Besov Regularity for the Poisson Equation in Smooth and Polyhedral Cones

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Abstract

This paper is concerned with the regularity of the solutions to Dirichlet and Neumann problems in smooth and polyhedral cones contained in $\mathbf{R}^3$. Especially, we consider the specific scale $B^s_\tau(L_\tau), 1/\tau = s/3 + 1/2$, of Besov spaces. The regularity of the solution in these Besov spaces determines the order of approximation that can be achieved by adaptive and nonlinear numerical schemes. We show that the solutions are much smoother in the specific Besov scale than in the ususal L_2-Sobolev scale which justifies the use of adaptive schemes. The proofs are performed by combining weighted Sobolev estimates with characterizations of Besov spaces by wavelet expansions.

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1 Introduction

We investigate the regularity of solutions of the Poisson equation in smooth and polygonal cones $\mathcal{K} \subset \mathbf{R}^3$, respectively, within Besov spaces $B^s_\tau(L_\tau(\mathcal{K}))$ with $0 < \tau < 2$. The motivation for these studies can be explained as follows.

In recent years, the numerical treatment of operator equations by adaptive numerical algorithms has become a field of increasing importance, with many applications in science and engineering. Especially, adaptive finite element schemes have been very successfully developed and implemented, and innumerable numercial experiments impressively confine their excellent performance. Complementary to this, also adaptive algorithms based on wavelets have become more and more in the center of attraction during the last years, for the following reason. The strong analytical properties of wavelets can be used to derive adaptive strategies which are guaranteed to converge for a huge class of elliptic operator equations, involving operators of negative order [4, 10]. Moreover, these algorithms are optimal in the sense that they asymptotically realize the convergence order of the optimal (but not directly implementable) approximation scheme, i.e., the order of best n-term wavelet approximation. Moreover, the number of arithmetic operations that is needed stays proportional to the number of degrees of freedom [4]. By now, various generalizations to non-elliptic equations [5], saddle point problems [11] and also nonlinear operator equations [6] exist. For finite element schemes, rigorous statements of these forms have been rather rare, although, inspired by the results for wavelet schemes, the situation has changed during the last years [2, 16]. Although the above mentioned results are quite impressive, in the realm of adaptivity one is always faced with the following question: does adaptivity really pay for the problem under consideration, i.e., does our favorite adaptive scheme really provide a substantial gain of efficiency compared to more conventional nonadaptive schemes which are usually much easier to implement? At least for the case of adaptive wavelet schemes, it is possible to give a quite rigorous answer. A reasonable comparison would be to compare the performance of wavelet algorithms with classical, nonadaptive schemes which consist of approximations by linear spaces that are generated by uniform grid refinements. It is well-known that, under natural assumptions, the approximation order that can be achieved by such a uniform method depends on the smoothness of the exact solution as measured in the classical L_2-Sobolev scale [9] (in what follows called *Sobolev regularity*). On the other hand, as already outlined above, for adaptive wavelet methods the best n-term approximation serves as the benchmark scheme. It is well-known that the convergence order that can be achieved by best n-term approximations also depends on the smoothness of the object we want to approximate, but now the smoothness has to be measured in specific Besov spaces, usually corresponding to L_τ-spaces with $0 < \tau < 2$. Therefore, we can make the following statement: the use of adaptive wavelet schemes is completely justified if the Besov smoothness of the unknown solution of our operator equation is higher compared to its regularity in the Sobolev scale.

At this point, the shape of the domain comes into play. As the classical model problem of elliptic operator equations, let us discuss the Poisson equation. If the domain Ω is smooth, e.g., C^∞, say, then the problem is completely regular, i.e., if the right-hand side is contained in $H^s(\Omega), s \geq -1$, the solution is contained in $H^{s+2}(\Omega)$ [1, 19], and there is no reason why the Besov smoothness should be higher. However, on a nonsmooth domain, the situation is completely different. In this case, singularities near the boundary occur which significantly diminish the Sobolev regularity [20], and consequently the order of convergence of uniform methods drops down. Fortunately, in recent studies it has been shown that these singularities do not influence the Besov smoothness too much [9, 12], so that for certain nonsmooth domains the use of adaptive schemes is completely justified. For the specific case of polygonal domains contained in $\mathbf{R}^2$, even more can be said. Then, the Besov smoothness only depends on the smoothness of the right-hand side, so that for arbitrary smooth right-hand sides, one gets arbitrary high oder of convergence, at least in principle [7]. The proof of this result relies on the fact that for polygonal domains the exakt solution can be decomposed into a regular and a singular part correponding to reentrant corners [18]. Having these results in mind, it is quite natural to try to generalize them to the very important case of polyhedral domains in $\mathbf{R}^3$, and this is exactly the task we are concerned with in this paper. In the polyhedral case, the solution can also be decomposed into a singular and a regular part [18], however the situation is much more complicated since edge singularities as well as vertex singularities occur which have to be treated separately. For edge singularities, a first positive result has been shown in [8]. Therefore, in this paper, we concentrate on vertex singularities.

For vertex singularities in 3D, the situation is much more unclear compared to the 2D-setting since the singularity functions are not given explicitly but depend in a somewhat complicated way on the shape of the domain in the vicinity of the vertex [18]. A quite promising way to handle this difficulty is the following: reduce the problem to the case of a smooth or a polyhedral cone, and treat the cone case by using *weighted* smoothness spaces [22, 27]. The weight takes into account the distance to the vertex or, more general, the distance to parts of the boundary of the cone. Although the problem is not regular in the classical Sobolev spaces, one has regularity in these weighted spaces in the following sense: if the right-hand side has smoothness $l-2$ in the weighted scale, then the solution has smoothness l in the same scale, see [22, 27] and Appendix A for details. In this paper, we show that this regularity of the solution in weighted Sobolev spaces is sufficient to establish Besov smoothness (in the original unweighted sense). Consequently, the use of adaptive wavelet schemes for problems in polyhedral domains is also justified.

In the context of adaptive approximation for elliptic problems, also the recent work of Nitsche [30] should be mentioned. In his pioneering studies, Nitsche is primary concerned with approximations of singularity functions by anisotropic tensor product refinements, whereas in this paper we focus on isotropic wavelet approximations.

This paper is organized as follows. In Section 2, we first of all discuss the case of a smooth cone. We show that the regularity results in weighted Sobolev spaces are indeed sufficient to establish Besov regularity. The proof is based on the fact that smoothness norms such as Besov norms are equivalent to weighted sequence norms of wavelet expansion coefficients, and we use the weighted regularity results to estimate wavelet coefficients. In Section 3, we study polyhedral cones. In this case, the situation is more difficult since one has to deal with weights that include the distance to the vertex as well as the distance to the edges, but nevertheless the wavelet coefficients can again be estimated and Besov smoothness can be established. Additional information is presented in the Appendices A and B. In Appendix A, we collect the relevant facts concerning regularity theory for elliptic PDEs as far as they are needed for our purposes. Finally, in Appendix B, we recall the definition of Besov spaces and introduce their characterizations by wavelet expansions.

2 A Regularity Result for a Smooth Cone

Let $\mathcal{K} \subset \mathbf{R}^3$ be an infinite cone with vertex at the origin, i.e.,

$$\mathcal{K} := \{x \in \mathbf{R}^3 \ : \quad x = \rho\,\omega,\ 0 < \rho < \infty,\ \omega \in \Omega\}, \tag{2.1}$$

where Ω is a domain on the unit sphere S^2 with smooth boundary $\partial\Omega$ and ρ and ω are the spherical coordinates of x. For integer $l \geq 0$ and real β we define the weighted Sobolev spaces $V^l_{2,\beta}(\mathcal{K})$ as the closure of $C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ with respect to the norm

$$\|u\|_{V^l_{2,\beta}(\mathcal{K})} := \left(\int_{\mathcal{K}} \sum_{|\alpha|\leq l} \rho^{2(\beta-l+|\alpha|)}\, |D^\alpha u(x)|^2\, dx\right)^{1/2} . \tag{2.2}$$

If $l \geq 1$, then $V^{l-1/2}_{2,\beta}(\partial\mathcal{K})$ denotes the space of traces of functions from $V^l_{2,\beta}(\mathcal{K})$ on the boundary equipped with the norm

$$\|\, u\,\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K})} := \inf\left\{ \|\, v\,\|_{V^l_{2,\beta}(\mathcal{K})} \ : \quad v \in V^l_{2,\beta}(\mathcal{K})\,,\ v_{|\partial\mathcal{K}} = u \right\}.$$

A more explicit description of these trace classes, using differences and derivatives, is given in [22, Lem. 6.1.2]. Let us consider the Poisson equation

$$\begin{aligned} -\triangle u &= f \qquad \text{in } \mathcal{K}, \\ u|_{\partial\mathcal{K}} &= g\,. \end{aligned} \tag{2.3}$$

By $\mathcal{K}_0$ we denote an arbitrary truncated cone, i.e. there exists a positive real number r_0 such that

$$\mathcal{K}_0 = \{x \in \mathcal{K} \ : \quad |x| < r_0\}\,. \tag{2.4}$$

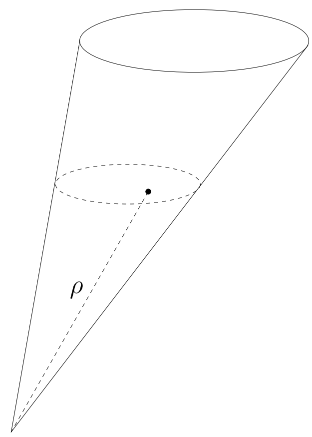

Figure 1: A smooth cone

Theorem 2.1 *Suppose that the right-hand side f is contained in $V^{l-2}_{2,\beta}(\mathcal{K}) \cap L_2(\mathcal{K}_0)$, where $l \geq 2$ is a natural number. Further we assume that $g \in V^{l-1/2}_{2,\beta}(\partial\mathcal{K})$. Let $\alpha_0 = \alpha_0(\mathcal{K})$ be the number defined in Remark 4.1. Then there exists a countable set E of complex numbers such that the following holds. If the real number β is chosen such that*

$$\Re e\,\lambda \neq -\beta + l - 3/2 \qquad \textit{for all} \quad \lambda \in E\,, \tag{2.5}$$

then the solution u of (2.3) satisfies

$$u \in B^s_\tau(L_\tau(\mathcal{K}_0)), \quad \frac{1}{\tau} = \frac{s}{3} + \frac{1}{2}, \quad s < \min\Big(l,\ \frac{3}{2}\alpha_0\Big)\,. \tag{2.6}$$

Remark 2.1 (i) Our set-up for the pde is taken from the monograph Kozlov, Maz'ya and Rossmann [22, Sect. 6.1]. It turns out that the exeptional set E coincides with the collection of the eigenvalues of the operator pencil associated to (2.3). In particular situations there are explicit formulas for E, we refer to [22, Lem. 6.6.3]. Furthermore, under the given restrictions there is an a priori estimate for u within the scale $V^l_{2,\beta}(\mathcal{K})$, see Theorem 6.1.1 in [22] or Proposition 4.3 in the Appendix A. However, for adaptive wavelet methods we need to know the regularity within unweighted Besov spaces $B^s_\tau(L_\tau(\mathcal{K}_0))$ with s as large as possible, compare with (ii). Curiously we can not use the regularity theory for (2.3) within unweighted Sobolev spaces, see e.g Dauge [14], for deriving the above regularity result. For us the investigations of (2.3) in weighted Sobolev spaces, seemingly started by Kondrat'ev [21] and continued by Maz'ya and Plamenevskij [25, 26], Koslov, Maz'ya and Rossmann [22] and Maz'ya and Rossmann [27], to mention at least a few, were most helpful.
(ii) *Best n-term approximation.* It is well-known that the order of convergence of best n-term wavelet approximation in $\mathbf{R}^3$ is determined by the regularity of the object one wants to approximate as measured in the specific Besov scale $B^s_\tau(L_\tau)$, $\frac{1}{\tau} = \frac{s}{3} + \frac{1}{2}$ introduced in (2.6), see again [9, 15] for details. As an immediate consequence of (2.6) we obtain, that for the solution u of (2.3) there exist subsets $\Gamma \subset \mathbf{Z}^3$ and $\Lambda \subset \{1, \ldots, 7\} \times \mathbf{N}_0 \times \mathbf{Z}^3$ such that $|\Gamma| + |\Lambda| \le n$ (here $|\Gamma|$, $|\Lambda|$ denotes the cardinality of the sets Γ and Λ, respectively) and

$$S_n u := \sum_{k\in\Gamma} \langle u, \widetilde{\varphi}_k \rangle \, \varphi_k + \sum_{(i,j,k)\in\Lambda} \langle u, \widetilde{\psi}_{i,j,k} \rangle \, \psi_{i,j,k} \tag{2.7}$$

satisfies

$$\| \, u - S_n u \, \|_{L_2(\mathcal{K}_0)} \lesssim \| \, u \, \|_{B^s_\tau(L_\tau(\mathcal{K}_0))} \, n^{-s/3} , \qquad s < \min\Big(l, \, \frac{3}{2}\alpha_0\Big) , \tag{2.8}$$

and s and τ are coupled as in (2.6). (We refer to Appendix B for the definition of $\varphi_k, \tilde{\varphi}_k, \psi_{i,j.k}$ and $\tilde{\psi}_{i,j,k}$. In this paper '$a \lesssim b$' always means that there exists a constant c such that $a \le cb$, independent of all context relevant parameters a and b may depend on). In contrary to this, the order of convergence of uniform methods is determined by the regularity in the L_2-Sobolev scale H^s. Therefore, since the critical Sobolev index α_0 is multiplied by 3/2, Theorem 2.1 implies that for l large enough the Besov smoothness is always higher compared to the Sobolev smoothness, so that the use of adaptive wavelet schemes is completely justified. In Figure 2 below we plotted the situation where $l \ge 3\alpha_0/2$ and $3\alpha_0/2 = 3(\frac{1}{\tau_0} - \frac{1}{2})$.
(iii) So far, we have discussed best n-term approximation in L_2. However, it is well–known that adaptive wavelet methods realize the order of best n-term approximation with respect to the energy norm, i.e., the H^1-norm would be more natural. Theorem 2.1 also implies a result in this direction. We refer to [9] and [13] where similar arguments have been used. Indeed, the following estimate for the best n-term approximation in the H^1-norm holds. For all $u \in B^s_{\tau_1}(L_{\tau_1})$, $\frac{1}{\tau_1} = \frac{(s-1)}{d} + \frac{1}{2}$ and all $n \in \mathbf{N}$ there exist subsets $\Gamma \subset \mathbf{Z}^3$ and $\Lambda \subset \{1, \ldots, 7\} \times \mathbf{N}_0 \times \mathbf{Z}^3$ such that $|\Gamma| + |\Lambda| \le n$ and $S_n u$ (defined as in (2.7)) satisfies

$$\| \, u - S_n(u) \, \|_{H^1(\mathcal{K}_0)} \lesssim \| \, u \, \|_{B^s_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))} \, n^{-(s-1)/3}, \qquad \frac{1}{\tau_1} = \frac{(s-1)}{3} + \frac{1}{2}. \tag{2.9}$$

We therefore have to estimate the Besov norm $B^s_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))$ of u. Let us for simplicity assume that $l \geq \frac{3}{2}\alpha_0$. We know that the solution is contained in the Sobolev space $H^\alpha(\mathcal{K}_0))$, $\alpha < \alpha_0$, as well as in the Besov space $B^{\bar{\alpha}}_{\tau_0}(L_{\tau_0}(\mathcal{K}_0)))$, $\frac{1}{\tau} = \frac{\bar{\alpha}}{3} + \frac{1}{2}$, $\bar{\alpha} < 3\alpha_0/2$. We continue by real interpolation

$$\Big(H^{s_0}(\mathcal{K}_0)), B^{s_1}_{\tau_0}(L_{\tau_0}(\mathcal{K}_0))\Big)_{\Theta,\tau_1} = B^s_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))$$

where $0 < \Theta < 1$,

$$\frac{1}{\tau_1} = \frac{1-\Theta}{2} + \frac{\Theta}{\tau_0} \qquad \text{and} \qquad s = (1-\Theta)\, s_0 + \Theta\, s_1\,,$$

see [38]. This shows that

$$u \in B^s_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))\,, \qquad s < \frac{3}{2}\alpha_0 - \frac{1}{2}, \quad \frac{1}{\tau_1} = \frac{(s-1)}{3} + \frac{1}{2}\,,$$

see Figure 2. There we plotted the situation where $l \geq 3\alpha_0/2$, $s_0 = \alpha_0$ and $s_1 = 3\alpha_0/2$.

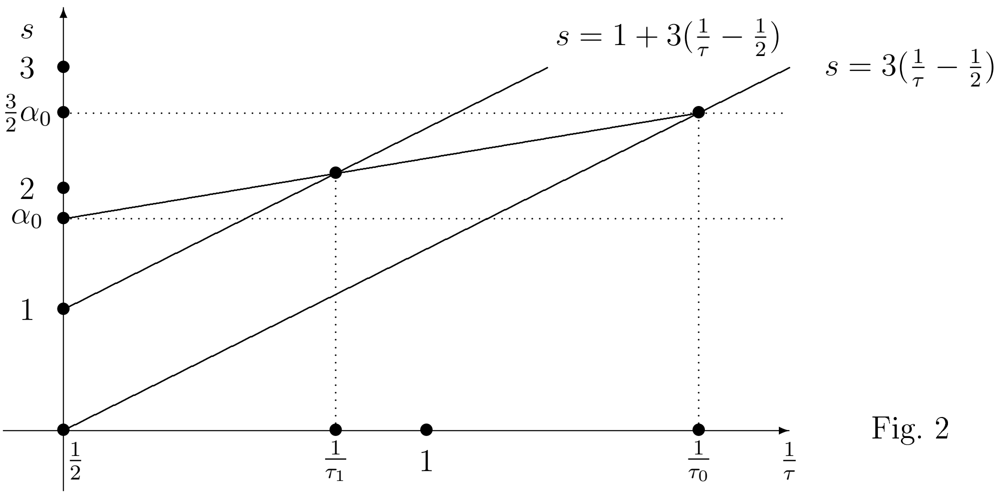

Fig. 2

(iv) Let $f \in C^\infty(\overline{\mathcal{K}})$ such that $\operatorname{supp} f \subset \mathcal{K}_0$. Then $f \in V^l_{2,\beta}(\mathcal{K})$ for all pairs (l, β) such that $\beta > l - 3/2$. Hence we can apply Theorem 2.1 with $s < 3\alpha_0/2$.

(v) At first sight, condition (2.5) looks restrictive. However, it is well-known that the set E consists of a countable number of isolated points, see again [22] for details. Therefore, by a minor modification of β, condition (2.5) is satisfied, and this minor modification does not change the arguments outlined below. This argument also shows that an explicite knowledge of E in our context is not necessary.

Proof of Theorem 2.1:

The proof is based on the characterizations of Besov spaces by wavelet expansions, see

Proposition 5.1. Therefore we estimate the wavelet coefficients of the solution u to (2.3) and show that they are contained in the weighted sequence spaces that are related to the scale $B^s_\tau(L_\tau(\mathcal{K}_0))$, $\quad \frac{1}{\tau} = \frac{s}{3} + \frac{1}{2}$.

Step 1. Preparations. First of all we make the following agreement concerning the wavelet characterization of Besov spaces on $\mathbf{R}^3$, see again Proposition 5.1: to each dyadic cube $I := 2^{-j}k + 2^{-j}[0,1]^3$ we associate the functions

$$\eta_I := \widetilde{\psi}_{i,j,k}\,, \qquad j \in \mathbf{N}\,, \quad k \in \mathbf{Z}^3, \quad i = 1, \ldots, 7\,,$$

by ignoring the dependence on i. In case $I = k + [0,1]^3$, i.e. $j = 0$, we shall use $\widetilde{\varphi}_k$ instead of $\widetilde{\psi}_{i,0,k}$, $k \in \mathbf{Z}^3$, $i = 1, \ldots, 7$. By η_I^* we denote the corresponding element of the dual basis. Since the wavelet basis is assumed to be compactly supported, there exists a cube Q, centered at the origin, such that $Q(I) := 2^{-j}k + 2^{-j}Q$ contains the support of η_I and of η_I^* for all I.

Step 2. Since $f \in L_2(\mathcal{K}_0)$ we a priori know $u \in H^s(\mathcal{K}_0)$ for some $s > 0$, see Proposition 4.2. We start by estimating the coefficients corresponding to interior wavelets, i.e., we estimate those coefficients $\langle u, \eta_I\rangle$, where $\operatorname{supp}\eta_I \subset \mathcal{K}_0$. Let ρ_I denote the distance of the cube $Q(I)$ to the vertex. We fix a refinement level j and introduce the sets

$$\begin{aligned} \Lambda_j &:= \{\, I \mid \operatorname{supp}\eta_I \subset \mathcal{K}_0\,,\ 2^{-3j} \le |I| \le 2^{-3j+2}\},\\ \Lambda_{j,k} &:= \{\, I \in \Lambda_j \mid k2^{-j} \le \rho_I < (k+1)2^{-j}\}\,, \quad j \in \mathbf{N}_0\,, \quad k \in \mathbf{N}_0\,. \end{aligned}$$

In this first step we deal with $k \ge 1$ only. Further we put

$$|u|_{W^l(L_2(Q(I)))} := \Big(\int_{Q(I)} |\nabla^l u(x)|^2\, dx \Big)^{1/2}.$$

Let P_I denote the polynomial of order at most l such that

$$\|u - P_I\|_{L_2(Q(I))} = \inf \Big\{ \|u - P\|_{L_2(Q(I))} :\ P \text{ is a polynomial of degree } \ \le l \Big\}.$$

Employing the vanishing moment properties of wavelets, see Appendix 5.3., and a classical Whitney–estimate yields

$$\begin{aligned} |\langle u, \eta_I\rangle| &\le \|u - P_I\|_{L_2}\|\eta_I\|_{L_2} \\ &\lesssim |I|^{l/3}|u|_{W^l(L_2(Q(I)))} \\ &\lesssim 2^{-lj}|u|_{W^l(L_2(Q(I)))}\,, \end{aligned} \tag{2.10}$$

if $I \in \Lambda_j$. Let $0 < \tau < 2$. Summing up over $I \in \Lambda_{j,k}$ we find

$$\begin{aligned} \sum_{I \in \Lambda_{j,k}} |\langle u, \eta_I\rangle|^\tau &\lesssim \sum_{I \in \Lambda_{j,k}} 2^{-lj\tau} \Big(\int_{Q(I)} |\nabla^l u|^2 dx \Big)^{\tau/2} \\ &\lesssim \sum_{I \in \Lambda_{j,k}} 2^{-lj\tau}\rho_I^{-\beta\tau} \Big(\int_{Q(I)} |\rho^\beta |\nabla^l u||^2 dx \Big)^{\tau/2} \\ &\lesssim (k\,2^{-j})^{-\beta\tau} \sum_{I \in \Lambda_{j,k}} 2^{-lj\tau} \Big(\int_{Q(I)} |\rho^\beta |\nabla^l u||^2 dx \Big)^{\tau/2}. \end{aligned}$$

The next step consists in a use of Hölder's inequality with $p = 2/(2-\tau)$ and $q = 2/\tau$. This yields

$$\sum_{I\in\Lambda_{j,k}} |\langle u,\eta_I\rangle|^\tau \lesssim (k\,2^{-j})^{-\beta\tau}\left(\sum_{I\in\Lambda_{j,k}} 2^{-\frac{2lj\tau}{2-\tau}}\right)^{\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_{j,k}}\int_{Q(I)} |\rho^\beta|\nabla^l u||^2 dx\right)^{\frac{\tau}{2}} .$$

Observe that for the cardinality $|\Lambda_{j,k}|$ of $\Lambda_{j,k}$ we have

$$|\Lambda_{j,k}| \lesssim k^2 , \qquad k \in \mathbf{N} ,$$

where the constant is independent of j but depending on the shape of the domain Ω. Therefore we further obtain

$$\begin{aligned}\sum_{I\in\Lambda_{j,k}} |\langle u,\eta_I\rangle|^\tau &\lesssim (k\,2^{-j})^{-\beta\tau}\left(k^2\,2^{-\frac{2lj\tau}{2-\tau}}\right)^{\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_{j,k}}\int_{Q(I)} |\rho^\beta|\nabla^l u||^2 dx\right)^{\frac{\tau}{2}} \\ &\lesssim k^{2-\tau-\beta\tau}2^{(\beta-l)j\tau}\left(\sum_{I\in\Lambda_{j,k}}\int_{Q(I)} |\rho^\beta|\nabla^l u||^2 dx\right)^{\frac{\tau}{2}} .\end{aligned}$$

Now we have to sum over the set Λ_j. Since we are restricting to a truncated cone there is a general number C such that

$$I \cap \mathcal{K}_0 = \emptyset \qquad \text{if} \quad I \in \Lambda_{j,k} , \quad k > C\,2^j . \tag{2.11}$$

Using Hölder's inequality once again and invoking (4.25) yields

$$\begin{aligned}\sum_{k=1}^{C\,2^j}\sum_{I\in\Lambda_{j,k}} |\langle u,\eta_I\rangle|^\tau &\lesssim \left(\sum_{k=1}^{C2^j} k^{(2-\tau-\beta\tau)\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}} 2^{-j(l-\beta)\tau}\left(\sum_{I\in\Lambda_j}\int_{Q(I)} |\rho^\beta|\nabla^l u||^2 dx\right)^{\frac{\tau}{2}} \\ &\lesssim 2^{-j(l-\beta)\tau}\,\|\,u\,\|^\tau_{V^l_{2,\beta}(\mathcal{K})}\begin{cases} 2^{j(3-\frac{3}{2}\tau-\beta\tau)} & \text{if} \quad 3\left(\frac{1}{\tau}-\frac{1}{2}\right) > \beta , \\ (1+j)^{\frac{2-\tau}{2}} & \text{if} \quad 3\left(\frac{1}{\tau}-\frac{1}{2}\right) = \beta , \\ 1 & \text{if} \quad 3\left(\frac{1}{\tau}-\frac{1}{2}\right) < \beta , \end{cases} \\ &\lesssim \|\,f\,\|^\tau_{V^{l-2}_{2,\beta}(\mathcal{K})}\begin{cases} 2^{j(3-\frac{3}{2}\tau-l\tau)} & \text{if} \quad 3\left(\frac{1}{\tau}-\frac{1}{2}\right) > \beta , \\ (1+j)^{\frac{2-\tau}{2}}\,2^{-j(l-\beta)\tau} & \text{if} \quad 3\left(\frac{1}{\tau}-\frac{1}{2}\right) = \beta , \\ 2^{-j(l-\beta)\tau} & \text{if} \quad 3\left(\frac{1}{\tau}-\frac{1}{2}\right) < \beta . \end{cases}\end{aligned}$$

This implies that the function

$$u^* := \sum_{j=0}^{\infty}\sum_{k=1}^{C2^j}\sum_{I\in\Lambda_{j,k}} \langle u,\eta_I\rangle\,\eta^*_I \tag{2.12}$$

belongs to

$$\begin{cases} B^l_\infty(L_\tau(\mathbf{R}^3)) & \text{if } \; 3\Big(\frac{1}{\tau}-\frac12\Big) > \beta\,, \\ B_\tau^{l-\beta-\delta-3(\frac12-\frac1\tau)}(L_\tau(\mathbf{R}^3)) & \text{if } \; 3\Big(\frac{1}{\tau}-\frac12\Big) = \beta\,, \quad \delta>0\,, \\ B_\infty^{l-\beta-3(\frac12-\frac1\tau)}(L_\tau(\mathbf{R}^3)) & \text{if } \; 3\Big(\frac{1}{\tau}-\frac12\Big) < \beta\,. \end{cases} \tag{2.13}$$

Now we consider the cases $\beta < l$ and $\beta \ge l$ separetely. For the first case we choose s (τ respectively) such that $\beta < s < l$ and s sufficiently close to l. Then, because of $s = 3\Big(\frac{1}{\tau}-\frac12\Big)$ we may use the first line in (2.13) and the continuous embedding $B^l_\infty(L_\tau(\mathbf{R}^3)) \hookrightarrow B^s_\tau(L_\tau(\mathbf{R}^3))$. In the second case we choose $s < \beta$ sufficiently close to β and argue by using the third line in (2.13). With

$$\beta - s = \beta - 3\Big(\frac{1}{\tau}-\frac12\Big) = \varepsilon > 0\,, \qquad \varepsilon < l - s\,,$$

we obtain $B^{l-\varepsilon}_\infty(L_\tau(\mathbf{R}^3)) \hookrightarrow B^s_\tau(L_\tau(\mathbf{R}^3))$ as in the first case.

Step 3. Estimate of the boundary layer. We recall the argument from Theorem 3.2 in [12]. The set $\Lambda_{j,0}$ can be empty (depending on the cone and on the wavelet system). If, then nothing is to do. If not, then we argue as follows. From the Lipschitz character of $\mathcal{K}_0$ it follows

$$|\Lambda_{j,0}| \;\lesssim\; 2^{2j}\,, \qquad j \in \mathbf{N}_0\,.$$

Let $0 < p < 2$. Using Hölder's inequality we find

$$\sum_{I\in\Lambda_{j,0}} |\langle u,\eta_I\rangle|^p \;\lesssim\; 2^{j2(1-p/2)} \left(\sum_{I\in\Lambda_{j,0}} |\langle u,\eta_I\rangle|^2\right)^{\frac p2}\,.$$

Summing up over $j \in \mathbf{N}_0$ we finally obtain

$$\begin{aligned} \sum_{j=0}^\infty 2^{j(s+3(\frac12-\frac1p))p} \sum_{I\in\Lambda_{j,0}} |\langle u,\eta_I\rangle|^p \;&\lesssim\; \sum_{j=0}^\infty 2^{j(s+3(\frac12-\frac1p))p}\, 2^{j(\frac2p-1)p} \left(\sum_{I\in\Lambda_{j,0}} |\langle u,\eta_I\rangle|^2\right)^{\frac p2} \\ &\lesssim\; \|\, u\,\|^p_{B_p^{s+\frac12-\frac1p}(L_2(\mathbf{R}^3))} \\ &\lesssim\; \|\, u\,\|^p_{B_2^{s+\frac12-\frac1p}(L_2(\mathbf{R}^3))}\,, \end{aligned}$$

since $p < 2$, see Appendix B for the last step. Choosing s and p such that

$$s := \frac{3\,\alpha}{2} \qquad \text{and} \qquad \frac1p := \frac s3 + \frac12\,, \qquad \text{i.e.} \quad s = 3\Big(\frac1p - \frac12\Big)\,,$$

we get $\alpha = \frac2p - 1$ as well as $\alpha = s + \frac12 - \frac1p$. This means we have proved that

$$u^{**} := \sum_{j=0}^\infty \sum_{I\in\Lambda_{j,0}} \langle u,\eta_I\rangle\, \eta_I^* \tag{2.14}$$

belongs to $B_p^{3\alpha/2}(L_p(\mathbf{R}^3))$ for all $\alpha < \alpha_0$.
Step 4. Finally we need to deal with those wavelets for which the support intersects with the boundary of the truncated cone. We put

$$\Lambda_j^{\#} := \{ \ I \mid \operatorname{supp} \eta_I \cap \overline{\mathcal{K}}_0 \neq \emptyset \, , \ 2^{-3j} \leq |I| \leq 2^{-3j+2} \} \, , \qquad j \in \mathbf{N}_0 \, .$$

Furthermore, since $\mathcal{K}_0$ is a bounded Lipschitz domain there exists a linear and bounded extension operator

$$\mathcal{E} : \ H^\alpha(\mathcal{K}_0) \to H^\alpha(\mathbf{R}^3) \, ,$$

which is simultaneously a bounded operator belonging to $\mathcal{L}(B_q^s(L_p(\mathcal{K}_0)), B_q^s(L_p(\mathbf{R}^3)))$ for all s, p, and q, cf. e.g. [33]. Defining

$$u^{\#} := \sum_{j=0}^{\infty} \sum_{I \in \Lambda_j^{\#}} \langle \mathcal{E}u, \eta_I \rangle \, \eta_I^* \tag{2.15}$$

we can argue as in Step 3, since

$$|\Lambda_j^{\#}| \ \lesssim \ 2^{2j} \, , \qquad j \in \mathbf{N}_0 \, .$$

This implies

$$\| \, u^{\#} \, \|_{B_p^{3\alpha/2}(L_p(\mathbf{R}^3))} \ \lesssim \ \| \, \mathcal{E} u \, \|^p_{B_2^\alpha(L_2(\mathbf{R}^3))} \ \lesssim \ \| \, u \, \|^p_{H^\alpha(\mathcal{K}_0))},$$

see Appendix B for the last step. Adding up the finiteley many functions of type u^*, u^{**}, and $u^{\#}$, see Step 1, we end up with a function which belongs to $B_\tau^s(L_\tau(\mathbf{R}^3))$ (where s satisfies the restrictions in (2.6)) and which coincides with u on $\mathcal{K}_0$. Hence $u \in B_\tau^s(L_\tau(\mathcal{K}_0))$.
□

Remark 2.2 Observe, that the estimates of the parts u^{**}, see (2.14), and $u^{\#}$, see (2.15), only depend on the Lipschitz character of the cone $\mathcal{K}_0$ and on the number α_0 associated via Proposition 4.1 to the cone $\mathcal{K}$.

3 Besov Regularity for the Neumann Problem

Let

$$\mathcal{K} = \{ x \in \mathbf{R}^3 \ : \ x = \rho \, \omega \, , \quad 0 < \rho < \infty \, , \, \omega \in \Omega \} \tag{3.16}$$

be a polyhedral cone with faces $\Gamma_j = \{ x \ : \ x/|x| \in \gamma_j \}$ and edges $M_j, j = 1, \ldots, n$. Here Ω is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, \ldots, \gamma_n$. The angle at the edge M_j will be denoted by θ_j. We consider the problem

$$-\triangle u = f \quad \text{in } \mathcal{K}, \qquad \frac{\partial u}{\partial n} = g_j \text{ on } \Gamma_j \, , \ j = 1, \ldots, n. \tag{3.17}$$

We denote by $\rho(x) = |x|$ the distance to the vertex of the cone and by $r_j(x)$ the distance to the edge M_j. Let $\beta \in \mathbf{R}$ and $\vec{\delta} = (\delta_1, \ldots, \delta_n) \in \mathbf{R}^n$ such that $\delta_j > -1$ for all j. We shall use the abbreviation $|\vec{\delta}| := \delta_1 + \ldots + \delta_n$ without assuming that the components δ_j of $\vec{\delta}$ are positive. Then the weighted Sobolev space $W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})$ is defined as the collection of all functions $u \in H^{l,\ell oc}(\mathcal{K})$ such that

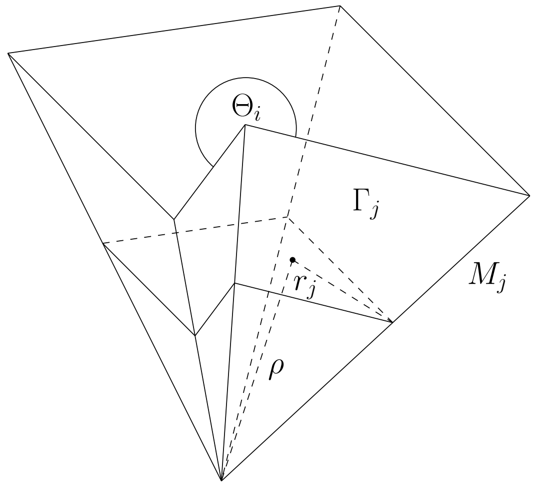

Figure 3: A polyhedral cone

$$\|u\|_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} := \left(\int_{\mathcal{K}} \sum_{|\alpha| \le l} \rho^{2(\beta - l + |\alpha|)} \left(\prod_{j=1}^{n} (\frac{r_j}{\rho})^{\delta_j} \right)^2 |D^\alpha u(x)|^2 \, dx \right)^{1/2} < \infty \, . \tag{3.18}$$

If $\vec{\delta} = \vec{0}$, then we are back in case of (2.2). If $l \ge 1$, then $W^{l-1/2,2}_{\beta,\vec{\delta}}(\Gamma_j)$ denotes the space of traces of functions from $W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})$ on the face Γ_j equipped with the norm

$$\| u \|_{W^{l-1/2,2}_{\beta,\vec{\delta}}(\Gamma_j)} := \inf \left\{ \| v \|_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} \; : \quad v \in W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K}) \, , \; v_{|\Gamma_j} = u \right\} .$$

As in the previous section $\mathcal{K}_0$ denotes an arbitrary truncated cone, see (2.4).

Theorem 3.1 *Suppose that the right-hand side* $f \in W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap L_2(\mathcal{K})$*, where* $l \ge 2$ *is a natural number. Further we assume that* $g_j \in W^{l-3/2,2}_{\beta,\vec{\delta}}(\Gamma_j)$*,* $j = 1, \ldots, n$*. Let* $\alpha_0 = \alpha_0(\mathcal{K})$ *be the number defined in Proposition 4.1. Then there exists a countable set* E *of complex numbers such that the following holds. If the real number* β *and the vector* $\vec{\delta}$ *are chosen such that* $\beta < l$*,*

$$\lambda \neq l - \beta - \frac{3}{2} \qquad \textit{for all} \quad \lambda \in E \, ,$$

and

$$\max(l - \frac{\pi}{\theta_j}, 0) < \delta_j + 1 < l, \quad j = 1, \ldots, n \, , \tag{3.19}$$

then the solution u *of (3.17) satisfies*

$$u \in B^s_\tau(L_\tau(\mathcal{K}_0)), \quad \frac{1}{\tau} = \frac{s}{3} + \frac{1}{2}, \quad s < \min\Big(l, \, \frac{3}{2}\alpha_0, \, 3\,(l - |\delta|)\Big) \, . \tag{3.20}$$

Remark 3.1 (i) In contrary to Section 2, here we have formulated the main result for the Neumann problem and not for the Dirichlet problem. The reason is that the analysis in this section heavily relies on the results in [27]. In that paper, the weighted Sobolov estimates are in particular tuned to the Neumann problem, compare with Appendix A, Proposition 4.3. However, by suitable modifications, also similar results for the Dirichlet problem can be shown [32].
(ii) Since again the critical Sobolev index α_0 is multiplied by $3/2$, it turns out that also for the Neumann problem the use of adaptive schemes is completely justified.
(iii) By using real interpolation arguments as outlined in Remark 2.1, again a result for best n-term approximation in H^1 can be derived.
(iv) We comment on the additional restriction $s < 3(l - |\vec{\delta}|)$ for s in (3.20) compared with (2.6). This restriction comes into play if $|\vec{\delta}| > 2l/3$. We will be forced to take such an vector $\vec{\delta}$ if there are some large angles θ_j, see (3.19). However, also the relation between l and the number of faces n plays a role. To see this we suppose $|\vec{\delta}| > 2l/3$ and choose all δ_j as small as possible in (3.19). Further, by $k \in \{1, \ldots, n\}$ we denote the number of angles θ_j such that $\theta_j > \pi/l$. Observe that $k = 0$ is impossible. Without loss of generality we assume $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$. As a consequence we obtain

$$\frac{2}{3}\, l \leq -n + \sum_{j=1}^{k} \left(l - \frac{\pi}{\theta_j}\right) \qquad \text{(see (3.19))}\,.$$

This implies

$$l \geq \frac{n + \pi \sum_{j=1}^{k} \frac{1}{\theta_j}}{k - 2/3}\,.$$

Using the trivial inequality $\theta_j < 2\pi$ we conclude

$$l > \frac{2n + k}{2(k - 2/3)}\,. \tag{3.21}$$

This inequality allows different interpretations. For example, if there is only one large angle (i.e. $k = 1$), then (3.21) implies $l > 10$ (since n must be at least 3). However, on this way the geometry of the polyhedral cone enters once again but we do not know whether this is caused by our method.
(v) Both, the Poisson equation with Dirichlet boundary conditions (2.3) as well as the Poisson equation with Neumann boundary conditions (3.17) are to understand as model cases. Since we did not use any specific property besides the existence, uniqueness and regularity of the solution both, Theorem 2.1 and Theorem 3.1, extend to much more general classes of elliptic differential equations, see Theorem 6.1.1 in [22] and [27] for details.

Proof of Theorem 3.1:
The proof is organized as the proof of Theorem 2.1. We shall use the same agreements concerning the wavelets as in the proof of this theorem. Using Remark 2.2 it will be

sufficient to concentrate on the estimate of the interior wavelets.
Let ρ_I denote the distance of the cube $Q(I)$ to the vertex and let

$$r_I := \min_{j=1,\dots,n} \min_{x\in Q(I)} r_j(x)\,.$$

Similar as above we will work with the following decomposition of the set of interior wavelets:

$$\begin{aligned} \Lambda_j &:= \{\, I \mid \operatorname{supp}\eta_I \subset \mathcal{K}_0\,,\ 2^{-3j} \le |I| \le 2^{-3j+2}\}, \\ \Lambda_{j,k} &:= \{\, I \in \Lambda_j \mid k2^{-j} \le \rho_I < (k+1)2^{-j}\}\,, \quad j \in \mathbf{N}_0\,, \quad k \in \mathbf{N}\,, \\ \Lambda_{j,k,m} &:= \{\, I \in \Lambda_{j,k} \mid 2^{-j}m \le r_I < 2^{-j}(m+1)\}\,, \quad m \in \mathbf{N}\,. \end{aligned}$$

Elementary arguments yield

$$|\Lambda_{j,k}| \ \lesssim\ k^2 \qquad \text{and} \qquad |\Lambda_{j,k,m}| \ \lesssim\ m \tag{3.22}$$

independent of j,k and m. Let $0 < \tau < 2$. Using the Whitney estimate (2.10) first we obtain

$$\begin{aligned} \sum_{I\in\Lambda_{j,k}} |\langle u,\eta_I\rangle|^\tau \ &\lesssim\ \sum_{I\in\Lambda_{j,k}} 2^{-lj\tau} \Big(\int_{Q(I)} |\nabla^l u|^2 dx\Big)^{\tau/2} \\ &\lesssim\ \sum_{I\in\Lambda_{j,k}} 2^{-lj\tau}\, r_I^{-\tau|\vec{\delta}|}\, \rho_I^{-\tau(\beta-|\delta|)} \Big(\int_{Q(I)} \rho^{2(\beta-|\vec{\delta}|)} \Big(\prod_{t=1}^n r_t^{\delta_j}\Big)^2 |\nabla^l u|^2 dx\Big)^{\tau/2}. \end{aligned}$$

We put

$$u_I := \int_{Q(I)} \rho^{2(\beta-|\vec{\delta}|)} \Big(\prod_{t=1}^n r_t^{\delta_j}\Big)^2 |\nabla^l u|^2 dx\,.$$

To continue our estimate we concentrate first on the set $\Lambda_{j,k,m}$. We use Hölder's inequality with $p = 2/\tau,\ q = 2/(2-\tau)$ and the fact that the layer $\Lambda_{j,k,m}$ contains of order m cubes, see (3.22). This yields

$$\begin{aligned} \sum_{I\in\Lambda_{j,k,m}} |\langle u,\eta_I\rangle|^\tau \ &\lesssim\ 2^{-l\tau j}(k2^{-j})^{-\tau(\beta-|\vec{\delta}|)} \Big(\sum_{I\in\Lambda_{j,k,m}} r_I^{-\tau|\vec{\delta}|\frac{2}{2-\tau}}\Big)^{\frac{2-\tau}{2}} \Big(\sum_{I\in\Lambda_{j,k,m}} u_I\Big)^{\tau/2} \\ &\lesssim\ 2^{-l\tau j}(k2^{-j})^{-\tau(\beta-|\vec{\delta}|)} \Big(\sum_{I\in\Lambda_{j,k,m}} (m2^{-j})^{-\tau|\vec{\delta}|\frac{2}{2-\tau}}\Big)^{\frac{2-\tau}{2}} \Big(\sum_{I\in\Lambda_{j,k,m}} u_I\Big)^{\tau/2} \\ &\lesssim\ 2^{\tau j(\beta-l)}\, k^{-\tau(\beta-|\vec{\delta}|)}\, m^{-\tau|\vec{\delta}|+\frac{2-\tau}{2}} \Big(\sum_{I\in\Lambda_{j,k,m}} u_I\Big)^{\tau/2}. \end{aligned}$$

The next step is to exploit the fact that there are of order k sets $\Lambda_{j,k,m}$ in each layer $\Lambda_{j,k}$ (the distance of a point in $\mathcal{K}_0$ to the edges can not be much larger than the distance to the vertex). Together with Hölder's inequality, this leads us to

$$\sum_{I\in\Lambda_{j,k}} |\langle u,\eta_I\rangle|^\tau \;\lesssim\; 2^{j\tau(\beta-l)}\, k^{-\tau(\beta-|\vec{\delta}|)} \left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}|\frac{2}{2-\tau}+1}\right)^{\frac{2-\tau}{2}} \Big(\sum_{I\in\Lambda_{j,k}} u_I\Big)^{\tau/2}, \tag{3.23}$$

where C is an appropriate constant depending on $\mathcal{K}_0$ only. Observe that

$$\left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}|\frac{2}{2-\tau}+1}\right)^{\frac{2-\tau}{2}} \;\lesssim\; \begin{cases} k^{-\tau|\vec{\delta}|+2-\tau} & \text{if} \quad 2>\tau(1+|\vec{\delta}|)\,,\\ (\log(1+k))^{\frac{2-\tau}{2}} & \text{if} \quad 2=\tau(1+|\vec{\delta}|)\,,\\ 1 & \text{if} \quad 2<\tau(1+|\vec{\delta}|)\,. \end{cases}$$

Inserting this into (3.23) we obtain

$$\begin{aligned} \sum_{I\in\Lambda_{j,k}} |\langle u,\eta_I\rangle|^\tau \quad &\lesssim \quad 2^{j\tau(\beta-l)}\Big(\sum_{I\in\Lambda_{j,k}} u_I\Big)^{\tau/2} \\ &\times \begin{cases} k^{-\tau(\beta+1)+2} & \text{if} \quad 2>\tau(1+|\vec{\delta}|)\,,\\ k^{-\tau(\beta-|\vec{\delta}|)}\,(\log(1+k))^{\frac{2-\tau}{2}} & \text{if} \quad 2=\tau(1+|\vec{\delta}|)\,,\\ k^{-\tau(\beta-|\vec{\delta}|)} & \text{if} \quad 2<\tau(1+|\vec{\delta}|)\,. \end{cases} \end{aligned}$$

To simplify notation we denote these functions of k in the second line by a_k. For each refinement level j, we have to take $C2^j$ layers $\Lambda_{j,k}$ into account, see (2.11). Therefore, by using Hölder's inequality for another time and Proposition 4.4, we finally get

$$\begin{aligned} \sum_{I\in\Lambda_j} |\langle u,\eta_I\rangle|^\tau \quad &\lesssim \quad 2^{j\tau(\beta-l)}\Big(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\Big)^{\frac{2-\tau}{2}}\Big(\sum_{I\in\Lambda_j} u_I\Big)^{\tau/2} \\ &\lesssim \quad 2^{j\tau(\beta-l)}\Big(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\Big)^{\frac{2-\tau}{2}} \|u\|^\tau_{V^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})}\,. \end{aligned}$$

To complete the estimate we have to sum with respect to $j\in\mathbf{N}_0$. Formally the discussion of this splits into nine cases. However, by using the abbreviation from (2.12) we end up with

$$\|\, u^*\,\|_{B^s_\tau(L_\tau(\mathbf{R}^3))} \;\lesssim\; \|u\|_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})}\,, \qquad \frac{1}{\tau}=\frac{s}{3}+\frac{1}{2}\,, \tag{3.24}$$

if one of the following conditions is satisfied:

$$\begin{aligned}
3\left(\frac{1}{\tau}-\frac{1}{2}\right) &< l && \text{if} \quad \tau\,(1+|\vec{\delta}|) < 2 \quad \text{and} \quad \beta < 3\left(\frac{1}{\tau}-\frac{1}{2}\right), \\
\beta &< l && \text{if} \quad \tau\,(1+|\vec{\delta}|) < 2 \quad \text{and} \quad \beta \geq 3\left(\frac{1}{\tau}-\frac{1}{2}\right), \\
\frac{3}{2}\,|\vec{\delta}| &< l && \text{if} \quad \tau\,(1+|\vec{\delta}|) = 2 \quad \text{and} \quad \beta < \frac{3}{2}\,|\vec{\delta}|, \\
\beta &< l && \text{if} \quad \tau\,(1+|\vec{\delta}|) = 2 \quad \text{and} \quad \beta \geq \frac{3}{2}\,|\vec{\delta}|, \\
\frac{1}{\tau}-\frac{1}{2} &< l-|\vec{\delta}| && \text{if} \quad \tau\,(1+|\vec{\delta}|) > 2 \quad \text{and} \quad \frac{1}{\tau}-\frac{1}{2} > \beta-|\vec{\delta}|, \\
\beta &< l && \text{if} \quad \tau\,(1+|\vec{\delta}|) > 2 \quad \text{and} \quad \frac{1}{\tau}-\frac{1}{2} \leq \beta-|\vec{\delta}|.
\end{aligned}$$

Observe that $\beta < l$ is necessary in all six cases. If $\beta < l$ and if $|\vec{\delta}| < 2l/3$, then, according to the first case, we can choose $\beta < s < l$, s arbitrary close to l. Now, let $|\vec{\delta}| \geq 2l/3$. We employ case five in our list of sufficient conditions above. Using $s = 3(\frac{1}{\tau} - \frac{1}{2})$ we can reformulate this as follows:

$$\beta - |\vec{\delta}| < s < 3\,(l - |\vec{\delta}|) \qquad \text{and} \qquad s < \frac{3}{2}\,|\vec{\delta}|\,.$$

Since $|\vec{\delta}| \geq 2l/3$ implies $3\,(l - |\vec{\delta}|) \leq 3\,|\vec{\delta}|/2$ we have found the third restriction for s in (3.20). But the second originates from the estimates of those terms connected with the boundary, see Remark 2.2. This proves the theorem. □

4 Appendix A – Regularity of Solutions of the Poisson Equation

First of all we recall a result of Grisvard, see [18, Cor. 2.6.7].

Proposition 4.1 *Let Ω be any bounded polyhedral open subset of $\mathbf{R}^3$. Then there exists a number $\alpha_0 > 3/2$ such that for every $f \in L_2(\Omega)$ the variational solution u of the Poisson equation (either with Dirichlet boundary conditions or with Neumann boundary conditions) belongs to $H^s(\Omega)$ for every $s < \alpha_0$.*

A second result which will be of certain use for us is taken from Jerison and Kenig [20, Thm. 0.5,Thm. 5.1].

Proposition 4.2 *Let Ω be any bounded Lipschitz domain in $\mathbf{R}^3$. Let $1/2 < \alpha < 3/2$. Suppose $f \in H^{\alpha-2}(\Omega)$ and $g \in H^{\alpha-1/2}(\partial\Omega)$. Then the Poisson problem (2.3) has a unique solution $u \in H^\alpha(\Omega)$.*

Remark 4.1 Summarizing, for bounded smooth and polyhedral cones there exists a number $\alpha_0 \geq 3/2$ such that for all

$$(f,g) \in L_2(\mathcal{K}) \times H^{\alpha-1/2}(\partial\mathcal{K})$$

the solution u of (2.3) belongs to $H^\alpha(\mathcal{K})$ as long as $\alpha < \alpha_0$.

Next we quote an a priori estimate from [22, Thm. 6.1.1]. It will be the basis of our treatment in Section 2.

Proposition 4.3 *Let $\mathcal{K}$ be a smooth cone as defined in (2.1). Suppose that the right-hand side f is contained in $V^{l-2}_{2,\beta}(\mathcal{K})$, where $l \geq 2$ is a natural number. Further we assume that $g \in V^{l-1/2}_{2,\beta}(\partial\mathcal{K})$. Then there exists a countable set E of complex numbers such that the following holds. If the real number β is chosen such that*

$$\Re e\, \lambda \neq -\beta + l - 3/2 \qquad \text{for all} \quad \lambda \in E\,,$$

then the solution u of (2.3) satisfies

$$\|u\|_{V^l_{2,\beta}(\mathcal{K})} \lesssim \Big(\|f\|_{V^{l-2}_{2,\beta}(\mathcal{K})} + \|g\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K})} \Big)\,. \tag{4.25}$$

Finally, the following result of Maz'ya and Rossmann [27] plays a fundamental role in Section 3.

Proposition 4.4 *Let $\mathcal{K}$ be a polyhedral cone as defined in (3.16). Suppose that the right-hand side $f \in W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap L_2(\mathcal{K})$, where $l \geq 2$ is a natural number. Further we assume that $g_j \in W^{l-3/2,2}_{\beta,\vec{\delta}}(\Gamma_j)$, $j = 1,\ldots,n$. Then there exists a countable set E of complex numbers such that the following holds. If the real number β and the vector $\vec{\delta}$ are chosen such that*

$$\lambda \neq l - \beta - \frac{3}{2} \qquad \text{for all} \quad \lambda \in E\,,$$

and

$$\max(l - \frac{\pi}{\theta_j}, 0) < \delta_j + 1 < l, \quad j = 1,\ldots,n\,,$$

then the solution u of (3.17) satisfies

$$\|u\|_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} \lesssim \Big(\|f\|_{W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K})} + \sum_{j=1}^n \|g_j\|_{W^{l-1/2,2}_{\beta,\vec{\delta}}(\Gamma_j)} \Big)\,. \tag{4.26}$$

5 Appendix B – Function Spaces

We take it for granted that the reader is familiar with Sobolev and Besov spaces on $\mathbf{R}^d$. There are different approaches to spaces defined on domains. We make a few remarks in this direction.

5.1 Besov Spaces on Domains

Let $\Omega \subset \mathbf{R}^d$ be a bounded open nonempty set. Then we define $B^s_q(L_p(\Omega))$ to be the collection of all distributions $f \in D'(\Omega)$ such that there exists a tempered distribution $g \in B^s_q(L_p(\mathbf{R}^d))$ satisfying

$$f(\varphi) = g(\varphi) \qquad \text{for all} \quad \varphi \in D(\Omega)\,,$$

i.e. $g|_\Omega = f$ in $D'(\Omega)$. We put

$$\| \, f \,| B^s_q(L_p(\Omega))\| := \inf \, \| \, g \,| B^s_q(L_p(\mathbf{R}^d))\| \,,$$

where the infimum is taken with respect to all distributions g as above.

5.2 Sobolev Spaces on Domains

Let Ω be a bounded Lipschitz domain. Let $m \in \mathbf{N}$. As usual $H^m(\Omega)$ denotes the collection of all functions f such that the distributional derivatives $D^\alpha f$ of order $|\alpha| \le m$ belong to $L_2(\Omega)$. The norm is defined as

$$\| \, f \,| H^m(\Omega)\| := \sum_{|\alpha| \le m} \| \, D^\alpha f \,| L_2(\Omega)\| \,.$$

It is well-known that $H^m(\mathbf{R}^d) = B^m_2(L_2(\mathbf{R}^d))$ in the sense of equivalent norms, cf. e.g. [36]. As a consequence of the existence of a bounded linear extension operator for Sobolev spaces on bounded Lipschitz domains, cf. [34, p. 181] or [33], it follows

$$H^m(\Omega) = B^m_2(L_2(\Omega)) \qquad \text{(equivalent norms)}\,,$$

for such domains. For fractional $s > 0$ we introduce the classes by complex interpolation. Let $0 < s < m$, $s \notin \mathbf{N}$. Then, following [24, 9.1], we define

$$H^s(\Omega) := \Big[H^m(\Omega), L_2(\Omega) \Big]_\Theta \,, \qquad \Theta = 1 - \frac{s}{m}\,.$$

This definition does not depend on m in the sense of equivalent norms, cf. [38]. The outcome $H^s(\Omega)$ coincides with $B^s_2(L_2(\Omega))$, cf. [38, 39] for further details.

5.3 Besov Spaces and Wavelets

Here we collect some properties of Besov spaces which have been used in the text before. For general information on Besov spaces we refer to the monographs [29, 31, 35, 36, 37, 39]. For the construction of biorthogonal wavelet bases as considered below we refer to the recent monograph of Cohen [3, Chapt. 2]. Let φ be a compactly supported scaling function of sufficiently high regularity and let ψ_i, $i = 1, \ldots 2^d - 1$ be corresponding wavelets. More exactly, we suppose for some $N > 0$ and $r \in \mathbf{N}$

$$\begin{aligned}
&\operatorname{supp} \varphi\,,\ \operatorname{supp} \psi_i \quad \subset \quad [-N, N]^d\,, \qquad i = 1, \ldots, 2^d - 1\,,\\
&\varphi, \psi_i \in C^r(\mathbf{R}^d)\,, \qquad i = 1, \ldots, 2^d - 1\,,\\
&\int x^\alpha\, \psi_i(x)\, dx = 0 \qquad \text{for all} \quad |\alpha| \le r\,, \qquad i = 1, \ldots, 2^d - 1\,,
\end{aligned}$$

and

$$\varphi(x-k),\ 2^{jd/2}\, \psi_i(2^j x - k)\,, \qquad j \in \mathbf{N}_0\,, \quad k \in \mathbf{Z}^d\,, \quad i = 1, \ldots, 2^d - 1\,,$$

is a Riesz basis in $L_2(\mathbf{R}^d)$. We shall use the standard abbreviations

$$\psi_{i,j,k}(x) = 2^{jd/2}\, \psi_i(2^j x - k) \qquad \text{and} \qquad \varphi_k(x) = \varphi(x-k)\,.$$

Further, the dual Riesz basis should fulfill the same requirements, i.e., there exist functions $\widetilde{\varphi}$ and $\widetilde{\psi}_i$, $i = 1, \ldots, 2^d - 1$, such that

$$\begin{aligned}
\langle \widetilde{\varphi}_k, \psi_{i,j,k} \rangle &= \langle \widetilde{\psi}_{i,j,k}, \varphi_k \rangle = 0\,,\\
\langle \widetilde{\varphi}_k, \varphi_\ell \rangle &= \delta_{k,\ell} \qquad \text{(Kronecker symbol)}\,,\\
\langle \widetilde{\psi}_{i,j,k}, \psi_{u,v,\ell} \rangle &= \delta_{i,u}\, \delta_{j,v}\, \delta_{k,\ell}\,,\\
\operatorname{supp} \widetilde{\varphi}\,, \qquad & \operatorname{supp} \widetilde{\psi}_i \quad \subset \quad [-N, N]^d\,, \qquad i = 1, \ldots, 2^d - 1\,,\\
\widetilde{\varphi}, \widetilde{\psi}_i &\in C^r(\mathbf{R}^d)\,, \qquad i = 1, \ldots, 2^d - 1\,,\\
\int x^\alpha\, \widetilde{\psi}_i(x)\, dx &= 0 \qquad \text{for all} \quad |\alpha| \le r\,, \qquad i = 1, \ldots, 2^d - 1\,.
\end{aligned}$$

For $f \in S'(\mathbf{R}^d)$ we put

$$\langle f, \psi_{i,j,k} \rangle = f(\overline{\psi_{i,j,k}}) \qquad \text{and} \qquad \langle f, \varphi_k \rangle = f(\overline{\varphi_k})\,, \tag{5.27}$$

whenever this makes sense.

Proposition 5.1 *Let $s \in \mathbf{R}$ and $0 < p, q \le \infty$. Suppose*

$$r > \max\Big(s, d\, \max(0, \frac{1}{p} - 1) - s\Big)\,. \tag{5.28}$$

Then $B^s_q(L_p(\mathbf{R}^d))$ is the collection of all tempered distributions f such that f is representable as

$$f = \sum_{k \in \mathbf{Z}^d} a_k\, \varphi_k + \sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}^d} a_{i,j,k}\, \psi_{i,j,k} \qquad (\textit{convergence in} \quad S')$$

with

$$\|\, f\, |B^s_q(L_p(\mathbf{R}^d))\|^* := \Big(\sum_{k \in \mathbf{Z}^d} |a_k|^p\Big)^{1/p} + \Big(\sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \Big(\sum_{k \in \mathbf{Z}^d} |a_{i,j,k}|^p\Big)^{q/p}\Big)^{1/q} < \infty\,,$$

if $q < \infty$ *and*

$$\| f \,|B^s_\infty(L_p(\mathbf{R}^d))\|^* := \Big(\sum_{k\in\mathbf{Z}^d} |a_k|^p \Big)^{1/p} + \sup_{i=1,\ldots,2^d-1} \sup_{j=0,\ldots} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))} \Big(\sum_{k\in\mathbf{Z}^d} |a_{i,j,k}|^p \Big)^{1/p} < \infty \,.$$

The representation is unique and

$$a_{i,j,k} = \langle f, \widetilde{\psi}_{i,j,k}\rangle \qquad \text{and} \qquad a_k = \langle f, \widetilde{\varphi}_k\rangle$$

hold. Further $J : f \mapsto \{\langle f, \widetilde{\varphi}_k\rangle, \langle f, \widetilde{\psi}_{i,j,k}\rangle\}$ *is an isomorphic map of* $B^s_q(L_p(\mathbf{R}^d))$ *onto the sequence space (equipped with the quasi-norm* $\| \cdot |B^s_q(L_p(\mathbf{R}^d))\|^*$*), i.e.* $\| \cdot |B^s_q(L_p(\mathbf{R}^d))\|^*$ *may serve as an equivalent quasi-norm on* $B^s_q(L_p(\mathbf{R}^d))$.

A proof of Proposition 5.1 has been given in [40], see also [23] for a homogeneous version. A different proof, but restricted to $s > d(\frac{1}{p} - 1)_+$, is given in [3, Thm. 3.7.7]. However, there are many forerunners with some restrictions on s, p and q.

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