SHEARLET COORBIT SPACES: TRACES AND EMBEDDINGS IN HIGHER DIMENSIONS – EXTENDED VERSION

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Abstract. This papers examines structural properties of the recently developed shearlet coorbit spaces in higher dimensions. We prove embedding theorems for subspaces of shearlet coorbit spaces resembling shearlets on the cone in three dimensions into Besov spaces. The results are based on general atomic decompositions of Besov spaces. Furthermore, we establish trace results for these subspaces with respect to the coordinate planes. It turns out that in many cases these traces are contained in lower dimensional shearlet coorbit spaces.

1. Introduction

This paper is concerned with the investigation of structural properties of shearlet coorbit spaces. In recent years it has turned out that shearlets have the potential to retrieve directional information so that they became interesting for many applications, see [13, 16, 18]. Moreover, quite surprisingly, the shearlet transform has the outstanding property to stem from a square integrable group representation [2]. This remarkable fact provides the opportunity to design associated canonical smoothness spaces by applying the general coorbit theory derived by Feichtinger and Gröchenig [6, 7, 8, 11]. Indeed, in [3, 4] the above relationships have been clarified and new smoothness spaces, the so-called shearlet coorbit spaces, have been established. In particular, it has been shown that all the conditions needed in the context of the coorbit space theory to obtain atomic decompositions and Banach frames can satisfied by the shearlet setting.

However, once these abstract smoothness space are established some natural questions arise. Of course one would like to know how these spaces look like and how they are related to other known classical smoothness spaces such as Besov or Triebel-Lizorkin spaces. Moreover, one would like to understand the structure of these new spaces. That is, it would be desirable to know how these new scales of shearlet coorbit spaces behave under embeddings, trace and interpolation operations.

For the two-dimensional case, first results in this direction have been obtained in [5]. In [5] it has been shown that shearlet coorbit spaces of function on $\mathbb{R}^2$ can be embedded into Besov spaces and that the traces on the real axes are also contained in Besov spaces. Moreover, a first embedding result of Sobolev type has been established. The present paper can be interpreted as a continuation of this work in the sense that we study similar questions in the three-dimensional setting. We will prove that as in the two-dimensional case there exist embedding results for subspaces of shearlet coorbit spaces resembling shearlets on the cone. However, the trace spaces turn out to be much more involved. In the higher dimensional case it cannot be expected that all the trace spaces are again contained in Besov spaces since the shear parameter plays a much more important role. Indeed, we will see that certain traces of shearlet coorbit spaces are again shearlet coorbit spaces! To establish these results new techniques become necessary since linear combinations of the traces of

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analyzing shearlets might not be again admissible shearlets. To overcome this difficulty we decided to use the more general concept of coorbit molecules developed by Gröchenig and Piotrowski in [12]. Their concept of molecules provides more flexibility than atomic decompositions. It turns out that specific linear combinations of traces of shearlets can be interpreted as coorbit molecules for lower dimensional shearlet coorbit spaces.

Organization of the paper: In Section 2 we review the basic setting of the shearlet transform and the associated coorbit space theory as it is needed for our purposes. Then, in Section 3 we provide the concepts of atomic decompositions for Besov spaces and molecular decompositions for shearlet coorbit spaces. The main results are contained in Section 4, where we use the machinery explained in the previous sections to prove various trace results. Finally, in Section 5 we establish three-dimensional embedding results of shearlet coorbit spaces into Besov spaces.

In the remaining paper, we use the notation $f \lesssim g$ for the relation $f \leq C\, g$ with some generic constant $C \geq 0$, and the notation '$\sim$' stands for equivalence up to constants which are independent of the involved parameters.

2. Shearlets on $\mathbb{R}^d$

In this section, we recall basic results about the shearlet group on $\mathbb{R}^d$, $d \geq 2$, its square integrable representations and shearlet coorbit spaces from [4]. While [4] deals only with band-limited shearlets, we will see that also compactly supported shearlets can serve as so-called analyzing vectors for shearlet coorbit spaces.

2.1. **Shearlet Group and Shearlet Transform.**

For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}^{d-1}$, let

$$A_a := \begin{pmatrix} a & 0_{d-1}^{\mathrm{T}} \\ 0_{d-1} & \operatorname{sgn}(a)|a|^{\frac{1}{d}}\, I_{d-1} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s^{\mathrm{T}} \\ 0_{d-1} & I_{d-1} \end{pmatrix}$$

be the *parabolic scaling matrix* and the *shear matrix*, respectively, where $\operatorname{sgn}(a)$ denotes the sign of a. The *(full) shearlet group* $\mathbb{S}$ is defined to be the set $\mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ endowed with the group operation

$$(a, s, t)\,(a', s', t') = (aa', s + |a|^{1-1/d} s', t + S_s A_a t').$$

A left-invariant and right-invariant Haar measure of $\mathbb{S}$ is given by

$$\mu_{\mathbb{S},l} = \frac{da}{|a|^{d+1}}\, ds\, dt \quad \text{and} \quad \mu_{\mathbb{S},r} = \frac{da}{|a|}\, ds\, dt,$$

respectively, and the modular function of $\mathbb{S}$ by $\Delta(a, s, t) = 1/|a|^d$. In the following, we use the left-invariant Haar measure $\mu_{\mathbb{S}} = \mu_{\mathbb{S},l}$.

Recall that a *unitary representation* of a locally compact group G on a Hilbert space $\mathcal{H}$ is a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ from G into the group of unitary operators $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$ which is continuous with respect to the strong operator topology. For the shearlet group the mapping $\pi : \mathbb{S} \to \mathcal{U}(L_2(\mathbb{R}^d))$ defined by

$$\pi(a, s, t)\, \psi(x) := |\det A_a|^{-\frac{1}{2}} \psi(A_a^{-1} S_s^{-1}(x - t)) \tag{1}$$

is a unitary representation of $\mathbb{S}$. The representation (1) is also *square integrable*, i.e., it is irreducible and there exists a nontrivial *admissible* function $\psi \in L_2(\mathbb{S})$ fulfilling the *admissibility condition*

$$\int_{\mathbb{S}} |\langle f, \pi(a, s, t)\psi\rangle|^2\, d\mu(a, s, t) < \infty.$$

Let the Fourier transform be defined by

$$\mathcal{F}f(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}^2} f(x) e^{-2\pi i \langle \omega, x\rangle}\, dx.$$

Then straightforward computation yields

$$\hat{\psi}_{a,s,t}(\omega) = |a|^{1-\frac{1}{2d}} e^{-2\pi i t\omega} \hat{\psi}\left(A_a^{\mathrm{T}} S_s^{\mathrm{T}} \omega\right). \tag{2}$$

More precisely it turns out that $\psi \in L_2(\mathbb{S})$ is admissible if and only if

$$C_\psi := \int_{\mathbb{R}^d} \frac{|\hat{\psi}(\omega)|^2}{|\omega_1|^d}\, d\omega < \infty. \tag{3}$$

A function $\psi \in L_2(\mathbb{R}^d)$ fulfilling the admissibility condition (3) is called an *admissible shearlet* and the transform $\mathcal{SH}_\psi : L_2(\mathbb{R}^d) \to L_2(\mathbb{S})$ defined by

$$\mathcal{SH}_\psi f(a,s,t) := \langle f, \pi(a,s,t)\psi\rangle, \tag{4}$$

continuous shearlet transform. It is known that there exist both bandlimited and compactly supported shearlets, see [2, 5, 15, 17].

2.2. **Shearlet Coorbit Spaces.** Let w be real-valued, continuous, submultiplicative weight on $\mathbb{S}$, i.e., $w(gh) \le w(g)w(h)$ for all $g, h \in \mathbb{S}$. Furthermore, we assume that the weight function w satisfies all the coorbit-theory conditions as stated in [11, Section 2.2]. A function contained in

$$\mathcal{A}_w := \{\psi \in L_2(\mathbb{R}^d) : \mathcal{SH}_\psi(\psi) = \langle \psi, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}.$$

of is called an *analyzing vector*. For an analyzing vector ψ we can consider the space

$$\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^d) : \mathcal{SH}_\psi(f) = \langle f, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}, \tag{5}$$

with norm $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi f\|_{L_{1,w}(\mathbb{S})}$ and its anti-dual $\mathcal{H}_{1,w}^\sim$, the space of all continuous conjugate-linear functionals on $\mathcal{H}_{1,w}$. The spaces $\mathcal{H}_{1,w}$ and $\mathcal{H}_{1,w}^\sim$ are π-invariant Banach spaces with continuous embedding $\mathcal{H}_{1,w} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{1,w}^\sim$. Then the inner product on $L_2(\mathbb{R}^d)\times L_2(\mathbb{R}^d)$ extends to a sesquilinear form on $\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}$. Therefore for $\psi \in \mathcal{H}_{1,w}$ and $f \in \mathcal{H}_{1,w}^\sim$ the *extended representation coefficients*

$$\mathcal{SH}_\psi(f)(a,s,t) := \langle f, \pi(a,s,t)\psi\rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}}$$

are well-defined.

Let m be a w-moderate weight on $\mathbb{S}$ which means that $m(xyz) \le w(x)m(y)w(z)$ for all $x, y, z \in \mathbb{S}$. For $1 \le p \le \infty$, let

$$L_{p,m}(\mathbb{S}) := \{F \text{ measurable} : Fm \in L_p(\mathbb{S})\}.$$

We are interested in the following Banach spaces which are called *shearlet coorbit spaces*

$$\mathcal{SC}_{p,m} := \{f \in \mathcal{H}_{1,w}^\sim : \mathcal{SH}_\psi(f) \in L_{p,m}(\mathbb{S})\}, \quad \|f\|_{\mathcal{SC}_{p,m}} := \|\mathcal{SH}_\psi f\|_{L_{p,m}(\mathbb{S})}. \tag{6}$$

Note that the definition of $\mathcal{SC}_{p,m}$ is independent of the analyzing vector ψ and of the weight w, see [6, Theorem 4.2]. In applications, one may start with some sub-multiplicative weight m and use the symmetric weight $w(g) = m^\#(g) := m(g) + m(g^{-1})\Delta(g)$ for the definition of $\mathcal{A}_w$. Obviously, we have that such m is w-moderate.

To construct Banach frames in coorbit spaces, the following **b**etter subset $\mathcal{B}_w$ of $\mathcal{A}_w$ has to be non-empty:

$$\mathcal{B}_w := \{\psi \in L_2(\mathbb{R}^d) : \mathcal{SH}_\psi(\psi) \in \mathcal{W}^L(C_0, L_{1,w})\},$$

where $\mathcal{W}^L(C_0, L_{1,w})$ is the Wiener-Amalgam space

$$\mathcal{W}^L(C_0, L_{1,w}) := \{F : \|(L_x\chi_{\mathcal{Q}})F\|_\infty \in L_{1,w}\}, \quad \|(L_x\chi_{\mathcal{Q}})F\|_\infty = \sup_{y\in x\mathcal{Q}} |F(y)|$$

and $\mathcal{Q}$ is a relatively compact neighborhood of the identity element e in $\mathbb{S}$, see [11]. Note that in general $\mathcal{B}_w$ is defined with respect to the right version $\mathcal{W}^R(C_0, L_{1,w}) := \{F : \|(R_x\chi_Q)F\|_\infty = \sup_{y\in\mathcal{Q}x^{-1}} |F(y)| \in L_{1,w}\}$ of the Wiener-Amalgam space. Regarding that $\mathcal{SH}_\psi(\psi)(g) = \mathcal{SH}_\psi\psi(g^{-1})$ and assuming that $\mathcal{Q} = \mathcal{Q}^{-1}$ both definitions of $\mathcal{B}_w$ coincide. We want to show that $\mathcal{B}_w$ contains shearlets with compact support. To this end, we need the following lemma.

Lemma 2.1. *Let $y > 0$, $c > 0$, $\delta > 0$ and $d \in [-\delta, \delta]$. Then, for $r > 2$ and $f(x) := \frac{1}{(c+y|x+d|)^r}$, the following estimates hold true*

$$\int_{\mathbb{R}} f(x)\ dx \lesssim \frac{\delta}{c^r} + \frac{1}{y} \quad \textit{and} \quad \int_{\mathbb{R}} |x| f(x)\ dx \lesssim \frac{\delta^2}{c^r} + \frac{1}{y^2}.$$

Proof. For $|x| \le 2\delta$ we see that $f(x) \le \frac{1}{c^r}$ and for $|x| > 2\delta$ we obtain with $|x+d| \ge \frac{|x|}{2}$ that $f(x) \le \frac{1}{(c+\frac{y}{2}|x|)^r}$. Hence, we have

$$\begin{aligned}\int_0^\infty f(x)\ dx &\le \int_{-2\delta}^{2\delta} \frac{1}{c^r} dx + 2\int_\delta^\infty \frac{1}{(c+\frac{y}{2}x)^r} dx \ \le \ \frac{4\delta}{c^r} + 2\int_0^\infty \frac{1}{(c+\frac{y}{2}x)^r} dx \\ &\le \frac{4\delta}{c^r} + \frac{4}{y} C < \infty, \quad r > 1.\end{aligned}$$

The second integral can be estimated as

$$\begin{aligned}\int_{-\infty}^\infty |x| f(x)\ dx &\le \int_{-2\delta}^{2\delta} \frac{|x|}{c^r} dx + 2\int_0^\infty \frac{x}{(c+\frac{y}{2}x)^r} dx \\ &\le \frac{(4\delta)^2}{c^r} + \left(\frac{8}{y^2}\right) \int_0^\infty \frac{t}{(c+t)^r} dt \\ &\le \frac{(4\delta)^2}{c^r} + \left(\frac{8}{y^2}\right) C < \infty, \quad r > 2.\end{aligned}$$

□

By the following theorem, there exist compactly supported functions $\psi \in L_2(\mathbb{R}^d)$ which are contained in $\mathcal{B}_w$ for certain weights w.

Theorem 2.2. *Let $\psi(x) \in L_2(\mathbb{R}^d)$ fulfill $\operatorname{supp}\psi \in Q_D$, where $Q_D := [-D, D]^d$. Suppose that the weight function satisfies $w(a,s,t) = w(a) \le |a|^{-\rho_1} + |a|^{\rho_2}$ for $\rho_1, \rho_2 \ge 0$ and that*

$$|\hat{\psi}(\omega_1, \omega_2)| \lesssim \frac{|\omega_1|^n}{(1+|\omega_1|)^r} \prod_{k=2}^d \frac{1}{(1+|\omega_k|)^r} \tag{7}$$

for sufficiently large n and r. Then we have that $\psi \in \mathcal{B}_w$.

Proof. To keep technicalities at a reasonable level, we restrict ourselves to the case $w \equiv 1$. Let $\mathcal{Q} = \mathcal{Q}^{-1} \subset [\alpha^{\frac{1}{d}-1}, \alpha^{\frac{1}{d}}] \times [-\sigma, \sigma]^{d-1} \times [-\tau, \tau]^d$, where $\alpha > 1$, $\sigma, \tau > 0$. In the following, we restrict our attention to group elements of $\mathbb{S}$ with $a > 0$. The other case can be deduced in a similar way.

Let $(a_q, s_q, t_q) \in \mathcal{Q}$ and

$$(a', s', t') := (a, s, t)(a_q, s_q, t_q) = \left(aa_q, s + a^{1-\frac{1}{d}} s_q, \begin{pmatrix} t_1 + at_{q,1} + a^{\frac{1}{d}}(s_1 t_{q,2} + \cdots + s_{d-1} t_{q,d}) \\ t_2 + a^{\frac{1}{d}} t_{q,2} \\ \vdots \\ t_d + a^{\frac{1}{d}} t_{q,d} \end{pmatrix} \right). \tag{8}$$

We are interested in

$$G(a, s, t) := \sup_{(a_q, s_q, t_q) \in \mathcal{Q}} |\mathcal{SH}_\psi \psi(a', s', t')|.$$

With the support property of ψ and

$$\psi_{a',s',t'}(x) := \pi(a', s', t')\psi(x) = (a')^{-1+\frac{1}{2d}} \psi \begin{pmatrix} (a')^{-1}((x_1 - t'_1) - s_1(x_2 - t'_2) - \ldots - s_{d-1}(x_d - t'_d)) \\ (a')^{-\frac{1}{d}}(x_2 - t'_2) \\ \vdots \\ (a')^{-\frac{1}{d}}(x_d - t'_d) \end{pmatrix}$$

we obtain that $\mathcal{SH}_\psi \psi(a', s', t') = \langle \psi, \psi_{a',s',t'} \rangle \neq 0$ implies $(x_1, \ldots, x_d)^{\mathrm{T}} \in [-D, D]^d$ and

$$-D \leq (a')^{-\frac{1}{d}}(x_j - t'_j) \leq D, \quad j = 2, \ldots, d, \tag{9}$$

$$-a'D \leq x_1 - t'_1 - \sum_{j=2}^{d} s'_{j-1}(x_j - t'_j) \leq a'D. \tag{10}$$

With (8) it follows from (9) that

$$x_j - a^{\frac{1}{d}} t_{q,j} - (aa_q)^{\frac{1}{d}} D \leq t_j \leq x_j - a^{\frac{1}{d}} t_{q,j} + (aa_q)^{\frac{1}{d}} D, \quad j = 2, \ldots, d, \tag{11}$$

and from (10) that

$$x_1 - \sum_{j=2}^{d} s'_{j-1}(x_j - t'_j) - a'D \leq t'_1 \leq x_1 - \sum_{j=2}^{d} s'_{j-1}(x_j - t'_j) + a'D$$

and with $r := x_1 - \sum_{j=2}^d (s_{j-1} + a^{1-\frac{1}{d}} s_{q,j-1})(x_j - (t_j + a^{\frac{1}{d}} t_{q,j})) - at_{q,1} - a^{\frac{1}{d}} \sum_{j=2}^d s_{j-1} t_{q,j}$ further

$$r - aa_q D \leq t_1 \leq r + aa_q D.$$

Since $\mathcal{Q} \subset [\alpha^{\frac{1}{d}-1}, \alpha^{\frac{1}{d}}] \times [-\sigma, \sigma]^{d-1} \times [-\tau, \tau]^d$ we obtain from (11) that

$$-C(1 + a^{\frac{1}{d}}) \leq t_j \leq C(1 + a^{\frac{1}{d}}) \quad j = 2, \ldots, d, \; C := \max\{D, D\alpha^{\frac{1}{d}} + \tau\}. \tag{12}$$

For estimating t_1 we need an estimate for r:

$$
\begin{aligned}
r &= x_1 - \sum_{j=2}^{d}(s_{j-1} + a^{1-\frac{1}{d}} s_{q,j-1})(x_j - (t_j + a^{\frac{1}{d}} t_{q,j})) - a t_{q,1} - a^{\frac{1}{d}} \sum_{j=2}^{d} s_{j-1} t_{q,j} \\
&= x_1 - a t_{q,1} - \left[\sum_{j=2}^{d} s_{j-1}x_j - s_{j-1}t_j + a^{1-\frac{1}{d}} s_{q,j-1} x_j - a^{1-\frac{1}{d}} s_{q,j-1} t_j - a^{1-\frac{1}{d}} s_{q,j-1} a^{\frac{1}{d}} t_{q,j} \right] \\
&\leq D + \tau a + \sigma D(d-1) a^{1-\frac{1}{d}} + \sigma(d-1) C a^{1-\frac{1}{d}} + \sigma(d-1) C a + \sigma\tau(d-1) a + \sum_{j=2}^{d} |s_{j-1}|(D + C + C a^{\frac{1}{d}}) \\
&= P_d(a^{\frac{1}{d}}) + \left(\sum_{j=2}^{d} |s_{j-1}| \right) P_1(a^{\frac{1}{d}})
\end{aligned}
$$

where $P_k \in \Pi_k$ are polynomials with nonnegative coefficients depending on α, σ, τ and D. Similarly we conclude

$$
r \geq -P_d(a^{\frac{1}{d}}) - \left(\sum_{j=2}^{d} |s_{j-1}| \right) P_1(a^{\frac{1}{d}}).
$$

To keep the notation simple we use P_d for $P_d(a^{\frac{1}{d}}) + a a_q D$ again (just pointing to the degree of the polynomial) such that

$$
-P_d(a^{\frac{1}{d}}) - \left(\sum_{j=2}^{d} |s_{j-1}| \right) P_1(a^{\frac{1}{d}}) \leq t_1 \leq P_d(a^{\frac{1}{d}}) + \left(\sum_{j=2}^{d} |s_{j-1}| \right) P_1(a^{\frac{1}{d}}). \tag{13}
$$

With Plancherel's equality and the decay property of $\hat{\psi}$ (7) we obtain

$$
\begin{aligned}
|\mathcal{SH}_\psi \psi(a,s,t)| = |\langle \psi, \psi_{a,s,t} \rangle| \;&=\; |\langle \hat{\psi}, \hat{\psi}_{a,s,t} \rangle| \;=\; \left| \int_{\mathbb{R}^d} \hat{\psi}(\omega) \hat{\psi}_{a,s,t}(\omega) d\omega \right| \\
&\leq \int_{\mathbb{R}^d} |\hat{\psi}(\omega)| a^{1-\frac{1}{2d}} \left| \hat{\psi} \begin{pmatrix} a\omega_1 \\ a^{\frac{1}{d}}(s_1\omega_1 + \omega_2) \\ \vdots \\ a^{\frac{1}{d}}(s_{d-1}\omega_1 + \omega_d) \end{pmatrix} \right| d\omega \\
&\leq C a^{1-\frac{1}{2d}} \int_{\mathbb{R}^d} \frac{|\omega_1|^n}{(1+|\omega_1|)^r} \\
&\times \prod_{k=2}^{d} \frac{1}{(1+|\omega_k|)^r} \frac{|a\omega_1|^n}{(1+|a\omega_1|)^r} \prod_{k=2}^{d} \frac{1}{(1 + a^{\frac{1}{d}} |\omega_1 s_{k-1} + \omega_k|)^r} d\omega \\
&\leq C a^{1-\frac{1}{2d}+n} \int_{\mathbb{R}^d} \frac{|\omega_1|^n}{(1+|\omega_1|)^r} \frac{|\omega_1|^n}{(1+|a\omega_1|)^r} \\
&\times \prod_{k=2}^{d} \left(\int_{\mathbb{R}} \frac{1}{(1+|\omega_k|)^r} \frac{1}{(1 + a^{\frac{1}{d}} |\omega_1 s_{k-1} + \omega_k|)^r} d\omega_k \right) d\omega_1.
\end{aligned}
$$

The inner integrals can be estimated using [5, Lemma 3.1], which results in

$$
|\mathcal{SH}_\psi \psi(a,s,t)| \leq C\, J(a,s)
$$

where

$$J(a,s) = a^{1-\frac{1}{2d}+n} \int_{\mathbb{R}} \frac{|\omega_1|^n}{(1+|\omega_1|)^r} \frac{|\omega_1|^n}{(1+|a\omega_1|)^r} \prod_{k=2}^{d} \left(\frac{1}{a^{\frac{1}{d}}(1+|\omega_1 s_{k-1}|)^r} + \frac{1}{(1+a^{\frac{1}{d}}|\omega_1 s_{k-1}|)^r} \right) d\omega_1$$
$$= 2a^{1-\frac{1}{2d}+n} \int_0^\infty \frac{\omega_1^n}{(1+\omega_1)^r} \frac{\omega_1^n}{(1+a\omega_1)^r} \prod_{k=2}^{d} \left(\frac{1}{a^{\frac{1}{d}}(1+\omega_1 |s_{k-1}|)^r} + \frac{1}{(1+a^{\frac{1}{d}}\omega_1 |s_{k-1}|)^r} \right) d\omega_1.$$

Hence, we obtain for $G(a,s,t)$ that $|G(a,s,t)| \le C\, J(a',s')$. To conclude that $G \in L_{1,w}$ we have to show that the following integral is finite:

$$I := \int_{\mathbb{S}} |G(a,s,t)| dt\, ds\, \frac{da}{|a|^{d+1}}.$$

Since $|G(a,s,t)|$ is zero, except the t_j are in the intervals given by (12) and (13), we have that

$$I \lesssim \int_0^\infty \int_{\mathbb{R}^{d-1}} \left(P_d(a^{\frac{1}{d}}) + \left(\sum_{j=2}^{d} |s_{j-1}|\right) P_1(a^{\frac{1}{d}})\right)(1+a^{\frac{1}{d}})^{d-1} J(a',s') ds\, \frac{da}{a^{d+1}}$$
$$\lesssim \int_0^\infty \int_{\mathbb{R}^{d-1}} \left(P_{2d-1}(a^{\frac{1}{d}}) + \left(\sum_{j=2}^{d} |s_{j-1}|\right) P_d(a^{\frac{1}{d}})\right)(aa_q)^{n+1-\frac{1}{2d}} \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+aa_q\omega_1)^r}$$
$$\times \prod_{k=2}^{d} \left(\frac{1}{(aa_q)^{\frac{1}{d}}(1+\omega_1|s_{k-1}+a^{1-\frac{1}{d}}s_{q,k-1}|)^r} + \frac{1}{(1+(aa_q)^{\frac{1}{d}}\omega_1|s_{k-1}+a^{1-\frac{1}{d}}s_{q,k-1}|)^r} \right) d\omega_1 ds\, \frac{da}{a^{d+1}}$$
$$\lesssim \int_0^\infty \int_0^\infty a^{n-\frac{1}{2d}-d} \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+aa_q\omega_1)^r}$$
$$\times \int_{\mathbb{R}^{d-1}} \left[\prod_{k=2}^{d} \left(\frac{1}{(aa_q)^{\frac{1}{d}}(1+\omega_1|s_{k-1}+a^{1-\frac{1}{d}}s_{q,k-1}|)^r} + \frac{1}{(1+(aa_q)^{\frac{1}{d}}\omega_1|s_{k-1}+a^{1-\frac{1}{d}}s_{q,k-1}|)^r} \right) \right]$$
$$\times \left(P_{2d-1}(a^{\frac{1}{d}}) + \left(\sum_{j=2}^{d} |s_{j-1}|\right) P_d(a^{\frac{1}{d}}) \right) ds\, da\, d\omega_1.$$

We consider the inner integral $I^\star$ given by

$$I^\star := \int_{\mathbb{R}^{d-1}} \left(P_{2d-1}(a^{\frac{1}{d}}) + \left(\sum_{j=2}^{d} |s_{j-1}|\right) P_d(a^{\frac{1}{d}}) \right)$$
$$\times \left[\prod_{k=2}^{d} \left(\frac{1}{(aa_q)^{\frac{1}{d}}(1+\omega_1|s_{k-1}+a^{1-\frac{1}{d}}s_{q,k-1}|)^r} + \frac{1}{(1+(aa_q)^{\frac{1}{d}}\omega_1|s_{k-1}+a^{1-\frac{1}{d}}s_{q,k-1}|)^r} \right) \right] ds.$$

Setting

$$A_1(x) := \frac{1}{(aa_q)^{\frac{1}{d}}(1+\omega_1|x+a^{1-\frac{1}{d}}x_q|)^r} \quad \text{and} \quad A_2(x) := \frac{1}{(1+(aa_q)^{\frac{1}{d}}\omega_1|x+a^{1-\frac{1}{d}}x_q|)^r},$$

we obtain with the symmetry in s and Lemma 2.1 the following estimates

$$\int_{-\infty}^{\infty} A_1(x) + A_2(x) dx \lesssim a^{-\frac{1}{d}} \left(a^{1-\frac{1}{d}}\sigma + \frac{1}{\omega_1} \right) + a^{1-\frac{1}{d}}\sigma + \frac{1}{\omega_1 a^{\frac{1}{d}}}$$

and

$$\int_{-\infty}^{\infty} |x|(A_1(x)+A_2(x))dx \lesssim a^{-\frac{1}{d}}\left(\left(a^{1-\frac{1}{d}}\sigma\right)^2+\left(\frac{1}{\omega_1}\right)^2\right)+\left(a^{1-\frac{1}{d}}\sigma\right)^2+\left(\frac{1}{\omega_1 a^{\frac{1}{d}}}\right)^2.$$

We can rewrite $I^\star$ in the following way

$$\begin{aligned}
I^\star &= \int_{\mathbb{R}^{d-1}} \left(P_{2d-1}(a^{\frac{1}{d}})+\left(\sum_{j=2}^{d}|s_{j-1}|\right)P_d(a^{\frac{1}{d}})\right)\left[\prod_{k=2}^{d}\left(A_1(s_{k-1})+A_2(s_{k-1})\right)\right] ds\\
&= P_{2d-1}(a^{\frac{1}{d}})\int_{\mathbb{R}^{d-1}}\prod_{k=2}^{d}\left(A_1(s_{k-1})+A_2(s_{k-1})\right) ds\\
&+ P_d(a^{\frac{1}{d}})\int_{\mathbb{R}^{d-1}}\left(\sum_{j=2}^{d}|s_{j-1}|\right)\prod_{k=2}^{d}\left(A_1(s_{k-1})+A_2(s_{k-1})\right) ds\\
&= P_{2d-1}(a^{\frac{1}{d}})\prod_{k=2}^{d}\int_{-\infty}^{\infty}\left(A_1(s_{k-1})+A_2(s_{k-1})\right) ds_{k-1}\\
&+ P_d(a^{\frac{1}{d}})\sum_{j=2}^{d}\int_{-\infty}^{\infty}|s_{j-1}|(A_1(s_{j-1})+A_2(s_{j-1}))\,ds_{j-1}\prod_{\substack{k=2\\k\neq j}}^{d}\int_{-\infty}^{\infty}\left(A_1(s_{k-1})+A_2(s_{k-1})\right) ds_{k-1}.
\end{aligned}$$

Together with the above estimates we obtain

$$\begin{aligned}
I^\star &\lesssim P_{2d-1}(a^{\frac{1}{d}})\left(a^{-\frac{1}{d}}\left(a^{1-\frac{1}{d}}\sigma+\frac{1}{\omega_1}\right)+a^{1-\frac{1}{d}}\sigma+\frac{1}{\omega_1 a^{\frac{1}{d}}}\right)^{d-1}\\
&+ P_d(a^{\frac{1}{d}})\left(a^{-\frac{1}{d}}\left(a^{1-\frac{1}{d}}\sigma+\frac{1}{\omega_1}\right)+a^{1-\frac{1}{d}}\sigma+\frac{1}{\omega_1 a^{\frac{1}{d}}}\right)^{d-2}(d-1)\\
&\times\left(a^{-\frac{1}{d}}\left(\left(a^{1-\frac{1}{d}}\sigma\right)^2+\left(\frac{1}{\omega_1}\right)^2\right)+\left(a^{1-\frac{1}{d}}\sigma\right)^2+\left(\frac{1}{\omega_1 a^{\frac{1}{d}}}\right)^2\right)\\
&\lesssim P_{2d-1}(a^{\frac{1}{d}})\left(a^{1-\frac{2}{d}}\sigma+a^{1-\frac{1}{d}}\sigma+\frac{2}{\omega_1 a^{\frac{1}{d}}}\right)^{d-1}\\
&+ P_d(a^{\frac{1}{d}})\left(a^{1-\frac{2}{d}}\sigma+a^{1-\frac{1}{d}}\sigma+\frac{2}{\omega_1 a^{\frac{1}{d}}}\right)^{d-2}\left(a^{2-\frac{3}{d}}\sigma^2+\frac{1}{\omega_1^2 a^{\frac{1}{d}}}+a^{2-\frac{2}{d}}\sigma^2+\frac{1}{\omega_1^2 a^{\frac{2}{d}}}\right).
\end{aligned}$$

We then have for I that

$$
\begin{aligned}
I \lesssim & \int_0^\infty \int_0^\infty a^{n-\frac{1}{2d}-d} \frac{\omega_1^{2n}}{(1+\omega_1)^r(1+aa_q\omega_1)^r} \left(P_{2d-1}(a^{\frac{1}{d}}) \left(a^{1-\frac{2}{d}}\sigma + a^{1-\frac{1}{d}}\sigma + \frac{2}{\omega_1 a^{\frac{1}{d}}} \right)^{d-1} \right. \\
& \left. + P_d(a^{\frac{1}{d}}) \left(a^{1-\frac{2}{d}}\sigma + a^{1-\frac{1}{d}}\sigma + \frac{2}{\omega_1 a^{\frac{1}{d}}} \right)^{d-2} \left(a^{2-\frac{3}{d}}\sigma^2 + \frac{1}{\omega_1^2 a^{\frac{1}{d}}} + a^{2-\frac{2}{d}}\sigma^2 + \frac{1}{\omega_1^2 a^{\frac{2}{d}}} \right) \right) da\, d\omega_1 \\
\lesssim & \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r} \int_0^\infty a^{n-\frac{1}{2d}-d} \frac{1}{(\frac{1}{a_q}+a\omega_1)^r} \left(P_{2d-1}(a^{\frac{1}{d}}) \left(a^{1-\frac{2}{d}}\sigma + a^{1-\frac{1}{d}}\sigma + \frac{2}{\omega_1 a^{\frac{1}{d}}} \right)^{d-1} \right. \\
& \left. + P_d(a^{\frac{1}{d}}) \left(a^{1-\frac{2}{d}}\sigma + a^{1-\frac{1}{d}}\sigma + \frac{2}{\omega_1 a^{\frac{1}{d}}} \right)^{d-2} \left(a^{2-\frac{3}{d}}\sigma^2 + \frac{1}{\omega_1^2 a^{\frac{1}{d}}} + a^{2-\frac{2}{d}}\sigma^2 + \frac{1}{\omega_1^2 a^{\frac{2}{d}}} \right) \right) da\, d\omega_1 \\
\lesssim & \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r} \int_0^\infty a^{n-\frac{1}{2d}-d} \frac{1}{(\frac{1}{a_q}+a\omega_1)^r} \left(P_{2d-1}(a^{\frac{1}{d}}) a^{-\frac{d-1}{d}} \left(a^{1-\frac{1}{d}}\sigma + a\sigma + \frac{2}{\omega_1} \right)^{d-1} \right. \\
& \left. + P_d(a^{\frac{1}{d}}) a^{-1} \left(a^{1-\frac{1}{d}}\sigma + a\sigma + \frac{2}{\omega_1} \right)^{d-2} \left(a^{2-\frac{1}{d}}\sigma^2 + \frac{a^{\frac{1}{d}}}{\omega_1^2} + a^2\sigma^2 + \frac{1}{\omega_1^2} \right) \right) da\, d\omega_1
\end{aligned}
$$

and since $(x+y)^d \leq (x+1)^d(y+1)^2$ further

$$
\begin{aligned}
\lesssim & \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r} \int_0^\infty a^{n-\frac{1}{2d}-d} \frac{1}{(\frac{1}{a_q}+a\omega_1)^r} \Big(P_{2d-1}(a^{\frac{1}{d}}) a^{-\frac{d-1}{d}} P_{d^2-d}(a^{\frac{1}{d}}) P_{d-1}(\omega_1^{-1}) \\
& + P_d(a^{\frac{1}{d}}) a^{-1} P_{d^2-2d}(a^{\frac{1}{d}}) P_{d-2}(\omega_1^{-1}) \left(P_{2d}(a^{\frac{1}{d}}) + (1+a^{\frac{1}{d}}) \frac{1}{\omega_1^2} \right) \Big) da\, d\omega_1 \\
\lesssim & \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r} P_{d-1}(\omega_1^{-1}) \int_0^\infty a^{n-\frac{1}{2d}-d-\frac{d-1}{d}} \frac{1}{(\frac{1}{a_q}+a\omega_1)^r} P_{d^2+d-1}(a^{\frac{1}{d}})\, da\, d\omega_1 \\
& + \int_0^\infty \frac{\omega_1^{2n}}{(1+\omega_1)^r} P_{d-2}(\omega_1^{-1}) \int_0^\infty a^{n-\frac{1}{2d}-d-1} \frac{1}{(\frac{1}{a_q}+a\omega_1)^r} P_{d^2+d}(a^{\frac{1}{d}})\, da\, d\omega_1 \\
& + \int_0^\infty \frac{\omega_1^{2n-2}}{(1+\omega_1)^r} P_{d-2}(\omega_1^{-1}) \int_0^\infty a^{n-\frac{1}{2d}-d-1} \frac{1}{(\frac{1}{a_q}+a\omega_1)^r} P_{d^2-d+1}(a^{\frac{1}{d}})\, da\, d\omega_1.
\end{aligned}
$$

Substituting $b := a\omega_1$ with $db = \omega_1 da$ we finally get

$$
\begin{aligned}
\lesssim & \int_0^\infty \frac{\omega_1^{2n-1}}{(1+\omega_1)^r} P_{d-1}(\omega_1^{-1}) \int_0^\infty \left(\frac{b}{\omega_1} \right)^{n-\frac{1}{2d}-d-\frac{d-1}{d}} \frac{1}{(\frac{1}{a_q}+b)^r} P_{d^2+d-1}\left(\left(\frac{b}{\omega_1} \right)^{\frac{1}{d}} \right) db\, d\omega_1 \\
& + \int_0^\infty \frac{\omega_1^{2n-1}}{(1+\omega_1)^r} P_{d-2}(\omega_1^{-1}) \int_0^\infty \left(\frac{b}{\omega_1} \right)^{n-\frac{1}{2d}-d-1} \frac{1}{(\frac{1}{a_q}+b)^r} P_{d^2+d}\left(\left(\frac{b}{\omega_1} \right)^{\frac{1}{d}} \right) db\, d\omega_1 \\
& + \int_0^\infty \frac{\omega_1^{2n-3}}{(1+\omega_1)^r} P_{d-2}(\omega_1^{-1}) \int_0^\infty \left(\frac{b}{\omega_1} \right)^{n-\frac{1}{2d}-d-1} \frac{1}{(\frac{1}{a_q}+b)^r} P_{d^2-d+1}\left(\left(\frac{b}{\omega_1} \right)^{\frac{1}{d}} \right) db\, d\omega_1
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \int_0^\infty \frac{\omega_1^{n-\frac{1}{2d}+d}}{(1+\omega_1)^r} P_{d-1}(\omega_1^{-1}) \int_0^\infty \frac{b^{n+\frac{1}{2d}-d-1}}{(\frac{1}{a_q}+b)^r} P_{d^2+d-1}\left(\left(\frac{b}{\omega_1}\right)^{\frac{1}{d}}\right) db\, d\omega_1 \\
&+ \int_0^\infty \frac{\omega_1^{n+d+\frac{1}{2d}}}{(1+\omega_1)^r} P_{d-2}(\omega_1^{-1}) \int_0^\infty \frac{b^{n-\frac{1}{2d}-d-1}}{(\frac{1}{a_q}+b_1)^r} P_{d^2+d}\left(\left(\frac{b}{\omega_1}\right)^{\frac{1}{d}}\right) db\, d\omega_1 \\
&+ \int_0^\infty \frac{\omega_1^{n-2+d+\frac{1}{2d}}}{(1+\omega_1)^r} P_{d-2}(\omega_1^{-1}) \int_0^\infty \frac{b^{n-\frac{1}{2d}-d-1}}{(\frac{1}{a_q}+b)^r} P_{d^2-d+1}\left(\left(\frac{b}{\omega_1}\right)^{\frac{1}{d}}\right) db\, d\omega_1.
\end{aligned}
$$

Since $P_k \in \Pi_k$ are polynomials we see that the integrals are finite for sufficiently large $n \geq f_1(d)$ and $r \geq f_2(n,d)$. □

A (countable) family $X = \{g_i = (a_i, s_i, t_i) : i \in \mathcal{I}\}$ in $\mathbb{S}$ is said to be *U-dense* if $\bigcup_{i\in\mathcal{I}} g_i U = \mathbb{S}$, and *separated* if for some compact neighborhood Q of e we have $g_iQ \cap g_jQ = \emptyset, i \neq j$, and *relatively separated* if X is a finite union of separated sets. Let $\alpha > 1$ and $\beta, \tau > 0$ be defined such that $[1/\alpha, \alpha) \times [-\beta, \beta)^{d-1} \times Q_\tau \subset U$. Then it was shown in [4] that for a neighborhood

$$
U \supseteq [\alpha^{\frac{1}{d}-1}, \alpha^{\frac{1}{d}}) \times [-\frac{\beta}{2}, \frac{\beta}{2})^{d-1} \times [-\frac{\tau}{2}, \frac{\tau}{2})^d, \quad \alpha > 1,\ \beta, \tau > 0 \tag{14}
$$

of the identity, the set

$$
X := \left\{(\varepsilon\alpha^{-j}, \beta\alpha^{-j(1-\frac{1}{d})}k, S_{\beta\alpha^{-j(1-\frac{1}{d})}k} A_{\alpha^{-j}}\tau l) : j \in \mathbb{Z}, k \in \mathbb{Z}^{d-1}, l \in \mathbb{Z}^d,\ \varepsilon \in \{-1, 1\}\right\} \tag{15}
$$

is U-dense and relatively separated. The following theorem collects results about the existence of atomic decompositions and Banach frames from [3, 6].

Theorem 2.3. *Let $1 \leq p \leq \infty$ and $\psi \in \mathcal{B}_w$, $\psi \neq 0$. Then there exists a (sufficiently small) neighborhood U of e so that for any U-dense and relatively separated set $X = \{g_i = (a_i, s_i, t_i) : i \in \mathcal{I}\}$ the set $\{\pi(g_i)\psi)\}$ provides an atomic decomposition and a Banach frame for $\mathcal{SC}_{p,m}$:*
Atomic Decompositions*: If $f \in \mathcal{SC}_{p,m}$, then*

$$
f = \sum_{i\in\mathcal{I}} c_i(f)\pi(g_i)\psi, \tag{16}
$$

where the sequence of coefficients depends linearly on f and satisfies

$$
\|(c_i(f))_{i\in\mathcal{I}}\|_{\ell_{p,m}} \lesssim \|f\|_{\mathcal{SC}_{p,m}} \tag{17}
$$

with $\ell_{p,m}$ being defined by

$$
\ell_{p,m} := \{c = (c_i)_{i\in\mathcal{I}} : \|c\|_{\ell_{p,m}} := \|c\, m\|_{\ell_p} < \infty\},
$$

where $m = (m(g_i))_{i\in\mathcal{I}}$. Conversely, if $(c_i(f))_{i\in\mathcal{I}} \in \ell_{p,m}$, then $f = \sum_{i\in\mathcal{I}} c_i\pi(g_i)\psi$ is in $\mathcal{SC}_{p,m}$ and

$$
\|f\|_{\mathcal{SC}_{p,m}} \lesssim \|(c_i(f))_{i\in\mathcal{I}}\|_{\ell_{p,m}}. \tag{18}
$$

Banach Frames: *The set $\{\pi(g_i)\psi : i \in \mathcal{I}\}$ is a Banach frame for $\mathcal{SC}_{p,m}$ which means that*

i)
$$
\|f\|_{\mathcal{SC}_{p,m}} \sim \|(\langle f, \pi(g_i)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{i\in\mathcal{I}}\|_{\ell_{p,m}}, \tag{19}
$$

ii) *there exists a bounded, linear reconstruction operator $\mathcal{R}$ from $\ell_{p,m}$ to $\mathcal{SC}_{p,m}$ such that $\mathcal{R}\left((\langle f, \psi(g_i)\psi\rangle_{\mathcal{H}^\sim_{1,w}\times\mathcal{H}_{1,w}})_{i\in\mathcal{I}}\right) = f$.*

3. Characterization of Coorbit Spaces and Besov Spaces

In the next section, we will show that traces of shearlet coorbit spaces onto certain hyperplanes are contained in Besov spaces or again in shearlet coorbit spaces. The proof of these trace theorems will heavily rely on the characterization

- of Besov spaces via atomic decompositions,
- of coorbit spaces via expansions of molecules.

The following subsections provide the results which will be necessary for our analysis.

3.1. **Atoms in Besov Spaces.** We start by the characterization of homogeneous Besov spaces $B^{\sigma}_{p,q}$ from [9], see also [14, 20]. For inhomogeneous Besov spaces we refer to [19]. For $\alpha > 1$, $D > 1$ and $K \in \mathbb{N}_0$, a K times differentiable function ϕ on $\mathbb{R}^d$ is called a K-*atom* if the following two conditions are fulfilled:

A1) $\operatorname{supp} \phi \subset DQ_{j,l}(\mathbb{R}^d)$ for some $l \in \mathbb{R}^d$,
where $DQ_{j,l}(\mathbb{R}^d)$ denotes the cube in $\mathbb{R}^d$ centered at $\alpha^{-j}l$ with sides parallel to the coordinate axes and side length $2\alpha^{-j}D$.

A2) $|\mathrm{D}^{\gamma}\phi(x)| \le \alpha^{|\gamma|j}$ for $|\gamma| \le K$.

Now the homogeneous Besov spaces can be characterized as follows.

Theorem 3.1. *Let $D > 1$ and $K \in \mathbb{N}_0$ with $K \ge 1 + \lfloor \sigma \rfloor$, $\sigma > 0$ be fixed. Let $1 \le p \le \infty$. Then $f \in B^{\sigma}_{p,q}$ if and only if it can be represented as*

$$f(x) = \sum_{j \in Z} \sum_{l \in \mathbb{Z}^d} \lambda(j,l)\phi_{j,l}(x), \tag{20}$$

where the $\phi_{j,l}$ are K-atoms with $\operatorname{supp} \phi_{j,l} \subset DQ_{j,l}(\mathbb{R}^d)$ *and*

$$\|f\|_{B^{\sigma}_{p,q}} \sim \inf\Big(\sum_{j \in \mathbb{Z}} \alpha^{j(\sigma - \frac{d}{p})q} \big(\sum_{l \in \mathbb{Z}^d} |\lambda(j,l)|^p \big)^{\frac{q}{p}} \Big)^{\frac{1}{q}}$$

where the infimum is taken over all admissible representations (20).

3.2. **Molecules in Shearlet Coorbit Spaces.** Further, we will make use of the recently introduced molecules in general coorbit spaces, see [12]. We summarize the results needed from [12] for our shearlet coorbit spaces. Let $\psi \in \mathcal{B}_w$, $\psi \neq 0$ and let $X := \{g_i\}_{i \in \mathcal{I}}$ be a U-dense, relatively separated family in $\mathbb{S}$. A collection of functions $\{\phi_i\}_{i \in \mathcal{I}}$ from $L_2(\mathbb{R}^d)$ is called a *set of molecules*, if there exists an envelope function $H \in \mathcal{W}^R(L_\infty, L_{1,w})$ such that

$$|\mathcal{SH}_\psi \phi_i(g)| \le H(g_i^{-1}g), \quad i \in \mathcal{I}.$$

This definition of the molecules does not depend on the particular choice of $\psi \in \mathcal{B}_w$. For a fixed $\psi \in \mathcal{B}_w$ and $H \in \mathcal{W}^R(L_\infty, L_{1,w})$, let

$$\mathcal{C} := \{\phi \in L_2(\mathbb{R}^d) : |\mathcal{SH}_\psi \phi(g)| \le H(g)\}.$$

Then, for $\phi_i \in \mathcal{C}$, the family $\{\pi(g_i)\phi_i : i \in \mathcal{I}\}$ is a set of molecules since

$$|\mathcal{SH}_\psi(\pi(g_i)\phi_i)(g))| = |\langle \pi(g_i)\phi_i, \pi(g)\psi \rangle| = |\mathcal{SH}_\psi \phi_i(g_i^{-1}g)| \le H(g_i^{-1}g).$$

The following synthesis property was proved in [12] for general coorbit spaces.

Theorem 3.2. *Let $\{\phi_i\}_{i \in \mathcal{I}}$ be a set of molecules subordinated to $H \in \mathcal{W}^R(L_\infty, L_{1,w})$. If $(c_i)_{i \in \mathcal{I}} \in \ell_{p,m}$, $p \in [1,\infty]$, then $f := \sum_{i \in \mathcal{I}} c_i \phi_i \in \mathcal{SC}_{p,m}$ and*

$$\|f\|_{\mathcal{SC}_{p,m}} \lesssim \|(c_i)_{i \in \mathcal{I}}\|_{\ell_{p,m}} \ .$$

4. Traces of Shearlet Coorbit Spaces

In this section, we are interested in traces of shearlet coorbit spaces. Traces of shearlet coorbit spaces on $\mathbb{R}^2$ onto the real axes were considered in [5]. It turned out that such traces of subspaces of shearlet coorbit spaces resembling shearlets on the cone are contained in Besov spaces. We will see that in higher dimensions the shear parameter will play an important role. More precisely, traces onto $d-1$-dimensional hyperplanes containing the x_1-axis will be contained in shearlet coorbit spaces again.

To keep the technicalities at a reasonable level, we restrict ourselves to the practically most important case of three dimensions. Moreover, we are only interested in weights

$$m(a,s,t) = m(a) := |a|^{-r}, r \geq 0$$

and use the abbreviation

$$\mathcal{SC}_{p,r} := \mathcal{SC}_{p,m}.$$

By (15), the set

$$\{(\varepsilon\alpha^{-j}, \beta\alpha^{\frac{-2j}{3}}k, S_{\beta\alpha^{\frac{-2j}{3}}k} A_{\alpha^{-j}}\tau l) : j \in \mathbb{Z}, k \in \mathbb{Z}^2, l \in \mathbb{Z}^3, \, \varepsilon \in \{-1,1\}\}. \tag{21}$$

is U-dense and relatively separated for U defined as in (14). We restrict ourselves to the case $a > 0$ such that $\varepsilon = +1$. The case $a < 0$ (and $\varepsilon = -1$) can be handled analogously. For $a := \alpha^{-j}$, $s := \beta\alpha^{-\frac{2j}{3}}(k_1,k_2)^{\mathrm{T}}$ and $t := S_{\beta\alpha^{\frac{-2j}{3}}k} A_{\alpha^{-j}}\tau l$ we use the abbreviation $\psi_{j,k,l} := \pi(a,s,t)\psi$. By straightforward computation we obtain that

$$\psi_{j,k,l}(x) = \alpha^{\frac{5j}{6}} \psi \begin{pmatrix} \alpha^j x_1 - \tau l_1 - \alpha^{\frac{j}{3}}\beta(k_1x_2 + k_2x_3) \\ \alpha^{\frac{j}{3}}x_2 - \tau l_2 \\ \alpha^{\frac{j}{3}}x_3 - \tau l_3 \end{pmatrix}. \tag{22}$$

Replacing $f(x)$ by $f^\tau(x) := f(\tau x)$ and $\psi(x)$ by $\psi^\tau(x) := \psi(\tau x)$, we see that we can work without loss of generality with $\tau := 1$. In the following, we restrict our attention to this case. Note that if $\psi^\tau(x)$ has support in $[-D,D]^3$ then ψ has support in $[-\tau D, \tau D]^3$.

By Theorem 2.3, any $f \in \mathcal{SC}_{p,r}$ can be written as

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2} \sum_{l\in\mathbb{Z}^3} c(j,k,l)\psi_{j,k,l}(x). \tag{23}$$

To derive reasonable trace and embedding theorems, it is necessary to introduce the following subspaces of $\mathcal{SC}_{p,r}$. For fixed $\psi \in B_w$, we denote by $\mathcal{SC}_{p,r}^{(\eta)}$, $\eta \in \{0,1\}^2$ the closed subspace of $\mathcal{SC}_{p,r}$ consisting of those functions which are representable as in (23) but with integers $|k_i| \leq \alpha^{\frac{2j}{3}}$ if $\eta_i = 1$. We want to investigate the traces of functions lying in the subspaces $\mathcal{SC}_{p,r}^{(\eta)}$ with respect to the coordinate planes. For symmetry reasons we can restrict our attention to the x_1x_2-plane and to the x_2x_3-plane. We start with the latter one, where we prove that the traces are contained in Besov spaces.

Theorem 4.1. *Let* $\mathrm{Tr}_{x_1} f$ *denote the restriction of* f *to the* x_2x_3*-plane, i.e.,* $(\mathrm{Tr}_{x_1} f)(x_2,x_3) := f(0,x_2,x_3)$*. Then the embedding* $Tr_{x_1}(\mathcal{SC}_{p,r}^{(1,1)}(\mathbb{R}^3)) \subset B_{p,p}^{\sigma_1}(\mathbb{R}^2) + B_{p,p}^{\sigma_2}(\mathbb{R}^2)$ *holds true, where* $\sigma_1 + 2\lfloor\sigma_1\rfloor = 3r - \frac{21}{2} + \frac{8}{p}$ *and* $\sigma_2 = 3r - \frac{1}{2} - \frac{5}{2} + \frac{2}{p}$.

Proof. We split f into $f = f_1 + f_2$ as follows:

$$f_1(x_1,x_2,x_3) := \sum_{j\geq 0} \sum_{|k_1|\leq \alpha^{2j/3}} \sum_{|k_2|\leq \alpha^{2j/3}} \sum_{l\in\mathbb{Z}^3} c(j,k,l)\psi_{j,k,l}(x_1,x_2,x_3) \tag{24}$$

$$f_2(x_1,x_2,x_3) := \sum_{j<0} \sum_{l\in\mathbb{Z}^3} c(j,0,l)\psi_{j,k,l}(x_1,x_2,x_3). \tag{25}$$

By Theorem 2.2, the analyzing function ψ can be chosen compactly supported in $[-D,D]^3$ for some $D > 1$. For our σ_i, $i = 1,2$ defined in the statement of theorem, let $K_i := 1 + \lfloor \sigma_i \rfloor$, $i = 1,2$ and $K := \max\{K_1, K_2\}$. We normalize ψ such that its derivatives of order $0 \leq |\gamma| \leq K$ are not larger than $1/\max\{1, \beta^K\}$. By the support assumption on ψ we have that

$$\begin{aligned}
\alpha^{-\frac{j}{3}}(l_3 - D) \leq x_3 \leq \alpha^{-\frac{j}{3}}(l_3 + D)\\
\alpha^{-\frac{j}{3}}(l_2 - D) \leq x_2 \leq \alpha^{-\frac{j}{3}}(l_2 + D)\\
-D - \alpha^{\frac{j}{3}}\beta(k_1x_2 + k_2x_3) \leq l_1 \leq D - \alpha^{\frac{j}{3}}\beta(k_1x_2 + k_2x_3).
\end{aligned}$$

Consequently, we obtain that

$$-\beta(k_1l_2 + k_2l_3) - D(1 + \beta(|k_1| + |k_2|)) \leq l_1 \leq -\beta(k_1l_2 + k_2l_3) + D(1 + \beta(|k_1| + |k_2|)).$$

Let

$$I := I(k_1,k_2,l_1,l_2) := \{n \in \mathbb{Z} : |n + \beta(k_1l_2 + k_2l_3)| \leq D(1 + \beta(|k_1| + |k_2|))\}.$$

For $j \geq 0$, we set

$$\lambda(j,l_2,l_3) := \alpha^{\frac{5+4K_1}{6}j} \sum_{|k_1|\leq\alpha^{2j/3}} \sum_{|k_2|\leq\alpha^{2j/3}} \sum_{l_1\in I} |c(j,k,l)|$$

and

$$\phi_{j,l_2,l_3}(x_2,x_3) := \lambda(j,l_2,l_3)^{-1}\alpha^{\frac{2K_1}{3}j} \sum_{|k_1|\leq\alpha^{2j/3}} \sum_{|k_2|\leq\alpha^{2j/3}} \sum_{l_1\in I} c(j,k,l)\alpha^{-\frac{2K_1}{3}j}\psi_{j,k,l}(0,x_2,x_3)$$

if $\lambda(j,l_2,l_3) \neq 0$ and for $j < 0$ analogously

$$\lambda(j,l_2,l_3) := \alpha^{\frac{5}{6}j} \sum_{l_1\in I} |c(j,0,l)|$$

and

$$\phi_{j,l_2,l_3}(x_2,x_3) := \lambda(j,l_2,l_3)^{-1} \sum_{l_1\in I} c(j,0,l)\psi_{j,k,l}(0,x_1,x_2)$$

if $\phi_{j,l_2,l_3} \neq 0$. In both cases we set $\phi_{j,l_2,l_3} := 0$ if $\lambda(j,l_2,l_3) = 0$. Now we can write

$$\begin{aligned}
\mathrm{Tr}_{x_1} f(x_2,x_3) &= f(0,x_2,x_3)\\
&= \sum_{j\in\mathbb{Z}} \sum_{|k_1|\leq\alpha^{2j/3}} \sum_{|k_2|\leq\alpha^{2j/3}} \sum_{(l_2,l_3)\in\mathbb{Z}^2} \sum_{l_1\in I} c(j,k,l)\psi_{j,k,l}(0,x_2,x_1)\\
&= \sum_{j\geq 0} \sum_{(l_2,l_3)\in\mathbb{Z}^2} \lambda(j,l_2,l_3)\phi_{j,l_2,l_3}(x_2,x_3) + \sum_{j<0} \sum_{(l_2,l_3)\in\mathbb{Z}^2} \lambda(j,l_2,l_3)\phi_{j,l_2,l_3}(x_2,x_3)\\
&= \mathrm{Tr}_{x_1} f_1(x_2,x_3) + \mathrm{Tr}_{x_1} f_2(x_2,x_3).
\end{aligned}$$

We want to show that the ϕ_{j,l_2,l_3} are K_i-atoms. First, we have that

$$\operatorname{supp}\psi_{j,k,l}(0,x_2,x_3) = \operatorname{supp}\psi\begin{pmatrix} -\alpha^{\frac{j}{3}}\beta(k_1x_2+k_2x_3)-l_1 \\ \alpha^{\frac{j}{3}}x_2-l_2 \\ \alpha^{\frac{j}{3}}x_3-l_3 \end{pmatrix} \subset DQ_{j,l_2,l_3}(\mathbb{R}^2)$$

with respect to the side length $2\alpha^{-\frac{j}{3}}D$. Since we sum over finite sets, this support property is also true for ϕ_{j,l_2,l_3}. Next, we conclude by $|k_i| \le \alpha^{\frac{2j}{3}}$, $i=1,2$ that for $j \ge 0$ the derivatives of ϕ_{j,l_2,l_3} can be estimated as

$$\alpha^{\frac{-5-4K_1}{6}j}\left|\alpha^{\frac{5}{6}j}\mathrm{D}^\gamma\psi\begin{pmatrix} -\alpha^{\frac{j}{3}}\beta(k_1x_2+k_2x_3)-l_1 \\ \alpha^{\frac{j}{3}}x_2-l_2 \\ \alpha^{\frac{j}{3}}x_3-l_3 \end{pmatrix}\right| \le \alpha^{-K_1\frac{2j}{3}}(\alpha^{\frac{j}{3}}\alpha^{\frac{2j}{3}})^{|\gamma|} \le \alpha^{\frac{j}{3}|\gamma|}, \quad |\gamma| \le K_1.$$

For $j<0$ we have similarly that

$$\alpha^{-\frac{5}{6}j}\left|\alpha^{\frac{5}{6}j}\mathrm{D}^\gamma\psi\begin{pmatrix} -l_1 \\ \alpha^{\frac{j}{3}}x_2-l_2 \\ \alpha^{\frac{j}{3}}x_3-l_3 \end{pmatrix}\right| \le \alpha^{\frac{j}{3}|\gamma|}, \quad |\gamma| \le K_2.$$

Thus, by their definition, the ϕ_{j,l_2,m_3} are K_i-atoms. By Theorem 3.1 and Theorem 2.3 we get

$$\begin{aligned}\|\mathrm{Tr}_{x_1}f_1\|^p_{B^{\sigma_1}_{p,p}(\mathbb{R}^2)} &\lesssim \sum_{j\ge 0}\alpha^{\frac{j}{3}(\sigma_1-\frac{2}{p})p}\sum_{(l_2,l_3)\in\mathbb{Z}^2}|\lambda(j,l_2,l_3)|^p \\ &\lesssim \sum_{j\ge 0}\alpha^{\frac{j}{3}(\sigma_1-\frac{2}{p})p}\alpha^{\frac{5+4K_1}{6}jp}\sum_{(l_2,l_3)\in\mathbb{Z}^2}\Big(\sum_{|k_1|\le\alpha^{2j/3}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{l_1\in I}|c(j,k,l)|\Big)^p\end{aligned}$$

and since $(\sum_{i=1}^N |z_i|)^p \le N^{p-1}\sum|z_i|^p$ further

$$\begin{aligned}\|\mathrm{Tr}_{x_1}f_1\|^p_{B^{\sigma_1}_{p,p}(\mathbb{R}^2)} &\lesssim \sum_{j\ge 0}\alpha^{\frac{j}{3}(\sigma_1-\frac{2}{p})p}\alpha^{\frac{5+4K_1}{6}jp}\alpha^{2j(p-1)}\sum_{(l_2,l_3)\in\mathbb{Z}^2}\sum_{|k_1|\le\alpha^{2j/3}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{l_1\in I}|c(j,k,l)|^p \\ &\lesssim \sum_{j\ge 0}\alpha^{jpr}\sum_{|k_1|\le\alpha^{2j/3}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{l_1\in I}|c(j,k,l)|^p \\ &\lesssim \|f\|^p_{\mathcal{SC}_{p,r}(\mathbb{R}^3)}\end{aligned}$$

with $r = \frac{1}{3}(\sigma_1 + 2\lfloor\sigma_1\rfloor + \frac{21}{2} - \frac{8}{p})$. Analogously we can compute

$$\begin{aligned}\|\mathrm{Tr}_{x_1}f_2\|^p_{B^{\sigma_2}_{p,p}(\mathbb{R}^2)} &\lesssim \sum_{j<0}\alpha^{\frac{j}{3}(\sigma_2-\frac{2}{p})p}\sum_{(l_2,l_3)\in\mathbb{Z}^2}|\lambda(j,l_2,l_3)|^p \\ &\lesssim \sum_{j<0}\alpha^{\frac{j}{3}(\sigma_2-\frac{2}{p})p}\alpha^{\frac{5}{6}jp}\sum_{(l_2,l_3)\in\mathbb{Z}^2}\Big(\sum_{l_1\in I}|c(j,k,l)|\Big)^p \\ &\lesssim \sum_{j<0}\alpha^{jpr}\sum_{l}|c(j,k,l)|^p \\ &\lesssim \|f\|^p_{\mathcal{SC}_{p,r}(\mathbb{R}^3)}\end{aligned}$$

with $r = \frac{1}{3}(\sigma_2 + \frac{5}{2} - \frac{2}{p})$. $\square$

Let us now turn to traces on the x_1x_2-plane. In this case the shear parameter will play an additional role so that the traces will again be contained in shearlet coorbit spaces. For the proof of the theorem, we need the following auxiliary lemma.

Remark 4.2. Consider the representation $\pi((a,s,t)|\mathbb{R}^2)\psi(x) := |a|^{-\frac{2}{3}}\psi(A_a^{-1}S_s^{-1}(x-t))$ of the shearlet group on $L_2(\mathbb{R}^2)$ with

$$A_a := \begin{pmatrix} a & 0 \\ 0 & \operatorname{sgn}(a)|a|^{\frac{1}{3}} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

such that

$$\pi((a,s,t)|\mathbb{R}^2)\psi(x) = |a|^{-\frac{2}{3}}\psi\begin{pmatrix} a^{-1}(x_1 - t_1 - s(x_2-t_2)) \\ \operatorname{sgn}(a)|a|^{-\frac{1}{3}}(x_2-t_2) \end{pmatrix} \tag{26}$$

which is slightly different from (1). The representation (26) can be interpreted as the restriction of (1) for $d=3$ to the two dimensional case. This representation preserves all the properties shown in Section 2, in particular, we have that for a neighborhood

$$U \supseteq [\alpha^{-\frac{2}{3}}, \alpha^{\frac{1}{3}}) \times [-\frac{\beta}{2}, \frac{\beta}{2}) \times [-\frac{\tau}{2}, \frac{\tau}{2})^2, \quad \alpha > 1, \beta, \tau > 0$$

of the identity, the set

$$X := \left\{(\varepsilon\alpha^j, \alpha^{\frac{2j}{3}}\beta k_1, S_{\alpha^{\frac{2j}{3}}\beta k_1} A_{\alpha^j}\tau l) : j \in \mathbb{Z}, k_1 \in \mathbb{Z}, l \in \mathbb{Z}^2, \varepsilon \in \{-1,1\}\right\}$$

is U-dense and relatively separated.

Proof. The group properties, the representation and the U-density can be obtained by straightforward computation.

Another way to see this is by identifying $\mathbb{R}^2 \simeq \mathbb{R}^2 \times 0$ (and $\mathbb{Z}^2 \simeq \mathbb{Z}^2 \times 0$) and using these sets with the original setting in Section 2 omitting the third components (i. e. rows and columns). □

Using the setting from Lemma 4.2 to define our shearlet coorbit spaces on $\mathbb{R}^2$, we can prove the following theorem.

Theorem 4.3. *Let* $\mathrm{Tr}_{x_3}f$ *denote the restriction of* f *to the* x_1x_2*-plane, i.e.,* $(\mathrm{Tr}_{x_3}f)(x_1,x_2) := f(x_1,x_2,0)$*. Then* $\mathrm{Tr}_{x_3}(\mathcal{SC}_{p,r}^{(0,1)}(\mathbb{R}^3)) \subset \mathcal{SC}_{p,r_1}(\mathbb{R}^2) + \mathcal{SC}_{p,r_2}(\mathbb{R}^2)$*, where* $r_1 = r - \frac{5}{6} + \frac{2}{3p}$ *and* $r_2 = r - \frac{1}{6}$.

Proof. We split f into $f = f_1 + f_2$ as follows:

$$f_1(x_1,x_2,x_3) := \sum_{j\geq 0}\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|\leq\alpha^{2j/3}}\sum_{l\in\mathbb{Z}^3} c(j,k,l)\psi_{j,k,l}(x_1,x_2,x_3), \tag{27}$$

$$f_2(x_1,x_2,x_3) := \sum_{j<0}\sum_{k_1\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^3} c(j,k_1,0,l)\psi_{j,k_1,0,l}(x_1,x_2,x_3). \tag{28}$$

Now $\mathrm{Tr}_{x_3}f$ can be written as

$$\begin{aligned} \mathrm{Tr}_{x_3}f(x_1,x_2) &= f(x_1,x_2,0) \\ &= \sum_{j\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|\leq\alpha^{2j/3}}\sum_{l\in\mathbb{Z}^3} c(j,k,l)\,\underbrace{\alpha^{\frac{5j}{6}}\psi\begin{pmatrix} \alpha^j x_1 - l_1 - \alpha^{\frac{j}{3}}\beta k_1 x_2 \\ \alpha^{\frac{j}{3}}x_2 - l_2 \\ -l_3 \end{pmatrix}}_{\psi_{j,k,l}(x_1,x_2,0)}. \end{aligned}$$

By Theorem 2.2 we can choose ψ compactly supported in $[-D,D]^3$ for some $D>1$. Consequently, we obtain

$$\begin{aligned}\mathrm{Tr}_{x_3} f(x_1,x_2) &= \sum_{j\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{(l_1,l_2)\in\mathbb{Z}^2}\sum_{|l_3|\le D} c(j,k,l)\psi_{j,k,l}(x_1,x_2,0)\\ &= \sum_{j\ge 0}\sum_{k_1\in\mathbb{Z}}\sum_{(l_1,l_2)\in\mathbb{Z}^2}\lambda(j,k_1,l_1,l_2)\phi_{j,k_1,l_1,l_2} + \sum_{j<0}\sum_{k_1\in\mathbb{Z}}\sum_{(l_1,l_2)\in\mathbb{Z}^2}\lambda(j,k_1,l_1,l_2)\phi_{j,k_1,l_1,l_2}\\ &= \mathrm{Tr}_{x_3} f_1(x_1) + \mathrm{Tr}_{x_3} f_2(x_1),\end{aligned}$$

where for $j\ge 0$,

$$\phi_{j,k_1,l_1,l_2}(x_1,x_2) := \lambda(j,k_1,l_1,l_2)^{-1}\alpha^{\frac{j}{6}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{|l_3|\le D} c(j,k,l)\alpha^{\frac{2j}{3}}\psi\begin{pmatrix}\alpha^j x_1 - l_1 - \alpha^{\frac{j}{3}}\beta k_1 x_2\\ \alpha^{\frac{j}{3}}x_2 - l_2\\ l_3\end{pmatrix}$$

if

$$\lambda(j,k_1,l_1,l_2) := \alpha^{\frac{j}{6}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{|l_3|\le D}|c(j,k,l)| \neq 0$$

and for $j<0$,

$$\phi_{j,k_1,l_1,l_2}(x_1,x_2) := \lambda(j,k_1,l_1,l_2)^{-1}\alpha^{\frac{j}{6}}\sum_{|l_3|\le D} c(j,k_1,0,l)\alpha^{\frac{2j}{3}}\psi\begin{pmatrix}\alpha^j x_1 - l_1 - \alpha^{\frac{j}{3}}\beta k_1 x_2\\ \alpha^{\frac{j}{3}}x_2 - l_2\\ l_3\end{pmatrix}$$

if

$$\lambda(j,k_1,l_1,l_2) := \alpha^{\frac{j}{6}}\sum_{|l_3|\le D}|c(j,k_1,0,l)| \neq 0.$$

In both cases we set $\phi_{j,k_1,l_1,l_2}(x_1,x_2) := 0$ if $\lambda(j,k_1,l_1,l_2)=0$.

The functions ϕ_{j,k_1,l_1,l_2} are molecules by the following reasons: We restrict our attention to $a>0$ and $\tau=1$. With

$$a := \alpha^{-j},\ s := \alpha^{-\frac{2j}{3}}\beta k_1 \quad\text{and}\quad \begin{pmatrix}t_1\\ t_2\end{pmatrix} := S_{\alpha^{-\frac{2j}{3}}\beta k_1} A_{\alpha^{-j}}\begin{pmatrix}l_1\\ l_2\end{pmatrix} = \begin{pmatrix}\alpha^{-j}l_1 + \alpha^{-j}l_2\beta k_1\\ \alpha^{-\frac{j}{3}}l_2\end{pmatrix} \tag{29}$$

representation (26) reads as

$$\pi((a,s,t)|\mathbb{R}^2)\psi(x) = \alpha^{\frac{2j}{3}}\psi\begin{pmatrix}\alpha^j x_1 - l_1 - \alpha^{\frac{j}{3}}\beta k_1 x_2\\ \alpha^{\frac{j}{3}}x_2 - l_2\end{pmatrix}.$$

With $g_i = (a_i, s_i, t_i)$ defined by (29) the functions $\phi_{j,k_1,l_1,l_2}(x_1,x_2)$ (for $j\ge 0$) can be written as

$$\begin{aligned}\phi_{j,k_1,l_1,l_2}(x_1,x_2) &= \lambda(j,k_1,l_1,l_2)^{-1}\alpha^{\frac{j}{6}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{|l_3|\le D} c(j,k,m)\pi(g_i|\mathbb{R}^2)\psi(x_1,x_2,l_3)\\ &= \pi(g_i|\mathbb{R}^2)\Big(\lambda(j,k_1,l_1,l_2)^{-1}\alpha^{\frac{j}{6}}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{|l_3|\le D} c(j,k,l)\psi(x_1,x_2,l_3)\Big)\\ &= \pi(g_i|\mathbb{R}^2)\phi_i(x_1,x_2),\end{aligned}$$

with $\phi_i(x_1,x_2) := \lambda(j,k_1,l_1,l_2)^{-1}\alpha^{\frac{j}{6}} \sum_{|k_2|\le \alpha^{2j/3}} \sum_{|l_3|\le D} c(j,k,l)\psi(x_1,x_2,l_3)$. Let $\tilde{\psi} := \psi(x_1,x_2,0)$ and $H(g) := \max_{|l_3|\le D} |\mathcal{SH}_{\tilde{\psi}}\psi(\cdot,l_3)(g)|$. Then we know by Theorem 2.2 that $H \in \mathcal{W}^R(L_\infty, L_{1,w})$. Further we obtain

$$\begin{aligned} |\mathcal{SH}_{\tilde{\psi}}(\phi_i)(g)| &\le \lambda(j,k_1,l_1,l_2)^{-1}\alpha^{\frac{j}{6}} \sum_{|k_2|\le\alpha^{2j/3}} \sum_{|l_3|\le D} |c(j,k,l)| \underbrace{|\mathcal{SH}_{\tilde{\psi}}\psi(\cdot,l_3)(g)|}_{\le H(g)} \\ &\le |H(g)|. \end{aligned}$$

Hence, ϕ_{j,k_1,l_1,l_2} is a molecule for $j \ge 0$. For $j < 0$ it can be shown similarly that ϕ_{j,k_1,l_1,l_2} is a molecule.

Finally, we obtain by Theorem 3.2 and Theorem 2.3 the desired trace estimate for $f \in \mathcal{SC}^{(0,1)}_{p,r}(\mathbb{R}^3)$):

$$\begin{aligned} \|\mathrm{Tr}_{x_3} f_1\|^p_{\mathcal{SC}_{p,r_1}(\mathbb{R}^2)} &\lesssim \sum_{j\ge 0}\sum_{k_1\in\mathbb{Z}}\sum_{(l_1,l_2)\in\mathbb{Z}^2} \alpha^{jpr_1}|\lambda(j,k_1,l_1,l_2)|^p \\ &\lesssim \sum_{j\ge 0}\alpha^{jp(r_1+\frac{1}{6})}\sum_{k_1\in\mathbb{Z}}\sum_{(l_1,l_2)\in\mathbb{Z}^2}\Big|\sum_{|k_2|\le\alpha^{2j/3}}\sum_{|l_3|\le D}|c(j,k_1,k_2,l)|\Big|^p, \\ &\lesssim \sum_{j,k_1,l_1,l_2}\alpha^{jp(r_1+\frac{1}{6})}\alpha^{\frac{2j}{3}(p-1)}\sum_{|k_2|\le\alpha^{2j/3}}\sum_{|l_3|\le D}|c(j,k_1,k_2,l)|^p \\ &\lesssim \|f\|^p_{\mathcal{SC}_{p,r}(\mathbb{R}^3)} \end{aligned}$$

with $r = r_1 + \frac{5}{6} - \frac{2}{3p}$. In the same way we obtain

$$\begin{aligned} \|\mathrm{Tr}_{x_3} f_2\|^p_{\mathcal{SC}_{p,r_2}(\mathbb{R}^2)} &\lesssim \sum_{j<0}\sum_{k_1\in\mathbb{Z}}\sum_{(l_1,l_2)\in\mathbb{Z}^2} \alpha^{jpr_2}|\lambda(j,k_1,l_1,l_2)|^p \\ &\lesssim \sum_{j,k_1,l_1,l_2}\alpha^{jp(r_2+\frac{1}{6})}\Big|\sum_{|l_3|\le D}|c(j,k_1,k_2,l)|\Big|^p \\ &\lesssim \sum_{j,k_1,l_1,l_2}\alpha^{jp(r_2+\frac{1}{6})}\sum_{|l_3|\le D}|c(j,k_1,k_2,l)|^p \\ &\lesssim \|f\|^p_{\mathcal{SC}_{p,r}(\mathbb{R}^2)} \end{aligned}$$

with $r = r_2 + \frac{1}{6}$. This completes the proof. □

5. Embeddings into Besov Spaces

In this section, we prove the following embedding result of certain subspaces of shearlet coorbit spaces in three dimensions into (sums of) homogeneous Besov spaces. We like to mention that embedding results in Besov spaces have also been shown for the curvelet setting by Borup and Nielsen [1]. However, the technique used by these authors is completely different since they work in the frequency domain.

Theorem 5.1. *The embedding $\mathcal{SC}^{(1,1)}_{p,r}(\mathbb{R}^3) \subset B^{\sigma_1}_{p,p}(\mathbb{R}^3) + B^{\sigma_2}_{p,p}(\mathbb{R}^3)$, holds true, where*

$$\sigma_1 + 2\lfloor\sigma_1\rfloor = 3r - \frac{21}{2} + \frac{9}{p} \quad \textit{and} \quad \sigma_2 - \frac{2}{3}\lfloor\sigma_2\rfloor = r + \frac{5}{3p} + \frac{7}{6}.$$

Proof. By (23) we know that $f \in \mathcal{SC}_{p,r}^{(1,1)}$ can be written as

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{|k_1|\le\alpha^{2j/3}} \sum_{|k_2|\le\alpha^{2j/3}} \sum_{l\in\mathbb{Z}^3} c(j,k,l)\psi_{j,k,l(x)}$$

By Theorem 2.2, the analyzing function ψ can be chosen compactly supported in $[-D,D]^3$ for some $D > 1$. For our σ_i, $i = 1,2$ defined in the theorem, let $K_i := 1 + \lfloor\sigma_i\rfloor$, $i = 1,2$, and $K := \max\{K_1, K_2\}$.

We split $f \in \mathcal{SC}_{p,r}^{(1,1)}$ as in (27) and (28) into f_1 and f_2 and restrict our attention to $\tau = 1$ and $\beta = 1$. We normalized ψ such that its derivatives of order $0 \le |\gamma| \le K$ are not larger than 1. With the index transform $l_1 = r_1 - (k_1l_2 + k_2l_3)$ we obtain

$$\begin{aligned} f_1(x) =& \sum_{j\ge0} \sum_{|k_1|\le\alpha^{2j/3}} \sum_{|k_2|\le\alpha^{2j/3}} \sum_{(l_2,l_3)\in\mathbb{Z}^2} \sum_{n_1\in\mathbb{Z}} \sum_{r_1\in I(j,n_1)} c(j,k_1,k_2,r_1-k_1l_2-k_2l_3,l_2,l_3) \\ &\times \alpha^{\frac{5j}{6}}\psi\begin{pmatrix} \alpha^j x_1 - r_1 + k_1l_2 + k_2l_3 - \alpha^{\frac{j}{3}}(k_1x_2+k_2x_3) \\ \alpha^{\frac{j}{3}}x_2 - l_2 \\ \alpha^{\frac{j}{3}}x_3 - l_3 \end{pmatrix}, \end{aligned}$$

where $I(j,n_1) := \{r \in \mathbb{Z} : \alpha^{\frac{2j}{3}}(n_1-1) < r \le \alpha^{\frac{2j}{3}}n_1\}$. For $j \ge 0$ we set

$$\begin{aligned} \phi_{j,n_1,l_2,l_3}(x) :=& \lambda(j,n_1,l_2,l_3)^{-1}\alpha^{\frac{5+4K_1}{6}j} \sum_{|k_1|\le\alpha^{2j/3}} \sum_{|k_2|\le\alpha^{2j/3}} \sum_{r_1\in I(j,n_1)} c(j,k_1,k_2,r_1-k_1l_2-k_2l_3,l_2,l_3) \\ &\times \alpha^{-\frac{2K_1}{3}j}\psi\begin{pmatrix} \alpha^j x_1 - r_1 + k_1l_2 + k_2l_3 - \alpha^{\frac{j}{3}}(k_1x_2+k_2x_3) \\ \alpha^{\frac{j}{3}}x_2 - l_2 \\ \alpha^{\frac{j}{3}}x_3 - l_3 \end{pmatrix}, \end{aligned}$$

if $\lambda(j,n_1,l_2,l_3) \ne 0$ and $\phi_{j,n_1,l_2,l_3}(x) := 0$ otherwise, where

$$\lambda(j,n_1,l_2,l_3) := \alpha^{\frac{5+4K_1}{6}j} \sum_{|k_1|\le\alpha^{2j/3}} \sum_{|k_2|\le\alpha^{2j/3}} \sum_{r_1\in I(j,n_1)} |c(j,k_1,k_2,r_1-k_1l_2-k_2l_3,l_2,l_3)|.$$

Then we see that

$$f_1(x) = \sum_{j\ge0} \sum_{n_1\in\mathbb{Z}} \sum_{(l_2,l_3)\in\mathbb{Z}^2} \lambda(j,n_1,l_2,l_3)\phi_{j,n_1,l_2,l_3}(x).$$

By the support assumption on ψ, the functions appearing in the definition of ϕ_{j,n_1,l_2,l_3} can only be non-zero if the following conditions are satisfied:

$$\begin{aligned} &-D \le \alpha^{\frac{j}{3}}x_i - l_i \le D, \quad \text{i.e.,} \quad \alpha^{-\frac{j}{3}}(l_i - D) \le x_i \le \alpha^{-\frac{j}{3}}(l_i + D), \quad i = 2,3 \\ &-D \le \alpha^j x_1 - r_1 + k_1l_2 + k_2l_3 - \alpha^{\frac{j}{3}}(k_1x_2 - k_2x_3) \le D \end{aligned}$$

such that

$$\begin{aligned} x_1 &\le \alpha^{-j}r_1 + \alpha^{-j}k_1(\alpha^{\frac{j}{3}}x_2 - l_2) + \alpha^{-j}k_2(\alpha^{\frac{j}{3}}x_3 - l_3) + \alpha^{-j}D \\ &\le \alpha^{-j}r_1 + \alpha^{-\frac{j}{3}}(3D) \le \alpha^{-\frac{j}{3}}n_1 + \alpha^{-\frac{j}{3}}(3D) \end{aligned}$$

and similarly $x_1 \ge \alpha^{-j}r_1 - \alpha^{-\frac{j}{3}}(3D) \ge \alpha^{-\frac{j}{3}}n_1 - \alpha^{-\frac{j}{3}}(4D)$. And together

$$\alpha^{-\frac{j}{3}}n_1 - \alpha^{-\frac{j}{3}}(4D) \le x_1 \le \alpha^{-\frac{j}{3}}n_1 + \alpha^{-\frac{j}{3}}(3D)$$

Thus, ϕ_{j,n_1,l_2,l_3} is supported in $8DQ_{j,n_1,l_2,l_3}$, where the cube is considered with respect to side length $2\alpha^{\frac{j}{3}}$. The bounds $|D^\gamma \phi_{j,n_1,l_2,l_3}| \le \alpha^{\frac{j}{3}|\gamma|}$, $|\gamma| \le K_1$ can be derived as in the proof of Theorem 4.1. Hence the functions ϕ_{j,n_1,l_2,l_3} are K_1-atoms. Now we obtain by Theorem 3.1 that

$$
\begin{aligned}
\|f_1\|^p_{B^{\sigma_1}_{p,p}(\mathbb{R}^3)} &\lesssim \sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{3}(\sigma_1-\frac{3}{p})p} \sum_{n_1\in\mathbb{Z}} \sum_{(l_2,l_3)\in\mathbb{Z}^2} |\lambda(j,n_1,l_2,l_3)|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{3}(\sigma_1-\frac{3}{p})p} \alpha^{(\frac{5+4K_1}{6})jp} \sum_{(n_1,l_2,l_3)\in\mathbb{Z}^3} \Big| \sum_{|k_1|,|k_2|\le\alpha^{2j/3}} \sum_{r_1\in I(j,n_1)} |c(j,k_1,k_2,r_1-k_1l_2-k_2l_3,l_2,l_3)| \Big|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{3}(\sigma_1-\frac{3}{p})p+(\frac{5+4K_1}{6})jp+2j(p-1)} \sum_{(n_1,l_2,l_3)\in\mathbb{Z}^3} \sum_{|k_1|,|k_2|\le\alpha^{2j/3}} \sum_{r_1\in I(j,n_1)} |c(j,k_1,k_2,r_1-k_1l_2-k_2l_3,l_2,l_3)|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{l\in\mathbb{Z}^3} \sum_{|k_1|\le\alpha^{2j/3}} \sum_{|k_2|\le\alpha^{2j/3}} |c(j,k_1,k_2,l_1,l_2,l_3)|^p \\
&\lesssim \|f\|^p_{\mathcal{SC}_{p,r}(\mathbb{R}^3)}.
\end{aligned}
$$

with $r = \frac{1}{3}(\sigma_1 + 2\lfloor\sigma_1\rfloor + \frac{21}{2} - \frac{9}{p})$. In the case $j<0$ we obtain with $J(j,n_i) := \{r : \alpha^{-\frac{2j}{3}}(n_i-1) < r \le \alpha^{-\frac{2j}{3}} n_i\}$, $i=2,3$, that

$$
\begin{aligned}
f_2(x) &= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{l_2\in\mathbb{Z}}\sum_{l_3\in\mathbb{Z}} c(j,0,0,l_1,l_2,l_3)\alpha^{\frac{5j}{6}} \psi \begin{pmatrix} \alpha^j x_1 - l_1 \\ \alpha^{\frac{j}{3}} x_2 - l_2 \\ \alpha^{\frac{j}{3}} x_3 - l_3 \end{pmatrix} \\
&= \sum_{j<0}\sum_{l_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}} \sum_{r_2\in J(j,n_2)} \sum_{r_3\in J(j,n_3)} c(j,0,0,l_1,r_2,r_3)\alpha^{\frac{5j}{6}} \psi \begin{pmatrix} \alpha^j x_1 - l_1 \\ \alpha^{\frac{j}{3}} x_2 - r_2 \\ \alpha^{\frac{j}{3}} x_3 - r_3 \end{pmatrix} \\
&= \sum_{j<0} \sum_{(l_1,n_2,n_3)\in\mathbb{Z}^3} \lambda(j,l_1,n_2,n_3)\phi_{j,l_1,n_2,n_3}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_{j,l_1,n_2,n_3}(x) &:= \lambda(j,l_1,n_2,n_3)^{-1}\alpha^{\frac{5-4K_2}{6}j} \sum_{r_2\in J(j,n_2)} \sum_{r_3\in J(j,n_3)} c(j,0,0,l_1,r_2,r_3)\psi \begin{pmatrix} \alpha^j x_1 - l_1 \\ \alpha^{\frac{j}{3}} x_2 - r_2 \\ \alpha^{\frac{j}{3}} x_3 - r_3 \end{pmatrix} \\
\lambda(j,n_1,l_2,l_3) &:= \alpha^{\frac{5-4K_2}{6}j} \sum_{r_2\in J(j,n_2)} \sum_{r_3\in J(j,n_3)} |c(j,0,0,l_1,r_2,r_3)|
\end{aligned}
$$

if $\lambda(j,n_1,l_2,l_3) \neq 0$ and $\phi_{j,n_1,l_2,l_3}(x) := 0$ if $\lambda(j,n_1,l_2,l_3)=0$. By the support assumption on ψ we get

$$\alpha^{-j}(l_1 - D) \le x_1 \le \alpha^{-j}(l_1+D)$$

and since $\alpha^{-j/3} < \alpha^{-j}$ for $j<0$ we further get for $i=2,3$

$$\alpha^{-\frac{j}{3}}(r_i - D) \le x_i \le \alpha^{-\frac{j}{3}}(r_i + D) \quad \Rightarrow \quad \alpha^{-j}(n_i - 2D) \le x_i \le \alpha^{-j}(n_i+D).$$

Consequently, ϕ_{j,n_1,l_2,l_3} is supported in $4DQ_{j,n_1,l_2,l_3}$ with respect to side length $2\alpha^j$. Since $1 \ge \alpha^{j|\gamma|/3} \ge \alpha^{j|\gamma|} \ge \alpha^{jK_2}$ for $0 \le |\gamma| \le K_2$ and $j<0$ we obtain that $|D^\gamma \phi_{j,l_1,n_2,n_3}| \le \alpha^{j|\gamma|/3}\alpha^{2jK_2/3} \le$

$\alpha^{j|\gamma|}$, $0 \le |\gamma| \le K_2$ so that ϕ_{j,l_1,n_2,n_3} are K_2-atoms. Thus,

$$\begin{aligned}
\|f_2\|^p_{B^{\sigma_2}_{p,p}} &\lesssim \sum_{j\in\mathbb{Z}} \alpha^{j(\sigma_2-\frac{3}{p})p} \sum_{(l_1,n_2,n_3)\in\mathbb{Z}^3} |\lambda(j,l_1,n_2,n_3)|^p \\
&\lesssim \sum_{j<0} \alpha^{j(\sigma_2-\frac{3}{p})p} \alpha^{\frac{5-4K_2}{6}jp} \sum_{(l_1,n_2,n_3)\in\mathbb{Z}^3} \Big| \sum_{r_2\in J(j,n_2)} \sum_{r_3\in J(j,n_3)} c(j,0,0,l_1,n_2,n_3)\Big|^p \\
&\lesssim \sum_{j<0} \alpha^{j(\sigma_2-\frac{3}{p})p} \alpha^{\frac{5-4K_2}{6}jp} \alpha^{-\frac{4j}{3}(p-1)} \sum_{l\in\mathbb{Z}^3} |c(j,0,0,l)|^p \\
&\lesssim \sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}^2} \sum_{l\in\mathbb{Z}^3} |c(j,k,l)|^p \\
&\lesssim \|f\|^p_{\mathcal{SC}_{p,r}},
\end{aligned}$$

where $r = \sigma_2 - \frac{2}{3}\lfloor\sigma_2\rfloor - \frac{5}{3p} - \frac{7}{6}$. □

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