Biorthogonal Box Spline Wavelet Bases

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Abstract

Some specific box splines are refinable functions with respect to $n \times n$ expanding integer scaling matrices M satisfying $M^n = 2I$. Therefore they can be used to define a multiresolution analysis and a wavelet basis associated with these scaling matrices. In this paper, we construct biorthogonal wavelet bases for this special subclass of box splines. These specific bases can also be used to derive wavelets with respect to classical dyadic scaling matrices.

Key Words: biorthogonal wavelets, multiresolution analysis, box splines

AMS Subject classification: 41A15, 41A30, 41A63

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*The work of the first author has been supported by Deutsche Forschungsgemeinschaft (DA360/1-1).

†The second author would like to thank the Institut für Geometrie und Praktische Mathematik for its hospitality and support. He also acknowledges partial support by NSF grant DMS-9306430.

1 Introduction

In recent years, wavelet analysis has become a useful tool for both theoretical lines of developments and practical applications. Therefore the construction of appropriate wavelet bases is a field of increasing importance. Special emphasis was put on the construction of wavelets by using spline functions, since spline functions are well-understood, allow for explicit computations, and possess additional structure. First examples of one-dimensional B-spline wavelets were constructed by Lemarié [16] and Battle [1] and later investigated in detail by Chui and Wang [5], [6], see also Micchelli [18]. In higher dimensions, a first approach using box splines was given by Riemenschneider and Shen, see also [4]. In [19], they constructed an orthonormal box spline wavelet basis. However, the resulting wavelet basis is not compactly supported since the integer translates of a box spline are in general not orthonormal. This problem can be avoided by considering pre-wavelets. For box splines, this concept was also developed by Riemenschneider and Shen [20]. But again, when dealing with pre-wavelets, at least one of the associated filters will be infinite, so that the usefulness of pre-wavelets is limited. One way to get around this difficulty is to use a biorthogonal wavelet basis, i.e., two systems of functions $\{\psi_i\}_{i\in I}$, $\{\tilde{\psi}_i\}_{i\in I}$ satisfying for some expanding integer scaling matrix the biorthogonality condition

$$\langle m^{j/2}\psi_i(M^j\cdot-\alpha), m^{j'/2}\tilde{\psi}_i(M^{j'}\cdot-\alpha')\rangle = \delta_{ii'}\delta_{\alpha\alpha'}\delta_{j,j'}\ , \quad m := |\det M|\ . \tag{1.1}$$

In this setting, the wavelets can be chosen to have compact support, and all the calculations and corresponding numerical algorithms are as simple and convenient as in the orthonormal case.

In general, a system of wavelets can be constructed by using the multiresolution analysis technique introduced by Mallat [17]. A multiresolution analysis consists of a nested sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ such that their union is dense and their intersection is zero. Furthermore, we assume that the space V_0 is spanned by the integer translates of one function φ called the **generator** and that the spaces V_j are related with each other by a scaling process with respect to an expanding integer scaling matrix, i.e.,

$$f(\cdot)\in V_j \Longleftrightarrow f(M\cdot)\in V_{j+1}\ . \tag{1.2}$$

To ensure the nestedness and condition (1.2), a natural candidate for φ would be a **refinable function**, i.e., a function satisfying a two-scale-relation of the form

$$\varphi(x) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha\varphi(Mx-\alpha)\ , \quad \alpha\in\mathbb{Z}^n\ , \tag{1.3}$$

for some suitable sequence $\{a_\alpha\}_{\alpha\in\mathbb{Z}^n}\in\ell_2(\mathbb{Z}^n)$. In the biorthogonal setting, one has to deal with **two** sequences $\{V_j\}_{j\in\mathbb{Z}}$, $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ and tries to find the systems $\{\psi_i\}_{i\in I}$, $\{\tilde{\psi}_i\}_{i\in I}$ as bases functions for the specific complement spaces W_0 of V_0 in V_1, and $\tilde{W}_0$ of $\tilde{V}_0$ in $\tilde{V}_1$, that satisfy

$$W_0\perp\tilde{V}_0,\ \tilde{W}_0\perp V_0, \qquad V_1 = W_0\oplus V_0,\ \tilde{V}_1=\tilde{W}_0\oplus\tilde{V}_0\ . \tag{1.4}$$

The essential step in doing this consists in the extension of a row vector $(a_{e_1}(z), ..., a_{e_m}(z))$, of trigonometric polynomials to a matrix with constant determinant on $T^n := \{z \in \mathbb{C}^n | \, |z_i| = 1\,, i = 1, \dots n\}$. If m is large, this problem can be highly nontrivial, see e.g. [15] for details. On the other hand, for scaling matrices satisfying $|\det M| = 2$, the extension problem becomes quite simple. Furthermore, the case $|\det M| = 2$ is important, because only one wavelet is needed to generate a wavelet basis.

Fortunately, some box splines which are refinable with respect to a special subclass of these scaling matrices can be found and therefore may serve as generators for a multiresolution analysis. The reason for this is given by the intimate relations between box splines and regular self–affine tilings. In the sequel, we will always restrict ourselves to expanding integer scaling matrices $M \in \mathbb{Z}^{n\times n}$ satisfying

$$M^n = 2I. \tag{1.5}$$

It is well–known that these scaling matrices can produce regular self– affine lattice tilings, i.e., parallelepipeds Q that satisfy

$$|Q| = 1, \quad \bigcup_{\alpha \in \mathbb{Z}^n} (Q+\alpha) \cong \mathbb{R}^n, \quad Q \cap (Q+\alpha) \cong \emptyset,\ \alpha \in \mathbb{Z}^n \setminus \{0\}\ , \tag{1.6}$$

$$Q \cong \bigcup_{d^j \in R} M^{-1}(Q + d^j), \tag{1.7}$$

where R is a complete set of representatives for $\mathbb{Z}^n/M\mathbb{Z}^n$, see e.g. [14] and [11] for details. Observe that the number of cosets in $\mathbb{Z}^n/M\mathbb{Z}^n$ is given by $m = |\det M|$. Therefore in our case we only have to deal with two representatives. We always choose 0 to be the representative of $M\mathbb{Z}^n$ and denote the representative of $\mathbb{Z}^n/M\mathbb{Z}^n$ by d.

Let us now briefly recall the definition of a box spline. To this end, let $X := \{x^1, \dots, x^s\} \subset \mathbb{Z}^n \setminus \{0\}$ denote a set of not necessarily distinct vectors satisfying $s > n$ and

$$\langle X \rangle := \text{span } X = \mathbb{R}^n\ .$$

Then the **box spline** $B(\,\cdot\,|X)$ is defined by requiring that the equation

$$\int_{\mathbb{R}^n} f(y)B(y|X)dy = \int_{[0,1]^s} f(Xu)du \tag{1.8}$$

holds for any continuous function f on $\mathbb{R}^n$. The vectors $x^1, \dots, x^s$ are called the **direction vectors** of $B(\,\cdot\,|X)$. From (1.8) it is easy to derive a formula for the Fourier transform of $B(\,\cdot\,|X)$,

$$\widehat{B(\,\cdot\,|X)}(\xi) = \prod_{x^l \in X} \left(\frac{1 - e^{-ix^l\cdot\xi}}{ix^l \cdot \xi} \right)\ . \tag{1.9}$$

For further information, the reader is referred to the book of de Boor, Höllig and Riemenschneider [3]. Equation (1.9) illuminates the intimate relation between box splines and self-affine tilings since it can be used to check that the convolution product $\underbrace{\chi_Q * \chi_Q * \ldots * \chi_Q}_{K-times}$ for some $K \in I\!N$ where Q is a self-affine set satisfying (1.6) and (1.7) is the box spline $B(\,\cdot\,|X)$, $X = (\underbrace{d\ d \ldots d}_{K-times}\underbrace{Md\ Md \ldots Md}_{K-times}\ldots\underbrace{M^{n-1}d\ M^{n-1}d \ldots M^{n-1}d}_{K-times})$, $d = d^2$. Then equation (1.7) implies that $B(\,\cdot\,|X)$ is refinable with respect to M, see [9]. In this paper, we will study this particular class of box splines in Section 2 and construct a corresponding compactly supported biorthogonal wavelet basis in Section 3.

2 Dual Box Spline Pairs

To construct a biorthogonal wavelet basis, we have to find the approximation sequences $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ or equivalently we have to find suitable generators φ and $\tilde{\varphi}$. It turns out that φ and $\tilde{\varphi}$ have to form a **dual pair**, i.e.,

$$\langle\varphi(\cdot), \tilde{\varphi}(\cdot - \alpha)\rangle = \delta_{0\alpha}\ , \quad \alpha \in \mathbb{Z}^n \tag{2.10}$$

see e.g. [7] for details. Therefore our first step to construct a biorthogonal box spline wavelet basis is to find a dual function for the box spline $\varphi(\cdot) = B(\,\cdot\,|\underbrace{d\ d \ldots d}_{K-times}\underbrace{Md\ Md \ldots Md}_{K-times}\ldots\underbrace{M^{n-1}d\ M^{n-1}d \ldots M^{n-1}d}_{K-times})$. We want to find the dual function by means of a reduction principle. Indeed, it turns out that the problem can be converted to a simple tensor–product setting where the dual pair can be built from univariate dual functions for cardinal B-splines. These specific dual functions are well–known and e.g. studied in [7]. To carry out the program, we need a factorization result for the special scaling matrices we are dealing with. This fact is studied in the following lemma.

Lemma 2.1 *Suppose that M is an $n \times n$ expanding integer scaling matrix with the property*

$$M^n = \pm 2\,I\ .$$

If there exists a representative $d \in \mathbb{Z}^n/M\mathbb{Z}^n$, so that the matrix $(d, Md, \ldots, M^{n-1}d)$ is unimodular, then M possesses the factorization

$$M = AS\Pi A^{-1}$$

where $A \in SL(n, \mathbb{Z})$, $S =$ diag $(\pm 2, \pm 1, \ldots, \pm 1)$ and Π is an irreducible permutation matrix.

Proof: We assume that $M^n = 2I$. The case $M^n = -2I$ is argued in a similar fashion.

It follows from the assumptions that M can be diagonalized over $\mathbb{C}$ and that its eigenvalues are contained in $\{2^{1/n}e^{2\pi il/n}, l = 0, \ldots, n-1\}$. (or $\{2^{1/n}e^{2\pi i(l+1/2)/n}, l = 0, \ldots, n-1\}$, if $M^n = -2I$). From this it is easy to see that the vectors $M^jx, j = 0, \ldots, n-1$, are linearly independent for any $x \in I\!R^n$, $x \neq 0$.

Given $d \in \mathbb{Z}^n/M\mathbb{Z}^n$ with $\det(d, Md, \ldots, M^{n-1}d) = \pm 1$ we consider the parallelepiped $Q = \{x \in \mathbb{R}^n | x = \sum_{j=0}^{n-1} c_j M^j d,\ 0 \le c_j \le 1\}$ of volume 1. Then MQ consists of all points $y = \sum_{j=1}^{n-1} c_{j-1} M^j d + 2c_{n-1} d$. Clearly, $y \in Q$, if and only if $0 \le c_{n-1} \le 1/2$ and $y \in d + Q$, if and only if $1/2 \le c_{n-1} \le 1$. Therefore

$$MQ = Q \cup (d + Q)$$

$(MQ = (-2d+Q) \cup (-d+Q)$ if $M^n = -2I)$ and Q is a M-self–affine tile of measure 1. The structure of scaling matrices that permit parallelepipeds as tiles is well-understood and Lemma 9 of [14] asserts that M factors as $M = AD\Pi A^{-1}$, for some unimodular matrix A with integer entries, a diagonal matrix D with integer entries $\hat{\delta}_j$, and a permutation matrix Π, such that $\prod_{k=1}^{n} |\hat{\delta}_{\pi^k(i)}| > 1$ for $i = 1, \ldots, n$. Since $|\det M| = 2$, D must be of the form $\operatorname{diag}(\pm 2, \pm 1, \ldots, \pm 1)$. Moreover, the permutation P is irreducible, otherwise M would act as an isometry on a proper invariant subspace of $\mathbb{R}^n$, which is impossible. $\square$

Now we are able to state and to prove the main result of this section.

Theorem 2.1 *Let M be an $n \times n$ expanding integer scaling matrix satisfying $M^n = 2I$. Suppose that d is chosen such that the matrix $(d, Md, \ldots, M^{n-1}d)$ is unimodular and let* $\varphi(\cdot) := B(\ \cdot\ | \underbrace{d\, d \ldots d}_{K-times}\ \underbrace{Md\ Md \ldots Md}_{K-times} \ldots \underbrace{M^{n-1}d\ M^{n-1}d \ldots M^{n-1}d}_{K-times})$, $K \in \mathbb{N}$, *be an associated box spline. Furthermore, let $\tilde{\phi}(\cdot)$ denote a dual basis for the univariate cardinal B-spline* $N_K := \underbrace{\chi_{[0,1)} * \ldots * \chi_{[0,1)}}_{K-times}$. *Then there exists a unimodular integer matrix V and an integer scaling matrix $\tilde{M}$ which also satisfies $\tilde{M}^n = 2I$ such that*

(i) φ and $\tilde{\varphi}(\cdot) := \underbrace{\tilde{\phi} \otimes \tilde{\phi} \otimes \cdots \otimes \tilde{\phi}}_{n-times}(V^{-1}\cdot)$ *form a dual pair, i.e.,* $\langle \varphi(\cdot - \alpha), \tilde{\varphi}(\cdot) \rangle = \delta_{0\alpha}$ *for $\alpha \in \mathbb{Z}^n$, and*

(ii) both φ and $\tilde{\varphi}$ are refinable with respect to $\tilde{M}$.

Proof: Let us first consider the case of the special family of scaling matrices $S_\Pi = S\Pi$ where $S = \operatorname{diag}(\pm 2, \pm 1, \ldots \pm 1)$ and Π is an irreducible permutation matrix. The general case can later on be reduced to this special setting. By $\ell \in \{1, \ldots n\}$ we denote the index with $(S)_{\ell\ell} = \pm 2$. Then every representative d of $\mathbb{Z}^n/S_\Pi\mathbb{Z}^n$ with respect to S_Π is of the form $(m_1, m_2, \ldots m_{\ell-1}, 2m_\ell + 1, m_{\ell+1}, \ldots m_n)$ for some $m_1, \ldots, m_n \in \mathbb{Z}$. A canonical choice is $d = (\underbrace{0, \ldots, 0}_{(\ell-1)-times}, 1, \underbrace{0, \ldots, 0}_{(n-\ell)-times}) \in \{0,1\}^n$. Then $\{0, d, S_\Pi d, \ldots, S_\Pi^{n-1} d\} = \{0,1\}^n$ and the same arguments as in the proof of Lemma 2.1 show that the corresponding self–affine set is the unit cube $\tilde{Q} := [0,1)^n$. There exist other possible choices, for instance for $n = 2$ and $S_\Pi = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ we may take $\tilde{d} = \binom{5}{7}$, then $S_\Pi \tilde{d} = \binom{7}{10}$, i.e., $\det(\tilde{d}, S_\Pi \tilde{d}) = 1$. We first reduce these exotic cases to the canonical one. Observe that a noncanonical self–affine set P can be mapped onto $\tilde{Q}$ by means of a unimodular transformation U, i.e., $\tilde{Q} = U^{-1}P$. We set

$$X \;:=\; (\underbrace{d\;d\ldots d}_{K-times}\underbrace{S_\Pi d\; S_\Pi d\ldots\; S_\Pi d}_{K-times}\ldots\underbrace{S_\Pi^{n-1}d\; S_\Pi^{n-1}d\ldots\; S_\Pi^{n-1}d}_{K-times})$$

and

$$\tilde{X} \;:=\; (\underbrace{\tilde{d}\;\tilde{d}\ldots\tilde{d}}_{K-times}\underbrace{S_\Pi\tilde{d}\; S_\Pi\tilde{d}\ldots S_\Pi\tilde{d}}_{K-times}\ldots\underbrace{S_\Pi^{n-1}\tilde{d}\; S_\Pi^{n-1}\tilde{d}\ldots S_\Pi^{n-1}\tilde{d}}_{K-times}).$$

Then

$$\begin{aligned} B(\,\cdot\,|\tilde{X}) &= \underbrace{\chi_P * \chi_P * \ldots * \chi_P}_{K-times}(\cdot) = \underbrace{\chi_P * \chi_P * \ldots * \chi_P}_{K-times}(UU^{-1}\cdot) \\ &= \underbrace{\chi_{U^{-1}P} * \ldots * \chi_{U^{-1}P}}_{K-times}(U^{-1}\cdot) = B(U^{-1}\cdot|X). \end{aligned}$$

Let us now suppose that we have found a biorthogonal basis $\eta(\cdot)$ for $B(\,\cdot\,|X)$. Then

$$\langle B(\,\cdot\,|\tilde{X}), \eta(U^{-1}(\cdot-\alpha))\rangle = \langle B(U^{-1}\cdot)|X), \eta(U^{-1}(\cdot-\alpha)\rangle = \langle B(\,\cdot\,|X), \eta(\cdot-\beta)\rangle = \delta_{0\beta},$$

$\beta = U^{-1}\alpha$, i.e., $\eta(U^{-1}\cdot)$ is a dual basis for $B(\,\cdot\,|\tilde{X})$. Moreover, $\eta(U^{-1}\cdot)$ and $B(\,\cdot\,|\tilde{X})$ are refinable functions with respect to $\tilde{S_\Pi} = US_\Pi U^{-1}$ since one has for some appropriate sequences $\{c_\alpha\}_{\alpha\in\mathbb{Z}^n}$ and $\{a_\alpha\}_{\alpha\in\mathbb{Z}^n}$

$$\begin{aligned} \eta(U^{-1}x) &= \sum_{\alpha\in\mathbb{Z}^n} c_\alpha \eta(S_\Pi U^{-1}x-\alpha) \\ &= \sum_{\alpha\in\mathbb{Z}^n} c_\alpha \eta(U^{-1}US_\Pi U^{-1}x-\alpha) \\ &= \sum_{\alpha\in\mathbb{Z}^n} c_\alpha \eta(U^{-1}(US_\Pi U^{-1}x-U\alpha)) \\ &= \sum_{\beta\in\mathbb{Z}^n} c_{U^{-1}\beta} \eta(U^{-1}(\tilde{S_\Pi}x-\beta)) \end{aligned}$$

and

$$\begin{aligned} B(\,x\,|\tilde{X}) &= B(\,U^{-1}x\,|X) \\ &= \sum_{\alpha\in\mathbb{Z}^n} a_\alpha B(S_\Pi U^{-1}x-\alpha|X) \\ &= \sum_{\alpha\in\mathbb{Z}^n} a_\alpha B(U^{-1}(US_\Pi U^{-1}x-U\alpha)|X) \\ &= \sum_{\beta\in\mathbb{Z}^n} a_{U^{-1}\beta} B(U^{-1}(\tilde{S_\Pi}x-\beta))|X) \\ &= \sum_{\beta\in\mathbb{Z}^n} a_{U^{-1}\beta} B(\tilde{S_\Pi}x-\beta|\tilde{X}). \end{aligned}$$

Consequently, i) and ii) hold with $V = U$, $\tilde{M} = \tilde{S_\Pi} = US_\Pi U^{-1}$. To finish the case of the special matrices S_Π it remains to construct a biorthogonal basis for $B(\,\cdot\,|X)$.

Observe that $B(\cdot\,|X)$ is the tensor product cardinal B-spline $N_K(x_1)N_K(x_2)\ldots N_K(x_n)$. Therefore $\tilde{\varphi}(x_1,\ldots,x_n) = \tilde{\phi}(x_1)\tilde{\phi}(x_2)\ldots\tilde{\phi}(x_n)$ is a dual function for $B(\cdot\,|X)$. If K is even then the dual function $\tilde{\phi}$ can be chosen in such a way that $\tilde{\phi}(-x) = \tilde{\phi}(x)$, see [7] Section 6.A for details. Let i_0 be the index with $(S_\Pi)_{\ell,i_0} = \pm 2$ then we obtain for some finite sequence $\{c_k\}_{k\in\mathbb{Z}}$

$$\begin{aligned}
\tilde{\varphi}(x_1,\ldots x_n) &= \tilde{\phi}(x_1)\tilde{\phi}(x_2)\ldots\tilde{\phi}(x_n) \\
&= \sum_{k\in\mathbb{Z}} c_k\tilde{\phi}(x_1)\ldots\tilde{\phi}(x_{i_0-1})\tilde{\phi}(x_{i_0+1})\ldots\tilde{\phi}(x_n)\tilde{\phi}(2x_{i_0}-k) \qquad (2.11)\\
&= \sum_{k\in\mathbb{Z}} c_k\tilde{\varphi}(S_\Pi x - \beta_k)
\end{aligned}$$

with

$$(\beta_k)_i := \begin{cases} \pm k\ , & i=\ell \\ 0\ , & i\neq\ell\ , \end{cases}$$

which shows that $\tilde{\varphi}$ is refinable with respect to S_Π. If K is odd one has $\tilde{\phi}(-x) = \tilde{\phi}(1+x)$ and similar, but slightly more technical arguments as in (2.11) show that $\tilde{\varphi}$ is also refinable with respect to S_Π.

Now let M denote an arbitrary scaling matrix satisfying $M^n = 2I$ and let the associated representative d be chosen such that $|\det(d, Md, \ldots, M^{n-1}d)| = 1$. According to Lemma 2.1 M possesses a factorization

$$M = AS_\Pi A^{-1},$$

where A is a unimodular matrix and S_Π is one of the canonical matrices from above. Let P denote the tile associated with M and d. Then $\tilde{P} = A^{-1}P$ is a self–affine tile with respect to S_Π since

$$\begin{aligned}
S_\Pi(A^{-1}P) &= A^{-1}MA(A^{-1}P) \\
&= A^{-1}MP \\
&= A^{-1}(P\cup(P+d)) \\
&= A^{-1}P\cup(A^{-1}P + A^{-1}d).
\end{aligned}$$

Observe that $A^{-1}d$ is an admissible representative, for if $A^{-1}d = S_\Pi e$ for some $e\in\mathbb{Z}^n$, then $A^{-1}d = A^{-1}MAe$ so that $d = MAe$ which is a contradiction. Therefore with

$$\begin{aligned}
Y &:= (\underbrace{d\ d\ldots d}_{K-times}\underbrace{Md\ Md\ldots Md}_{K-times}\ldots\underbrace{M^{n-1}d\ M^{n-1}d\ldots M^{n-1}d}_{K-times}),\\
Z &:= (\underbrace{A^{-1}d\ A^{-1}d\ldots A^{-1}d}_{K-times}\underbrace{S_\Pi A^{-1}d\ S_\Pi A^{-1}d\ldots S_\Pi A^{-1}d}_{K-times}\ldots\underbrace{S_\Pi^{n-1}A^{-1}d\ S_\Pi^{n-1}A^{-1}d\ldots S_\Pi^{n-1}A^{-1}d}_{K-times})
\end{aligned}$$

we obtain

$$\begin{aligned}
B(\cdot\,|Y) &= \underbrace{\chi_P * \chi_P * \ldots * \chi_P}_{K-times}(\cdot) \\
&= \underbrace{\chi_{\tilde{P}} * \chi_{\tilde{P}} * \ldots * \chi_{\tilde{P}}}_{K-times}(A^{-1}\cdot) \\
&= B(A^{-1}\cdot\ |Z).
\end{aligned}$$

Therefore if $\tilde{\eta}(\cdot)$ is a dual function for $B(\,\cdot\,|Z)$, then $\tilde{\eta}(A^{-1}\cdot)$ is a dual function for $B(\,\cdot\,|Y)$. $\tilde{\eta}(\cdot)$ is refinable with respect to $\tilde{U}S_\Pi\tilde{U}^{-1}$ for some suitable unimodular matrix $\tilde{U}$, so that $\tilde{\eta}(A^{-1}\cdot)$ is refinable with respect to $\tilde{M} = (A\tilde{U})S_\Pi(A\tilde{U})^{-1}$ since for some sequence $\{b_\alpha\}_{\alpha\in\mathbb{Z}^n}$

$$\begin{aligned}
\tilde{\eta}(A^{-1}x) &= \sum_{\alpha\in\mathbb{Z}^n} b_\alpha\tilde{\eta}(\tilde{U}S_\Pi\tilde{U}^{-1}A^{-1}x-\alpha) \\
&= \sum_{\alpha\in\mathbb{Z}^n} b_\alpha\tilde{\eta}(A^{-1}((A\tilde{U})S_\Pi(A\tilde{U})^{-1}x-A\alpha)) \\
&= \sum_{\beta\in\mathbb{Z}^n} b_{A^{-1}\beta}\tilde{\eta}(A^{-1}(\tilde{M}x-\beta)).
\end{aligned}$$

It remains to check that $B(\,\cdot\,|Y)$ is also refinable with respect to $\tilde{M}$. We obtain

$$\begin{aligned}
B(\,\cdot\,|Y) &= B(A^{-1}\cdot|Z) \\
&= \sum_{\alpha\in\mathbb{Z}^n} a_\alpha B(\tilde{U}S_\Pi\tilde{U}^{-1}A^{-1}x-\alpha|Z) \\
&= \sum_{\alpha\in\mathbb{Z}^n} a_\alpha B(A^{-1}((A\tilde{U})S_\Pi(A\tilde{U})^{-1}x-A\alpha)|Z) \\
&= \sum_{\beta\in\mathbb{Z}^n} a_{A^{-1}\beta} B(\tilde{M}x-\beta|Y),
\end{aligned}$$

so that indeed i) and ii) hold with $V = A\tilde{U}$ and $\tilde{M} = (A\tilde{U})S_\Pi(A\tilde{U})^{-1}$. □

3 Biorthogonal Box Spline Wavelets

Once we have found a dual pair, we want to construct the associated wavelet basis. For a given scaling matrix M and a refinable function φ we introduce the subsymbols

$$a_\rho(z) := \sum_{\alpha\in\mathbb{Z}^n} a_{M\alpha+\rho}z^\alpha, \qquad \rho\in R, \tag{3.12}$$

where once again R denotes a complete set of representatives of $\mathbb{Z}^n/M\mathbb{Z}^n$. Then

$$a(z) = \sum_{\rho\in R} z^\rho a_\rho(z^M).$$

It is easy to check that φ and $\tilde{\varphi}$ indeed give rise to a multiresolution analysis, see e.g. [2]. Therefore to construct a biorthogonal basis, we have to find a basis for the complement spaces W_0 and $\tilde{W}_0$, respectively.

A general procedure is the following. One has to extend the row $(a_{\rho_1}(z),\ldots,a_{\rho_m}(z))$ to a matrix

$$A(z) = (a^\rho_{\tilde{\rho}}(z))_{\rho,\tilde{\rho}\in R}$$

satisfying

$$\det A(z) = \text{const} \qquad \text{for all} \quad z\in T^n.$$

Then one has to solve the equation

$$A(z) \cdot B(z) = mI, \qquad B(z) = (\overline{b^{\rho}_{\tilde{\rho}}}(z^{-1}))_{\tilde{\rho},\rho \in R}.$$

If there exists a refinable function $\eta(\cdot)$ satisfying

$$\eta(x) = \sum_{\alpha \in \mathbb{Z}^n} b^0_{\alpha} \eta(Mx - \alpha)$$

and

$$\langle \eta(\cdot), \varphi(\cdot - \alpha) \rangle = \delta_{0,\alpha},$$

then the functions

$$\psi^{\rho}(x) := \sum_{\alpha \in \mathbb{Z}^n} a^{\rho}_{\alpha} \varphi(Mx - \alpha), \qquad \tilde{\psi}^{\rho'}(x) := \sum_{\alpha' \in \mathbb{Z}^n} b^{\rho'}_{\alpha'} \eta(Mx - \alpha'), \qquad \rho, \rho' \in R \backslash \{0\},$$

satisfy

$$\langle m^{j/2} \psi^{\rho}(M^j \cdot - \alpha), m^{j'/2} \tilde{\psi}^{\rho'}(M^{j'} \cdot - \alpha') \rangle = \delta_{\rho,\rho'} \delta_{j,j'} \delta_{\alpha,\alpha'}, \tag{3.13}$$

i.e., the system $\{\psi^{\rho}\}_{\rho \in R \backslash \{0\}}, \{\tilde{\psi}^{\rho'}\}_{\rho' \in R \backslash \{0\}}$ forms a biorthogonal wavelet basis. For further information concerning biorthogonal bases and suitable extensions the reader is referred e.g. to Cohen and Daubechies [8] and to Dahmen and Micchelli [12].

In general, the extension problem is nontrivial. However, in our special case, the solution is quite simple and can always be given explicitly. Unless otherwise stated, all symbols and subsymbols are assumed to be defined with respect to the scaling matrix $\tilde{M}$ from Theorem 2.1.

Theorem 3.1 *Suppose that the conditions of Theorem 2.1 are satisfied and let $a(z)$ and $b(z)$ denote the symbols of φ and $\tilde{\varphi}$, respectively. Furthermore, let $c(z)$ denote the symbol of $\underbrace{\tilde{\phi} \otimes \cdots \otimes \tilde{\phi}}_{n-times}$ with respect to a canonical scaling matrix S_{Π}. Then the symbols $a^{\rho}(z)$ and $b^{\rho}(z)$ can be chosen as*

$$a^{\rho}(z) := -\overline{c}_{(A \cdot U)^{-1}\rho}(z^{-A \cdot U \cdot S_{\Pi}}) + z^{\rho} \overline{c_0}(z^{-A \cdot U \cdot S_{\Pi}}), \qquad b^{\rho}(z) = -\overline{a}_{\rho}(z^{-M}) + z^{\rho} \overline{a}_0(z^{-M}) \ ,$$

and the functions

$$\psi(x) = \sum_{\alpha \in \mathbb{Z}^n} a^{\rho}_{\alpha} \varphi(\tilde{M}x - \alpha), \qquad \tilde{\psi}(x) = \sum_{\alpha \in \mathbb{Z}^n} b^{\rho}_{\alpha} \tilde{\varphi}(\tilde{M}x - \alpha)$$

generate a biorthogonal wavelet basis.

Proof: As stated above, we have to extend the row $(a_0(z), a_{\rho}(z))$ to a matrix with constant determinant on T^n. However, in Theorem 2.1 we have already constructed a dual function $\tilde{\varphi}$. The duality relation

$$\langle \varphi(\cdot), \tilde{\varphi}(\cdot - \alpha) \rangle = \delta_{0,\alpha}$$

necessarily implies

$$a_0(z)\overline{b_0}(z^{-1}) + a_\rho(z)\overline{b_\rho}(z^{-1}) = 2,$$

so that one possible extension is given by

$$\begin{pmatrix} a_0(z) & a_\rho(z) \\ -\overline{b_\rho}(z^{-1}) & \overline{b_0}(z^{-1}) \end{pmatrix} \cdot \begin{pmatrix} \overline{b_0}(z^{-1}) & -a_\rho(z) \\ \overline{b_\rho}(z^{-1}) & a_0(z) \end{pmatrix} = 2I,$$

i.e.,

$$a_0^\rho(z) = -\overline{b_\rho}(z^{-1}), \quad a_\rho^\rho(z) = \overline{b_0}(z^{-1}).$$

It was already shown above that

$$\tilde{\varphi}(x) = \sum_{\alpha \in \mathbb{Z}^n} c_{U^{-1}A^{-1}\alpha} \tilde{\varphi}(\tilde{M}x - \alpha),$$

so that

$$\begin{aligned} \overline{b_\rho}(z^{-1}) &= \sum_{\alpha \in \mathbb{Z}^n} \overline{c}_{(A\cdot U)^{-1}\tilde{M}\alpha + (A\cdot U)^{-1}\rho} z^{-\alpha} = \sum_{\alpha \in \mathbb{Z}^n} \overline{c}_{S_\Pi \cdot (A\cdot U)^{-1}\alpha + (A\cdot U)^{-1}\rho} z^{-\alpha} \\ &= \sum_{\beta \in \mathbb{Z}^n} \overline{c}_{S_\Pi \beta + (A\cdot U)^{-1}\rho} z^{-(A\cdot U)\beta} = \overline{c}_{(A\cdot U)^{-1}\rho}(z^{-A\cdot U}), \\ \overline{b_0}(z^{-1}) &= \overline{c_0}(z^{-A\cdot U}). \end{aligned}$$

This yields

$$a^\rho(z) = a_0^\rho(z^{\tilde{M}}) + z^\rho a_\rho^\rho(z^{\tilde{M}}) = -\overline{c}_{(A\cdot U)^{-1}\rho}(z^{-A\cdot U\cdot S_\Pi}) + z^\rho \overline{c_0}(z^{-A\cdot U\cdot S_\Pi}).$$

□

It is well-known and easy to check from (1.9) that every box spline satisfies a two-scale-relation with respect to the scaling matrix $M = 2I$. In some applications it may be interesting to find a biorthogonal wavelet basis also for the dyadic scaling. A classical construction of these wavelets would require the extension of a 2^n-dimensional vector as described above. If the set of direction vectors is judiciously chosen, then the Theorem of Quillen and Suslin ensures that such an extension exists, see e.g. [15] for details. Nevertheless, its explicit determination can be very complicated. It turns out that the construction presented here provides a way to circumvent this difficulty. Indeed, at least for our special family of box splines a biorthogonal wavelet basis with respect to dyadic scaling matrices can be always constructed explicitly by using the wavelet basis given in Theorem 3.1.

Theorem 3.2 *Suppose that the assumptions of Theorem 3.1 are satisfied. Let ψ and $\tilde{\psi}$ be the corresponding biorthogonal box spline wavelets and let $R_k := \{d_1^k, \ldots, d_{2^k}^k\}$, $k \in \{1, \ldots n-1\}$ be full sets of representatives of the cosets of $\mathbb{Z}^n / M^k \mathbb{Z}^n$. Moreover, define for $k \in \{1, \ldots, n-1\}$, $i \in \{1, \ldots, 2^k\}$*

$$\psi_{d_i^k}(\cdot) := 2^{k/2}\psi(M^k(\cdot - M^{-k}d_i^k)) \quad \text{and} \quad \tilde{\psi}_{d_i^k}(\cdot) := 2^{k/2}\tilde{\psi}(M^k(\cdot - M^{-k}d_i^k)) \,. \tag{3.14}$$

Then $\{\psi, \psi_{d_i^k}\}_{\substack{k\in\{1,\ldots,n-1\} \\ i\in\{1,\ldots,2^k\}}}$ and $\{\tilde{\psi}, \tilde{\psi}_{d_i^k}\}_{\substack{k\in\{1,\ldots,n-1\} \\ i\in\{1,\ldots,2^k\}}}$ form a biorthogonal wavelet basis with respect to scaling by 2.

Proof: We want to prove this theorem by using the concept of multiresolution analysis for the construction of wavelets. Let $\{V_j\}_{j\in\mathbb{Z}}$ and $\{V_j^*\}_{j\in\mathbb{Z}}$ be the multiresolution analysis generated by the box spline φ with respect to scaling by M and 2, respectively, that is,

$$V_j = \overline{\operatorname{span}\{\varphi(M^j\cdot-\alpha)|\ \alpha\in\mathbb{Z}^n\}} \text{ and } V_j^* = \overline{\operatorname{span}\{\varphi(2^j\cdot-\alpha)|\ \alpha\in\mathbb{Z}^n\}}\ .$$

Thus, we have $V_0 = V_0^*$ and since $M^n = 2I$, one easily concludes that $V_n = V_1^*$. Let W_j and W_j^* be the complement spaces of V_j in V_{j+1} and of V_j^* in V_{j+1}^*, respectively. The spaces W_j are spanned by the translates of $\psi(M^j\cdot)$, i.e.,

$$W_j := \overline{\operatorname{span}\{\psi(M^j\cdot-\alpha)|\ \alpha\in\mathbb{Z}^n\}}$$

and one has

$$W_0^* = \bigoplus_{k=0}^{n-1} W_k\ . \tag{3.15}$$

Therefore it remains to find suitable bases for the spaces W_k. As one would expect, the system $\{\psi,\psi_{d_i^k}\}_{i\in\{1,\ldots,2^k\}}$ naturally does the job since for some $f\in W_k,\ k\geq 1$

$$\begin{aligned} f(x) &= \sum_{\alpha\in\mathbb{Z}^n} c_\alpha\psi(M^k x-\alpha) = \sum_{i=1}^{2^k}\sum_{\alpha\in\mathbb{Z}^n} c_{M^k\alpha+d_i^k}\psi(M^k x-(M^k\alpha+d_i^k)) \\ &= \sum_{i=1}^{2^k}\sum_{\alpha\in\mathbb{Z}^n} c_{M^k\alpha+d_i^k}\psi(M^k(x-\alpha-M^{-k}d_i^k)) = \sum_{i=1}^{2^k}\sum_{\alpha\in\mathbb{Z}^n} c_{M^k\alpha+d_i^k}\psi_{d_i^k}(x-\alpha)\ . \end{aligned}$$

Consequently,

$$W_0^* = \overline{\operatorname{span}\{\psi(\cdot-\alpha)\ ,\psi_{d_i^k}(\cdot-\alpha)|\ k\in\{1,\ldots,n-1\},\ i\in\{1,\ldots,2^k\},\ \ \alpha\in\mathbb{Z}^n\}}\ .$$

It can be shown analogously that the system $\{\tilde\psi,\tilde\psi_{d_i^k}\}_{ik}$ has similar properties. It remains to prove the duality of $\psi_{d_i^k}$ and $\tilde\psi_{d_j^k}$. We obtain for $k\in\{1,\ldots,n-1\}$

$$\begin{aligned} \langle\psi_{d_i^k}(\cdot-\alpha),\tilde\psi_{d_j^k}(\cdot)\rangle &= \int_{\mathbb{R}^n} 2^{k/2}\psi(M^k(x-\alpha-M^{-k}d_i^k))2^{k/2}\overline{\tilde\psi(M^k(x-M^{-k}d_j^k))}dx \\ &= \int_{\mathbb{R}^n} 2^k\psi(M^k x-M^k\alpha-d_i^k)\overline{\tilde\psi(M^k x-d_j^k)}dx \\ &= \frac{1}{|\det M|^k}\int_{\mathbb{R}^n} 2^k\psi(x-(M^k\alpha+d_i^k))\overline{\tilde\psi(x-d_j^k)}dx \\ &= \delta_{d_j^k,M^k\alpha+d_i^k} = \delta_{i,j}\delta_{0,\alpha}\ . \end{aligned}$$

$\langle\psi_{d_i^k},\tilde\psi_{d_j^l}\rangle = 0$ for $k\neq l$, since V_k is biorthogonal fo $\tilde V_l$. This proves the assertion. $\square$

4 Examples

We finish this note by some simple examples to indicate the applicability of the construction presented above. For simplicity, we will restrict ourselves to the case $n = 2$. The claim is to find dual pairs for which both functions φ and $\tilde{\varphi}$ are at least C_0^1. Therefore, our examples are based on

$$N_3(x) = \chi_{[0,1)} * \chi_{[0,1)} * \chi_{[0,1)} = \begin{cases} x^2/2 & 0 \le x \le 1, \\ -(x-3/2)^2 + 3/4 & 1 \le x \le 2, \\ (x-3)^2/2 & 2 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

A family of symmetric dual functions ${}_{K,\tilde{K}}\tilde{\phi}(\cdot)$ for $N_K(\cdot)$ with arbitrarily high regularity was computed by Cohen, Daubechies and Feauveau in [7]. It turns out that the function ${}_{K,\tilde{K}}\tilde{\phi}$ is C^m if $\tilde{K} > 0.2401K + 1.2401(m+1)$. Therefore, in our case, C^1-continuity requires at least $\tilde{K} \ge 4$. Choosing $\tilde{K} = 5$, a combination of the results in [7] and (2.11) yields the following expression for $c(z)$:

$$c_{\binom{0}{-4}} = -\tfrac{10}{512},\quad c_{\binom{0}{-3}} = \tfrac{30}{512},\quad c_{\binom{0}{-2}} = \tfrac{38}{512},\quad c_{\binom{0}{-1}} = -\tfrac{194}{512},\quad c_{\binom{0}{0}} = -\tfrac{26}{256},\quad c_{\binom{0}{1}} = \tfrac{350}{256},$$
$$c_{\binom{0}{2}} = \tfrac{350}{256},\quad c_{\binom{0}{3}} = -\tfrac{26}{256},\quad c_{\binom{0}{4}} = -\tfrac{194}{512},\quad c_{\binom{0}{5}} = \tfrac{38}{512},\quad c_{\binom{0}{6}} = \tfrac{30}{512},\quad c_{\binom{0}{7}} = -\tfrac{10}{512}.$$

Example 4.1 For the first example, let the scaling matrix M be given by

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

M produces the quincunx-grid Γ, i. e.,

$$\alpha = (\alpha_1, \alpha_2) \in \Gamma \ \text{ if and only if } \ \alpha_1 + \alpha_2 \text{ is even},$$

and therefore we may choose $d = \binom{1}{0}$. The resulting self–affine tile P is the parallelepiped with vertices $\binom{0}{0}, \binom{1}{0}, \binom{1}{1}$ and $\binom{2}{1}$ which produces the box spline

$$\varphi(\cdot) = B(\ \cdot \mid \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}),$$

see Figure 4.1.

Insert Figure 1

It is easy to check that M factors as

$$M = AS_\Pi A^{-1} \qquad \text{for} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_\Pi = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

In this special case, $\tilde{P} = A^{-1}P$ is the unit cube so that $\tilde{U} = I$ and $\tilde{M} = M$. The symbol of the dual generator $\tilde{\varphi}(\cdot) = \phi \otimes \phi(A^{-1}\cdot)$ is given by

$$b(z) = \sum_{\alpha \in \mathbb{Z}^2} c_{A^{-1}\alpha} z^\alpha = \sum_{\beta \in \mathbb{Z}^2} c_\beta z^{A\beta} = c(z).$$

In Figure 4.2, we plotted $\tilde{\varphi}$.

Insert Figure 2

It remains to construct the wavelet basis. According to Theorem 3.1 we first have to compute the subsymbols of $c(z)$ with respect to S_Π and $A^{-1}\rho$ for some suitable representative ρ. If we choose $\rho = \binom{1}{0}$ we obtain

$$\begin{aligned}
c_0(z) &= \sum_{\alpha\in Z^2} c_{S_\Pi\alpha} z^\alpha = -\frac{10}{512}z_1^{-2} + \frac{38}{512}z_1^{-1} - \frac{26}{256} + \frac{350}{256}z_1 - \frac{194}{512}z_1^2 + \frac{30}{512}z_1^3, \\
c_{\binom{0}{1}}(z) &= \sum_{\alpha\in Z^2} c_{S_\Pi\alpha+\binom{0}{1}} z^\alpha = \frac{30}{512}z_1^{-2} - \frac{194}{512}z_1^{-1} + \frac{350}{256} - \frac{26}{256}z_1^1 + \frac{38}{512}z_1^2 - \frac{10}{512}z_1^3.
\end{aligned}$$

Therefore, since

$$A \cdot S_\Pi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},$$

we get

$$\begin{aligned}
a^\rho(z) &= -\overline{c}_{\binom{0}{1}}(z^{-A\cdot S_\Pi}) + z^{(1,0)}\overline{c}_0(z^{-A\cdot S_\Pi}) \\
&= \frac{10}{512}z_1^{-6} + \frac{30}{512}z_1^{-5} - \frac{38}{512}z_1^{-4} - \frac{194}{512}z_1^{-3} + \frac{26}{256}z_1^{-2} + \frac{350}{256}z_1^{-1} \\
&\quad -\frac{350}{256} - \frac{26}{256}z_1 + \frac{194}{512}z_1^2 + \frac{38}{512}z_1^3 - \frac{30}{512}z_1^4 - \frac{10}{512}z_1^5.
\end{aligned}$$

The corresponding function $\psi(\cdot)$ is plotted in Figure 4.3.

Insert Figure 3

Finally, we have to compute $\tilde{\psi}(\cdot)$. This is now very easy since the symbol $a(z)$ of $\varphi(\cdot)$ with respect to M is clearly given by

$$a(z) = \frac{1}{4}(1+z_1)^3 = \frac{1}{4}(1 + 3z_1 + 3z_1^2 + z_1^3),$$

so that

$$a_0(z) = \frac{1}{4} + \frac{3}{4}z_1z_2, \qquad a_\rho(z) = \frac{3}{4} + \frac{1}{4}z_1z_2.$$

Consequently,

$$b^\rho(z) = -\overline{a}_\rho(z^{-M}) + z^\rho\overline{a}_0(z^{-M}) = -\frac{1}{4}z_1^{-2} + \frac{3}{4}z_1^{-1} - \frac{3}{4} - \frac{1}{4}z_1.$$

The resulting wavelet is shown in Figure 4.4.

Insert Figure 4

Example 4.2 For the second example, let us consider the canonical matrix $S_\Pi = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and the self–affine tile P produced by $d = \binom{5}{7}$, i.e., the parallelepiped with vertices 0, $\binom{5}{7}$, $\binom{7}{10}$ and $\binom{12}{17}$. Then $A = I$, $U = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix}$ and

$$\tilde{M} = US_\Pi U^{-1} = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 10 & -7 \\ -7 & 5 \end{pmatrix} = \begin{pmatrix} 105 & -73 \\ 151 & -105 \end{pmatrix}.$$

The symbol $b(z)$ of the dual generator is given by

$$b(z) = \sum_{\alpha \in \mathbb{Z}^2} c_{U^{-1}\alpha} z^\alpha = \sum_{\alpha \in \mathbb{Z}^2} c_\beta z^{U\beta} = \sum_{\beta \in \mathbb{Z}^2} c_\beta z_1^{5\alpha_1 + 7\alpha_2} z_2^{7\alpha_1 + 10\alpha_2}.$$

We may choose $d = \binom{1}{0}$ as a second representative for $\mathbb{Z}^2 / \tilde{M}\mathbb{Z}^2$. Then $U^{-1}\binom{1}{0} = \binom{10}{-7}$ so that

$$c_{\binom{10}{-7}}(z) = \frac{30}{512} z_1^2 z_2^{-10} - \frac{194}{512} z_1^3 z_2^{-10} + \frac{350}{256} z_1^4 z_2^{-10} - \frac{26}{256} z_1^5 z_2^{-10} + \frac{38}{512} z_1^6 z_2^{-10} - \frac{10}{512} z_1^7 z_2^{-10}.$$

By the fact that

$$U \cdot S_\Pi = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 5 \\ 20 & 7 \end{pmatrix}$$

this yields

$$\begin{aligned} a^\rho(z) &= -\overline{c}_{\binom{10}{-7}}(z^{-U \cdot S_\Pi}) + z_1 \overline{c}_0(z^{-U \cdot S_\Pi}) \\ &= -\frac{30}{512} z_1^{22} z_2^{30} + \frac{194}{512} z_1^8 z_2^{10} - \frac{350}{256} z_1^{-6} z_2^{-10} + \frac{26}{256} z_1^{-20} z_2^{-30} - \frac{38}{512} z_1^{-34} z_2^{-50} + \frac{10}{512} z_1^{-48} z_2^{-70} \\ &\quad - \frac{10}{512} z_1^{29} z_2^{40} + \frac{38}{512} z_1^{15} z_2^{20} - \frac{26}{256} z_1 + \frac{350}{256} z_1^{-13} z_2^{-20} - \frac{194}{512} z_1^{-27} z_2^{-40} + \frac{30}{512} z_1^{-41} z_2^{-60}. \end{aligned}$$

It remains to compute the symbol b^ρ of $\tilde{\psi}$. Clearly, the symbol of $B(\,\cdot\,|\binom{0}{1}\binom{0}{1}\binom{0}{1}S_\Pi\binom{0}{1}S_\Pi\binom{0}{1}S_\Pi\binom{0}{1})$ with respect to S_Π is given by

$$\tilde{a}(z) = \frac{1}{4}(1 + 3z_2 + 3z_2^2 + z_2^3) \ .$$

Since $a(z) = \sum_{\alpha \in \mathbb{Z}^2} \tilde{a}_{U^{-1}\alpha} z^\alpha$ we obtain

$$a(z) = \frac{1}{4}(1 + 3z_1^7 z_2^{10} + 3z_1^{14} z_2^{20} + z_1^{21} z_2^{30}) \ ,$$

so that

$$a_0(z) = \frac{1}{4}(1 + 3z_1^5 z_2^7), \ a_{\binom{0}{1}} = \frac{1}{4}(3z_1^{39} z_2^{56} + z_1^{44} z_2^{63}) \ .$$

Finally, we get

$$b^{\binom{0}{1}}(z) = -\overline{a}_{\binom{0}{1}}(z^{-M}) + z^{\binom{0}{1}} \overline{a}_0(z^{-M}) = \frac{1}{4}(-1 + z_1^{-20} z_2^{-29} - 3z_1^{-14} z_2^{-20} + 3z_1^{-6} z_2^{-9}) \ .$$

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