

Besov Regularity for Elliptic Boundary Value Problems

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Abstract

This paper studies the regularity of solutions to boundary value problems for Laplace's equation on Lipschitz domains Ω in \mathbf{R}^d and its relationship with adaptive and other nonlinear methods for approximating these solutions. The smoothness spaces which determine the efficiency of such nonlinear approximation in $L_p(\Omega)$ are the Besov spaces $B_\tau^\alpha(L_\tau(\Omega))$, $\tau := (\alpha/d + 1/p)^{-1}$. Thus, the regularity of the solution in this scale of Besov spaces is investigated with the aim of determining the largest α for which the solution is in $B_\tau^\alpha(L_\tau(\Omega))$. The regularity theorems given in this paper build upon the recent results of Jerison and Kenig [JK]. The proof of the regularity theorem uses characterizations of Besov spaces by wavelet expansions.

Key Words: Besov spaces, elliptic boundary value problems, potential theory, adaptive methods, nonlinear approximation, wavelets

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1 Introduction

This paper is concerned with the regularity of solutions to second order elliptic boundary value problems. We shall consider two related model problems. The first is the following boundary value problem for Laplace's equation:

$$\begin{aligned} -\Delta u &= f & \text{on } \Omega \subset \mathbf{R}^d, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The second is the Dirichlet problem for harmonic functions on Ω :

$$\begin{aligned} \Delta v &= 0 & \text{on } \Omega \subset \mathbf{R}^d, \\ v &= g & \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

We assume throughout this paper that Ω is a bounded, simply-connected, Lipschitz domain contained in \mathbf{R}^d (see [A] for the definition of Lipschitz domains).

We shall prove regularity theorems for the solution to (1.1) and (1.2) in a certain scale of Besov spaces. The particular scale of Besov spaces that we consider is of interest to us because it is connected to the rate of convergence of nonlinear and adaptive methods of approximation as we shall now explain.

We consider for example (1.1) in the weak formulation

$$a(u, v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega), \tag{1.3}$$

where $H_0^1(\Omega)$ is the subspace of the Sobolev space $H^1(\Omega) = W^1(L_2(\Omega))$ which reflects the homogeneous boundary conditions.

The numerical treatment of (1.3) is generally performed by means of a **Galerkin** approach, i.e., we consider a nested sequence $\{S_j\}_{j \geq 0}$ of finite dimensional linear subspaces of H_0^1 whose union is dense in H_0^1 and project (1.3) onto the spaces S_j . One then has to solve the problems

$$a(u_j, v) = (f, v), \quad v \in S_j, \tag{1.4}$$

for $u_j \in S_j$, which corresponds to solving a finite-dimensional linear system. Typical choices for $\{S_j\}_{j \geq 0}$ are **finite elements spaces** consisting of certain piecewise polynomials on partitions of the domain Ω , or a ladder of spaces generated by **multiresolution analysis** (see [D] for a discussion of multiresolution analysis).

The approximation order provided by such a Galerkin scheme is related to the smoothness of the solution u to (1.3) and the approximation properties of the spaces S_j . Consider, for example, approximation in $L_2(\Omega)$. If the domain is sufficiently smooth and if $f \in L_2(\Omega)$, then the weak solution is in the Sobolev space $W^2(L_2(\Omega))$, see Wloka [W] for details. Therefore, a Galerkin scheme using suitable finite element spaces obtained by **uniform grid refinement** provides an approximation $u_j \in S_j$ which satisfies

$$\|u - u_j\|_{L_2(\Omega)} \leq C 2^{-2j} |u|_{W^2(L_2(\Omega))}, \quad j = 0, 1, \dots,$$

with C a constant independent of u and j (see e.g. Johnson [J] for details). This can be restated in terms of the dimension $n_j = O(2^{jd})$ of S_j as

$$\|u - u_j\|_{L_2(\Omega)} = O(n_j^{-2/d}). \quad (1.5)$$

We refer to such numerical methods as **linear** since the approximation u_j comes from the linear space S_j .

An estimate of the form (1.5) does not hold in general for nonsmooth domains, e.g, for domains with edges and corners, for then the smoothness of u could decrease significantly due to singularities near the boundary, see e.g. Grisvard [G] or Kondrat'ev and Oleinik [KO] for details. For example, for Laplace's equation on a general Lipschitz domain, we can only expect the solution to be in the Sobolev space $W^\alpha(L_2(\Omega))$ if $\alpha \leq 3/2$ and therefore $2/d$ needs to be replaced by $3/2d$ in (1.5).

One can actually characterize the functions F which can be approximated with order $O(n^{-\alpha/d})$ in the metric $L_p(\Omega)$ by typical sequences of finite element spaces of dimension n . Indeed, we have this order of approximation if and only if F is in the Besov space $B_\infty^\alpha(L_p(\Omega))$. Thus the maximum smoothness α^* of the solution u to (1.1) in the Besov scale $B_p^\alpha(L_p(\Omega))$ limits the efficiency that a finite element method can have.

One way to possibly increase numerical efficiency in recovering the solution to (1.3) is to use **adaptive methods**. In this case, the underlying grid is refined only in regions where the solution lacks smoothness and the approximation u_j is still "far away" from the exact solution u . To implement such a strategy, one clearly needs some **a-posteriori error estimators** which give some information about the local error of the approximation u_j . We will not discuss here the problem of how to construct a-posteriori error estimators. Let us only remark that for finite elements several error estimators have been developed in the last years, see e.g. Bank and Weiser [BW], Babuska and Rheinboldt [BR] and Erikson and Johnson [EJ]. Furthermore, for the wavelet setting, a first approach was given by Dahlke, Dahmen, Hochmuth and Schneider [DDHS], see also Bertoluzza [B].

We are interested in the question of whether adaptive methods as described above can indeed provide increased efficiency in numerically recovering the solution to (1.3). The question then becomes firstly what is the regularity of a function F which governs its approximation by such an adaptive method and secondly does the solution u to (1.3) possess this regularity. As we shall now describe, the regularity which determines the efficiency of adaptive methods and related wavelet methods is determined by the smoothness of u as measured in certain scale of Besov spaces.

In general, an adaptive method can be interpreted as a method of **non-linear approximation**. In nonlinear approximation, we do not approximate by elements from a linear space but rather from a nonlinear manifold. The dimension n of the linear space is then replaced by the dimension (number of parameters) of the manifold. We first briefly describe the theory as it applies to nonlinear approximation by wavelet sums. The theory is rather fully developed in this case and the known results for adaptive approximation are analogous (although somewhat weaker - as we shall explain).

We shall restrict our discussion of nonlinear wavelet approximation to the case of approximation in $L_p(\mathbf{R}^d)$, $1 < p < \infty$, using the orthogonal wavelets of Daubechies.

Similar results hold in other settings [DJP],[DY]. The proofs of the results stated below are particularly trivial (see e.g. [DT]) in the case $p = 2$.

Daubechies (see [D]) has constructed a univariate family D_m , $m = 1, 2, \dots$, of compactly supported wavelets. When $m = 1$, D_1 is the Haar function. Larger values of m correspond to higher smoothness of the wavelet D_m ; the smoothness of D_m increases without bound as m increases to infinity, as does the support of D_m . The wavelet D_m has m vanishing moments. We fix an arbitrary value of m and let $\phi = \phi_m$ be the univariate scaling function which generates the wavelet $\psi = D_m$. We define $\psi^0 := \phi$ and $\psi^1 := \psi$. Further, let E denote the nontrivial vertices of the square $[0, 1]^d$. Then, the set Ψ of the $2^d - 1$ functions

$$\psi^e(x_1, \dots, x_d) := \prod_{j=1}^d \psi^{e_j}(x_j), \quad e \in E, \quad (1.6)$$

generate by shifts and dilates an orthonormal (wavelet) basis for $L_2(\mathbf{R}^d)$. Namely, let $\mathcal{D} := \mathcal{D}(\mathbf{R}^d)$ denote the set of dyadic cubes in \mathbf{R}^d . Each cube $I \in \mathcal{D}$ is of the form $I = 2^{-j}k + 2^{-j}[0, 1]^d$ with $k \in \mathbf{Z}^d$, $j \in \mathbf{Z}$. The functions

$$\eta_I := \eta_{j,k} := 2^{jd/2} \eta(2^j \cdot -k), \quad I = 2^{-j}k + 2^{-j}[0, 1]^d, \quad k \in \mathbf{Z}^d, j \in \mathbf{Z}, \eta \in \Psi, \quad (1.7)$$

form an orthonormal basis for $L_2(\mathbf{R}^d)$.

In nonlinear wavelet approximation, we approximate a function $F \in L_p(\mathbf{R}^d)$ by linear combinations of n of the basis elements η_I . Namely, let \mathcal{M}_n denote the non-linear manifold of all functions

$$S = \sum_{(I,\eta) \in \Lambda} a_{I,\eta} \eta_I$$

with $\Lambda \subset \mathcal{D} \times \Psi$ of cardinality n and let

$$\sigma_n(F)_{L_p(\mathbf{R}^d)} := \inf_{S \in \mathcal{M}_n} \|F - S\|_{L_p(\mathbf{R}^d)}. \quad (1.8)$$

In this setting, one has the following characterization (see [DJP]) for $1 < p < \infty$,

$$\sum_{n=1}^{\infty} [n^{\alpha/d} \sigma_n(F)_{L_p(\mathbf{R}^d)}]^\tau \frac{1}{n} < \infty \iff F \in B_\tau^\alpha(L_\tau(\mathbf{R}^d)), \quad \tau = (\alpha/d + 1/p)^{-1}, \quad (1.9)$$

and the $B_\tau^\alpha(L_\tau(\mathbf{R}^d))$ are the **Besov** spaces (see §2 for the definition of Besov spaces).

Let us make a few remarks comparing (1.9) with the analogous case of linear approximation (1.5). As we have already noted, for linear approximation, the requirement for approximation order like $O(n^{-\alpha/d})$ in $L_p(\mathbf{R}^d)$ is that F has smoothness of order α in $L_p(\mathbf{R}^d)$. In the case of nonlinear approximation, the smoothness of F is measured in $L_\tau(\mathbf{R}^d)$, $\tau = (\alpha/d + 1/p)^{-1}$. Since $\tau < p$, F may have a higher order of smoothness α when measured in L_τ than it does when measured in L_p . It is precisely for these types of functions that nonlinear methods will perform better than linear methods.

There are analogous results to (1.9) for adaptive approximation; although the situation here is not as fully developed. For example, for adaptive approximation by piecewise

polynomials (see [DY]), it is known that for each function $F \in B_\tau^{\alpha+\epsilon}(L_\tau(\mathbf{R}^d))$, $\epsilon > 0$, $\tau := (\alpha/d + 1/p)^{-1}$, adaptive approximation with n parameters will approximate F to order $O(n^{-\alpha/d})$. In other words, assuming slightly more smoothness than in the case of nonlinear wavelet approximation, we obtain the same order of approximation. One should note that adaptive approximation is more restrictive than nonlinear wavelet approximation. A comparable type of nonlinear wavelet approximation would require that the approximation only uses sets Λ that have a tree like structure - whenever $(I, \eta) \in \Lambda$, then (I', η) must also be in Λ for the parent I' of I .

A similar theory of nonlinear approximation also holds on domains although the results here are not as complete and need to be developed further. For the case of nonlinear wavelet approximation, one needs the development of orthogonal (or stable) wavelet bases for domains. For simple domains such as cubes this is done. For more general domains first steps have been made (see e.g. [CDD]). The adaptive theory should carry over as well although the only paper known to the authors is restricted to domains that are cubes [DY]. Note however that some results can be concluded from the case of cubes by using the fact that any function F in a Besov space $B_q^\alpha(L_p(\Omega))$ with Ω Lipschitz, can be extended to a function on all of \mathbf{R}^d with the same Besov regularity.

These results on nonlinear approximation lead us to ask what is the regularity of the solution u to an elliptic equation as measured in the scale of Besov spaces $B_\tau^\alpha(L_\tau(\Omega))$, $\tau = (\alpha/d + 1/p)^{-1}$? In particular, does the solution u have a higher smoothness order α in this scale of Besov spaces then it does when the smoothness is measured in $L_p(\Omega)$? One of the main results (Theorem 4.1) of the present paper shows that this is indeed the case. For example, in the case $p = 2$, we show that the solution u is in the Besov space $B_\tau^\alpha(L_\tau(\Omega))$, $\tau = (\alpha/d + 1/2)^{-1}$, for $\alpha < 3d/(2d - 2)$, provided that $f \in W^\gamma(L_2(\Omega))$, $\gamma = (4 - d)/(2d - 2)$. Similar results hold for other values of p . In other words, we show that u has the smoothness necessary that allows adaptive or other forms of nonlinear approximation to perform better than linear methods.

Our results for the regularity of (1.1) are proved by reducing the problem (in a standard manner) to the regularity of harmonic functions which are solutions to the Dirichlet problem (1.2). We prove that if the solution v to (1.2) is in the Besov space $B_p^\lambda(L_p(\Omega))$, then v is also in the Besov space $B_\tau^\alpha(L_\tau(\Omega))$, $\tau = (\alpha/d + 1/p)^{-1}$, for every $0 < \alpha < \frac{\lambda d}{(d-1)}$. In other words, the regularity of v in the Besov scale for nonlinear approximation is always greater by a factor $d/(d - 1)$ than its smoothness in the scale for linear approximation.

Our regularity results are closely related to the work of Jerison and Kenig [JK] who proved several deep theorems about the Besov regularity of the solutions to the two model problems (1.1) and (1.2). In fact, in some cases our results can be derived from theirs. In general, however, our results are new - primarily because we consider (as is necessary) Besov spaces with smoothness measured in L_τ where $\tau < 1$. We remark further on the connections between our results and those in [JK] in §§3,4.

We want to make clear that the present paper is concerned with the regularity (i.e. the smoothness) of solutions to elliptic equations. Our motivation for the type of regularity we study are adaptive and other forms of nonlinear approximation. However,

we do not construct in this paper a numerical scheme for solving (1.3). Indeed, the nonlinear approximation schemes described above require full knowledge of the solution u to construct the approximation which is not available when numerically solving (1.3).

An outline of this paper is that in §2, we state the properties we need of Besov spaces and wavelet decompositions; in §3 we prove the regularity theorem for the Dirichlet problem (1.2); and in §4 we discuss the regularity of the solution to (1.1).

2 Besov spaces and wavelet decompositions

In this section, we define the Besov spaces and give their characterization in terms of wavelet decompositions.

Let Ω be a Lipschitz domain. If $h \in \mathbf{R}^d$, we denote by Ω_h the set of all $x \in \Omega$ such that the line segment $[x, x + h]$ is contained in Ω . The modulus of smoothness $\omega_r(F, t)_{L_p(\Omega)}$ of a function $F \in L_p(\Omega)$, $0 < p \leq \infty$, is defined by

$$\omega_r(F, t)_{L_p(\Omega)} := \sup_{|h| \leq t} \|\Delta_h^r(F, \cdot)\|_{L_p(\Omega_{rh})}, \quad t > 0,$$

with Δ_h^r the r -th difference with step h . For $\alpha > 0$ and $0 < q, p \leq \infty$, the Besov space $B_q^\alpha(L_p(\Omega))$ is defined as the space of all functions F for which

$$|F|_{B_q^\alpha(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty [t^{-\alpha} \omega_r(F, t)_{L_p(\Omega)}]^q dt / t \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t \geq 0} t^{-\alpha} \omega_r(F, t)_{L_p(\Omega)}, & q = \infty, \end{cases} \quad (2.1)$$

is finite with $r := [\alpha] + 1$. Then, (2.1) is a (quasi-)semi-norm for $B_q^\alpha(L_p(\Omega))$. If we add $\|F\|_{L_p(\Omega)}$ to (2.1), we obtain a (quasi-)norm for $B_q^\alpha(L_p(\Omega))$.

It is also possible to characterize Besov spaces by wavelet decompositions. Let ϕ be a univariate Daubechies' scaling function and $\psi = D_m$ be the corresponding wavelet. These functions have compact support. As noted earlier, the function $\psi = D_m$ has m vanishing moments and the smoothness of the D_m increase without bound as m grows. The functions (1.7) are an orthonormal basis for $L_2(\mathbf{R}^d)$ and they also form an unconditional basis for $L_p(\mathbf{R}^d)$, $1 < p < \infty$. Each $F \in L_p(\mathbf{R}^d)$, $1 < p < \infty$ has the wavelet decomposition

$$F = \sum_{I \in \mathcal{D}} \sum_{\eta \in \Psi} \langle F, \eta_I \rangle \eta_I \quad (2.2)$$

with convergence in $L_p(\mathbf{R}^d)$.

We can also restrict the wavelet expansion (2.2) to those η_I with $|I| \leq 1$. For this, we define S_0 to be the closure in $L_2(\mathbf{R}^d)$ of the finite linear combinations of the integer shifts of the function $\phi(x_1) \cdots \phi(x_d)$ and let P_0 be the orthogonal projector which maps $L_2(\mathbf{R}^d)$ onto S_0 . Then, P_0 has an extension as a projector to $L_p(\mathbf{R}^d)$, $1 \leq p \leq \infty$. For each $F \in L_p(\mathbf{R}^d)$, we have

$$F = P_0(F) + \sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} \langle F, \eta_I \rangle \eta_I \quad (2.3)$$

with \mathcal{D}^+ the set of dyadic cubes with measure ≤ 1 .

The Besov spaces $B_q^\alpha(L_p(\mathbf{R}^d))$ can be characterized by wavelet coefficients provided the parameters α, p, q satisfy certain restrictions. We shall only need the case $q = p$. In describing this characterization it is convenient to use a normalization for the wavelets which depend on p . If $0 < p \leq \infty$, we define

$$\eta_{I,p} := |I|^{1/2-1/p}\eta_I. \quad (2.4)$$

Then, $\|\eta_{I,p}\|_{L_p(\mathbf{R}^d)} = \|\eta\|_{L_p(\mathbf{R}^d)}$ is constant. We can then rewrite (2.3) as

$$F = P_0(F) + \sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} \langle F, \eta_{I,p'} \rangle \eta_{I,p} \quad (2.5)$$

with p' the conjugate index to p , $1/p + 1/p' = 1$. Note that p' is negative if $p < 1$.

Proposition 2.1 *Let ϕ and ψ be in $C^r(\mathbf{R})$. If $0 < p \leq \infty$ and $r > \alpha > d(1/p - 1)$, then a function F is in the Besov space $B_p^\alpha(L_p(\mathbf{R}^d))$, if and only if,*

$$F = P_0(F) + \sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} \langle F, \eta_{I,p'} \rangle \eta_{I,p} \quad (2.6)$$

with

$$\|P_0(F)\|_{L_p(\mathbf{R}^d)} + \left(\sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} |I|^{-\alpha p/d} |\langle F, \eta_{I,p'} \rangle|^p \right)^{1/p} < \infty \quad (2.7)$$

and (2.7) provides an equivalent (quasi-)norm for $B_p^\alpha(L_p(\mathbf{R}^d))$.

In the case $p \geq 1$, this is a standard result and can be found for example in Meyer [M] (§10 of Chapter 6). For the general case of p , this can be deduced from general results in Littlewood-Paley theory (see e.g. §4 of Frazier and Jawerth [FJ]) or proved directly (see Kyriazis [Ky]). The condition that $\alpha > d(1/p - 1)$ implies that the Besov space $B_p^\alpha(L_p(\mathbf{R}^d))$ is embedded in $L_s(\mathbf{R}^d)$ for some $s > 1$ so that the wavelet decomposition of F is defined. Also, with this restriction on α , the Besov space $B_p^\alpha(L_p(\mathbf{R}^d))$ is equivalent to the non-homogeneous Besov spaces $B_{p,p}^\alpha$ defined via Fourier transforms and Littlewood-Paley theory.

We now fix a value of $1 < p \leq \infty$ and consider the scale of spaces $B_\tau^\alpha(L_\tau(\mathbf{R}^d))$, $\tau = (\alpha/d + 1/p)^{-1}$, $\alpha > 0$. Using the fact that $\eta_{I,\tau'} = |I|^{1/p'-1/\tau'}\eta_{I,p'}$, a simple computation gives

$$|I|^{-\alpha\tau/d} |\langle F, \eta_{I,\tau'} \rangle|^\tau = |\langle F, \eta_{I,p'} \rangle|^\tau.$$

This gives the following equivalent characterization of $B_\tau^\alpha(L_\tau(\mathbf{R}^d))$.

Proposition 2.2 *Let ϕ and ψ be in $C^r(\mathbf{R})$. If $1 < p \leq \infty$ and $r > \alpha > 0$ and $\tau = (\alpha/d + 1/p)^{-1}$, then a function F is in the Besov space $B_\tau^\alpha(L_\tau(\mathbf{R}^d))$, if and only if,*

$$F = P_0(F) + \sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} \langle F, \eta_{I,p'} \rangle \eta_{I,p} \quad (2.8)$$

with

$$\|P_0(F)\|_{L_\tau(\mathbf{R}^d)} + \left(\sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} |\langle F, \eta_{I,p'} \rangle|^\tau \right)^{1/\tau} < \infty \quad (2.9)$$

and (2.9) provides an equivalent (quasi-)norm for $B_\tau^\alpha(L_\tau(\mathbf{R}^d))$.

3 Regularity of the solution to the Dirichlet boundary value problem

In this section, we shall study the regularity of harmonic functions on Ω . Our main result shows that whenever an harmonic function v on Ω is known to be in a Besov space $B_p^\lambda(L_p(\Omega))$, then it automatically has additional smoothness in a scale of Besov spaces associated to p and λ . This added smoothness is nontrivial in the sense that general nonharmonic functions do not possess this property.

We shall utilize certain maximal functions which measure smoothness that have been extensively studied in [DS]. Let $\Pi = \Pi_m$ be a bounded projector from $L_1([0, 1]^d)$ onto the space \mathcal{P}_m of polynomials of total degree at most m . Such a projector gives by change of scale a projector Π_Q from $L_1(Q)$ onto \mathcal{P}_m for each cube Q (all cubes are taken with sides parallel to the coordinate axis). If $\beta > 0$, we take $m := [\beta]$ and define for each $F \in L_1(\Omega)$,

$$F_\beta^\sharp(x) := \sup_{\Omega \supset Q \ni x} \frac{1}{|Q|^{1+\beta/d}} \int_Q |F - \Pi_Q F|. \quad (3.1)$$

It was shown by DeVore and Sharpley [DS] (see Theorem 7.1 and Corollary 11.6¹) that for $1 \leq p \leq \infty$ the following inequality holds for each $F \in B_p^\beta(L_p(\Omega))$,

$$\|F_\beta^\sharp\|_{L_p(\Omega)} \leq C \|F\|_{B_p^\beta(L_p(\Omega))}, \quad (3.2)$$

with a constant C which depends at most on Ω and β .

The following theorem is an extension of a result of Jerison and Kenig (see Theorem 4.1 in [JK]).

Theorem 3.1 *Let $1 \leq p \leq \infty$, $\beta > 0$, and let $k > \beta$ be an integer. Then there is a constant $C > 0$ depending only on k , β , and Ω such that whenever v is an harmonic function on Ω which is in $B_p^\beta(L_p(\Omega))$ we have*

$$\|\delta(x)^{k-\beta} |\nabla^k v(x)|\|_{L_p(\Omega)} \leq C \|v\|_{B_p^\beta(L_p(\Omega))}, \quad \delta(x) := \text{dist}(x, \partial\Omega), \quad (3.3)$$

where $\nabla^k v$ denotes the vector of all k th order derivatives of v and $|\nabla^k v|$ is its Euclidean length.

¹while Corollary 11.6 of [DS] is stated for $1 < p < \infty$ the same proof is valid for $p = 1, \infty$

Proof: We first consider the case when $0 < \beta \leq 1$. For any function F defined on \mathbf{R}^d , we define its dilates $F_\delta(y) := \delta^{-d}F(y/\delta)$, $\delta > 0$. Let φ be a fixed C^∞ function on \mathbf{R}^d which is radial and supported in the unit ball. It follows from the mean-value property of harmonic functions that for any γ and for $\delta := \delta(x)$, we have

$$D^\gamma v(x) = \int_{\mathbf{R}^d} v(y)(\delta/2)^{-|\gamma|} (D^\gamma \varphi)_{\delta/2}(x-y) dy, \quad x \in \Omega, \quad (3.4)$$

see e.g. Stein [S], Appendix C for details. We use (3.4) with $|\gamma| = k$ to obtain

$$\begin{aligned} |D^\gamma v(x)| &\leq \sup_{y \in B(x, \delta/2)} (\delta/2)^{-|\gamma|} |(D^\gamma \varphi)_{\delta/2}(x-y)| \int_{B(x, \delta/2)} |v(y)| dy \\ &\leq (\delta/2)^{-d-k} \sup_{y \in \mathbf{R}^d} |D^\gamma \varphi(y)| \int_{B(x, \delta/2)} |v(y)| dy \\ &\leq C \delta^{-k+\beta} \frac{1}{(\delta/2)^{d+\beta}} \int_{B(x, \delta/2)} |v(y)| dy, \quad x \in \Omega. \end{aligned}$$

We can replace v by $v - \Pi_Q v$ where Π is the projector corresponding to $m = 0$ if $\beta < 1$ and to $m = 1$ if $\beta = 1$ (because $\Pi_Q v$ is also harmonic). In this way we obtain the inequality

$$\delta(x)^{k-\beta} |D^\gamma v(x)| \leq C v_\beta^\sharp(x), \quad x \in \Omega. \quad (3.5)$$

Taking a norm with respect to $L_p(\Omega)$ and using (3.2) establishes (3.3) in the case $0 < \beta \leq 1$.

Consider now any $\beta > 0$ and write $\beta = \ell + \alpha$ with $0 < \alpha \leq 1$. Since $k - \beta = k - \ell - \alpha$, we can apply what we have already proved to any of the functions $D^\gamma v$, $|\gamma| = \ell$ and obtain

$$\|\delta(x)^{k-\beta} |\nabla^{k-\ell} D^\gamma v(x)|\|_{L_p(\Omega)} \leq C \|D^\gamma v\|_{B_p^\alpha(L_p(\Omega))} \leq C \|v\|_{B_p^\beta(L_p(\Omega))}, \quad (3.6)$$

where the last inequality follows from the reduction theorem for Besov spaces (see e.g. Theorem 6.2.7 in Bergh and Löfström [BL]). Since $|\gamma| = \ell$ is arbitrary, we have proved (3.3). \square

The following is the main result of this paper.

Theorem 3.2 *Let Ω be a bounded Lipschitz domain in \mathbf{R}^d . If v is an harmonic function on Ω which is in the Besov class $B_p^\lambda(L_p(\Omega))$, for some $1 < p < \infty$ and $\lambda > 0$, then*

$$v \in B_\tau^\alpha(L_\tau(\Omega)), \quad \tau = \left(\frac{\alpha}{d} + \frac{1}{p} \right)^{-1}, \quad 0 < \alpha < \frac{\lambda d}{(d-1)}. \quad (3.7)$$

Proof: We fix τ and α as in the statement of the theorem. We will denote by m an integer which depends on α , λ , and d and whose value will be specified during the course of the proof of this theorem. Because Ω is a Lipschitz domain, we can extend v to all of \mathbf{R}^d with the extension in $B_p^\lambda(L_p(\mathbf{R}^d))$. We denote this extension also by v .

We can represent v with respect to the wavelet basis (1.7). Let $\psi := D_m$ be the Daubechies' wavelet with parameter m and let Ψ be the collection of multivariate wavelets defined by (1.6). We require that m is large enough that the functions ϕ and ψ are in $C^s(\mathbf{R})$, $s := [d\lambda/(d-1)] + 1$. Since ψ and its generating function ϕ have compact support, there is a cube $Q \subset \mathbf{R}^d$, centered at the origin, such that $\text{supp } \eta \subset Q$ for all $\eta \in \Psi$. By shifts and dilates we obtain the cubes $Q(I) := 2^{-j}k + 2^{-j}Q$, $I = 2^{-j}k + 2^{-j}[0, 1]^d$, which contain $\text{supp } \eta_I$, for all $I \in \mathcal{D}$, $\eta \in \Psi$. We recall our notation \mathcal{D}^+ for the dyadic cubes of measure at most 1. Let Λ denote the set of pairs (I, η) , $I \in \mathcal{D}^+$, $\eta \in \Psi$, for which $Q(I) \cap \Omega \neq \emptyset$. Then, on Ω , we have

$$v = P_0 v + v_0, \quad v_0 := \sum_{(I, \eta) \in \Lambda} \langle v, \eta_I \rangle \eta_I, \quad (3.8)$$

where P_0 is the projector introduced in §2. The function $P_0 v|_\Omega$ is in $C^s(\Omega)$ because it is a finite linear combination of shifts of $\phi(x_1) \cdots \phi(x_d)$. To complete the proof of the theorem, we shall show that $v_0 \in B_\tau^\alpha(L_\tau(\mathbf{R}^d))$ from which the theorem follows.

It will be (notationally) convenient to use the L_p -normalized wavelets $\eta_{I,p}$ introduced in §2. We have

$$v_0 := \sum_{(I, \eta) \in \Lambda} \langle v, \eta_{I,p'} \rangle \eta_{I,p} \quad (3.9)$$

According to Proposition 2.2, we are left with showing

$$\left(\sum_{(I, \eta) \in \Lambda} |\langle v, \eta_{I,p'} \rangle|^\tau \right)^{1/\tau} < \infty. \quad (3.10)$$

To prove (3.10), we shall use Theorem 3.1 which says that that $\delta(x)^{m-\lambda} |\nabla^m v(x)| \in L_p(\Omega)$.

For $I \in \mathcal{D}^+$, let

$$\delta_I := \inf_{x \in Q(I)} \delta(x).$$

There is a polynomial P_I of total degree $< m$ such that

$$\|v - P_I\|_{L_p(Q(I))} \leq C |Q(I)|^{m/d} |v|_{W^m(L_p(Q(I)))} \leq C |I|^{m/d} |v|_{W^m(L_p(Q(I)))}.$$

The constants C which appear here and later in this proof depend only on α , d , m , and the Lipschitz character of Ω . Recall that $\eta_{I,p'}$ is orthogonal to any polynomial of total degree $< m$. Hence,

$$\begin{aligned} |\langle v, \eta_{I,p'} \rangle| &\leq |\langle v - P_I, \eta_{I,p'} \rangle| \leq \|v - P_I\|_{L_p(Q(I))} \|\eta_{I,p'}\|_{L_{p'}(\mathbf{R}^d)} \\ &\leq C |I|^{m/d} |v|_{W^m(L_p(Q(I)))} \\ &\leq C |I|^{m/d} \delta_I^{\lambda-m} \left(\int_{Q(I)} |\delta(x)^{m-\lambda} |\nabla^m v|^p dx \right)^{1/p} \\ &= C |I|^{m/d} \delta_I^{\lambda-m} \mu_I, \end{aligned} \quad (3.11)$$

where μ_I is defined by the last equality.

Let Λ_j denote the set of those pairs $(I, \eta) \in \Lambda$ with $|I| = 2^{-jd}$. For each $k = 0, 1, \dots$, let $\Lambda_{j,k} \subset \Lambda_j$ be the set of those $(I, \eta) \in \Lambda_j$ such that

$$k2^{-j} \leq \delta_I < (k+1)2^{-j}.$$

From the Lipschitz character of Ω , it follows that

$$|\Lambda_{j,k}| \leq C2^{j(d-1)}, \quad j, k = 0, 1, \dots \quad (3.12)$$

Also note that since the domain Ω is bounded, we have $\Lambda_{j,k} = \emptyset$ if $k \geq C2^j$. Let $\Lambda_j^\circ := \Lambda_j \setminus \Lambda_{j,0}$. We now fix j with $0 \leq j < \infty$ and estimate the portion of the sum in (3.10) corresponding to $(I, \eta) \in \Lambda_j^\circ$. For each $(I, \eta) \in \Lambda_j^\circ$, the cube $Q(I)$ is contained strictly in Ω . It follows from (3.11) that

$$\sum_{(I, \eta) \in \Lambda_j^\circ} |\langle v, \eta_{I,p'} \rangle|^\tau \leq C \sum_{(I, \eta) \in \Lambda_j^\circ} 2^{-mj\tau} \delta_I^{(\lambda-m)\tau} \mu_I^\tau.$$

We use Hölder's inequality with exponents $\frac{p}{\tau}$ and $\frac{p}{p-\tau}$, to find

$$\sum_{(I, \eta) \in \Lambda_j^\circ} |\langle v, \eta_{I,p'} \rangle|^\tau \leq C \left(\sum_{(I, \eta) \in \Lambda_j^\circ} 2^{\frac{-pm\tau j}{p-\tau}} \delta_I^{\frac{(\lambda-m)p\tau}{p-\tau}} \right)^{\frac{p-\tau}{p}} \left(\sum_{(I, \eta) \in \Lambda_j^\circ} \mu_I^p \right)^{\frac{\tau}{p}}.$$

Now, a point $x \in \Omega$ appears in at most C of the cubes $Q(I)$, $I \in \Lambda_j^\circ$. Using Theorem 3.1, we obtain

$$\begin{aligned} \left(\sum_{(I, \eta) \in \Lambda_j^\circ} \mu_I^p \right)^{\frac{\tau}{p}} &= \left(\sum_{(I, \eta) \in \Lambda_j^\circ} \int_{Q(I)} |\delta(x)^{m-\lambda} |\nabla^m v(x)||^p dx \right)^{\frac{\tau}{p}} \\ &\leq C \left(\int_{\Omega} |\delta(x)^{m-\lambda} |\nabla^m v(x)||^p dx \right)^{\frac{\tau}{p}} \leq C \|v\|_{B_p^\lambda(L_p(\Omega))}^\tau \leq C. \end{aligned}$$

Therefore, using (3.12), and summing over the sets $\Lambda_{j,k}$, $k = 1, 2, \dots$, gives

$$\begin{aligned} \sum_{(I, \eta) \in \Lambda_j^\circ} |\langle v, \eta_{I,p'} \rangle|^\tau &\leq C \left(\sum_{k=1}^{C2^j} \sum_{(I, \eta) \in \Lambda_{j,k}} 2^{\frac{-pm\tau j}{p-\tau}} \delta_I^{\frac{(\lambda-m)p\tau}{p-\tau}} \right)^{\frac{p-\tau}{p}} \\ &\leq C \left(\sum_{k=1}^{C2^j} 2^{j(d-1)} \cdot 2^{-\frac{pm\tau j}{p-\tau}} \cdot (k2^{-j})^{\frac{(\lambda-m)p\tau}{p-\tau}} \right)^{\frac{p-\tau}{p}} \\ &\leq C \left(2^{j(d-1-\frac{p\lambda\tau}{p-\tau})} \sum_{k=1}^{C2^j} k^{\frac{((\lambda-m)p\tau)}{p-\tau}} \right)^{\frac{p-\tau}{p}}. \end{aligned}$$

We now choose m large enough that $(m - \lambda)\tau > 1 - \tau/p$ and obtain

$$\sum_{(I,\eta) \in \Lambda_j^\circ} |\langle v, \eta_{I,p'} \rangle|^\tau \leq C 2^{j(\frac{(d-1)(p-\tau)}{p} - \lambda\tau)}.$$

We now define $\Lambda^\circ := \cup_{j=0}^\infty \Lambda_j^\circ$ and sum our last inequalities over all dyadic levels $j = 0, 1, \dots$ to find

$$\sum_{(I,\eta) \in \Lambda^\circ} |\langle v, \eta_{I,p'} \rangle|^\tau \leq C \sum_{j=0}^\infty 2^{j(\frac{(d-1)(p-\tau)}{p} - \lambda\tau)} \leq C$$

provided

$$\frac{(d-1)(p-\tau)}{p} - \lambda\tau < 0, \quad \text{i.e.,} \quad \tau > \frac{p(d-1)}{p\lambda + d - 1}. \quad (3.13)$$

This condition on τ is equivalent to the condition on α given in the statement of the theorem.

Finally, we need to estimate the sum of wavelet coefficients corresponding to the sets $\Lambda_{j,0}$, $j = 0, 1, \dots$. Using Hölder's inequality, and the fact that $|\Lambda_{j,0}| \leq C 2^{j(d-1)}$ gives

$$\begin{aligned} \sum_{(I,\eta) \in \Lambda_{j,0}} |\langle v, \eta_{I,p'} \rangle|^\tau &\leq C 2^{j(d-1)(1-\tau/p)} \left(\sum_{(I,\eta) \in \Lambda_{j,0}} |\langle v, \eta_{I,p'} \rangle|^p \right)^{\tau/p} \\ &\leq C 2^{j(d-1)(1-\tau/p)} 2^{-j\lambda\tau} \left(\sum_{(I,\eta) \in \Lambda_{j,0}} 2^{\lambda pj} |\langle v, \eta_{I,p'} \rangle|^p \right)^{\tau/p}. \end{aligned}$$

Hence, summing over all dyadic levels j and using Hölder's inequality again, we find

$$\sum_{j=0}^\infty \sum_{(I,\eta) \in \Lambda_{j,0}} |\langle v, \eta_{I,p'} \rangle|^\tau \leq C \left(\sum_{j \geq 0} \sum_{(I,\eta) \in \Lambda_{j,0}} 2^{\lambda pj} |\langle v, \eta_{I,p'} \rangle|^p \right)^{\tau/p} \left(\sum_{j \geq 0} 2^{(-\frac{p\lambda\tau j}{p-\tau} + (d-1)j)} \right)^{\frac{p-\tau}{p}}.$$

From Proposition (2.1), the first sum on the right side is bounded by $C \|v\|_{B_p^\lambda(L_p(\mathbf{R}^d))}$ which is in turn bounded by $C \|v\|_{B_p^\lambda(L_p(\Omega))}$. The second sum on the right side is finite if the exponent of 2^j is negative or what is the same thing if

$$\tau > \frac{p(d-1)}{p\lambda + d - 1}.$$

This is the same restriction we had on τ in (3.13) and corresponds to the condition on α stated in the theorem. We have therefore completed the verification of (3.10) and have therefore proved the theorem. \square

We use the remainder of this section to explain how Theorem 3.2 can be used and to bring out the connections between this theorem and the results of Jerison and Kenig [JK].

Jerison and Kenig (Theorem 5.1 of [JK]) have shown that for each $1 < p < \infty$, there is a range of values s such that if the boundary condition g is in the Besov space $B_p^s(L_p(\partial\Omega))$, then the solution v to (1.2) is in the Besov space $B_p^{s+1/p}(L_p(\Omega))$. Thus, under these same conditions on g , Theorem 3.2 implies that v is in $B_\tau^\alpha(L_\tau(\Omega))$, provided $\alpha < \frac{d(s+1/p)}{d-1}$. In certain cases (namely if $\tau > 1$), the results of Theorem 3.1 follow from their results.

We single out for further mention only the case $p = 2$ in the following corollary.

Corollary 3.1 *Let Ω be a bounded Lipschitz domain in \mathbf{R}^d . If v is a solution to the Dirichlet problem (1.2) with $g \in W^1(L_2(\partial\Omega))$ then*

$$v \in B_\tau^\alpha(L_\tau(\Omega)), \quad \tau = \left(\frac{\alpha}{d} + \frac{1}{2}\right)^{-1}, \quad 0 \leq \alpha < \frac{3d}{2(d-1)}. \quad (3.14)$$

Proof: If $g \in W^1(L_2(\partial\Omega))$, then $v \in W^{3/2}(L_2(\Omega))$ (see [JK]). The Corollary then follows from Theorem 3.2. \square

Note that if $d = 2, 3$, then α is permitted to be larger than 2 and $\tau < 1$. For $d \geq 4$, the corollary can be also derived from Theorem 5.1 of [JK].

Theorem 3.2 says that if an harmonic function v is in a Besov space $B_p^\lambda(L_p(\Omega))$, then it is automatically in the Besov space $B_\tau^\alpha(L_\tau(\Omega))$ of that theorem. By interpolation and embeddings for Besov spaces, we can conclude that v is in a family of Besov spaces $B_q^s(L_q(\Omega))$ for a certain range of the parameters q and s . This is depicted in Figure 1 for the special case $\lambda = 3/2$, $p = 2$, $d = 2$ of Corollary 3.1. If $v \in B_2^{3/2}(L_2(\Omega))$, then it is in $B_q^s(L_q(\Omega))$ whenever $(1/q, s)$ is in the interior of the quadrilateral with vertices $(1/2, 0)$, $(1/2, 3/2)$, $(2, 0)$, $(2, 3)$. The heavy line connecting $(1/2, 0)$ to $(2, 3)$ corresponds to the spaces $B_\tau^\alpha(L_\tau(\Omega))$ of Theorem 3.2.

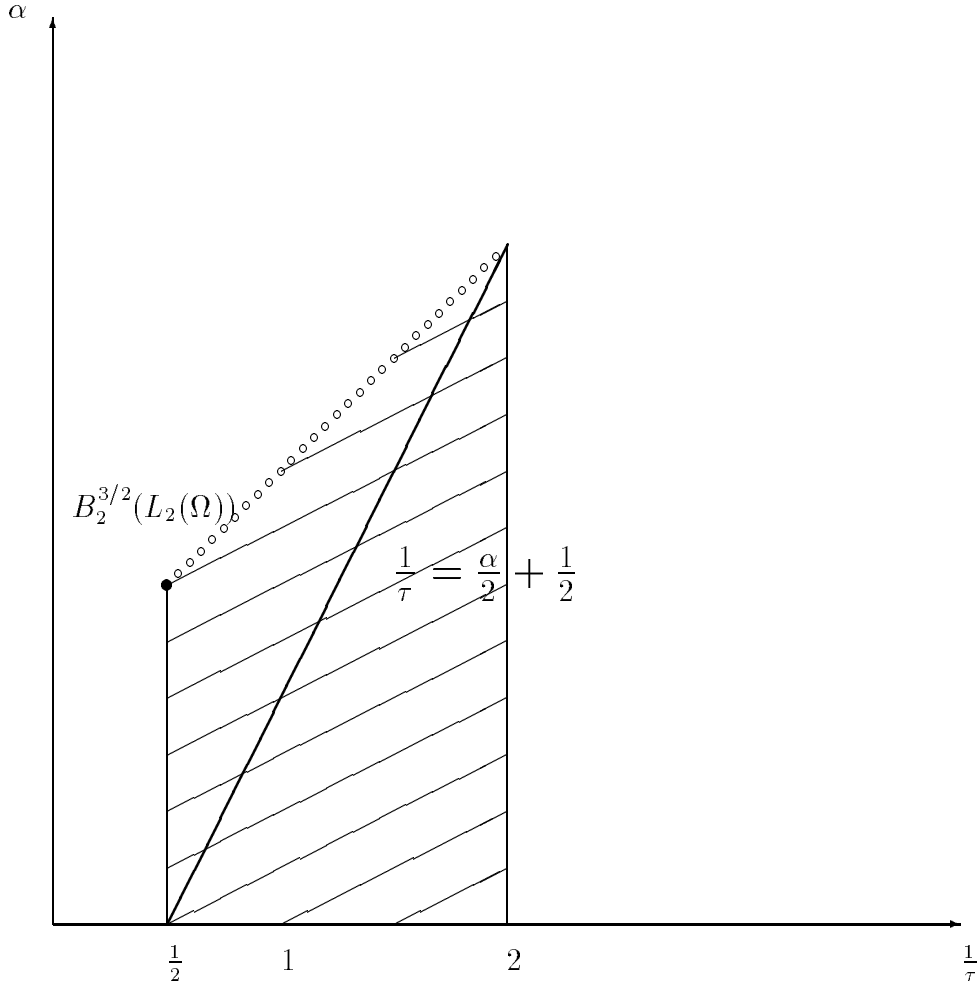


Figure 1: Regularity spaces when $g \in B_2^1(L_2(\partial\Omega))$, $v \in B_2^{3/2}(L_2(\Omega))$, $\Omega \subset \mathbf{R}^2$.

4 Regularity Estimates for Laplace's equation

There is a general strategy for reducing the boundary value problem (1.1) to the Dirichlet problem (1.2) for harmonic functions which proceeds as follows. Suppose that f is in some space $X^\alpha(\Omega)$ which can be a smoothness space like $B_p^\alpha(L_p(\Omega))$ or $W^\alpha(L_p(\Omega))$ or an $L_p(\Omega)$ space (in the case $\alpha = 0$). We can extend f to a compactly supported function \tilde{f} defined on all of \mathbf{R}^d which is in the space $X^\alpha(\mathbf{R}^d)$. We solve the problem (1.1) with f replaced by \tilde{f} and with Ω replaced by a C^∞ domain $\tilde{\Omega}$ which strictly contains Ω . For suitable X^α , the solution \tilde{u} will be in $X^{\alpha+2}(\tilde{\Omega})$. We can write the solution u to (1.1) as

$$u = \tilde{u} - v \quad \text{on } \Omega, \quad (4.1)$$

where v is the solution to the Dirichlet problem

$$\begin{aligned} \Delta v &= 0 \quad \text{on } \Omega, \\ v|_{\partial\Omega} &= \tilde{u}|_{\partial\Omega} =: g. \end{aligned} \quad (4.2)$$

We use a trace theorem to infer smoothness of g on $\partial\Omega$. In this way, a regularity theorem can be deduced for u from regularity theorems for v . We give one example of this approach which employs our regularity results. Several new variants are possible.

Theorem 4.1 *Let Ω be a bounded Lipschitz domain in \mathbf{R}^d . Then, there is an $0 < \epsilon < 1$ depending only on the Lipschitz character of Ω such that whenever u is a solution to*

$$\begin{aligned} -\Delta u &= f & \text{on } \Omega \subset \mathbf{R}^d, \\ u &= 0 & \text{on } \partial\Omega \end{aligned} \tag{4.3}$$

with $f \in B_p^{\lambda-2}(L_p(\Omega))$, $\lambda := \frac{d}{d-1}(1 + \frac{1}{p})$, $1 < p \leq 2 + \epsilon$, then $u \in B_\tau^\alpha(L_\tau(\Omega))$, $\tau = (\alpha/d + 1/p)^{-1}$, for all $0 < \alpha < \lambda$.

Proof: Using the approach outlined above, we have $\tilde{u} \in B_p^\lambda(L_p(\tilde{\Omega}))$. Hence, by the embeddings of Besov spaces: $B_p^\lambda(L_p(\Omega)) \hookrightarrow B_p^\lambda(L_\tau(\Omega)) \hookrightarrow B_\tau^\alpha(L_\tau(\Omega))$, we have $\tilde{u} \in B_\tau^\alpha(L_\tau(\Omega))$ for any α, τ as in the statement of the theorem. Since $\lambda > 1/p$, we can use the trace theorem of Jonsson and Wallin (p. 209 in [JW]) to conclude that $g \in B_p^\beta(L_p(\partial\Omega))$ for every $\beta < 1$. From the regularity theorem of Jerison and Kenig (Theorem 5.1 of [JK]), v is in $B_p^{\beta+1/p}(L_p(\Omega))$ for every $\beta < 1$. Theorem 3.2 now implies that $v \in B_\tau^\alpha(L_\tau(\Omega))$ for every α and τ as in the statement of the theorem. Since $u = \tilde{u} - v$, the theorem follows. \square

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