COORBIT SPACE THEORY FOR THE TOEPLITZ SHEARLET TRANSFORM

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Abstract. In this paper we are concerned with the continuous shearlet transform in arbitrary space dimensions where the shear operation is of Toeplitz type. In particular, we focus on the construction of associated shearlet coorbit spaces and on atomic decompositions and Banach frames for these spaces.

1. Introduction

One of the most important tasks in modern applied analysis is the analysis and synthesis of signals. To this end, usually the first step is to decompose the signal with respect to suitable building blocks which are well–suited for the specific application and allow a fast and efficient extraction of the relevant information. In this context, one particular problem which is currently in the center of interest is the analysis of *directional* information. In recent studies, several approaches have been suggested such as ridgelets [1], curvelets [2], contourlets [7], shearlets [14] and many others. For a general approach see also [13]. Among all these approaches, the shearlet transform stands out because it is related to group theory, i.e., this transform can be derived from a square-integrable representation $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^2))$ of a certain group $\mathbb{S}$, the so-called *shearlet group*, see [3]. Therefore, in the context of the shearlet transform, all the powerful tools of group representation theory can be exploited.

So far, the shearlet transform and associated analysis and synthesis algorithms are well developed for problems in $\mathbb{R}^2$. Quite recently, a first generalization to higher dimensions was given in [5]. This generalization is very close to the two–dimensional approach which is based on translations, anisotropic dilations and specific shear matrices. It has been shown that the associated integral transform originates from a square-integrable representation of a group, the full n-variate shearlet group. Moreover, a very useful link to the important coorbit space theory developed by Feichtinger and Gröchenig [8, 9, 10] has been established, see [4], and the potential to detect singularities has been demonstrated, see [5].

A different shearlet transform for arbitrary space dimensions was established in [6]. This approach deviates from [5] mainly by the choice of the shear component. Instead of the block form used in [5], the authors of [6] deal with a suitable subgroup of Toeplitz matrices of the group of upper triangular matrices. Moreover, in contrary to the anisotropic (parabolic) dilation employed in [5], the dilation operation used in [6] is isotropic. Nevertheless, similar to [5], it could be shown that the associated integral transform stems from a square-integrable group representation of a specific group. Moreover, it was demonstrated that the established Toeplitz shearlet transform has the potential to detect singularities. The goal of this paper is to develop for the Toeplitz shearlet

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transform the associated coorbit space theory including atomic decompositions and Banach frames for them.

Organization of the remaining paper: We review in Section 2 the Toeplitz shearlet transform and admissibility conditions established in [6]. In Section 3 we present our main results: Toeplitz shearlet coorbit spaces and associated atomic decompositions and Banach frames.

2. The Toeplitz Shearlet Transform

The so-called Toeplitz shearlet transform on $L_2(\mathbb{R}^d)$ uses upper triangular Toeplitz matrices as shear matrices. For $d = 2$ this coincides with the usual shearlet transform. For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}^{d-1}$, we set

$$A_a := \begin{pmatrix} a & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a \end{pmatrix} = aI_d \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s_1 & s_2 & \dots & s_{d-1} \\ 0 & 1 & s_1 & s_2 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & s_1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

The upper triangular Toeplitz matrix S_s will play the role of the shear matrix used in the ordinary shearlet transform. Note that the inverse of an upper triangular Toeplitz matrix is again an upper triangular Toeplitz matrix. The set $\mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ endowed with the operation

$$(a,s,t) \circ (a',s',t') = (aa', [S_{s'}^T S_s^T]_1, t + A_a S_s t')\ ,$$

where the bracket operation $[\cdot]_1$ extracts the last $d-1$ elements of the first column, is a locally compact group $\mathbb{S}$. We call this group *Toeplitz shearlet group*. The left and right Haar measures on $\mathbb{S}$ are given by

$$d\mu_l(a,s,t) = \frac{1}{|a|^{d+1}}\, da\, ds\, dt \quad \text{and} \quad d\mu_r(a,s,t) = \frac{1}{|a|}\, da\, ds\, dt.$$

In the following, we will apply only the left Haar measure and use the abbreviation $d\mu = d\mu_l$. For $f \in L_2(\mathbb{R}^d)$, we define the mapping $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^d))$ from $\mathbb{S}$ into the group $\mathcal{U}(L_2(\mathbb{R}^d))$ of unitary operators on $L_2(\mathbb{R}^d)$ by

$$(\pi(a,s,t)f)(x) := f_{a,s,t}(x) =: |a|^{-d/2} f(A_a^{-1} S_s^{-1}(x-t)). \tag{1}$$

This mapping π is a unitary representation of $\mathbb{S}$, i.e., a homomorphism π from $\mathbb{S}$ into the group of unitary operators $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$ which is continuous with respect to the strong operator topology. The Fourier transform of $f_{a,s,t}$ is given by

$$\begin{aligned} (\hat{\pi}(a,s,t)\hat{f})(\omega) &= |a|^{-d/2} \left(f(A_a^{-1} S_s^{-1}(\cdot - t)) \right)^{\wedge}(\omega) \\ &= |a|^{-d/2} |\det(A_a^{-1} S_s^{-1})|^{-1} \hat{f}(S_s^T A_a^T \omega) e^{-2\pi i \langle t, \omega\rangle} \\ &= |a|^{d/2} \hat{f}(S_s^T A_a^T \omega) e^{-2\pi i \langle t,\omega\rangle}. \end{aligned} \tag{2}$$

A function $\psi \in L_2(\mathbb{R}^d)$ is admissible if and only if it fulfills the admissibility condition

$$0 < C_\psi := \int_{\mathbb{R}^d} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^d}\, d\omega < \infty. \tag{3}$$

Then, for any $f \in L_2(\mathbb{R}^d)$, the following equality holds true:

$$\int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2\, d\mu(a,s,t) = C_\psi\ \|f\|^2_{L_2(\mathbb{R}^d)}\,. \tag{4}$$

In particular, the unitary representation π is irreducible and hence square integrable.

We call the transform $\mathcal{SH}: L_2(\mathbb{R}^d) \to L_2(\mathbb{S})$ defined by

$$\mathcal{SH}_f(a,s,t) := \langle f, \psi_{a,s,t}\rangle = \langle f, |a|^{-d/2}\psi(A_a^{-1}S_s^{-1}(\cdot - t))\rangle = f * \psi^*_{a,s,0}(t), \tag{5}$$

where $\psi^*_{a,s,t}(x) := \overline{\psi_{a,s,t}(-x)}$, *Toeplitz shearlet transform.* It was shown in [6] that the Toeplitz shearlet transform at hyperplane singularities has similar decay properties as the usual shearlet transform.

3. Toeplitz Shearlet Coorbit Theory

Within this section we establish the coorbit space theory for the Toeplitz shearlet group and its associated square integrable representation (1). We mainly follow the lines of [5]. For further details on coorbit space theory, the reader is referred to [8, 9, 10, 11, 12].

3.1. **Toeplitz Shearlet Coorbit Spaces.** Let w be real-valued, continuous, submultiplicative weight on $\mathbb{S}$, i.e., $w(gh) \le w(g)w(h)$ for all $g, h \in \mathbb{S}$. Furthermore, we assume that the weight function w satisfies all the coorbit-theory conditions as stated in [12, Section 2.2]. A function contained in

$$\mathcal{A}_w := \{\psi \in L_2(\mathbb{R}^d) : \mathcal{SH}_\psi(\psi) = \langle \psi, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}.$$

of is called an *analyzing vector.* We want to show that $\mathcal{A}_w$ contains shearlets that are compactly supported in Fourier domain. To this end, we need the following auxiliary lemma on the support of $\hat{\psi}$.

Lemma 3.1. *Let $a_1 > a_0 \ge \alpha > 0$ and $b = (b_1, \dots, b_{d-1})^T$ be a vector with positive components. Suppose that supp $\hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b$ where $Q_b := [-b_1, b_1] \times \cdots \times [-b_{d-1}, b_{d-1}]$. Then $\hat{\psi}\hat{\psi}_{a,s,0} \not\equiv 0$ implies $a \in \left[-\frac{a_1}{a_0}, -\frac{a_0}{a_1}\right] \cup \left[\frac{a_0}{a_1}, \frac{a_1}{a_0}\right]$ and $s_i \in [-c_i, c_i]$ for alle $i = 1, \dots, d-1$, where*

$$c_i := \frac{b_i}{a_0}\left(1 + \frac{a_1}{a_0}\right) + \frac{1}{a_0}\sum_{j=2}^{i} b_{j-1}|s_{i-j+1}|.$$

In other words: If $a \notin \left[-\frac{a_1}{a_0}, -\frac{a_0}{a_1}\right] \cup \left[\frac{a_0}{a_1}, \frac{a_1}{a_0}\right]$ or $s_i \notin [-c_i, c_i]$ for one $i \in \{1, \dots, d-1\}$ then

$$\hat{\psi}\hat{\psi}_{a,s,0} \equiv 0.$$

Proof. First we take a look at the case $a > 0$. By (2) the following conditions are necessary for $\hat{\psi}\hat{\psi}_{a,s,0} \not\equiv 0$:

(i) $a_0 \le \omega_1 \le a_1$ and $a_0 \le a\omega_1 \le a_1$
  or
  $-a_1 \le \omega_1 \le -a_0$ and $-a_1 \le a\omega_1 \le -a_0$,

(ii) $-b_i \le \omega_{i+1} \le b_i$ and $-b_i \le a(\omega_{i+1} + \sum_{j=1}^{i} \omega_j s_{i-j+1}) \le b_i$ for all $i = 1, \dots, d-1$.

Let us first discuss the case $a_0 \le \omega_1 \le a_1$. The second condition in (i) implies

$$\frac{a_0}{\omega_1} \le a \le \frac{a_1}{\omega_1}.$$

Therefore, together with the first condition in (i) we get

$$\frac{a_0}{a_1} \le a \le \frac{a_1}{a_0},$$

hence

$$a \in \left[\frac{a_0}{a_1}, \frac{a_1}{a_0}\right].$$

The second condition in (ii) is equivalent to

$$-\frac{b_i}{a} \leq \omega_{i+1} + \omega_1 s_i + \sum_{j=2}^{i} \omega_j s_{i-j+1} \leq \frac{b_i}{a},$$

resp.

$$-\frac{b_i}{a} - \omega_{i+1} - \sum_{j=2}^{i} \omega_j s_{i-j+1} \leq \omega_1 s_i \leq \frac{b_i}{a} - \omega_{i+1} - \sum_{j=2}^{i} \omega_j s_{i-j+1}.$$

The interval becomes largest for $a = \frac{a_0}{a_1}$ so that

$$-\frac{b_i a_1}{a_0} - \omega_{i+1} - \sum_{j=2}^{i} \omega_j s_{i-j+1} \leq \omega_1 s_i \leq \frac{b_i a_1}{a_0} - \omega_{i+1} - \sum_{j=2}^{i} \omega_j s_{i-j+1}, \tag{6}$$

and therefore

$$-\frac{1}{\omega_1}\left(\frac{b_i a_1}{a_0} + \omega_{i+1} + \sum_{j=2}^{i} \omega_j s_{i-j+1}\right) \leq s_i \leq \frac{1}{\omega_1}\left(\frac{b_i a_1}{a_0} - \omega_{i+1} - \sum_{j=2}^{i} \omega_j s_{i-j+1}\right).$$

The first condition in (ii) is $-b_i \leq \omega_{i+1} \leq b_i$. On the left hand side the interval becomes largest for $\omega_{i+1} = b_i$ and $\omega_j = \operatorname{sgn}(s_{i-j+1}) b_{j-1}$ and on the right hand side for $\omega_{i+1} = -b_i$ and $\omega_j = -\operatorname{sgn}(s_{i-j+1}) b_{j-1}$, i.e.,

$$-\frac{1}{\omega_1}\left(\frac{b_i a_1}{a_0} + b_i + \sum_{j=2}^{i} b_{j-1}|s_{i-j+1}|\right) \leq s_i \leq \frac{1}{\omega_1}\left(\frac{b_i a_1}{a_0} + b_i + \sum_{j=2}^{i} b_{j-1}|s_{i-j+1}|\right). \tag{7}$$

Finally, we replace ω_1 with its smallest value a_0

$$-\left(\frac{b_i}{a_0}\left(\frac{a_1}{a_0} + 1\right) + \frac{1}{a_0}\sum_{j=2}^{i} b_{j-1}|s_{i-j+1}|\right) \leq s_i \leq \underbrace{\frac{b_i}{a_0}\left(\frac{a_1}{a_0} + 1\right) + \frac{1}{a_0}\sum_{j=2}^{i} b_{j-1}|s_{i-j+1}|}_{=:c_i}.$$

Similarly, the result can be shown for the other cases. Regarding the case $a < 0$ we additionally get

$$a \in \left[-\frac{a_1}{a_0}, -\frac{a_0}{a_1}\right].$$

□

Based on this Lemma we can prove the required property of $\mathcal{SH}_\psi(\psi)$ by following exactly the lines in [5].

Theorem 3.2. *Let ψ be a Schwartz function such that $\operatorname{supp}\hat{\psi} \subseteq ([-a_1, -a_0] \cup [a_0, a_1]) \times Q_b$. Then we have that $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$, i.e.,*

$$\|\langle \psi, \pi(\cdot)\psi \rangle\|_{L_{1,w}(\mathbb{S})} = \int_{\mathbb{S}} |\mathcal{SH}_\psi(\psi)(a,s,t)|\, w(a,s,t)\, d\mu(a,s,t) < \infty.$$

For an analyzing vector ψ we now consider the space

$$\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^d) : \mathcal{SH}_\psi(f) = \langle f, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}, \tag{8}$$

with norm $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi f\|_{L_{1,w}(\mathbb{S})}$ and its anti-dual $\mathcal{H}_{1,w}^\sim$, the space of all continuous conjugate-linear functionals on $\mathcal{H}_{1,w}$. The spaces $\mathcal{H}_{1,w}$ and $\mathcal{H}_{1,w}^\sim$ are π-invariant Banach spaces with continuous embedding $\mathcal{H}_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{1,w}^\sim$. Then the inner product on $L_2(\mathbb{R}^d)\times L_2(\mathbb{R}^d)$ extends to a sesquilinear form on $\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}$. Therefore for $\psi \in \mathcal{H}_{1,w}$ and $f \in \mathcal{H}_{1,w}^\sim$ the *extended representation coefficients*

$$\mathcal{SH}_\psi(f)(a,s,t) := \langle f, \pi(a,s,t)\psi\rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}}$$

are well-defined.

Let now m be a *w-moderate weight* on $\mathbb{S}$ which means that $m(xyz) \le w(x)m(y)w(z)$ for all $x,y,z \in \mathbb{S}$. For $1 \le p \le \infty$, let

$$L_{p,m}(\mathbb{S}) := \{F \text{ measurable} : Fm \in L_p(\mathbb{S})\}.$$

We are interested in the following Banach spaces which are called *shearlet coorbit spaces*

$$\mathcal{SC}_{p,m} := \{f \in \mathcal{H}_{1,w}^\sim : \mathcal{SH}_\psi(f) \in L_{p,m}(\mathbb{S})\}, \quad \|f\|_{\mathcal{SC}_{p,m}} := \|\mathcal{SH}_\psi f\|_{L_{p,m}(\mathbb{S})}. \tag{9}$$

Note that the definition of $\mathcal{SC}_{p,m}$ is independent of the analyzing vector ψ and of the weight w, see [8, Theorem 4.2]. In applications, one may start with some sub-multiplicative weight m and use the symmetric weight $w(g) := m^\#(g) := m(g) + m(g^{-1})\Delta(g)$ for the definition of $\mathcal{A}_w$. Obviously, we have that such m is w-moderate.

In the remaining parts of this paper, we will restrict ourselves to weights w that only depend on a and s, but not on the translation parameter t, i.e., $w = w(a,s)$.

3.2. **Toeplitz Shearlet Banach Frames.** The Feichtinger-Gröchenig theory provides us with a machinery to construct atomic decompositions and Banach frames for our shearlet coorbit spaces $\mathcal{SC}_{p,w}$. In a first step, we have to determine, for a compact neighborhood U of $e \in \mathbb{S}$ with non-void interior, so-called U–dense sets. A (countable) family $X = ((a,s,t)_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{S}$ is said to be *U-dense* if $\cup_{\lambda\in\Lambda}(a,s,t)_\lambda U = \mathbb{S}$, and *separated* if for some compact neighborhood Q of e we have $(a_i,s_i,t_i)Q \cap (a_j,s_j,t_j)Q = \emptyset, i \ne j$, and *relatively separated* if X is a finite union of separated sets.

Lemma 3.3. *Let U be a neighborhood of the identity $e = (1,0,0)$ in $\mathbb{S}$, and let $\alpha > 1$ and $\beta, \gamma > 0$ be defined such that*

$$\left[\alpha^{\frac{1}{d}-1}, \alpha^{\frac{1}{d}}\right) \times \left[-\frac{\beta}{2}, \frac{\beta}{2}\right)^{d-1} \times \left[-\frac{\gamma}{2}, \frac{\gamma}{2}\right)^d \subseteq U. \tag{10}$$

Then the sequence

$$\left\{(\varepsilon\alpha^j, \beta k, S_{\beta k}A_{\alpha^j}\gamma n) : j \in \mathbb{Z}, k \in \mathbb{Z}^{d-1}, n \in \mathbb{Z}^d, \varepsilon \in \{-1,1\}\right\} \tag{11}$$

is U-dense and relatively separated.

Proof. Set

$$U_0 := \left[\alpha^{\frac{1}{d}-1}, \alpha^{\frac{1}{d}}\right) \times \left[-\frac{\beta}{2}, \frac{\beta}{2}\right)^{d-1} \times \left[-\frac{\gamma}{2}, \frac{\gamma}{2}\right)^d.$$

It is sufficient to prove that the sequence (11) is U_0-dense.

For this, fix any $(x,y,z) \in \mathbb{S}$. In the following we assume that $x \in \mathbb{R}^+$ in which case we have to set $\varepsilon = 1$. If $x < 0$, the same arguments apply while choosing $\varepsilon = -1$. We have that

$$(\alpha^j, \beta k, S_{\beta k}A_{\alpha^j}\gamma n) \circ U_0 = \left\{(\alpha^j u, [S_v^T S_{\beta k}^T]_1, S_{\beta k}A_{\alpha^j}\gamma n + S_{\beta k}A_{\alpha^j}\varpi) : (u,v,\varpi) \in U_0\right\}$$

We have to prove that there are unique $j \in \mathbb{Z}$, $k \in \mathbb{Z}^{d-1}$ and $n \in \mathbb{Z}^d$ such that

$$(x,y,z) = (\alpha^j u, [S_v^T S_{\beta k}^T]_1, S_{\beta k} A_{\alpha^j}(\gamma n + \varpi)) \tag{12}$$

for some $(u, v, \varpi) \in U_0$.

For given $x \in \mathbb{R}^*$ there exists a unique integer $j \in [\log_\alpha x - \frac{1}{d}, \log_\alpha x + 1 - \frac{1}{d})$ and for this j a unique $u \in [\alpha^{\frac{1}{d}-1}, \alpha^{\frac{1}{d}})$ such that $x = \alpha^j u$. For given $y \in \mathbb{R}^{d-1}$, we have

$$\begin{aligned} y &= [S_v^T S_{\beta k}^T]_1 \\ &= v + \beta k_1 \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ \vdots \\ v_{d-2} \end{pmatrix} + \beta k_2 \begin{pmatrix} 0 \\ 1 \\ v_1 \\ \vdots \\ v_{d-3} \end{pmatrix} + \cdots + \beta k_{d-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Starting with $y_1 = v_1 + \beta k_1$ we see that $v_1 \in [-\frac{\beta}{2}, \frac{\beta}{2})$, $k_1 \in \mathbb{Z}$ are uniquely determined. In general we have

$$y_i - \beta \sum_{j=1}^{i-1} k_j v_{i-j} = v_i + \beta k_i$$

such that $v_i \in [-\frac{\beta}{2}, \frac{\beta}{2})$ and $k_i \in \mathbb{Z}$ are uniquely determined by $y_1, \ldots, y_i$.

Finally, we consider $z = S_{\beta k} A_{\alpha^j}(\gamma n + \varpi)$ or $\tilde{z} := A_{\alpha^j}^{-1} S_{\beta k}^{-1} z = \gamma n + \varpi$. Clearly there exists unique $\varpi_i \in [-\frac{\gamma}{2}, \frac{\gamma}{2})$ and $n_i \in \mathbb{Z}$ such that $m_i \gamma + \varpi_i = \tilde{z}_i$, $i = 1, \ldots, d-1$. $\square$

Next we define the U*−oscillation* as

$$\operatorname{osc}_{\psi,U}(a,s,t) := \sup_{u \in U} |\mathcal{SH}_\psi(\psi)(u \circ (a,s,t)) - \mathcal{SH}_\psi(\psi)(a,s,t)|. \tag{13}$$

Then, the following decomposition theorem, which was proved in a general setting in [8, 9, 10, 11, 12], says that discretizing the representation by means of an U-dense set produces an atomic decomposition for $\mathcal{SC}_{p,m}$.

Theorem 3.4. *Assume that the irreducible, unitary representation π is w-integrable and let $\psi \in L_2(\mathbb{R}^d)$ which fulfills*

$$M\langle \psi, \pi(a,s,t)\rangle := \sup_{u \in (a,s,t)U} |\langle \psi, \pi(u)\psi\rangle| \in L_{1,w}(\mathbb{S}) \tag{14}$$

be given. Choose a neighborhood U of e so small that

$$\| \operatorname{osc}_{\psi,U} \|_{L_{1,w}(\mathbb{S})} < 1. \tag{15}$$

Then for any U-dense and relatively separated set $X = ((a,s,t)_\lambda)_{\lambda \in \Lambda}$ the space $\mathcal{SC}_{p,m}$ has the following atomic decomposition*: If $f \in \mathcal{SC}_{p,m}$, then*

$$f = \sum_{\lambda \in \Lambda} c_\lambda(f) \pi((a,s,t)_\lambda)\psi \tag{16}$$

where the sequence of coefficients depends linearly on f and satisfies

$$\|(c_\lambda(f))_{\lambda \in \Lambda}\|_{\ell_{p,m}} \le C \|f\|_{\mathcal{SC}_{p,m}} \tag{17}$$

with a constant C depending only on ψ and with $\ell_{p,w}$ being defined by

$$\ell_{p,m} := \{c = (c_\lambda)_{\lambda \in \Lambda} : \|c\|_{\ell_{p,m}} := \|cm\|_{\ell_p} < \infty\},$$

where $m = (m((a,s,t)_\lambda))_{\lambda\in\Lambda}$. *Conversely, if* $(c_\lambda(f))_{\lambda\in\Lambda} \in \ell_{p,m}$, *then* $f = \sum_{\lambda\in\Lambda} c_\lambda \pi((a,s,t)_\lambda)\psi$ *is in* $\mathcal{SC}_{p,m}$ *and*

$$\|f\|_{\mathcal{SC}_{p,m}} \le C' \|(c_\lambda(f))_{\lambda\in\Lambda}\|_{\ell_{p,m}}. \tag{18}$$

Given such an atomic decomposition, the problem arises under which conditions a function f is completely determined by its *moments* $\langle f, \pi((a,s,t)_\lambda)\psi\rangle$ and how f can be reconstructed from these moments. This is answered by the following theorem which establishes the existence of Banach frames.

Theorem 3.5. *Impose the same assumptions as in Theorem* 3.4*. Choose a neighborhood* U *of* e *such that*

$$\| \operatorname{osc}_{\psi,U} \|_{L_{1,w}(\mathbb{S})} < 1/\|\mathcal{SH}_\psi(\psi)\|_{L_{1,w}(\mathbb{S})}. \tag{19}$$

Then, for every U*-dense and relatively separated family* $X = ((a,s,t)_\lambda)_{\lambda\in\Lambda}$ *in* $\mathbb{S}$ *the set* $\{\pi((a,s,t)_\lambda)\psi : \lambda\in\Lambda\}$ *is a Banach frame for* $\mathcal{SH}_{p,m}$*. This means that*

i) $f \in \mathcal{SC}_{p,m}$ *if and only if* $(\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda} \in \ell_{p,w}$*;*

ii) *there exist two constants* $0 < D \le D' < \infty$ *such that*

$$D\,\|f\|_{\mathcal{SC}_{p,m}} \le \|(\langle f, \pi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\|_{\ell_{p,m}} \le D'\,\|f\|_{\mathcal{SC}_{p,m}}; \tag{20}$$

iii) *there exists a bounded, linear reconstruction operator* $\mathcal{S}$ *from* $\ell_{p,m}$ *to* $\mathcal{SC}_{p,m}$ *such that* $\mathcal{S}\left((\langle f, \psi((a,s,t)_\lambda)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{\lambda\in\Lambda}\right) = f.$

To apply the whole machinery of Theorems 3.4 and 3.5 to our Toeplitz shearlet group setting it remains as in [5] to prove that $\| \operatorname{osc}_{\psi,U} \|_{L_{1,w}(\mathbb{S})}$ becomes arbitrarily small for a sufficiently small neighborhood U of e.

Theorem 3.6. *Let* ψ *be a function contained in the Schwartz space* $\mathcal{S}$ *with* $\operatorname{supp}\hat\psi \subseteq ([-a_1,-a_0]\cup[a_0,a_1]) \times Q_b$*. Then, for every* $\varepsilon > 0$*, there exists a sufficiently small neighborhood* U *of* e *so that*

$$\| \operatorname{osc}_{\psi,U} \|_{L_{1,w}(\mathbb{S})} \le \varepsilon. \tag{21}$$

Proof. By Theorem 3.2 we have that $\mathcal{SH}_\psi(\psi) \in L_{1,w}(\mathbb{S})$. Moreover, it is easy to check that $\mathcal{SH}_\psi(\psi)$ is continuous on $\mathbb{S}$. It remains to show that $\operatorname{osc}_{\psi,U} \in L_{1,w}(\mathbb{S})$ for some compact neighborhood of e. By definition of $\operatorname{osc}_{\psi,U}$ and Parseval's identity we have that

$$\begin{aligned}
\operatorname{osc}_{\psi,U}(a,s,t) &= \sup_{(\alpha,\beta,\gamma)\in U} |\langle\hat\psi, \hat\psi_{a,s,t}\rangle - \langle\hat\psi, \hat\psi_{(\alpha,\beta,\gamma)(a,s,t)}\rangle| \\
&= \sup_{(\alpha,\beta,\gamma)\in U} \Big| |a|^{\frac{d}{2}} \mathcal{F}(\hat\psi(A_a S_s^T\cdot)\overline{\hat\psi})(t) - |a\alpha|^{\frac{d}{2}} \mathcal{F}(\hat\psi(A_{a\alpha} \underbrace{S^T_{[S_s^T S_\beta^T]_1}}_{=S_s^T S_\beta^T} \cdot)\overline{\hat\psi})(\gamma + S_\beta A_\alpha t)\Big|.
\end{aligned}$$

From Lemma 3.1 we can conclude that for α, β in a sufficient small neighborhood of $(1, 0_{d-1})$ both summands become zero except for a in two finite intervals away from zero and s in a finite interval. Thus, it remains to show that $\int_{\mathbb{R}^d} \operatorname{osc}_{\psi,U}(a,s,t)dt \le C(a,s)$ with some finite constant $C(a,s)$. We split the integral into three parts

$$\int_{\mathbb{R}^d} \operatorname{osc}_{\psi,U}(a,s,t)dt \le |a|^{\frac{d}{2}}(I_1 + I_2 + I_3),$$

where

$$I_1 := \int_{\mathbb{R}^d} \sup_{(\alpha,\beta,\gamma)\in U} \left| |1-\alpha^{\frac{d}{2}}| \mathcal{F}\left(\hat{\psi}(A_a S_s^T \cdot)\overline{\hat{\psi}}\right)(t) \right| dt,$$

$$I_2 := \int_{\mathbb{R}^d} \sup_{(\alpha,\beta,\gamma)\in U} \alpha^{\frac{d}{2}} \left| \mathcal{F}(\hat{\psi}(A_a S_s^T \cdot)\overline{\hat{\psi}})(t) - \mathcal{F}(\hat{\psi}(A_a S_s^T \cdot)\overline{\hat{\psi}})(\gamma + S_\beta A_\alpha t) \right| dt,$$

$$I_3 := \int_{\mathbb{R}^d} \sup_{(\alpha,\beta,\gamma)\in U} \alpha^{\frac{d}{2}} \left| \mathcal{F}(\hat{\psi}(A_a S_s^T \cdot)\overline{\hat{\psi}})(\gamma + S_\beta A_\alpha t) - \mathcal{F}(\hat{\psi}(A_{a\alpha} S_s^T S_\beta^T \cdot)\overline{\hat{\psi}})(\gamma + S_\beta A_\alpha t) \right| dt.$$

The integrals I_1 and I_3 can be handled in the same way as in [5]. Only for I_2 the proof has to be modified. We consider

$$G_{a,s}(t) = G(t) = \mathcal{F}(\hat{\psi}(A_a S_s^T \cdot)\overline{\hat{\psi}})(t) \in \mathcal{S}.$$

By Taylor expansion we obtain

$$\begin{aligned} |G(\gamma + S_\beta A_\alpha t) - G(t)| &= |\nabla G(t + \theta(\gamma + S_\beta A_\alpha t - t))^T (\gamma + S_\beta A_\alpha t - t)| \\ &\le \|\nabla G(t + \theta(\gamma + S_\beta A_\alpha t - t))^T\|_2 \|\gamma + S_\beta A_\alpha t - t\|_2 \\ &\le \|\nabla G(t + \theta(\gamma + S_\beta A_\alpha t - t))^T\|_2 (\|S_\beta A_\alpha - I\|_2 \|t\|_2 + \|\gamma\|_2) \end{aligned}$$

where $\theta \in [0,1)$. For any $\varepsilon > 0$, there exists a sufficiently small neighborhood U of e such that $\|S_\beta A_\alpha - I\|_2 \le \varepsilon$ and $\|\gamma\|_2 \le \varepsilon$ for all $(\alpha,\beta,\gamma) \in U$. Thus, since $\|\nabla G\|_2 \le \|\nabla G\|_1$, we conclude that

$$I_2 \le \int_{\mathbb{R}^d} \sup_{(\alpha,\beta,\gamma)\in U} \alpha^{\frac{d}{2}} \left(\sum_{j=1}^{d} |G_j(t + \theta(S_\beta A_\alpha t - t + \gamma))| \right) (\|t\|_2 + 1) dt,$$

where $G_j(t) := \frac{\partial}{\partial t_j} G$. Now $G_j \in \mathcal{S}$, $j = 1,\dots,d$ implies for all $r > 0$ and sufficiently small γ that

$$\begin{aligned} |G_j(t + \theta(S_\beta A_\alpha t - t + \gamma))| &\le C_j(a,s)(1 + \|t + \theta(S_\beta A_\alpha t - t + \gamma)\|_2^2)^{-r} \\ &\le \tilde{C}_j(a,s)(1 + \|t + \theta(S_\beta A_\alpha t - t)\|_2^2)^{-r}. \end{aligned}$$

To show that $I_2 \le C(a,s)$, it is sufficient to prove that

$$\sup_{(\alpha,\beta,0_d)\in U} (1 + \|t + \theta(S_\beta A_\alpha t - t)\|_2^2)^{-1} \le \tilde{C}(1 + \|t\|_2^2)^{-1},$$

resp., that

$$C_0 + C_1 \|t\|_2^2 \le 1 + \|t + \theta(S_\beta A_\alpha t - t)\|_2^2$$

for some $C_0, C_1 > 0$ and for all $(\alpha,\beta,0_n) \in U$. Now

$$\|t + \theta(S_\beta A_\alpha t - t)\|_2^2 = \|(I - \theta(I - \alpha S_\beta))t\|_2^2 = \|T_{\alpha,\beta} t\|_2^2$$

with

$$T_{\alpha,\beta} := \begin{pmatrix} 1-\theta(1-\alpha) & \theta\alpha\beta_1 & \cdots & \theta\alpha\beta_{d-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \theta\alpha\beta_1 \\ 0 & & & 1-\theta(1-\alpha) \end{pmatrix}.$$

Since $\theta \in [0,1)$ and α is in the neighborhood of 1 we have that $1-\theta(1-\alpha) > 0$ so that $T_{\alpha,\beta}$ is an invertible upper triangular Toeplitz matrix. Now we get

$$\|T_{\alpha,\beta}t\|_2^2 \geq \frac{1}{\|T_{\alpha,\beta}^{-1}\|_2^2}\|t\|_2^2.$$

We set $p := 1-\theta(1-\alpha)$ and $q := \theta\alpha$. With recursively defined $b_0 := \frac{1}{p}$ and $b_i := -\frac{q}{p}(\beta_1 b_{i-1} + \beta_2 b_{i-2} + \cdots + \beta_i b_0)$ for $i = 1, \ldots, d-1$ we obtain

$$T_{\alpha,\beta}^{-1} = \begin{pmatrix} b_0 & b_1 & \cdots & b_{d-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & b_1 \\ 0 & & & b_0 \end{pmatrix}$$

and consequently

$$\|T_{\alpha,\beta}^{-1}\|_2 \leq \sqrt{d}\|T_{\alpha,\beta}^{-1}\|_\infty = \sqrt{d}\sum_{j=0}^{d-1}|b_j|.$$

Let U be chosen such that $|\beta_i| \leq \tilde{\beta}$ for all $i = 1, \ldots, n-1$ with some fixed sufficiently small $\tilde{\beta}$. Using the fact that $(1+x)^d = 1 + x + x(1+x) + x(1+x)^2 + \cdots + x(1+x)^{d-1}$ we get recursively that

$$\begin{aligned}
|b_0| &= \frac{1}{p}, \\
|b_1| &\leq \frac{q}{p}\tilde{\beta}\cdot\frac{1}{p} = \frac{q}{p^2}\tilde{\beta}, \\
|b_2| &\leq \frac{q}{p}\tilde{\beta}\cdot\left(\frac{1}{p} + \frac{q}{p}\tilde{\beta}\cdot\frac{1}{p}\right) = \frac{q}{p^2}\tilde{\beta}\cdot\left(1+\frac{q}{p}\tilde{\beta}\right), \\
|b_3| &\leq \frac{q}{p}\tilde{\beta}\cdot\left(\frac{q}{p^2}\tilde{\beta}\cdot\left(1+\frac{q}{p}\tilde{\beta}\right) + \frac{q}{p}\tilde{\beta}\cdot\frac{1}{p} + \frac{1}{p}\right) = \frac{q}{p^2}\tilde{\beta}\cdot\left(1 + \frac{q}{p}\tilde{\beta} + \frac{q}{p}\tilde{\beta}\left(1+\frac{q}{p}\tilde{\beta}\right)\right) = \frac{q}{p^2}\tilde{\beta}\cdot\left(1+\frac{q}{p}\tilde{\beta}\right)^2, \\
&\vdots \\
|b_i| &\leq \frac{q}{p^2}\tilde{\beta}\cdot\left(1+\frac{q}{p}\tilde{\beta}\right)^{i-1}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\sum_{j=0}^{d-1}|b_j| &\leq \frac{1}{p}\left(1 + \frac{q}{p}\tilde{\beta} + \frac{q}{p}\tilde{\beta}(1+\frac{q}{p}\tilde{\beta}) + \frac{q}{p}\tilde{\beta}(1+\frac{q}{p}\tilde{\beta})^2 + \cdots + \frac{q}{p}\tilde{\beta}(1+\frac{q}{p}\tilde{\beta})^{d-2}\right) \\
&= \frac{1}{p}\left(1+\frac{q}{p}\tilde{\beta}\right)^{d-1}
\end{aligned}$$

such that

$$\|T_{\alpha,\beta}^{-1}\|_2 \leq \frac{\sqrt{d}}{p}\left(1+\frac{q}{p}\tilde{\beta}\right)^{d-1}, \quad \frac{1}{\|T_{\alpha,\tilde{\beta}}^{-1}\|_2} \geq \frac{p}{\sqrt{d}\left(1+\frac{q}{p}\tilde{\beta}\right)^{d-1}}.$$

Since $\frac{q}{p} < 1$, we get with a fixed δ, $0 < \delta < 1$ and $\alpha \in [1-\delta, 1+\delta]$ that

$$\frac{1}{\|T^{-1}_{\alpha,\beta}\|_2} \geq \frac{1-\delta}{\sqrt{d}\left(1+\tilde{\beta}\right)^{d-1}} =: C_1.$$

Hence $1 + \|t + \theta(S_\beta A_\alpha t - t)\|_2^2 \geq 1 + \frac{1}{\|T^{-1}_{\alpha,\beta}\|_2}\|t\|_2^2 \geq 1 + C_1\|t\|_2^2$ and we are done.

□

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