{\mathcal{H}}). \end{aligned}$$

By using the fact that $\mathcal{A} = \mathcal{B}+\mathcal{C}$ implies $I+\mathcal{B}^{-1}\mathcal{C} = \mathcal{B}^{-1}\mathcal{A}$, this reduces to

$$\|\mathcal{A}_j u_j\|_{\mathcal{H}'} \geq 1/\|\mathcal{B}_j\|^{-1}\left(1/\|\mathcal{A}^{-1}\mathcal{B}\| - \|(T-T_j)\mathcal{C}\|\right)\|u_j\|_{\mathcal{H}}.$$

Because of (47) and the stability of $\mathcal{B}_j$ there exists a positive constant K, independent of j, such that for all $j \geq j_0$ the following holds:

$$\|\mathcal{A}_j u_j\|_{\mathcal{H}'} \geq K\|u_j\|_{\mathcal{H}} \quad \text{for all} \quad u_j \in S_j,$$

hence $\mathcal{A}_j$ is stable. □

5 Applications to Elliptic and Navier–Stokes Equations

In this section, we want to explain how the fundamental Theorem 4.3 can be used to obtain a stable discretization. This is achieved for elliptic and the Navier–Stokes equations essentially simultaneously. We consider the problems in their linearized form.

It is well–known that any elliptic operator $\mathcal{A}$ induces a coercive bilinear form $a(\cdot,\cdot)$. This can be split into the sum of an elliptic bilinear form $c(\cdot,\cdot)$ and its complement $\tilde{c}(\cdot,\cdot)$ s.t. the induced operators $\mathcal{B},\mathcal{C}$ satisfy $\mathcal{A} = \mathcal{B}+\mathcal{C}$ with a compact perturbation. Choose e.g.,

$$c(u,v) := a(u,v) + m\langle u,v\rangle \quad \text{with sufficiently large } m > 0.$$

For the Navier–Stokes equations the linearized form is stated in (18) and (19). Again the idea is to show that this problem, for moderate Reynolds numbers or sufficiently large ν, can be interpreted as a compact perturbation of the Stokes problem (13). Therefore stable discretizations for (13) also yield stable schemes for (18). Some preparations are necessary. We want to solve the problem

$$(G'(u,p)(w,r),v,q)_2 = \begin{pmatrix} \nu a(w,v)+d(u,w,v)+d(w,u,v)+b(r,v) \\ b(w,q) \end{pmatrix} = \begin{pmatrix} \langle f_1,v\rangle \\ \langle f_2,q\rangle \end{pmatrix} = F(v,q). \tag{49}$$

For fixed u, the continuous bilinear forms $d(u,\cdot,w)$ and $d(\cdot,u,v)$ define elements $d(u,\cdot,w)$, and $d(\cdot,u,v)$ in $H^{-1}(\Omega)$, hence they define linear continuous operators

$$D_1(v) := d(u,\cdot,v) \quad \text{and} \quad D_2(v) := d(\cdot,u,v). \tag{50}$$

Hence we observe that (49) can be written as

$$\mathcal{A}(w,r) = \mathcal{B}(w,r) + \mathcal{C}(w,r) = F, \tag{51}$$

where

$$\mathcal{B} \;=\; \begin{pmatrix} \nu A & B \\ B' & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} D_1 + D_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{52}$$

and A and B are defined according to (13), (14). We treat this problem by a uniform method, i.e., we consider multiscale spaces of the form $X_j := X_{J_j^X}$, $J_j^X = \{\lambda \in J^X,\ |\lambda| \leq j\}$, $M_{j'} := M_{J_{j'}^M}$, $J_{j'}^M = \{\mu \in J^M,\ |\mu| \leq j'\}$. Let Q_j and $P_{j'}$ denote the associated biorthogonal projectors as defined in (25) and (29), respectively. Then the resulting Galerkin scheme for (51) is given by

$$\begin{aligned} \nu Q_j' A w_j + Q_j' B r_{j'} + Q_j'(D_1 + D_2) w_j &= Q_j' f_1, \\ P_{j'}' B' w_j &= P_{j'}' f_2. \end{aligned} \tag{53}$$

In this setting, the main result reads as follows, compare [6, 25].

Theorem 5.1 *Let the multiscale spaces X_j and $M_{j'}$ be chosen in such a way that one of the conditions in Theorem 4.2 is satisfied. Then the linearized Navier–Stokes operator $\mathcal{A}$ in* (51) *represents, for sufficiently large ν, a compact perturbation of the Stokes operator $\mathcal{B}$ in* (52)*. For boundedly invertible $\mathcal{A}$, in particular for sufficiently large ν, the Galerkin scheme* (53) *yields stable $\mathcal{A}_j$.*

Proof. Compare [6, 25]. We only have to show the compact perturbation property of the Navier–Stokes equation. For fixed $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} \langle D_1 v, w\rangle &= d(u,w,v) = \sum_{i,j=1}^{d} \int_\Omega u_i (D_i w_j) v_j \, dx, \\ \langle D_2 v, w\rangle &= d(w,u,v) = \sum_{i,j=1}^{d} \int_\Omega w_i (D_i u_j) v_j \, dx. \end{aligned} \tag{54}$$

Now the embedding $I : H_0^1(\Omega) \longrightarrow L_2(\Omega)$ is continuous and compact and (54) shows that

$$D_1 v = D_1 I v \quad \text{for all} \quad v \in H_0^1(\Omega).$$

Therefore, as a product of a compact and a continuous operator, $D_1 = D_1 I$ is a compact operator. The same is correct for D_2 as well. Hence the operator $\mathcal{C}$ in (51) is compact. Moreover, since $\mathcal{B}$ is the Stokes operator, Theorem 4.2 implies that the Galerkin scheme

$$\begin{aligned} \nu Q_j' A w_j + Q_j' B r_{j'} &= Q_j' f_1, \\ P_{j'}' B' w_j &= P_{j'}' f_2, \end{aligned}$$

is stable. Furthermore, the existence of $\mathcal{A}^{-1}$, $\mathcal{B}^{-1}$ and the stability of the $\mathcal{B}_j$ imply the stability of the $\mathcal{A}_j$ by Theorem 4.3. It is well known that for large enough parameters ν the operator $\mathcal{A}$ is always invertible. This finishes the proof. □

Remark 5.2 *For general elliptic operators $\mathcal{A}$, Theorem 4.3 can be applied to obtain a stable Galerkin discretization $\mathcal{A}_j$. Similar results have also been shown in* [17].

6 Stability for Bordered Systems

To numerically compute bifurcation scenarios and later on center and inertial manifolds, extended and, in particular, bordered systems have been introduced by Keller and used by many authors, see, e.g., [23, 24]. In the mean time, the concept of *bordered systems,* obtained by few additional parameters and equations, see (57), is the method of choice. Again we can, for stability arguments, restrict the discussion to linear problems, see Section 1. We give a short introduction to this bordering and interpret it as compact perturbation of an invertible operator $\mathcal{B}$ with stable $\mathcal{B}_j$. This will yield the desired stability results for bordered systems.

Suppose, we have the following splitting of $\mathcal{H}, \mathcal{H}'$, see [23],

$$\mathcal{H} = \mathcal{N} \oplus \mathcal{M}, \quad \mathcal{H}' = \mathcal{N}' \oplus \mathcal{M}' \tag{55}$$

with m–dimensional subspaces $\mathcal{N}$ and $\mathcal{N}'$. They approximate the eigenspaces with purely imaginary eigenvalues of $\mathcal{A}$ and its dual $\mathcal{A}'$, e.g., the kernel and corange of the linearized operator $\mathcal{A}$, respectively, and closed complements $\mathcal{M}, \mathcal{M}'$. We choose orthogonal bases w.r.t. $\langle\cdot,\cdot\rangle$, for the $\mathcal{N}, \mathcal{N}'$ as

$$\mathcal{N} = [\xi_1, \ldots, \xi_m] \subset \mathcal{H}, \quad \mathcal{N}' = [\theta'_1, \ldots, \theta'_m] \subset \mathcal{H}'. \tag{56}$$

For a Fredholm operator $\mathcal{A}$ with index 0 we have to discretize the following equations, see [23]: We define $L \in \mathcal{L}(\mathcal{H} \times \mathbb{R}^m, \mathcal{H}' \times \mathbb{R}^m)$ as,

$$L = \begin{pmatrix} \mathcal{A} & \theta'_1 & ,\ldots, & \theta'_m \\ \langle\cdot,\xi_1\rangle & 0 & ,\ldots, & 0 \\ \vdots & \vdots & & \vdots \\ \langle\cdot,\xi_m\rangle & 0 & ,\ldots, & 0 \end{pmatrix}. \tag{57}$$

We apply L to $(u,\alpha)^T \in \mathcal{H} \times \mathbb{R}^m$ as in Linear Algebra and solve, with $(f,0)^T \in \mathcal{H}' \times \mathbb{R}^m$,

$$L \begin{pmatrix} u \\ \alpha \end{pmatrix} := \begin{pmatrix} \mathcal{A}u + \sum_{i=1}^m \alpha_i \theta'_i \\ \langle u, \xi_1\rangle \\ \vdots \\ \langle u, \xi_m\rangle \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{58}$$

Now we can treat equation (58) with the methods from Section 4. We obtain as special case of a result in [25] and Theorem 4.3 the following

Theorem 6.1 *Under the conditions of Theorem 3.1 let* $\mathcal{A} = \mathcal{B} + \mathcal{C}$ *with* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, $\mathcal{C}$ *be compact and* $\mathcal{B}$ *be boundedly invertible with stable* $\mathcal{B}_j$. *Then the following conditions 1., 2. are mutually equivalent and each implies 3:*

1. *for all* $f \in \mathcal{H}'$ *the equation* (58) *is uniquely solvable,*
2. $L^{-1} \in \mathcal{L}(\mathcal{H}' \times I\!R^m, \mathcal{H} \times I\!R^m)$,
3. *the discretization* L_j *of* L *is stable.*

The above condition 2. is exactly the analytic condition which is imposed in [23] to guarantee the contact equivalence for the bifurcation functions for all choices of splittings in (55).

Proof. We only indicate the proof for this special case of [6, 25]: With the operators $\Xi' \in \mathcal{L}(\mathcal{H}, I\!R^m)$ and $\Theta \in \mathcal{L}(I\!R^m, \mathcal{H}')$ by

$$\Xi' u := (\langle u, \xi_i \rangle)_{i=1}^m, \quad \text{and} \quad \Theta\alpha := \sum_{i=1}^m \alpha_i \theta_i', \quad \text{we define } L := \left(\begin{smallmatrix} \mathcal{A} & \Theta \\ \Xi' & 0 \end{smallmatrix}\right).$$

We write L in the form

$$L = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & I_{I\!R^m} \end{pmatrix} + \begin{pmatrix} \mathcal{C} & \Theta \\ \Xi' & -I_{I\!R^m} \end{pmatrix} =: \mathcal{B}_{ext} + \mathcal{C}_{ext}.$$

The bounded invertability of $\mathcal{B}$ and the stability of its discrete $\mathcal{B}_j$ imply immediately that $\mathcal{B}_{ext}$ and its discrete $\mathcal{B}_{ext,j}$ have a bounded inverse and are stable, since

$$\|(\mathcal{B}_{ext,j}|_{S_j \times I\!R^m})^{-1}\|_{S_j \times I\!R^m \leftarrow S_j \times I\!R^m} \leq \|((\mathcal{B}_j|_{S_j})^{-1}\|_{S_j \leftarrow S_j} + 1.$$

□

This result shows that the numerical Liapunov–Schmidt methods and its generalizations to center and inertial manifolds yield convergent results if wavelet discretizations are employed.

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